Hadamard’s variational formulas for some functionals defined in the class of convex bodies

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*Shape optimization problems and spectral theory*

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Convex bodies
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\[ F : \mathcal{C} \rightarrow \mathbb{R} . \]
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\[ F : C \rightarrow \mathbb{R}. \]

**Problem:** compute

\[
\lim_{\epsilon \to 0^+} \frac{F(K + \epsilon L) - F(K)}{\epsilon}.
\]
The volume functional

Let $F = V(n)$ dimensional Lebesgue measure.)

Easy case: assume that $L = B$ is the unit ball of $\mathbb{R}^n$:

$$B = \{ x \in \mathbb{R}^n : |x| \leq 1 \}.$$  

Then $K + \epsilon B = \{ x \in \mathbb{R}^n : \text{dist}(x, K) \leq \epsilon \}$, i.e. is a parallel set of $K$. In this case $$\lim_{\epsilon \to 0} V(K + \epsilon B) - V(K) \epsilon = \text{perimeter of } A.$$  

In the general case the limit can be computed in terms of $K$ and $L$, once two notions have been introduced:

▶ the support function;
▶ the area measure.
The volume functional

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In the general case the limit can be computed in terms of \( K \) and \( L \), once two notions have been introduced:

- the support function;
- the area measure.
The support function of a convex body

\[ h_K(y) := \sup \{ (x, y) : x \in K \}, \quad \forall y \in \mathbb{R}^n. \]

\[ h_K(y) = (y, N^{-1}K(y)) \text{ for } y \in S^{n-1}, \]

where \( N_K \) is the outer unit normal to \( K \).

a) \( h_K \) is 1–homogeneous;

b) \( h_K \) is convex in \( \mathbb{R}^n \).

Vice versa: if \( h : \mathbb{R}^n \to \mathbb{R} \) verifies a) and b), then \( \exists! K \in \mathbb{C} \) such that \( h = h_K \).

\{ convex bodies \} \rightleftharpoons \{ support functions \},

and this mapping is linear:

\[ h_{\alpha K + \beta L} = \alpha h_K + \beta h_L \forall K, L, \alpha, \beta. \]
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and this mapping is linear:

$$h_{\alpha K + \beta L} = \alpha h_K + \beta h_L \quad \forall K, L, \alpha, \beta.$$
The area measure of a convex body

Let $K$ be a convex body. For $\omega \subset S^{n-1}$ set:

$$N^{-1}K(\omega) = \{x \in \partial K : N_K(x) \text{ exists and } \in \omega\}$$

$N_K$ is the outer unit normal to $K$.

Then $\sigma_K(\omega) := \mu^{n-1}(N^{-1}K(\omega))$.

$\sigma_K$ is the push–forward of $\mu^{n-1}|_{\partial K}$ through $N_K$. 
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$\sigma_K$ is the push–forward of $\mathcal{H}^{n-1}_{\partial K}$ through $N_K$. 
Two special cases

▶ If $K = P$ is a convex polyhedron, then

$$\sigma_P = \sum_{i=1}^{m} \alpha_i \delta_{N_i},$$

where $N_1, \ldots, N_m$ are outer normals to the facets of $P$, and $\alpha_1, \ldots, \alpha_m$ are areas of the facets.

▶ If $K \in C^2_2$ (i.e. $\partial K \in C^2_2$ and Gauss curvature $> 0$), then

$$d\sigma_K(y) = G(N - K(y)) dH^{n-1}(y),$$

where $G$ is Gauss curvature.
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$\alpha_1, \ldots, \alpha_m = \text{areas of the facets}.$

▶ If $K \in C^2_+$ (i.e. $\partial K \in C^2$ and Gauss curvature $> 0$), then

$$d\sigma_K(y) = \frac{1}{G(N_K^{-1}(y))} d\mathcal{H}^{n-1}(y).$$

where $G = \text{Gauss curvature}.$
The variational formula for the volume

\[
\lim_{\epsilon \to 0} \epsilon \left( V(K + \epsilon L) - V(K) \right) = \int_{S} h \left( L(y) \right) \, d\sigma_{K}(y).
\]

**Remark.**

\[\text{r.h.s.} = \left( h \left( L \right), d\sigma_{K} \right) L_{2} \left( S, n-1 \right).\]

As \( h \left( L \right) \) is linear in \( L \), (1) suggests that \( \sigma_{K} \) is the first variation of \( V \) at \( K \).
The variational formula for the volume

For every $K, L \in \mathcal{C}$

$$\lim_{\epsilon \to 0^+} \frac{V(K + \epsilon L) - V(K)}{\epsilon} = \int_{S^{n-1}} h_L(y) \, d\sigma_K(y).$$  \hspace{1cm} (1)

Remark. r.h.s. = \left( h_L, d\sigma_K \right) \mathcal{L}^2 (S^{n-1}).

as $h_L$ is linear in $L$, (1) suggests that $\sigma_K$ is the first variation of $V$ at $K$. 

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For every $K, L \in \mathcal{C}$

$$\lim_{\epsilon \to 0^+} \frac{V(K + \epsilon L) - V(K)}{\epsilon} = \int_{\mathbb{S}^{n-1}} h_L(y) \, d\sigma_K(y). \quad (1)$$

**Remark.**

r.h.s. $= (h_L, d\sigma_K)_{L^2(\mathbb{S}^{n-1})}$. 
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For every \( K, L \in \mathcal{C} \)

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- **Remark.**

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as \( h_L \) is linear in \( L \), (1) suggests that \( \sigma_K \) is the first variation of \( V \) at \( K \).
A representation formula for the volume

\[ \int_{\sigma} S_{n-1} h_K d\sigma_K = \lim_{\epsilon \to 0} + V(K + \epsilon) - V(K) \epsilon = \lim_{\epsilon \to 0} + V((1 + \epsilon)K) - V(K) \epsilon = V(K) \lim_{\epsilon \to 0} + (1 + \epsilon)^{n-1} \epsilon = n V(K). \]

Hence \( V(K) = \frac{1}{n} \int_{\sigma} S_{n-1} h_K d\sigma_K \).

Note: this formula admits an elementary independent proof and it can be used to show the variational formula for \( V \) (see the next page).
A representation formula for the volume

Chose $L = K$:

$$\int_{\mathbb{S}^{n-1}} h_K \, d\sigma_K = \lim_{\epsilon \to 0^+} \frac{V(K + \epsilon K) - V(K)}{\epsilon}$$

Hence $V(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K \, d\sigma_K$.

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\[ d\sigma_K(t) = (h_K(t) + h''_K(t))dt , \quad (S^1 \text{ is identified with } [0, 2\pi]). \]
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\]
More examples

- Geometric functionals, other than the volume, like the perimeter, the intrinsic volumes (roughly speaking: integrals of elementary symmetric functions of the principal curvatures), etc.

- Functionals from the Calculus of Variations
  - First eigenvalue of $-\Delta$ with Dirichlet boundary conditions.
  - Torsional rigidity.
  - Capacity.
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- Torsional rigidity.
- Capacity.
The $p$–capacity of a convex body

Let $K \in \mathcal{C}$, $p \in (1, \infty)$. 

$$C^p(K) = \inf \left\{ \int_{\mathbb{R}^n} |\nabla v|^p \, dx : v \in C^1_c(\mathbb{R}^n), v \geq \chi_K \right\}.$$ 

Equivalently

$$C^p(K) = \int_{\mathbb{R}^n \setminus K} |\nabla u|^p \, dx,$$

where

$$\begin{cases} \Delta_p u = 0 \text{ in } \mathbb{R}^n \setminus K, \\ u = 1 \text{ on } \partial K, \\ \lim_{|x| \to \infty} u(x) = 0. \end{cases}$$

$u$ is called the $p$–equilibrium potential of $K$. 
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Let $K \in \mathcal{C}$, $p \in (1, n)$.

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The variational formula for $C_p$

For $K, L \in C$ and $p \in (1, n)$
\[
\lim_{\epsilon \to 0} C_p(K + \epsilon L) - C_p(K) \epsilon = \int_{\partial K} \left| \nabla u(N - 1)K(y) \right|^p d\sigma(K(y)).
\]

\[\text{▶ The case } p = 2 \text{ was proved by Jerison in a paper of 1996. In this paper Jerison started a systematic study of Hadamard type formulas for classical functionals in the Calculus of Variations, modeled on the one for the volume, and corresponding Minkowski type problems.}\\
\[\text{▶ The general case } p \in (1, n) \text{ was proved in collaboration with: Lutwak, Nyström, Salani, Xiao, Yang, Zhang.}\]
The variational formula for $C_p$

**Thm.** For $K, L \in \mathcal{C}$ and $p \in (1, n)$

$$
\lim_{\epsilon \to 0^+} \frac{C_p(K + \epsilon L) - C_p(K)}{\epsilon} = \int_{S^{n-1}} h_L(y) |\nabla u(N_K^{-1}(y))|^p \, d\sigma_K(y).
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The variational formula for $C_p$

**Thm.** For $K, L \in \mathbb{C}$ and $p \in (1, n)$

$$\lim_{\epsilon \to 0^+} \frac{C_p(K + \epsilon L) - C_p(K)}{\epsilon} = \int_{S^{n-1}} h_L(y) |\nabla u(N_{K}^{-1}(y))|^p \, d\sigma_K(y).$$

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\[ \text{r.h.s.} = (h_L, d\mu_p^p)_{L^2(\mathbb{S}^{n-1})}, \]

i.e. \( \mu_p^p \) represents the first variation of \( p \)-capacity at \( K \).
A representation formula

For $L = K$, using the $(n-p)$–homogeneity of $C^p$, we get

$$C^p(K) = 1 \frac{1}{n-p} \int_{S^{n-1}} h_K(y) \, d\mu_p K(y) \left( y = N_K(x) \right) = 1 \frac{1}{n-p} \int_{\partial K(x, N_K(x))} |\nabla u(x)|^p \, dH^{n-1}(x).$$

This formula admits an independent (but not so elementary) proof, based on the divergence theorem, when $K \in C^2 +$. 


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Scheme of the proof of the variational formula for $C_p$

Prove the variational formula when $K$ and $L$ are of class $C^2$, using the representation formula.

Prove the general case by approximation (each convex body can be approximated by a sequence of $C^2$ convex bodies).

This step is particularly delicate: we used recent results by Lewis and Nyström on boundary behavior of $p$–harmonic functions in Lipschitz domains as crucial tools.

The general scheme of the proof is the same as the one used by Jerison for $p = 2$; but its implementation in the non–linear setting required many additional efforts.
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Classical Minkowski problem. Find a convex body with prescribed area measure.

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- **Discrete Minkowski problem.** Find a convex polyhedron with prescribed outer normals to the facets, and prescribed areas of the facets.

- **Smooth Minkowski problem.** Find a smooth convex body with prescribed Gauss curvature as a function of the outer unit normal.
Solution of the classical Minkowski problem

Thm.

Let $\sigma$ be a Borel measure on $S^2_{n-1}$ such that

(A) the support of $\sigma$ is not contained in any great sub-sphere of $S^2_{n-1}$;

(B) $\int_{S^2_{n-1}} y \, d\sigma(y) = 0$.

Then there exists $K \in C^1$ such that $\sigma_K = \sigma$. $K$ is unique up to translation. Moreover

$\begin{align*}
\text{If } \sigma \text{ is purely atomic then } K \text{ is a polyhedron;}
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\text{if } \sigma \text{ has a positive and smooth density, then } \partial K \text{ is smooth.}
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Note: conditions (A) and (B) are necessary as well.
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The Minkowski problem for capacity

Let $p \in (1, n)$ and let $\mu$ be a Borel measure on $S^{n-1}$, verifying conditions (A) and (B). Find a convex body $K$ such that $\mu_K = \mu$.

For $p = 2$ this problem has been completely solved (existence, uniqueness, regularity) by Jerison (1996), also in collaboration with Caffarelli and Lieb (uniqueness issue).

Analogous problems have been considered when the capacity is replaced by the first eigenvalue of $-\Delta$, with Dirichlet boundary conditions (Jerison 1997, C. 2005), and by the torsional rigidity (C. & Fimiani 2009).
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The Minkowski problem for capacity, with $1 < p < 2$
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**Thm.** (CLNSXYZ, 2012) Let $1 < p < 2$, and let $\mu$ be a non–atomic measure on $\mathbb{S}^{n-1}$, verifying conditions (A) and (B). Then there exists a convex body $K$ such that $\mu^K_p = \mu$. $K$ is unique up to translations, and if $\mu$ has a positive and smooth density, $K$ has a correspondingly smooth boundary.