SYMPLECTIC SMALL DEFORMATIONS OF SPECIAL INSTANTON BUNDLE ON $\mathbb{P}^{2n+1}$

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ABSTRACT. Let $MI_{\text{Simp},\mathbb{P}^{2n+1}}(k)$ be the moduli space of stable symplectic instanton bundles on $\mathbb{P}^{2n+1}$ with second Chern class $c_2 = k$ (it is a closed subscheme of the moduli space $MI_{\mathbb{P}^{2n+1}}(k)$)
We prove that the dimension of its Zariski tangent space at a special (symplectic) instanton bundle is $2k(5n-1)+4n^2-10n+3, \quad k \geq 2$.

INTRODUCTION

Symplectic instanton bundles on $\mathbb{P}^{2n+1}$ are holomorphic bundles of rank 2n (see [**1**] ,[**4**] and [**6**]) that correspond to the self-dual solutions of Yang-Mills equations on $\mathbb{P}^n(\mathbb{H})$. They are given by some monads (see section 2 for precise definitions) and their only topological invariant is $c_2 = k$.
At present the dimension of their moduli space $MI_{\text{Simp},\mathbb{P}^{2n+1}}(k)$ is not known except in the cases n=1, where the dimension is $8k-3$ (see [**3**]), and in few other cases corresponding to small values of k.
$MI_{\text{Simp},\mathbb{P}^{2n+1}}(k)$ is a closed subscheme of $MI_{\mathbb{P}^{2n+1}}(k)$ and this last scheme parametrizes stable instanton bundles with structural group $GL(2n)$.
The class of special instanton bundles was introduced in [**8**].
Let $E \in MI_{\mathbb{P}^{2n+1}}(k)$ be a special symplectic instanton bundle. The tangent dimension $h^1(End(E))$ was computed in [**7**] and it is equal to $4(3n-1)k + (2n-5)(2n-1)$.
The Zariski tangent space of $MI_{\text{Simp},\mathbb{P}^{2n+1}}(k)$ at E is isomorphic to $H^1(S^2E)$ and in this paper we prove that

$$(1) \qquad h^2(S^2E) = \binom{k-2}{2} \cdot \binom{2n-1}{2} \qquad \forall\, k \geq 2$$

By the Hirzebruch-Riemann-Roch formula , since $h^0(S^2E) = 0$ and $h^i(S^2E) = 0 \ \forall\, i \geq 3$, it follows that:

$$\chi(S^2E) = h^2(S^2E) - h^1(S^2E) =$$
$$2n^2 + n + \tfrac{1}{2}\left[k^2 \binom{2n-1}{2} - k(10n^2-5n-1)\right]$$

and

Theorem 0.1. *Let E be a special symplectic instanton bundle.Then*

$$h^1(S^2E) = 2k(5n-1) + 4n^2 - 10n + 3 \qquad , k \geq 2$$

(for $n = 1$ it is well known that $h^1(S^2E) = 8k-3$ and, in the real case, for $n = 2$ the dimension $18k-1$ has been found in [**5**] by different techniques).

Now, since by the Kuranishi map $H^2(S^2E)$ is the space of obstructions to the smoothness at E of $MI_{\mathrm{Simp},\mathbb{P}^{2n+1}}(k)$, we obtain

Corollary 0.2. *$\forall k \geq 2$ the dimension of any irreducible component of $MI_{Simp,\mathbb{P}^{2n+1}}(k)$, containing a special symplectic instanton bundle is bounded by the value*
$2k(5n-1)+4n^2-10n+3$ *(linear in k)*

Corollary 0.3. *$\forall n$ $MI_{Simp,\mathbb{P}^{2n+1}}(3)$ is smooth at a special instanton bundle E,and the dimension of any irreducible component containing E is $4n^2+20n-3$.*

The main remark of this paper is that it is easier to compute $H^2(S^2E)$ and $H^2(\overset{2}{\wedge} E)$ together as SL(2)-modules (although this second cohomology space has a geometrical meaning only for orthogonal bundles) than to compute $H^2(S^2E)$ alone.

1. Preliminaries

Throughout this paper $\mathbb{K}$ denotes an algebraically closed field of characteristic zero. U denotes a 2-dimensional $\mathbb{K}$ vector space ($U =< s,t >$), $S_n = S^nU$ its n-th symmetric power ($S_n =< s^n, s^{n-1}t, \ldots, t^n >$) , $V_n = U \otimes S_n$ ($V_n =< s\otimes s^n, s\otimes s^{n-1}t, \ldots s\otimes t, \ldots t\otimes t^n >$) and $\mathbb{P}^{2n+1} = \mathbb{P}(V_n)$.

Definition 1.1. A vector bundle E on $\mathbb{P}^{2n+1}$ of rank $2n$ is called an **instanton bundle of quantum number** k if:

- E has Chern polinomial $c_t(E) = (1-t^2)^{-k}$;
- $E(q)$ has natural cohomology in the range $-(2n+1) \leq q \leq 0$, that is $H^i(E(q)) \neq 0$ for at most one $i = i(q)$.

By [**6**],[**2**],the Definition 1.1 is equivalent to :
i)E is the cohomology bundle of a monad:

$$0 \to O(-1)^k \to \Omega^1(1)^k \to O^{2n(k-1)} \to 0$$

or ii) E is the cohomology bundle of a monad:

$$0 \to O(-1)^k \xrightarrow{A} O^{2n+2k} \xrightarrow{B^t} O(1)^k \to 0$$

(where, after we have fixed a coordinate system, A and B can be identified with matrices in the space $Mat(k, 2n+2k, S_1)$)

Definition 1.2. An instanton bundle E is called **symplectic** if there is an isomorphism $\varphi : E \to E^\vee$ satisfying $\varphi = -\varphi^\vee$.

Definition 1.3. An instanton bundle is called **special** if it arises from a monad where the morfism B^t is defined in some system of homogeneous coordinates $x_0, \cdots x_n, y_0 \cdots y_n$ on $\mathbb{P}^{2n+1}$ by the trasposed of the matrix:

$$B = \begin{pmatrix} x_0 & \cdots & x_n & 0 & \cdots & 0 & y_0 & \cdots & y_n & 0 & \cdots & 0 \\ 0 & x_0 & \cdots & x_n & 0 & \cdots & 0 & y_0 & \cdots & y_n & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & x_0 & \cdots & x_n & 0 & \cdots & 0 & y_0 & \cdots & y_n & 0 \\ 0 & \cdots & 0 & x_0 & \cdots & x_n & 0 & \cdots & 0 & y_0 & \cdots & y_n \end{pmatrix}$$

The following lemma is well known (and easy to prove)

Lemma 1.4.

$$H^0(O(1)) \cong V^\vee$$

$$H^0(\Omega^1(2)) \cong \overset{2}{\wedge} V^\vee$$

$$H^i(\mathbb{P}^n, S^2\Omega^1(1)) = \begin{cases} 0 & se\ i \neq 1 \\ \overset{2}{\wedge} V^\vee & se\ i = 1 \end{cases}$$

2. Existence of a special symplectic instanton bundle

There is a natural exact sequence of GL(U)-equivariant maps for any $k, n \geq 1$ (Clebsch-Gordan sequence):

$$0 \to \overset{2}{\wedge} U \otimes S_{k-1} \otimes V_{n-1} \xrightarrow{\beta} S_k \otimes V_n \xrightarrow{\mu} V_{k+n} \to 0 \tag{2}$$

where μ is the multiplication map and β is defined by $(s \wedge t) \otimes f \otimes g \to (sf \otimes tg - tf \otimes sg)$

We can define (see [**7**]) the morphism

$$\tilde{b} : S^\vee_{k-1} \otimes \Omega^1(1) \to \overset{2}{\wedge} U^\vee \otimes S^\vee_{k-2} \otimes V^\vee_{n-1} \otimes O$$

and it is induced the complex

$$A \otimes O(-1) \xrightarrow{\tilde{a}} S^\vee_{k-1} \otimes \Omega^1(1) \xrightarrow{\tilde{b}} \overset{2}{\wedge} U^\vee \otimes S^\vee_{k-2} \otimes V^\vee_{n-1} \otimes O \tag{3}$$

where A is a k-dimensional subspace of $S^\vee_{2n+k-1} \otimes \overset{2}{\wedge} U^\vee$ such that (3) is a monad and the cohomology bundle E is a special symplectic instanton bundle. It was proved in [**7**] that

$$H^2(EndE) \cong Ker(\Phi^\vee)^\vee$$

where
$$\Phi^\vee : S^{\otimes 2}_{k-2} \otimes V^{\otimes 2}_{n-1} \to S^{\otimes 2}_{k-1} \otimes \overset{2}{\wedge} V_n$$
and there is an isomorphism of SL(2)-representations

$$\varepsilon : S^\vee_{k-3} \otimes S^\vee_{k-3} \otimes S^2 V^\vee_{n-2} \to Ker(\Phi^\vee)$$

3. How to identify $H^2(S^2E)$ and $H^2(\overset{2}{\wedge} E)$

Proposition 3.1. *Let E be special symplectic instanton bundle ,cohomology of monad (3) and $N = Ker\tilde{b}$.Then*

(i): $H^2(S^2E) \cong H^2(S^2N)$

(ii): $H^2(\overset{2}{\wedge} E) \cong H^2(\overset{2}{\wedge} N)$

Proof. We denote $B := S^\vee_{k-1}$ and $C := \overset{2}{\wedge} U^\vee \otimes S^\vee_{k-2} \otimes V^\vee_{n-1}$

The result follows from the two exact sequences given by monad (3) :

$$0 \to N \to B \otimes \Omega^1(1) \to C \otimes O \to 0 \tag{4}$$

$$0 \to A \otimes O(-1) \to N \to E \to 0 \tag{5}$$

In fact, by performing the second symetric and alternating power of sequence (4), we have

$$
\begin{array}{ccccccccccc}
0 \to S^2N \to & \tilde{A} & \to & B \otimes C \otimes \Omega^1(1) \to \overset{2}{\wedge} C \otimes O \to 0 \\
 & \searrow & & \nearrow \\
 & & M^1 & \\
 & \nearrow & & \searrow \\
 0 & & & & 0
\end{array}
$$

(6)

where $\tilde{A} := S^2(B \otimes \Omega^1(1)) = (S^2 B \otimes S^2(\Omega^1(1))) \oplus (\overset{2}{\wedge} B \otimes \Omega^2(2))$
and

$$
\begin{array}{ccccccccccc}
0 \to \overset{2}{\wedge} N \to & \overline{A} & \to & B \otimes C \otimes \Omega^1(1) \to O \otimes S^2C \to 0 \\
 & \searrow & & \nearrow \\
 & & M & \\
 & \nearrow & & \searrow \\
 0 & & & & 0
\end{array}
$$

(7)

where $\overline{A} := \overset{2}{\wedge} (B \otimes \Omega^1(1)) = (\overset{2}{\wedge} B \otimes S^2(\Omega^1(1))) \oplus (S^2 B \otimes \Omega^2(2))$

□

3.1. **Identifying $H^2(S^2N)$ and $H^2(\overset{2}{\wedge} N)$.** i) Diagram (6) gives the following two exact sequences:

(8)
$O \to H^0(M^1) \to H^1(S^2N) \to H^1(\tilde{A}) \to H^1(M^1) \to H^2(S^2(N)) \to H^2(\tilde{A}) \to \cdots$

$$O \to H^0(M^1) \to B \otimes C \otimes \underset{\overset{\|}{0}}{H^0(\Omega^1(1))} \to \overset{2}{\wedge} C \to H^1(M^1) \to B \otimes C \otimes \underset{\overset{\|}{0}}{H^1(\Omega^1(1))} \to \cdots$$

(9)
Sequence (9) implies:
$H^0(M^1) = 0 \quad$ and $\quad H^1(M^1) \cong \overset{2}{\wedge} C$
Then, by using the two formulas:
$H^1(\tilde{A}) = (S^2 B \otimes H^1(S^2 \Omega^1(1))) \oplus (\overset{2}{\wedge} B \otimes H^1(\Omega^2(2)) = S^2 B \otimes \overset{2}{\wedge} V^\vee$
and:
$H^2(\tilde{A}) = (S^2 B \otimes H^2(S^2 \Omega^1(1))) \oplus (\overset{2}{\wedge} B \otimes H^2(\Omega^2(2)) = 0$
sequence (8) becomes:

$$0 \to H^1(S^2N) \to H^1(\tilde{A}) \to H^1(M^1) \to H^2(S^2(N)) \to 0$$

i.e.

$$0 \to H^1(S^2N) \to S^2 B \otimes \overset{2}{\wedge} V^\vee \xrightarrow{\tilde{\Phi}} \overset{2}{\wedge} C \to H^2(S^2N) \to 0$$

$$\Longrightarrow \qquad H^2(S^2N) \cong \mathrm{Coker}(\tilde{\Phi}) = (\mathrm{Ker}(\tilde{\Phi}^\vee))^\vee$$

Then:

$$H^2(S^2N)^\vee = \text{Ker}\left[\overset{2}{\wedge}(S_{k-2}\otimes V_{n-1}) \xrightarrow{\tilde{\Phi}^\vee} S^2(S_{k-1})\otimes \overset{2}{\wedge} V_n\right]$$

ii)

Diagram (7) gives the following two exact sequences:

(10)

$$O \to H^0(M) \to H^1(\overset{2}{\wedge} N) \to H^1(\overline{A}) \to H^1(M) \to H^2(\overset{2}{\wedge} N) \to H^2(\overline{A}) \to \cdots$$

$$O \to H^0(M) \to \underset{\substack{\| \\ 0}}{B\otimes C\otimes H^0(\Omega^1(1))} \to \underset{\substack{\| \\ S^2C}}{S^2C\otimes H^0(O)} \to H^1(M) \to 0 \to \cdots$$

(11)

and, from sequence (11), we get

$H^0(M) = 0$ and $H^1(M) \simeq S^2C$

Then, since :

$H^1(\overline{A}) = (H^1(S^2(\Omega^1(1))\otimes \overset{2}{\wedge} B) \oplus (S^2B\otimes H^1(\Omega^2(2))) = \overset{2}{\wedge} B\otimes \overset{2}{\wedge} V^\vee$

and $H^2(\overline{A}) = 0$

sequence (10) becomes :

$$O \to \underset{\substack{\| \\ 0}}{H^0(M)} \to H^1(\overset{2}{\wedge} N) \to H^1(\overline{A}) \to H^1(M) \to H^2(\overset{2}{\wedge} N) \to 0$$

i.e. $\quad 0 \to H^1(\overset{2}{\wedge} N) \to \overset{2}{\wedge} B\otimes \overset{2}{\wedge} V^\vee \xrightarrow{\overline{\Phi}} S^2C \to H^2(\overset{2}{\wedge} N) \to 0$

$$\Longrightarrow \qquad H^2(\overset{2}{\wedge} N) \cong \text{Coker}(\overline{\Phi}) = (\text{Ker}(\overline{\Phi}^\vee))^\vee$$

Then we obtain :

$$(H^2(\overset{2}{\wedge} N))^\vee = \text{Ker}\left[S^2(S_{k-2}\otimes V_{n-1}) \xrightarrow{\overline{\Phi}^\vee} \overset{2}{\wedge} S_{k-1}\otimes \overset{2}{\wedge} V_n\right]$$

3.2. **Identifying $H^2(S^2E)$.** We have

$$H^2(S^2E)^\vee \cong \text{Ker }\tilde{\Phi}^\vee$$

where $\quad \tilde{\Phi}^\vee : \overset{2}{\wedge}(S_{k-2}\otimes V_{n-1}) \to S^2S_{k-1}\otimes \overset{2}{\wedge} V_n \quad$ is explicitly given by

$$\begin{aligned}\tilde{\Phi}^\vee((g\otimes v)\wedge(g^1\otimes v^1)) =& sg\cdot sg^1\otimes(tv\wedge tv^1) - sg\cdot tg^1\otimes(tv\wedge sv^1) + \\ & -tg\cdot sg^1\otimes(sv\wedge tv^1) + tg\cdot tg^1\otimes(sv\wedge sv^1)\end{aligned}$$

i.e. $\quad \tilde{\Phi}^\vee = \tilde{p}\circ(\overset{2}{\wedge}\beta)$

where $\quad \beta : \overset{2}{\wedge} U\otimes S_{k-2}\otimes V_{n-1} \to S_{k-1}\otimes V_n \quad$ is such that

$$(s\wedge t)\otimes(g\otimes v) \mapsto (sg\otimes tv) - (tg\otimes sv)$$

and

$$\begin{aligned}\tilde{p} : \overset{2}{\wedge}(S_{k-1}\otimes V_n) &\to S^2S_{k-1}\otimes \overset{2}{\wedge} V_n \\ \| \quad & \\ (\overset{2}{\wedge} S_{k-1}\otimes S^2V_n) \oplus (S^2S_{k-1}\otimes \overset{2}{\wedge} V_n) &\end{aligned}$$

is such that
$$(f\otimes u)\wedge(f'\otimes u^1)\mapsto f\cdot f'\otimes u\wedge u^1.$$
Now, we consider the $SL(2)$-equivariant morphism:
$$\tilde{\varepsilon}^1\ :\overset{2}{\wedge}(S_{k-3}\otimes V_{n-2})\to\overset{2}{\wedge}(S_{k-2}\otimes V_{n-1})$$
where, up to the order of factors, the map $\tilde{\varepsilon}^1:=\beta^1\wedge\beta^1$ and $\beta^1: S_{k-3}\otimes V_{n-2}\to S_{k-2}\otimes V_{n-1}$ is defined as β. Hence, $\tilde{\varepsilon}^1$ is injective.

Finally, we define
$$\tilde{\varepsilon}\ :\overset{2}{\wedge}S_{k-3}\otimes S^2V_{n-2}\to\overset{2}{\wedge}(S_{k-2}\otimes V_{n-1})$$
as $\tilde{\varepsilon}=\tilde{\varepsilon}^1\circ\tilde{i}$, where

$\tilde{i}\ :\overset{2}{\wedge}S_{k-3}\otimes S^2V_{n-2}\to\overset{2}{\wedge}(S_{k-3}\otimes V_{n-2})$ such that
$$f\wedge f'\otimes u\cdot u^1\longmapsto(f\otimes u)\wedge(f'\otimes u^1)+(f\otimes u^1)\wedge(f'\otimes u)$$

is an injective map. Then, also $\tilde{\varepsilon}$ is injective.

Lemma 3.2. *Im* $\tilde{\varepsilon}\subset$ *Ker* $\tilde{\Phi}^\vee$

Proof. Straightforward computation. □

3.3. **Identifying $H^2(\overset{2}{\wedge}E)$.** We have
$$H^2(\overset{2}{\wedge}E)^\vee\cong\text{Ker }\overline{\Phi}^\vee$$
where $\overline{\Phi}^\vee\ :\ S^2(S_{k-2}\otimes V_{n-1})\to\overset{2}{\wedge}S_{k-1}\otimes\overset{2}{\wedge}V_n$ is explicity given by
$$\begin{aligned}\overline{\Phi}^\vee((g\otimes v)\cdot(g^1\otimes v^1))=&sg\wedge sg^1\otimes(tv\wedge tv^1)-sg\wedge tg^1\otimes(tv\wedge sv^1)\\&-tg\wedge sg^1\otimes(sv\wedge tv^1)+(tg\wedge tg^1)\otimes(sv\wedge sv^1)\end{aligned}$$
i.e. $\overline{\Phi}^\vee=\overline{p}\circ(S^2\beta)$ where
$$\begin{aligned}\overline{p}\ :\ S^2(S_{k-1}\otimes V_n)&\to\overset{2}{\wedge}S_{k-1}\otimes\overset{2}{\wedge}V_n\\ \|&\\ (\overset{2}{\wedge}S_{k-1}\otimes\overset{2}{\wedge}V_n)&\oplus(S^2S_{k-1}\otimes S^2V_n)\end{aligned}$$
is such that
$$\overline{p}((f\otimes u)\cdot(f'\otimes u^1))=f\wedge f'\otimes u\wedge u^1$$
We consider the $SL(2)$-equivariant morphism:
$$\overline{\varepsilon}^1\ :\ S^2(S_{k-3}\otimes V_{n-2})\to S^2(S_{k-2}\otimes V_{n-1})$$
such that:
$$\begin{aligned}\overline{\varepsilon}^1((f\otimes u)\cdot(f'\otimes u^1))=&(sf\otimes tu)\cdot(sf'\otimes tu^1)-(sf\otimes su)\cdot(tf'\otimes tu^1)+\\&-(tf\otimes tu)\cdot(sf'\otimes su^1)+(sf\otimes tu)\cdot(sf'\otimes tu^1)\end{aligned}$$
($\overline{\varepsilon}^1=S^2\beta^1$ hence $\overline{\varepsilon}^1$ is injective). Finally, we define
$$\overline{\varepsilon}\ :\ S^2S_{k-3}\otimes S^2V_{n-2}\to S^2(S_{k-2}\otimes V_{n-1})$$
as $\overline{\varepsilon}=\overline{\varepsilon}^1\circ\overline{i}$ where

$\overline{i}\ :S^2S_{k-3}\otimes S^2V_{n-2}\to S^2(S_{k-3}\otimes V_{n-2})$ such that
$$f\cdot f'\otimes uu^1\mapsto(f\otimes u)(f'\otimes u^1)+(f\otimes u^1)(f'\otimes u)$$
is an injective map. Then, also $\overline{\varepsilon}$ is injective

Lemma 3.3. *Im* $\overline{\varepsilon} \subset$ *Ker* $\overline{\Phi}^{\vee}$

Proof. Straightforward computation. □

Theorem 3.4. *For any special symplectic instanton bundle E*

$$H^2(S^2E) \simeq \overset{2}{\wedge}(S_{k-3})^{\vee} \otimes S^2(V_{n-2})^{\vee}$$

Proof. By lemma 3.2 and 3.3 we have the following diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to & & H^2(\overset{2}{\wedge} N)^{\vee} & \to & S^2(S_{k-2} \otimes V_{n-1}) & \xrightarrow{\overline{\Phi}^{\vee}} & \overset{2}{\wedge} S_{k-1} \otimes \overset{2}{\wedge} V_n & \to & H^1(\overset{2}{\wedge} N)^{\vee} \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to & & H^2(N \otimes N)^{\vee} & \to & S_{k-2}^{\otimes 2} \otimes V_{n-1}^{\otimes 2} & \xrightarrow{\Phi^{\vee}} & S_{k-1}^{\otimes 2} \otimes \overset{2}{\wedge} V_n & \to & H^1(N \otimes N)^{\vee} \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to & & H^2(S^2 N)^{\vee} & \to & \overset{2}{\wedge}(S_{k-2} \otimes V_{n-1}) & \xrightarrow{\tilde{\Phi}^{\vee}} & S^2 S_{k-1} \otimes \overset{2}{\wedge} V_n & \to & H^1(S^2 N)^{\vee} \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0 & & 0
\end{array}$$

It was shown in [**7**] that:
$H^2(EndE) \simeq Ker\ \Phi^{\vee} = H^2(N \otimes N)^{\vee} \simeq S_{k-3}^{\otimes 2} \otimes S^2 V_{n-2}$
We have proved that there are two injective maps:

$$\tilde{\varepsilon} : \overset{2}{\wedge}(S_{k-3}) \otimes S^2 V_{n-2} \to \text{Ker}\ \tilde{\Phi}^{\vee} \simeq H^2(S^2N)^{\vee} \simeq H^2(S^2E)^{\vee}$$

$$\overline{\varepsilon} : S^2(S_{k-3}) \otimes S^2 V_{n-2} \to \text{Ker}\ \overline{\Phi}^{\vee} \simeq H^2(\overset{2}{\wedge} N)^{\vee} \simeq H^2(\overset{2}{\wedge} E)^{\vee}$$

Then, we can consider the following diagram:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & S^2 S_{k-3} \otimes S^2 V_{n-2} & \to & S_{k-3}^{\otimes 2} \otimes S^2 V_{n-2} & \to & \overset{2}{\wedge} S_{k-3} \otimes S^2 V_{n-2} & \to 0 \\
 & \downarrow \overline{\varepsilon} & & \downarrow \varepsilon & & \downarrow \tilde{\varepsilon} & \\
0 \to & H^2(\overset{2}{\wedge} E)^{\vee} & \to & H^2(EndE)^{\vee} & \to & H^2(S^2E)^{\vee} & \to 0 \\
 & & & \downarrow & & & \\
 & & & 0 & & &
\end{array}$$

and by the **Snake-Lemma** there is the exact sequence :

$$\begin{array}{ccccccccccccc}
0 \to & \text{Ker}\ \overline{\varepsilon} & \to & \text{Ker}\ \varepsilon & \to & \text{Ker}\ \tilde{\varepsilon} & \to & \text{Coker}\ \overline{\varepsilon} & \to & \text{Coker}\ \varepsilon & \to & \text{Coker}\ \tilde{\varepsilon} & \to 0 \\
 & \| & & \| & & \| & & & & \| & & & \\
 & 0 & & 0 & & 0 & & & & 0 & & &
\end{array}$$

$\Rightarrow$ Coker $\overline{\varepsilon} = 0 \Rightarrow \overline{\varepsilon}$ is an isomorphism $\Rightarrow \tilde{\varepsilon}$ is an isomorphism.
Thus:

$$H^2(S^2E)^{\vee} \cong \overset{2}{\wedge}(S_{k-3}) \otimes S^2(V_{n-2})$$

i.e. $H^2(S^2E) \simeq \overset{2}{\wedge}(S_{k-3})^{\vee} \otimes S^2(V_{n-2})^{\vee}$ as we wanted.

□

Remark 3.5. By this theorem formula 1 and theorem 0.1 are easily proved.

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