THE BINET-CAUCHY THEOREM FOR THE HYPERDETERMINANT OF BOUNDARY FORMAT MULTIDIMENSIONAL MATRICES

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Abstract. Let A, B be multidimensional matrices of boundary format respectively $\prod_{i=0}^{p}(k_i+1)$, $\prod_{j=0}^{q}(l_j+1)$. Assume that $k_p = l_0$ so that the convolution $A*B$ is defined. We prove that $Det(A*B) = Det(A)^{\alpha} \cdot Det(B)^{\beta}$ where $\alpha = \frac{l_0!}{l_1!\ldots l_q!}$, $\beta = \frac{(k_0+1)!}{k_1!\ldots k_{p-1}!(k_p+1)!}$ and Det is the hyperdeterminant. When A, B are square matrices this formula is the usual Binet-Cauchy Theorem computing the determinant of the product $A \cdot B$. It follows that $A*B$ is nondegenerate if and only if A and B are both nondegenerate. We show by a counterexample that the assumption of boundary format cannot be dropped.

1. Introduction

The Binet-Cauchy Theorem states that if A and B are square matrices then $det(A \cdot B) = det(A) \cdot det(B)$. The main result of this paper is a generalization of this theorem to multidimensional matrices A, B of boundary format (see definition 2.2), where the hyperdeterminant replaces the determinant (see the theorem (4.2) for the precise statement). The idea of the proof is quite simple, in fact we consider the hyperdeterminant of A as the determinant of a certain morphism ∂_A (see definition 3.2) as in [GKZ]. Then we compute ∂_{A*B} by means of ∂_A and ∂_B and we apply the usual Binet-Cauchy Theorem. The proof is better understood with the language of vector bundles in the setting of algebraic geometry, although we do not strictly need

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them. The study of multiplicative properties of hyperdeterminants was left as an open problem in [GKZ]. As a consequence (corollary 4.5), we prove that given two matrices A and B of boundary format then $A * B$ is nondegenerate if and only if A and B are both nondegenerate. We show by a counterexample (remark 4.6) that the assumption of boundary format cannot be dropped.

We remind how the definition of hyperdeterminant comes out. In chapter 14 of [GKZ] the hyperdeterminant is defined geometrically by considering the dual variety, that is by studying tangency conditions.

The nondegeneracy of a multidimensional matrix is algebraically equivalent to the absence of nontrivial solutions of a suitable system of equations containing some partial derivatives. With this approach the usual determinant of a square $n \times n$ matrix is realized as the equation of the dual variety to the Segre variety $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$.

A second well known approach is to define a square matrix to be nondegenerate if the associated linear system has only trivial solutions. In this paper we choose this second approach as the definition of nondegeneracy (2.1). The nondegenerate matrices fill up a codimension one subvariety exactly in the boundary format case. In this case the second approach is simpler and it allows us to compute the degree of the hyperdeterminant and to give an explicit formula for it directly from this definition of nondegeneracy. The above results were found in [GKZ] as consequences of a combinatorial statement (lemma 14.2.7) which needs a nontrivial proof about the irreducibility over $\mathbb{Z}$ of a certain polynomial (14.3.4, 14.3.5, 14.3.6 of [GKZ]). Following this approach, theorem 3.3 of [GKZ] comes quickly and the computation of the degree of the hyperdeterminant is a trivial consequence.

Our definition fits into invariant theory and does not depend on coordinates. The tools that we use are vector bundles over the product of projective spaces (as in [AO99] or [D]) and Künneth formula to compute their cohomology.

In the remark 3.6 we notice that an analog of the hyperdeterminant can be defined also in some cases where the variety of degenerate matrices has big codimension. This fact seems promising for other applications (see [CO]).

2. Notations and preliminaries

Let V_i for $i = 0, \dots, p$ be a complex vector space of dimension $k_i + 1$. We assume $k_0 = \max_i k_i$. It is not necessary to assume $k_0 \geq k_1 \geq \dots \geq k_p$ (see remark 3.5).

We remark that a multidimensional matrix $A \in V_0 \otimes \ldots \otimes V_p$ can be regarded as a map $V_0^\vee \to V_1 \otimes \cdots \otimes V_p$, hence taken the dual map $V_1^\vee \otimes \ldots \otimes V_p^\vee \to V_0$ (that we call also A), we can give the following definition:

2.1. Definition. A multidimensional matrix A is called *degenerate* if there are $v_i \in V_i^*$, $v_i \neq 0$ for $i = 1, \ldots, p$ such that $A(v_1 \otimes \ldots \otimes v_p) = 0$.

If $p = 1$ nondegenerate matrices are exactly the matrices of maximal rank.

If $k_0 \geq \sum_{i=1}^p k_i$ it is easy to check ([WZ] and also the proof of theorem (3.1)) that degenerate matrices fill an irreducible variety of codimension $k_0 - \sum_{i=1}^p k_i + 1$. If $k_0 < \sum_{i=1}^p k_i$ then all matrices in $V_0 \otimes \ldots \otimes V_p$ are degenerate.

2.2. Definition. If $k_0 = \sum_{i=1}^p k_i$ the matrices $A \in V_0 \otimes \ldots \otimes V_p$ are called *of boundary format.*

2.3. *Remark.* (see for instance [Hir]) For a vector space V of dimension n we denote $detV := \wedge^n V$. We recall that any linear map $\Phi \in Hom(V, W)$ between vector spaces of the same dimension induces the map $det\Phi \in Hom(detV, detW)$. If A and B are vector spaces of dimension a and b respectively, then there are canonical isomorphisms:

$$det(A \otimes B) \simeq (detA)^{\otimes b} \otimes (detB)^{\otimes a} \qquad det(S^k A) \simeq (detA)^{\otimes \binom{a+k-1}{a}}$$

$$\wedge^k A \simeq \wedge^{a-k} A^* \otimes (detA)$$

The above isomorphisms hold also if A and B are replaced by vector bundles over a variety X.

3. Hyperdeterminants

Let $A \in V_0 \otimes \ldots \otimes V_p$ be of boundary format and let $m_j = \sum_{i=1}^{j-1} k_i$ with the convention $m_1 = 0$.

We remark that the definition of m_i depends on the order we have chosen among the k_j's (see remark 3.5).

With the above notations the vector spaces $V_0^\vee \otimes S^{m_1} V_1 \otimes \ldots \otimes S^{m_p} V_p$ and $S^{m_1+1} V_1 \otimes \ldots \otimes S^{m_p+1} V_p$ have the same dimension $N = \frac{(k_0+1)!}{k_1! \ldots k_r!}$.

The following theorem is essentially equivalent to theorem 4.3 and lemma 4.4 of [GKZ1]. Since we want to make paper self-contained and since our proof of the irreducibility of the homogeneous polynomial Det does not need any combinatorial statement as in [GKZ] and [GKZ1], then we include the proof.

3.1. Theorem. (and definition of ∂_A). *Let $k_0 = \sum_{i=1}^{p} k_i$. Then the degenerate matrices fill an irreducible subvariety of degree $N = \frac{(k_0+1)!}{k_1!\ldots k_r!}$ whose equation is given by the determinant of the natural morphism*

$$\partial_A : V_0^\vee \otimes S^{m_1}V_1 \otimes \ldots \otimes S^{m_p}V_p \longrightarrow S^{m_1+1}V_1 \otimes \ldots \otimes S^{m_p+1}V_p$$

Proof. If A is degenerate then we get $A(v_1 \otimes \ldots \otimes v_p) = 0$ for some $v_i \in V_i^*$, $v_i \neq 0$ for $i = 1, \ldots, p$. Then $(\partial_A)^t \left(v_1^{\otimes m_1+1} \otimes \ldots \otimes v_p^{\otimes m_p+1}\right) = 0$.

Conversely if A is nondegenerate we get a surjective natural map of vector bundles over $X = \mathbb{P}(V_2) \times \ldots \times \mathbb{P}(V_p)$

$$V_0^\vee \otimes \mathcal{O}_X \xrightarrow{\phi_A} V_1 \otimes \mathcal{O}_X(1, \ldots, 1).$$

Indeed, by our definition, ϕ_A is surjective if and only if A is nondegenerate.

We construct a vector bundle S over $\mathbb{P}(V_2) \times \ldots \times \mathbb{P}(V_p)$ whose dual S^* is the kernel of ϕ_A so that we have the exact sequence

$$0 \longrightarrow S^* \longrightarrow V_0^\vee \otimes \mathcal{O} \longrightarrow V_1 \otimes \mathcal{O}(1, \ldots, 1) \longrightarrow 0. \tag{1}$$

After tensoring by $\mathcal{O}(m_2, \ldots, m_p)$ and taking cohomology we get

$$H^0(S^*(m_2, m_3, \ldots, m_p)) \longrightarrow V_0^\vee \otimes S^{m_1}V_1 \otimes \ldots \otimes S^{m_p}V_p \xrightarrow{\partial_A} S^{m_1+1}V_1 \otimes \ldots \otimes S^{m_p+1}V_p$$

and we need to prove

$$H^0(S^*(m_2, m_3, \ldots, m_p)) = 0. \tag{2}$$

Let $d = \dim(\mathbb{P}(V_2) \times \ldots \times \mathbb{P}(V_p)) = \sum_{i=2}^{p} k_i = m_{p+1} - k_1$.

Since $det(S^*) = \mathcal{O}(-k_1 - 1, \ldots, -k_1 - 1)$ and $rank\ S^* = d$ from remark 2.3 it follows that

$$S^*(m_2, m_3, \ldots, m_p) \simeq \wedge^{d-1} S(-1, -k_1 - 1 + m_3, \ldots, -k_1 - 1 + m_p) \tag{3}$$

Hence, by taking the $(d-1)$-th wedge power of the dual of the sequence (1), and using Künneth formula to calculate the cohomology as in [GKZ1], the result follows.

In order to prove the irreducibility of the subvariety D of degenerate matrices it is sufficient to construct the incidence variety

$$Z = \{(A, ([v_1], \dots, [v_p]) \in (V_0 \otimes \dots V_p) \times [\mathbb{P}(V_1) \times \dots \times \mathbb{P}(V_p)] \,| A(v_1 \otimes \dots \otimes v_p) = 0\}$$

Z is a vector bundle over $\mathbb{P}(V_1) \times \dots \times \mathbb{P}(V_p)$, hence it is irreducible and its projection over $V_0 \otimes \dots V_p$ is D. □

3.2. Definition. The hyperdeterminant of $A \in V_0 \otimes \dots \otimes V_p$ is the usual determinant of ∂_A, that is

$$Det(A) := det\partial_A \tag{4}$$

where $\partial_A = H^0(\phi_A)$ and $\phi_A : V_0^\vee \otimes \mathcal{O}_X \xrightarrow{\phi_A} V_1 \otimes \mathcal{O}_X(1, \dots, 1)$ is the sheaf morphism associated to A.

This is theorem 3.3 of chapter 14 of [GKZ]. Now, applying remark 2.3, we have a $GL(V_0) \times \dots \times GL(V_p)$-equivariant function

$$Det \colon V_0 \otimes \dots \otimes V_p \to \bigotimes_{i=0}^{p} (detV_i)^{\frac{N}{k_i+1}}$$

$$A \mapsto det(\partial_A)$$

3.3. Corollary. *Let $A \in V_0 \otimes \dots \otimes V_p$ of boundary format. A is nondegenerate if and only if $Det(A) \neq 0$*

3.4. *Remark.* Equality (4) is now proved without any ambiguity of the sign, while other methods give an answer modulo the choice of the sign (see [WZ] remark 7.2a).

3.5. *Remark.* Any permutation of the p numbers $k_1, \dots, k_p$ gives different m_i's and hence a different map ∂_A. As noticed by Gelfand, Kapranov and Zelevinsky, in all cases the determinant of ∂_A is the same by theorem 3.1.

3.6. *Remark.* The given definition of hyperdeterminant can be generalized to other cases where the codimension of the degenerate matrices is bigger than one, these cases are not covered in [GKZ]. If $k_0, \dots, k_p$ are nonnegative integers satisfying $k_0 = \sum_{i=1}^{p} k_i$ then we denote again $m_j = \sum_{i=1}^{j-1} k_i$ with the convention $m_1 = 0$.

Assume we have vector spaces $V_0, \dots, V_p$ and a positive integer q such that $\dim V_0 = q(k_0 + 1), \dim V_1 = q(k_1 + 1)$ and $\dim V_i = (k_i + 1)$ for $i = 2, \dots, p$. Then the vector spaces $V_0 \otimes S^{m_1}V_1 \otimes \ldots \otimes S^{m_p}V_p$ and $S^{m_1+1}V_1 \otimes \ldots \otimes S^{m_p+1}V_p$ still have the same dimension. In this case degenerate matrices form a subvariety of codimension bigger than 1.

The case $q = p = 2$ has been explored in [CO] leading to the proof that the moduli space of instanton bundles on $\mathbb{P}^3$ is affine.

4. The Binet-Cauchy theorem for hyperdeterminants of boundary format

Let $A = (a_{i_0,\ldots,i_p})$ a matrix of format $(k_0+1) \times \cdots \times (k_p+1)$ and $B = (b_{j_0,\ldots,j_q})$ of format $(l_0+1) \times \cdots \times (l_q+1)$, if $k_p = l_0$ it is defined (see [GKZ]) the convolution (or product) $A * B$ of A and B as the $(p+q-1)$-dimensional matrix C of format $(k_0+1) \times \cdots \times (k_{p-1}+1) \times (l_1+1) \times \cdots \times (l_q+1)$ with entries

$$c_{i_0,\ldots,i_{p-1},j_1,\ldots,j_q} = \sum_{h=0}^{k_p} a_{i_0,\ldots,i_{p-1},h} b_{h,j_1,\ldots,j_q}.$$

Similarly, we can define the convolution $A *_{r,s} B$ with respect to a pair of indices r, s such that $k_r = l_s$.

4.1. Proposition. [GKZ] *If A, B are degenerate then $A * B$ is also degenerate and if the hyperdeterminants of A, B and $A * B$ are non-trivial there exist polynomials $P(A,B)$ and $Q(A,B)$ in entries of A and B such that*

$$Det(A * B) = P(A,B)Det(A) + Q(A,B)Det(B)$$

In what follows we prove that in the case of boundary format matrices the hyperdeterminant of the convolution can be explicitly described by only the hyperdeterminants of the involved matrices.

4.2. Theorem. *If $A \in V_0 \otimes \cdots \otimes V_p$ and $B \in W_0 \otimes \cdots \otimes W_q$ are nondegenerate boundary format matrices with $dimV_i = k_i + 1$, $dimW_j = l_j + 1$ and $W_0^\vee \simeq V_p$ then $A * B$ is also nondegenerate and*

$$Det(A * B) = (DetA)^{\binom{l_0}{l_1,\ldots,l_q}} (DetB)^{\binom{k_0+1}{k_1,\ldots,k_{p-1},k_p+1}} \tag{5}$$

We remark that equation (5) generalizes the Binet-Cauchy theorem for determinant of usual square matrices.

Proof. We first observe that the convolution of boundary format matrices A and B is also boundary format, then by theorem 3.1 its hyperdeterminant is the usual determinant of ∂_{A*B}

We put

$$X_1 := \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_{p-1}); \quad X_2 := \mathbb{P}(W_1) \times \cdots \times \mathbb{P}(W_{q-1}) \quad \text{and} \quad X := X_1 \times X_2$$

Since A and B are nondegenerate matrices, they define vector bundles S_A and S_B respectively over X_1 and X_2 which verify the following exact sequences

$$0 \longrightarrow S_A^\vee \longrightarrow V_0^\vee \otimes \mathcal{O}_{X_1} \xrightarrow{\phi_A} V_p \otimes \mathcal{O}_{X_1}(1, \ldots, 1) \longrightarrow 0.$$

$$0 \longrightarrow S_B^\vee \longrightarrow W_0^\vee \otimes \mathcal{O}_{X_2} \xrightarrow{\phi_B} W_q \otimes \mathcal{O}_{X_2}(1, \ldots, 1) \longrightarrow 0.$$

Moreover the matrix $A * B$ defines the sheaf morphism

$$V_0^\vee \otimes \mathcal{O}_X \xrightarrow{\phi_{A*B}} W_q \otimes \mathcal{O}_X(1, \ldots, 1)$$

If the maps $\alpha : X_1 \times X_2 \to X_1$ and $\beta : X_1 \times X_2 \to X_2$ are the natural projections and $S_{A*B} = Ker(\phi_{A*B})$, we can construct the following commutative diagram:

(6)

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \alpha^* S_A^\vee \arrow[r, "f"] \arrow[d, equal] & S_{A*B}^\vee \arrow[r, "g"] \arrow[d] & \beta^* S_B^\vee(\underbrace{1, \ldots, 1}_{p-1}, 0, \ldots 0) \arrow[d] & \\
0 \arrow[r] & \alpha^* S_A^\vee \arrow[r] & V_0^\vee \otimes \mathcal{O}_X \arrow[r, "\alpha^*\phi_A"] \arrow[d, "\phi_{A*B}"] & V_p \otimes \mathcal{O}_X(\underbrace{1, \ldots, 1}_{p-1}, 0, \ldots 0) \arrow[r] \arrow[d, "\beta^*\phi_B \otimes id_{\mathcal{O}_X(1,\ldots,1,0,\ldots,0)}"] & 0 \\
 & & W_q \otimes \mathcal{O}_X(1, \ldots, 1) \arrow[r, equal] & W_q \otimes \mathcal{O}_X(1, \ldots, 1) \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

The surjectivity of maps $\beta^*\phi_B \otimes id_{\mathcal{O}_X(1,\ldots,1,0,\ldots,0)}$ and $\alpha^*\phi_A$ induce the surjectivity of g and ϕ_{A*B}, thus $A * B$ is nondegenerate and S_{A*B} is a vector bundle

Moreover, since

$$\phi_{A*B} = \beta^*\phi_B \otimes id_{\mathcal{O}_X(1,\ldots,1,0,\ldots,0)} \circ \alpha^*\phi_A$$

$$\text{and} \qquad \partial_{A*B} = H^0(\phi_{A*B} \otimes id_{\mathcal{O}(\sum_2^p k_i, \sum_3^p k_i, \ldots, k_p, \sum_2^q l_j, \sum_3^q l_j, \ldots, l_q)})$$

then

$$\partial_{A*B} = (\partial_B \otimes id_{S^{\sum_2^p k_i+1}V_1 \otimes \cdots \otimes S^{k_p+1}V_p}) \circ (\partial_A \otimes id_{S^{\sum_2^q l_j}W_1 \otimes \cdots \otimes S^{l_q}W_{q-1}})$$

i.e. by remark 2.3

$$det(\partial_{A*B}) = (det(\partial_A))^{\binom{l_0}{l_1,\ldots,l_q}} (det(\partial_B))^{\binom{k_0+1}{k_1,\ldots,k_p+1}}$$

as we wanted. $\square$

4.3. *Remark.* The degree of the hyperdeterminant of a boundary format $(k_0+1) \times \cdots \times (k_p+1)$ matrix A is given by the multinomial coefficient:

$$N_A = \binom{k_0+1}{k_1, \ldots, k_p}$$

This follows also from (3.2). Thus, (5) can be rewritten as

$$Det(A * B) = [(Det A)^{N_B} (Det B)^{N_A}]^{\frac{1}{l_0+1}}$$

4.4. *Remark.* The same result of the above theorem works for the convolution with respect to the pair of indices $(j, 0)$ with j varying in $\{1, \ldots p\}$. Indeed the condition $W_0^\vee \simeq V_j$ ensure that $A*_{j,0}$ is again of boundary format and we can arrange the indices as in the proof because for any permutation σ we have $Det(A) = Det(\sigma A)$.

4.5. Corollary. *If A and B are boundary format matrices then*

$$A \text{ and } B \text{ are nondegenerate} \iff A *_{0,j} B \text{ are nondegenerate}$$

The implication $\Longleftarrow$ of the previous corollary is true without the assumption of boundary format, see proposition 1.9 of [GKZ].

4.6. *Remark.* Theorem 4.2 and the implication $\Longrightarrow$ of the corollary 4.5 work only for boundary format matrices. Indeed, if, for instance, A and B are $2 \times 2 \times 2$ matrices with

$$\begin{aligned} a_{ijk} = 0 &\quad \text{for all} \quad (i,j,k) \notin \{(0,0,0),(1,1,1)\} \quad \text{and} \\ b_{krs} = 0 &\quad \text{for all} \quad (k,r,s) \notin \{(0,0,1),(1,1,0)\} \end{aligned}$$

then A and B are nondegenerate since, applying Cayley formula (see [Cay] pag.89 or [GKZ] pag.448), their hyperdeterminants are respectively:

$$Det(A) = a_{000}^2 a_{111}^2 \quad \text{and} \quad Det(B) = b_{001}^2 b_{110}^2$$

but the convolution $A * B$ is degenerate. In this case, by using Schläfli's method of computing hyperdeterminant ([GKZ]), it easy to find that $Det(A * B)$ corresponds to the discriminant of the polynomial $F(x_0, x_1) = a_{000}^2 a_{111}^2 b_{001}^2 b_{110}^2 x_0^2 x_1^2$ which obviously vanishes.

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