MINIMAL RESOLUTION OF GENERAL STABLE RANK-2 VECTOR BUNDLES ON $\mathbb{P}^2$

CARLA DIONISI AND MARCO MAGGESI

Sunto. – *Abbiamo studiato gli elementi generici degli spazi di moduli* $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ *dei fibrati vettoriali stabili di rango 2 su* $\mathbb{P}^2$ *e le loro risoluzioni libere minimali. Ne segue una dimostrazione piuttosto semplice dell'irriducibilità di* $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$.

Abstract. – *We study general elements of moduli spaces* $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ *of rank-2 stable holomorphic vector bundles on* $\mathbb{P}^2$ *and their minimal free resolutions. Incidentally, a quite easy proof of the irreducibility of* $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ *is shown.*

1. Introduction

We investigate stable rank-2 vector bundles on the complex projective plane $\mathbb{P}^2$ by means of their minimal free resolutions. Bohnhorst and Spindler in their paper [BS92] develop interesting techniques for the study of minimal free resolution of rank-n stable vector bundles on $\mathbb{P}^n$ of homological dimension 1. In this work we derive a number of consequences for rank-2 vector bundles on $\mathbb{P}^2$.

As Bohnhorst and Spindler observe, Betti numbers define a stratification of the moduli space $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ of stable holomorphic vector bundles on $\mathbb{P}^2$ by constructible subsets. We estimate the codimension of such strata and characterize Betti numbers of the general element of the moduli space. As a corollary, we get a simple proof of the irreducibility of $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$. Other proofs of the irreducibility of moduli spaces of stable vector bundles can be found in [Bar77a, Ell83, Hul79, HL93, Pot79, Mar78]. The irreducibility of $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ was proved for c_1 even in [Bar77a] and c_1 odd indipendently in [Pot79] and [Hul79].

We would like to thank G. Ottaviani for his invaluable guidance and V. Ancona for many useful discussions.

2. Admissible pairs and resolutions

Let $\mathcal{E}$ be a rank-2 vector bundle on $\mathbb{P}^2$. By Horrocks' theorem [Hor64], $\mathcal{E}$ has homological dimension at most 1, i.e., it has a free resolution of the form

$$0 \longrightarrow \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^2}(-a_i) \xrightarrow{\Phi} \bigoplus_{j=1}^{k+2} \mathcal{O}_{\mathbb{P}^2}(-b_j) \longrightarrow \mathcal{E} \longrightarrow 0. \tag{1}$$

We do not assume that such a resolution is minimal. In what follows we suppose that $\mathcal{E}$ has homological dimension 1 (so that $k > 0$); we are not interested in vector bundles of homological dimension 0, i.e., splitting vector bundles, since they are not stable.

We suppose that the two sequences a_i and b_i are indexed in nondecreasing order

$$\begin{aligned} &a_1 \le a_2 \le \cdots \le a_k, \\ &b_1 \le b_2 \le \cdots \le b_{k+2}; \end{aligned} \tag{2}$$

call $(a, b) = ((a_1, \dots, a_k), (b_1, \dots, b_{k+2}))$ the *associated pair* to the resolution (1). If this resolution is minimal, we call (a, b) the pair associated to the bundle $\mathcal{E}$. Notice that the associated pair and Betti numbers encode exactly the same information; in particular $\max(a_k - 1, b_{k+2})$ is the Castelnuovo-Mumford regularity of $\mathcal{E}$.

Chern classes c_1, c_2 of $\mathcal{E}$ are expressed in terms of the a_i, b_j by the formulas

$$c_1 = \sum_{i=1}^{k} a_i - \sum_{i=1}^{k+2} b_i, \tag{3}$$

$$2c_2 - c_1^2 = \sum_{i=1}^{k} a_i^2 - \sum_{i=1}^{k+2} b_i^2. \tag{4}$$

Definition 2.1. A pair (a, b) is said to be *admissible* if

$$a_i > b_{i+2} \qquad \text{for all } i = 1, \dots, k \tag{5}$$

For the sake of brevity, we say that the resolution (1) is admissible if its associated pair (a, b) is so.

More generally, we can consider the associated pair (a, b) to any vector bundle of homological dimension 1 on $\mathbb{P}^n$ with $n \geq 2$. In that case, we say that (a, b) is admissible if $a_i > b_{n+i}$ for $i = 1, \dots, k$, as in [BS92].

Let us restate the main results of Bohnhorst and Spindler on admissible pairs (proposition 2.3 and theorem 2.7 in [BS92]) in our settings.

Theorem 2.2. *Let $\mathcal{E}$ be the rank-2 vector bundle on $\mathbb{P}^2$ of resolution (1). Then*

1. *resolution (1) is minimal if and only if it is admissible and every constant entry of the matrix Φ is zero;*
2. *if resolution (1) is admissible then $\mathcal{E}$ is stable (resp. semistable) if and only if $b_1 > -\mu$ (resp. $b_1 \geq -\mu$) where $\mu = c_1/2$ is the slope of $\mathcal{E}$.*

We denote by $\mathfrak{I}$ the set of all admissible pairs (a, b) associated to rank-2 vector bundles on $\mathbb{P}^2$ with Chern classes c_1, c_2 which satisfy the condition

$$b_1 > -\mu = \frac{1}{2}\left(\sum a_i - \sum b_j\right).$$

Theorem 2.2 shows that the set $\mathfrak{I}$ contains the set of all possible associated pairs to a stable vector bundle in $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ and coincides exactly with it. Then

$$\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2) = \coprod_{(a,b) \in \mathfrak{I}} \mathfrak{M}(a, b) \tag{6}$$

where $\mathfrak{M}(a, b)$ is the constructible subset of $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ of vector bundles with associated pair (a, b).

Proposition 2.3. *For all $(a, b) \in \mathfrak{I}$, the closed set $\overline{\mathfrak{M}(a, b)}$ is an irreducible algebraic subset of $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ of dimension*

$$\begin{aligned} \dim \overline{\mathfrak{M}(a, b)} = {} & \dim \operatorname{Hom}(F_1, F_0) + \dim \operatorname{Hom}(F_0, F_1) + \\ & - \dim \operatorname{End}(F_1) - \dim \operatorname{End}(F_0) + 1 - \#\{(i, j) : a_i = b_j\}, \end{aligned} \tag{7}$$

where $F_0 = \oplus_{j=1}^{k+2} \mathcal{O}(-b_j)$, $F_1 = \oplus_{i=1}^{k} \mathcal{O}(-a_i)$.

Proof. Proposition 3.3 in [BS92]. □

In the following lemma we find an upper bound on the regularity of semistable vector bundles on $\mathbb{P}^2$ of rank-2. A lower bound is given in corollary 3.7.

Theorem 2.4. *A normalized semistable rank-2 bundle $\mathcal{E}$ on $\mathbb{P}^2$ is c_2-regular.*

Proof. For brevity's sake, we set $\xi_i := a_i - b_{i+2}$ and $t_i := b_{i+2} - b_2$. Obviously, $\xi_i \geq 1$ and $t_i \geq 0$. We rewrite (3) as

$$\sum_{i=1}^{k} \xi_i = b_1 + b_2 + c_1. \tag{8}$$

By equation (4) and theorem 2.2, using inequalities (2) and (5), we get

$$\begin{aligned} b_1^2 + b_2^2 + 2c_2 - c_1^2 &= \sum_{i=1}^{k}(a_i^2 - b_{i+2}^2) = \sum_{i=1}^{k} \xi_i(2b_2 + 2t_i + \xi_i) \geq \\ &\geq 2b_2 \sum_{i=1}^{k} \xi_i + \sum_{i=1}^{k}(2t_i + \xi_i) = \\ &= (2b_2 + 1)(b_1 + b_2 + c_1) + 2\sum_{i=1}^{k} t_i. \end{aligned} \tag{9}$$

If we suppose that $b_2 + \sum_{i=1}^{k} t_i \geq c_2 + 1$, we have

$$b_1^2 + b_2^2 + 2b_2 - c_1^2 - (2b_2 + 1)(b_1 + b_2 + c_1) \geq 2.$$

The left side of the above inequality is non-increasing with respect to b_1, hence it remains true after substituting $-c_1$ to b_1; but $b_2 - b_2^2 \geq 2$ has no solutions. Thus $\sum_{i=1}^{k} t_i$ must be at most $c_2 - b_2$ and in particular

$$b_{k+2} = b_2 + t_k \leq b_2 + \sum t_i \leq c_2. \tag{10}$$

Now, we must show that $a_k \leq c_2 + 1$. We rewrite (3) as $\sum_{i=1}^{k-1} \xi_i = b_1 + b_2 + b_{k+2} - a_k + c_1$ and by (4)

$$\begin{aligned} b_1^2 + b_2^2 + b_{k+2}^2 - a_k^2 + 2c_2 - c_1^2 &= \sum_{i=1}^{k-1}(a_i^2 - b_{i+2}^2) = \sum_{i=1}^{k-1} \xi_i(2b_2 + 2t_i + \xi_i) \geq \\ &\geq 2b_2 \sum_{i=1}^{k-1} \xi_i + \sum_{i=1}^{k-1} \xi_i \geq \\ &\geq (2b_2 + 1)(b_1 + b_2 + b_{k+2} - a_k + c_1) \end{aligned} \tag{11}$$

that can be put in the form

$$\begin{aligned} b_1^2 + b_2^2 - (2b_2 + 1)(b_1 + b_2 + c_1) + 2c_2 - c_1^2 &\geq \\ &\geq (a_k - b_{k+2})(a_k + b_{k+2} - 2b_2 - 1). \end{aligned} \tag{12}$$

Suppose that $a_k \geq c_2 + 2$. By (10) we have $a_k - b_{k+2} \geq c_2 + 2 - c_2 = 2$ and we observe also that $a_k + b_{k+2} - 2b_2 - 1 \geq c_2 - b_2 + 1$. Substituting and simplifying, equation (12) becomes

$$b_1^2 + b_2^2 - (2b_2 + 1)(b_1 + b_2 + c_1) - c_1^2 \geq c_1^2 + 2 \tag{13}$$

As before, we can restrict ourselves to the case $b_1 = -c_1$ obtaining

$$b_1^2 + b_2^2 - (2b_2 + 1)(b_1 + b_2) \geq 2$$

which has no solution for b_i positive. Then $a_k \leq c_2 + 1$. □

Remark 2.5. The above theorem is sharp. Indeed $((c_2 + 1), (0, 1, c_2))$ and $((c_2 + 1), (1, 1, c_2))$ are admissible pairs associated to rank-2 semistable bundles with Chern classes c_1, c_2 and regularity c_2.

Remark 2.6. It is also possible to prove that a semistable rank-2 bundle on $\mathbb{P}^2$ is c_2 regular if $c_1 = 0$ and $(c_2 + 1)$-regular if $c_1 = -1$ using the bounds on dimension of cohomology groups proved by Elencwajg and Forster (proposition 2.18 in [EF80]) and the Grauert-Mülich theorem.

Remark 2.7. From (8) and the thesis of the previous theorem, the value k in (1) is bounded by:

$$k \leq \sum_{i=1}^{k} \xi_i = b_1 + b_2 + c_1 \leq 2c_2 + c_1 \tag{14}$$

Hence, for fixed Chern classes c_1, c_2, there are only a finite number of admissible pairs of rank-2 vector bundles and we can write an algorithm to enumerate such pairs restricting the search to a finite domain.

3. Natural pairs and general vector bundles

We say that $(a, b) = ((a_1, \dots, a_k), (b_1, \dots, b_{k+2}))$ is a *natural pair* if it is admissible and

$$b_{k+2} < a_1, \qquad a_k \leq b_1 + 2. \tag{15}$$

The above inequalities imply $a_i \neq b_j$ for all i and j.

We observe that natural pairs are parametrized by three integers s, k, α such that

$$k \geq 1 \qquad \text{and} \qquad -k + 1 \leq \alpha \leq k + 2 \tag{16}$$

as follows: the pair $(a, b)_{s,k,\alpha}$ corresponding to the triple (s, k, α) is the pair associated to a resolution of the form

$$0 \to \mathcal{O}(-s-1)^k \to \mathcal{O}(-s)^\alpha \oplus \mathcal{O}(-s+1)^{k-\alpha+2} \to \mathcal{E} \to 0 \tag{17}$$

if $\alpha \geq 0$, or of the form

$$0 \to \mathcal{O}(-s-1)^{k+\alpha} \oplus \mathcal{O}(-s)^{-\alpha} \to \mathcal{O}(-s+1)^{k+2} \to \mathcal{E} \to 0 \tag{18}$$

if $\alpha < 0$. We have excluded the case $\alpha = -k$ so that s is the regularity of the pair, i.e., $s = \max(a_k - 1, b_{k+2})$.

In this section we are going to show that resolutions of general vector bundles have natural pairs.

Theorem 3.1. *One has* $\operatorname{codim} \mathfrak{M}(a, b) = 0$ *if and only if* (a, b) *is a natural pair.*

As a remarkable consequence we will derive a quite simple proof of the irreducibility of moduli spaces of rank-2 stable vector bundles on $\mathbb{P}^2$, and we will compute regularity and cohomology of their general elements.

We recall that, since $\dim \operatorname{Ext}^2(\mathcal{F}, \mathcal{F}) = 0$ for any stable vector bundle $\mathcal{F}$ on $\mathbb{P}^2$, the corresponding moduli space $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ is smooth of dimension

$$\dim \operatorname{Ext}^1(\mathcal{F}, \mathcal{F}) = 4c_2 - c_1^2 - 3. \tag{19}$$

Let us consider the function $A(t) := h^2(\mathcal{O}(t))$ and its finite differences of first and second order $(\Delta_u A)(t) := A(t+u) - A(t)$ and $(\Delta_v \Delta_u A)(t) := (\Delta_u A)(t+v) - (\Delta_u A)(t)$.

Lemma 3.2. *If $\mathcal{E} \in \mathfrak{M}_{\mathbb{P}^2}(2; c_1, c_2)$ has associated admissible pair (a, b), then*

$$\begin{aligned} \operatorname{codim} \overline{\mathfrak{M}(a,b)} &= h^1(\mathcal{E}(b_1)) + h^1(\mathcal{E}(b_2)) + \#\{(i,j) : a_i = b_j\} + \\ &\quad + \sum_{i,j=1}^{k} (\Delta_{b_{i+2}-a_i} \Delta_{b_{j+2}-a_j} A)(a_i - b_{i+2}). \end{aligned} \tag{20}$$

Proof. Let

$$0 \to F_1 \to F_0 \to \mathcal{E} \to 0 \tag{21}$$

be the minimal resolution of $\mathcal{E}$ where

$$F_0 = \bigoplus_{j=1}^{k+2} \mathcal{O}(-b_j), \qquad F_1 = \bigoplus_{i=1}^{k} \mathcal{O}(-a_i). \tag{22}$$

Stability of $\mathcal{E}$ ensures the vanishing $\dim(\operatorname{Ext}^2(\mathcal{E}, \mathcal{E})) = h^2(\mathcal{E}^* \otimes \mathcal{E}) = 0$ so that $h^2({F_0}^* \otimes \mathcal{E}) = h^2({F_1}^* \otimes \mathcal{E})$. Then from (21) we easily find the following data:

$$\begin{aligned} h^0({F_0}^* \otimes \mathcal{E}) &= h^0({F_0}^* \otimes F_0) - h^0({F_0}^* \otimes F_1), \\ h^0({F_1}^* \otimes \mathcal{E}) &= h^0({F_1}^* \otimes F_0) - h^0({F_1}^* \otimes F_1), \\ \dim(\operatorname{Ext}^1(\mathcal{E}, \mathcal{E})) &= h^1(\mathcal{E}^* \otimes \mathcal{E}) = \\ &= h^1({F_0}^* \otimes \mathcal{E}) - h^1({F_1}^* \otimes \mathcal{E}) + \\ &\quad + h^0({F_1}^* \otimes \mathcal{E}) - h^0({F_0}^* \otimes \mathcal{E}) + 1 \end{aligned} \tag{23}$$

and from (7) we have

$$\begin{aligned} \operatorname{codim} \overline{\mathfrak{M}(a,b)} &= \dim(\operatorname{Ext}^1(\mathcal{E}, \mathcal{E})) - \dim \overline{\mathfrak{M}(a,b)} = \\ &= h^1({F_0}^* \otimes \mathcal{E}) - h^1({F_1}^* \otimes \mathcal{E}) + \#\{(i,j) : a_i = b_j\} \end{aligned} \tag{24}$$

Now, by splitting F_0 as $\mathcal{O}(-b_1) \oplus \mathcal{O}(-b_2) \oplus \tilde{F_0}$ with $\tilde{F_0} := \oplus_{i=3}^{k+2} \mathcal{O}(-b_i)$, the above formula becomes

$$\begin{aligned} \operatorname{codim} \overline{\mathfrak{M}(a,b)} &= h^1(\mathcal{E}(b_1)) + h^1(\mathcal{E}(b_2)) + \#\{(i,j) : a_i = b_j\} + \\ &\quad + h^1(\tilde{F_0}^* \otimes \mathcal{E}) - h^1({F_1}^* \otimes \mathcal{E}). \end{aligned} \tag{25}$$

Since $h^2(\tilde{F_0}^* \otimes F_0) = h^2(\tilde{F_0}^* \otimes \tilde{F_0})$ and $h^2({F_1}^* \otimes F_0) = h^2({F_1}^* \otimes \tilde{F_0})$ the following identity holds:

$$\begin{aligned} & h^1(\tilde{F_0}^* \otimes \mathcal{E}) - h^1({F_1}^* \otimes \mathcal{E}) = \\ &= h^2(\tilde{F_0}^* \otimes F_1) - h^2(\tilde{F_0}^* \otimes F_0) - h^2({F_1}^* \otimes F_1) + h^2({F_1}^* \otimes F_0) = \\ &= h^2(\tilde{F_0}^* \otimes F_1) - h^2(\tilde{F_0}^* \otimes \tilde{F_0}) - h^2({F_1}^* \otimes F_1) + h^2({F_1}^* \otimes \tilde{F_0}) = \\ &= \sum_{i,j=1}^{2} \Big[h^2(\mathcal{O}(b_{i+2} - a_j)) - h^2(\mathcal{O}(b_{i+2} - b_{j+2})) + \\ &\qquad - h^2(\mathcal{O}(a_i - a_j)) + h^2(\mathcal{O}(a_i - b_{j+2})) \Big]. \end{aligned} \tag{26}$$

Then equation (20) follows by substitution of (26) in (25). ∎

Proof of theorem 3.1. It can be verified by direct computation from proposition 2.3 that, if $\mathcal{E}$ has natural pair, then the codimension of $\overline{\mathfrak{M}(a,b)}$ is zero. Conversely, let u, v be two non-negative integers. Since all finite differences $(\Delta_u A)(t) := A(t+u) - A(t)$ are non decreasing functions of t, then

$$(\Delta_v \Delta_u A)(t) \geq 0 \tag{27}$$

and by the previous lemma

$$\operatorname{codim} \overline{\mathfrak{M}(a,b)} \geq h^1(\mathcal{E}(b_1)) + h^1(\mathcal{E}(b_2)) + \#\{(i,j) : a_i = b_j\}. \tag{28}$$

If $\operatorname{codim} \overline{\mathfrak{M}(a,b)} = 0$, we have $a_k \leq b_1 + 2$ and $\#\{(i,j) : a_i = b_j\} = 0$, since $h^1(\mathcal{E}(b_1)) = 0$ implies $h^2(F_1(b_1)) = 0$. This forces (a,b) to be a natural pair. □

Proposition 3.3. *Suppose that $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ be nonempty and let*

$$s := \max\{\rho \in \mathbb{Z} : 2\rho^2 + 2c_1\rho - 2\rho \leq 2c_2 - c_1^2 + c_1 - 1\}, \tag{29}$$

If α and k are defined by

$$\begin{aligned} \alpha &:= 2c_2 - c_1^2 + 2 - 2s^2 - 2c_1 s, \\ k &:= (2s + c_1 - 2 + |\alpha|)/2, \end{aligned} \tag{30}$$

then $(a,b)_{s,k,\alpha}$ is the only natural pair of $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$.

Proof. Note that, since $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ is nonempty, from theorem 3.1 there exists a vector bundle associated to a natural pair. It is easy to verify that equations (3) and (4) are equivalent to (30) and conditions (16) are equivalent to

$$2s^2 + 2c_1 s - c_1 - 2s + 1 \leq 2c_2 - c_1^2 \leq 2s^2 + 2c_1 s + c_1 + 2s.$$

The intervals

$$[2s^2 + 2c_1 s - c_1 - 2s + 1, \quad 2s^2 + 2c_1 s + c_1 + 2s]$$

are disjoint for s varying in $\mathbb{Z}$. Hence s is uniquely determined from c_1, c_2 and satisfies (29). □

Remark 3.4. Equation (29) in proposition 3.3 is also equivalent to

$$s := \min\{\rho \in \mathbb{Z} : 2\rho^2 + 2c_1\rho + 2\rho \geq 2c_2 - c_1^2 - c_1\}. \tag{29bis}$$

Theorem 3.5. *Moduli spaces of stable rank-2 vector bundles on $\mathbb{P}^2$ are irreducible.*

Proof. Moduli spaces of stable rank-2 vector bundles on $\mathbb{P}^2$ are smooth. By theorem 3.1 and the above proposition they can have only one connected component. □

Corollary 3.6. *The general element of $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ has natural cohomology.*

The above corollary justify the terminology *"natural pair"*. A different proof for it, working also for rank greater than 2, can be found in [HL93], by using sophisticated techniques of stacks theory.

Using proposition 3.3 we are going to give some bounds on regularity and cohomology of stable vector bundles. In particular, for rank-2 vector bundles, the next two corollaries give respectively a refined version of corollary 5.4 in [Bru80] and proposition 7.1 in [Har78].

Corollary 3.7. *The general vector bundle $\mathcal{E}$ in $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ has regularity s given by (29).*

Corollary 3.8. *Let $[\mathcal{E}]$ be a vector bundle in $\mathfrak{M} = \mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ and let s be defined by (29). Then $\mathrm{H}^0(\mathcal{E}(t)) \neq 0$ if*

$$\begin{aligned} &t \geq s && \textit{when} \quad 2s^2 + 2c_1 s + 2s = 2c_2 - c_1^2 - c_1, \\ &t \geq s - 1 && \textit{otherwise.} \end{aligned}$$

The above inequality is sharp, in the sense that it gives a necessary and sufficient condition for $\mathcal{E}$ general.

Proof. Let $((a_1, \dots, a_k), (b_1, \dots, b_{k+2}))$ be the admissible pair associated to a vector bundle $\mathcal{E}$ in $\mathfrak{M}$. Then one has $\mathrm{H}^0(\mathcal{E}(t)) \neq 0$ if and only if $t - b_1 \geq 0$. By the semicontinuity of cohomology groups and theorem 3.5, it is enough to restrict ourselves to the case where $\mathcal{E}$ is general. So, by (17) and (18) one has $\mathrm{H}^0(\mathcal{E}(t)) \neq 0$ if and only if

$$\begin{aligned} &t \geq s && \text{when} \quad \alpha = k + 2 \\ &t \geq s - 1 && \text{otherwise} \end{aligned}$$

and the condition $\alpha = k + 2$ is equivalent to $2s^2 + 2c_1 s + 2s = 2c_2 - c_1^2 - c_1$ by (30). □

References

[Bar77a] W. Barth, *Moduli of vector bundles on the projective plane*, Invent. Math. (1977), no. 42, 63–91.

[Bar77b] W. Barth, *Some properties of stable rank 2 vector bundles on $\mathbb{P}^n$*, Math. Ann. (1977), no. 226, 125–150.

[BH78] W. Barth and K. Hulek, *Monads and moduli of vector bundles*, Manuscripta Math. (1978), no. 25, 323–347.

[Bru80] J. Brun, *Les fibrés stables de rang deux sur $\mathbb{P}^2$ et leur sections*, Bull. Soc. Math. France **4** (1980), no. 108, 457–473.

[BS92] G. Bohnhorst and H. Spindler, *The stability of certain vector bundles on $\mathbb{P}^n$*, Lecture Notes (1992), no. 1507, 39–50.

[Dio00] C. Dionisi, *Multidimensional matrices and minimal resolutions of vector bundles*, Ph.D. thesis, Dip. Matematica "R.Caccioppoli", Università di Napoli, 2000.

[DL85] J. M. Drezet and J. LePotier, *Fibrés stables et fibrés exceptionnels sur $\mathbb{P}^2$*, Ann. Sci. Ec. Norm. Supér. **IV** (1985), no. 18, 193–244.

[EF80] G. Elencwajg and O. Forster, *Bounding cohomology groups and vector bundles on $\mathbb{P}^n$*, Math. Ann. (1980), no. 246, 251–270.

[Ell83] G. Ellingsrud, *Sur l'irréducibilité du module des fibrès stable sur $\mathbb{P}^2$*, Math. Z. (1983), no. 182, 189–192.

[Gre89] M. L. Green, *Koszul cohomology and geometry*, Proceedings of the first college on Riemann surfaces held in Trieste (Italy) (M. Cornalba et al., eds.), World Scientific Publishing Co., November 1989, pp. 177–200.

[Har78] R. Hartshorne, *Stable vector bundles of rank 2 on $\mathbb{P}^3$*, Math. Ann. **238** (1978), 229–280.

[Har83] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, no. 52, Springer-Verlag, New York-Heidelberg-Berlin, 1983.

[HL93] A. Hirschowitz and Y. Laszlo, *Fibrés génériques sur le plan projectif*, Math. Ann. (1993), no. 297, 85–102.

[Hor64] G. Horrocks, *Vector bundles on the punctured spectrum of a local ring*, Proc. Lond. Math. Soc., III. Ser. **14** (1964), 689–713.

[Hor87] G. Horrocks, *Vector bundles on the punctured spectrum of a local ring II*, Vector bundles on algebraic varieties, Stud. Math., no. 11, Tata Inst. Fundam. Res., 1987, Bombay 1984, pp. 207–216.

[Hul79] K. Hulek, *Stable rank 2 vector bundles on $\mathbb{P}^2$ with c_2 odd*, Math. Ann. (1979), no. 242, 241–266.

[Hul80] K. Hulek, *On the classification of stable rank-k vector bundles over the projective plane*, A. Hirschowitz (ed): Vector bundles and differential equations, Progress in Mathematics, no. 7, Birkhäuser, Boston - Basel - Stuttgart, 1980, Nice 1979, pp. 113–144.

[Mag99] M. Maggesi, *Some results on holomorphic vector bundles over projective spaces*, Ph.D. thesis, Dip. Matematica "U.Dini" , Università di Firenze, 1999.

[Mar78] M. Maruyama, *Moduli of stable sheaves II*, J. Math. Kyoto Univ. (1978), no. 18, 557–614.

[OSS80] C. Okonek, M. Schneider, and H. Spindler, *Vector bundles on complex projective spaces*, Progress in Mathematics, no. 3, Birkhäuser, Boston - Basel - Stuttgart, 1980.

[Pot79] J. Le Potier, *Fibrés stables de rang 2 sur* $\mathbb{P}^2$, Math. Ann. (1979), no. 241, 217–256.

[Sch61] R. L. E. Schwarzenberger, *Vector bundles on the projective plane*, Proc. London Math. Soc. **3** (1961), no. 11, 623–640.

Carla Dionisi – Dipartimento di Matematica Applicata "G. Sansone"
Via S. Marta, 3, 50139, Firenze, Italy
email: `dionisi@math.unifi.it`

Marco Maggesi – Dipartimento di Matematica "U. Dini"
Viale Morgagni, 67/a, 50134, Firenze, Italy
email: `maggesi@math.unifi.it`