On entire solutions of the Hessian equation $S_k(D^2u) = 1$

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Abstract

We consider a solution u of the second order partial differential equation: $S_k(D^2u(x)) = 1$ in $\mathbf{R}^n$, $n \geq 2$; here $k \in \{1, \dots, n\}$ and $S_k(D^2(x))$ is the k–th elementary simmetric function of the eigenvalues of $D^2u(x)$. We prove that if u is convex and satisfies a certain uniform growth condition at infinity, then it is a second order polynomial.

1 Introduction

In the present paper we study the solutions of the following equation:

$$S_k(D^2u(x)) = 1\,, \quad \text{in} \quad \mathbf{R}^n\,. \tag{1}$$

Here k is an integer between 1 and n, $n \geq 2$, and $S_k(D^2u)$ is the k-th *Hessian operator* applied to u, i.e. it is the symmetric elementary function of order k of the eigenvalues of D^2u. We prove that any convex solution of the above equation, which satisfies a certain growth condition at infinity, is a second order polynomial.

The motivations of our research can be found in the following facts. Let $u \in C^2(\mathbf{R}^n)$ be a solution of the equation:

$$\Delta u = 1\,, \quad \text{in} \quad \mathbf{R}^n\,.$$

If u is *convex*, then it must be a second order polynomial. Indeed, the second derivatives u_{ii}, $i = 1, \dots, n$, are nonnegative (by the convexity of u), harmonic functions in the whole space; hence they are constant by the Liouville theorem.

Next consider the following Monge-Ampère equation:

$$\det(D^2u) = 1\,, \quad \text{in} \quad \mathbf{R}^n\,. \tag{2}$$

Theorem 1 (Calabi–Jörgens–Pogorelov) *Let $u \in C^2(\mathbf{R}^n)$ be a convex solution of (2); then u is a second order polynomial.*

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The proof of this result is not trivial; it was given by Jörgens [6] for $n = 2$, then by Calabi [2] for $n \leq 5$ and eventually by Pogorelov [7] for arbitrary n (see also [5] for recents developments). Note that in dimension two, Theorem 1 provides an elegant proof of Bernstein's theorem on minimal surfaces.

The Laplace and Monge–Ampère operators are the most relevant examples of Hessian operators, indeed $\Delta u = S_1(D^2u)$ and $\det(D^2u) = F_n(D^2u)$. The above results lead to wonder whether a solution of (1) must be a second order polynomial. Here we prove that the answer is affirmative for solutions having a *uniform radial growth* at infinity. More precisely, let $h(t) \in C([0, +\infty))$ be a real–valued, positive, strictly increasing function, verifying

$$\lim_{t\to+\infty} h(t) = +\infty\,,$$

and such that, for an arbitrary $\alpha \geq 1$, the ratio:

$$\frac{h^{-1}(\alpha s)}{h^{-1}(s)} \tag{3}$$

is uniformly bounded from above for $s \in [0, +\infty)$. Examples of such functions are $h(t) = t^p$, $p > 0$, and $h(t) = e^t$.

Theorem 2 *Let $k \in \{1, \dots n\}$ and $u \in C^2(\mathbf{R}^n)$ be a convex solution of equation (1). Assume that there exists a function $h(x)$, verifying the above assumptions, such that*

$$Ah(|x|) \leq u(x) \leq Bh(|x|) + C\,, \quad x \in \mathbf{R}^n \tag{4}$$

where A, B and C are constants, A, $B > 0$. Then u is a second order polynomial.

Hessian equations have been extensively studied in the last years. A great impetus was given by the paper by Caffarelli, Nirenberg and Spruck [1] and by various papers by Trudinger and Wang (see for instance [9]). Recently, Chou and Wang [3] established some a priori estimates for solutions of such equations, which extend analogous results obtained by Pogorelov [7] in its proof of Theorem 1. The estimates of Chou and Wang are among the main ingredients of the proof of Theorem 2.

2 Proof of Theorem 2

For every $x \in \mathbf{R}^n$ and $R > 0$, we set $B(x, R) = \{x : |x| \leq R\}$. Let $F = F(D^2u)$ be a (fully nonlinear) second order operator; for $i, j = 1 \dots n$, we set $F_{ij} = \frac{\partial F}{\partial u_{ij}}$. Let Ω be an open set in $\mathbf{R}^n$ and $u \in C^2(\Omega)$ be a solution of the equation:

$$F(D^2u) = C\,, \tag{5}$$

where C is a constant. Assume that the following conditions are fulfilled:

A) F is uniformly elliptic on u, which means that there exist two positive constants λ and Λ, $\lambda \leq \Lambda$, such that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} F_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2\,, \quad \forall\, \xi \in \mathbf{R}^n\,, \quad \forall x \in \Omega\,.$$

B) F is concave on u, i.e. F, as a real–valued mapping defined on matrices, is concave on the set $\{D^2u(x) \,:\, x \in \Omega\}$.

The following result can be readily deduced from the estimates proved in Section 17.4 of [4].

Lemma 3 *Under the above assumptions there exist two positive constants C and α, depending on n, λ and Λ, such that*

$$osc_{B(x,R)}D^2u \leq C \left(\frac{R}{R_0}\right)^{\alpha} osc_{B(x,R_0)}D^2u$$

for every $x \in \Omega$ and $0 \leq R \leq R_0$ with $B(x, R_0) \subset \Omega$.

Corollary 4 *Assume that u is a solution of equation (5) in $\Omega = \mathbf{R}^n$. If F is uniformly elliptic and concave on u in $\mathbf{R}^n$ and if D^2u is uniformly bounded in $\mathbf{R}^n$, then u is a second degree polynomial.*

The next result is a slight modification of assertion 1 in Section 6.4 of [7].

Lemma 5 *Let Ω be a bounded, open, convex set in $\mathbf{R}^n$, with boundary of class C^1, and take $x_0 \in \mathbf{R}^n$ and $R > 0$ such that $\Omega \subset B(x_0, R)$. Let $u \in C^2(\overline{\Omega})$ be a convex function, vanishing on $\partial\Omega$, such that*

$$\det(D^2u(x)) \leq C\,, \quad x \in \Omega\,,$$

for some nonnegative constant C; assume that $\min_\Omega u = -1$. Then there exists a constant C_1 depending on n, R and C, such that

$$|Du| \leq C_1$$

in the set

$$\Omega' = \left\{x \in \Omega \,:\, u \leq -\frac{1}{2}\right\}.$$

Proof. Let $x \in \partial\Omega'$ and $d = dist(x, \partial\Omega)$, clearly $d > 0$. By the convexity of u we have

$$|Du(x)| \leq \frac{-u(x)}{d} = \frac{1}{2d}\,. \tag{6}$$

Let w be the function defined in Ω, whose graph is the union of all the segments joining $(x, u(x))$ to the points of $\partial\Omega$; w is convex in Ω and verifies $u \leq w \leq 0$ in Ω, $u = w = 0$ on $\partial\Omega$. Consequently

$$\partial w(\Omega) \subset \partial u(\Omega)\,,$$

where ∂ denotes the subgradient map for convex functions. Thus

$$|\partial w(\Omega)| \leq |\partial u(\Omega)| = \int_\Omega \det(D^2u(x))dx \leq C|\Omega| = C'R^n\,. \tag{7}$$

Here $|\cdot|$ stands for the Lebesgue measure in $\mathbf{R}^n$ and $C' = C'(C, n)$.

Let us deduce some geometric features of the set $\partial w(\Omega)$; firstly, such set is convex and, as w has a minimum at $x \in \Omega$, it contains 0.

Secondly, let $y \in \partial\Omega$ be such that $|x-y| = d$; the tangent hyperplane to Ω at y is normal to $y - x$; this implies that $\frac{1}{2}(y-x) \in \partial w(\Omega)$. From this and the above fact, it follows that there exists a segment of length $\frac{1}{2d}$, with one end–point at the origin, entirely included in $\partial w(\Omega)$.

Furthermore, notice that:

$$\partial w(\Omega) = \frac{1}{2}\,\Omega^*_x = \left\{ z \,:\, z = \frac{v}{2} \text{ for some } v \in \Omega^*_x \right\} ,$$

where Ω^*_x denotes the *polar set of* Ω *with respect to* x (see [8], Section 1.6, for the notion of polar set). From standard property of polar sets we have:

$$\Omega \subset B(x_0, R) \quad \Rightarrow \quad \Omega^*_x \supset [B(x_0,R)]^*_x \,.$$

Finally, it is easy to check that the polar set of a ball of radius $r > 0$, with respect to any interior point, contains a ball of radius $\frac{1}{2r}$. Summarizing, $\partial w(\Omega)$ contains the convex hull of a point whose distance from the origin is $\frac{1}{2d}$ and a ball centered at the origin of radius $\frac{1}{4r}$. Consequently:

$$|\partial w(\Omega)| \geq C'' \frac{1}{2d} \left(\frac{1}{4R} \right)^{n-1} ,$$

where $C'' = C''(n)$. From (6), (7) and the above inequality, we obtain

$$|Du(x)| \leq C_1 \,, \quad x \in \partial\Omega' \,,$$

where C_1 depends on R, C and n. The assertion of the lemma follows taking into account that, as u is convex, $\max_{\Omega'} |Du|$ is attained on $\partial\Omega'$.

Proof of Theorem 2. Without loss of generality, we may assume that $u(0) = 0$ and $Du(0) = 0$ so that $u \geq 0$ in the whole space. Since $\lim_{|x|\to+\infty} u(x) = +\infty$, each sublevel set of u:

$$\Omega_t := \{x \in \mathbf{R}^n \,:\, u(x) \leq t\} \,, \quad t \geq 0 \,,$$

is a compact convex subset of $\mathbf{R}^n$, whose boundary is a smooth, closed, convex hypersurface.

Assumptions (3) and (4) implies the following fact: there exists a constant $\gamma \geq 1$, depending only on the function h, such that for every $t > 0$ we have

$$B(0,R) \subset \Omega_t \subset B(0, \gamma R) \,, \tag{8}$$

for some $R > 0$, depending on t.

We fix $t > 0$ and set $\Omega = \Omega_t$ and $w(x) = u(x) - t$, $x \in \Omega$. Let $R > 0$ be such that (8) is fulfilled for our particular t. Consider the function:

$$v_1(x) = c(n,k)(|x|^2 - R^2) \,,$$

where the costant $c(n,k)$ is chosen so that:

$$S_k(D^2 v_1(x)) \equiv 1 \,.$$

Since $v_1 = 0$ on $\partial B(0,R)$, by the maximum principle $w \leq v_1$ in $B(0,R)$. Analogously, the function

$$v_2(x) = \gamma^2 v_1\left(\frac{x}{\gamma}\right),$$

verifies $v_2 \leq w$ in $B(0,\gamma R)$. We deduce that there exixts $C>0$ depending on n, k and γ, such that

$$\frac{1}{C} \leq \frac{\max_\Omega |w|}{R^2} \leq C\,. \tag{9}$$

Next we rescale Ω and w. Let us define

$$\Omega_0 = \left\{\frac{x}{R}\,:\, x\in\Omega\right\};$$

clearly $B(0,1) \subset \Omega_0 \subset B(0,\gamma)$. Let

$$U(x) = \frac{1}{\max_\Omega |w|} w(Rx)\,, \quad x \in \Omega_0\,.$$

U is a convex function vanishing on $\partial\Omega_0$; by (9) we have the following relation between the Hessian matrices of U and w

$$\frac{1}{C} D^2U(x) \leq D^2 w(Rx) \leq C D^2 U(x)\,, \quad x\in\Omega_0\,. \tag{10}$$

Equation (1) imply that

$$S_k(D^2U(x)) = \left(\frac{R^2}{\max_\Omega |w|}\right)^k, \quad x\in\Omega_0\,. \tag{11}$$

In particular, by virtue of (9),

$$\det(D^2U(x)) \leq [S_k(D^2u)]^{n/k} \leq A\,.$$

where A is a positive constant depending only on n, k and γ.

Hence U verifies the assumptions of Lemma 5; as a consequence we obtain that $|DU|$ is bounded by a constant C', depending on n, k and γ, in the set

$$\Omega_0' = \{x\in\Omega_0\,:\, U \leq -1/2\}\,.$$

Notice also that

$$max_{\Omega_0}|U| = 1\,.$$

Now we apply the a priori estimates proved by Chou and Wang ([3], Theorem 1.5) to deduce that in the set

$$\Omega_0'' = \{x\in\Omega_0\,:\, U\leq -2/3\}\,,$$

the second derivatives of U are bounded, in absolute value, by a constant C'' depending on n, k and γ. Going back to the function u, this implies that in the set $\Omega_{t/3}$ all the second derivatives of u are bounded, in absolute value, by C''; as t was arbitrary, this is true in $\mathbf{R}^n$:

$$|D^2u(x)| \leq C = C(n,k,\gamma)\,, \quad \forall x\in\mathbf{R}^n\,. \tag{12}$$

Estimate (12) implies that the operator

$$F(D^2u) = [S_k(D^2u)]^{1/k}$$

is uniformly elliptic on u in the whole space. Indeed, fix a point x and let λ_i, $i = 1, \dots n$, be the eigenvalues of $D^2u(x)$; without loss of generality we may assume that

$$u_{ij}(x) = \lambda_i \delta_{ij}\,, \quad i, j = 1, \dots n\,,$$

where δ_{ij} denotes the Kronecker delta. Hence

$$\frac{\partial F}{\partial u_{ij}} = \frac{1}{k}[S_k(D^2u)]^{\frac{1}{k}-1}\frac{\partial S_k}{\partial u_{ij}} = \frac{1}{k}\delta_{ij}S^i_{k-1}\,, \tag{13}$$

where, for arbitrary $i \in \{1, \dots, n\}$ and $l \in \{1, \dots, n-1\}$, S^i_l denotes the l–th elementary symmetric function of $\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots \lambda_n$. On the other hand, for every i (provided $k \neq 1, n$),

$$1 = S_k(D^2u(x)) = \lambda_i S^i_{k-1} + S^i_k \leq C S^i_{k-1} + [S^i_{k-1}]^{k/(k-1)} \leq C' S^i_{k-1}\,,$$

where C' depends only on n, k and γ; here we have used the fact that the λ_i's are nonnegative, as u is convex. The last inequality shows that, for every $i = 1, \dots, n$, S^i_{k-1} is uniformly bounded away from zero; this information, together with (13) and (12), proves that F is unformly elliptic in $\mathbf{R}^n$.

Finally, notice that F is concave on u, see for instance [1]. Hence the conclusion of the theorem follows from Corollary 4.

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