ON ALGEBRAIC HYPERSURFACES OF $\mathbb{P}^r(\mathbb{C})$ MULTIPLE COMPONENTS OF THEIR HESSIAN

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Abstract. We partially solve a conjecture of A. Franchetta proving that an algebraic irreducible hypersurface of $\mathbb{P}^r(\mathbb{C})$ with 1-parabolic points, multiple component of its hessian and having scheme of foci of dimension $(r-2)$, if exists, has degree at least $(r-1)^2$.

Introduction

In this note we study the characterization of projective hypersurfaces multiple components of their hessian. This problem was pointed and partially solved by B. Segre in a first note [BS1] on 1951.

Since an hypersurface on $\mathbb{P}^r(\mathbb{C})$ with h-parabolic points (see definition 1.6) is contained in its hessian with multiplicity at least h, then the problem becomes to describe the hypersufaces with h-parabolic points contained in their hessian with multiplicity more than h. In [BS1] and [BS2], B. Segre gives various examples of hypersurfaces for which this case occurs. In [F1] A. Franchetta proved that the only irreducible algebraic hypersurface of $\mathbb{P}^3(\mathbb{C})$ with determined hessian (i.e. not a cone), contained with multiplicity in its hessian, is the developable surface circumscribed to a twisted cubic. Other results, that we will recall in the follows, concerning the hypersurfaces of $\mathbb{P}^r(\mathbb{C})$ with $r \geq 3$, are contained in [F1], [F2], [I1] and [I2].

B. Segre proved that if V_{r-1} is an irreducible algebraic hypersurface of $\mathbb{P}^r(\mathbb{C})$ with h-parabolic points then V_{r-1} is the locus of a system Σ of dimension $r-h-1$ of h-subspaces of $\mathbb{P}^r(\mathbb{C})$ such that, along the generic, the tangent hyperplane to V_{r-1} is fixed. In particular, if $h=1$, Σ is a system of dimension $r-2$ of lines and we can consider the locus W_d $(0 \leq d \leq r-2)$ of the foci of V_{r-1} (see definitions in [DePI]).

If $d=0$, V_{r-1} is a cone then its hessian is not determined.

For $d=1$ A. Franchetta in [F1] gives a completely description of the hypersurfaces V_3 multiple components of their hessian (their have also undetermined hessian).

For $d = 2$, and $r = 4$ in [F1] and [F2] A. Franchetta finds some conditions about the degree, the hessian and foci locus of these hypersurfaces and conjecture the analogous results for $d = 2$ and for any r. In [I1], the case $d = 2$ and $r = 4$ is completed proving that such hypersurfaces don't exist. In fact, also considering the results contained in [BS1], concerning hypersurfaces with 2-parabolic points, it follows that an algebraic irreducible hypersurface in $\mathbb{P}^4(\mathbb{C})$, having determined hessian, cannot be a multiple component of its hessian.

More in general, in [I2] it is proved that an hypersurface of $\mathbb{P}^r(\mathbb{C})$ with 1-parabolic points, multiple component of its hessian, and foci locus not ruled and of dimension $r - 2$, cannot be have hessian not determined.

In this note we prove that, if there exists an hypersurface V_4 in $\mathbb{P}^5(\mathbb{C})$ with 1-parabolic points, multiple component of its hessian and with foci locus of dimension $d = 3$, it has degree at least 16, and generalizing in $\mathbb{P}^r$ if an hypersurface V_r of this type exists ($d = r - 2$), it has degree at least $(r - 1)^2$. This result partially solve a conjecture due to A. Franchetta and contained in [F1].

1. Preliminaries

Let V be a vector space of dimension $r + 1$. Let $S^n(V)$ be its symmetric power. We may define the polarization map

$$pl_n^* : S^n(V) \to V^{\otimes n}$$

The image consists of symmetric tensors. We can consider the polarization map on V^*:

$$pl_n : S^n(V^*) \to (V^*)^{\otimes n}$$

whose image is $\mathrm{Sym}_n(V)$ of symmetric n-linear forms on V.By choosing a basis $u_0, \dots, u_r$ in V, and its dual basis $_0, \dots, x_r$ in V^*, we will identify the spaces $S^n(V)$ (resp. $S^n(V^*)$) with the space of homogeneous polynomials of degree n in $u_0, \dots, u_r$ (resp. $x_0, \dots, x_r$).The set of zeros of any non-zero polynomial $F \in S^n(V^*)$ is an hypersurface of degree n in the projective space $\mathbb{P}(V) = \mathbb{P}^r$ associated to V.

1.1. Definition. The *polarization* $pl_n(F)$ of a polynomial $F \in S^n(V^*)$ is the unique symmetric multi linear function $\tilde{F}(x, y, \dots, z)$ on V^n such that for all $x \in V$

$$F(x) = \tilde{F}(x, \dots, x)$$

1.2. Definition. We define the *k-th mixed polar* of F with respect to the points $a, b, \dots c$ as

$$P_{a,b,\dots,c}(F)(x) = \tilde{F}(a, b, \dots, c, x, \dots, x)$$

obtained by fixing the first k variables $a, b, \dots, c$ in $\tilde{F}$ and making equal the remaining ones.

For more details on polarity see [DK].
Let $F \in S^n(V^*)$ and let $\tilde{F} \in \mathrm{Sym}_n(V)$ be its full polarization. We consider the map

$$\phi : v \in V \to H(v) \in \mathrm{Sym}_2(V)$$

where $H(v)(a, b) = \tilde{F}(a, b, v, \dots, v) = P_{a,b}(F)(v) = P_{v^{n-2}}(F)(a, b)$

1.3. Definition. The *hessian* of F, denoted by $\mathrm{He}(F)$ is a polynomial function on V of degree $(r+1)(n-2)$, obtained composing the map ϕ with the discriminant map

$$d : H(v) \to \mathrm{discr}(H(v))$$

1.4. Lemma ([BS3] or [DK]). *Let $F \in S^n(V^*)$, $v \in V$ and $\tilde{F} \in Sym_n(V)$ be its full polarization. The following conditions are equivalent:*

1. *v is a singular point of F;*
2. *$v \in P_a(F)$ $\forall a \in V$;*
3. *$\tilde{F}(a, v, \dots, v) = 0$ $\forall a \in V$;*
4. *$P_{v^{n-1}}(F) = 0$*
5. *v is a singular point of $P_v(F)$.*

1.5. Lemma ([BS3] or [DK]). *Let $v \in V$. The following properties are equivalent:*

1. $\mathrm{He}(F)(v) = 0$;
2. *$\exists a \in V, a \neq 0$ such that $P_{v^{n-2},a}(F) = 0$;*
3. *the polar quadric $P_{v^{n-2}}(F)$ is singular;*
4. *$\exists a \in \mathbf{P}$ such that the hypersurface $P_a(F)$ has a singular point at v.*

We consider, now, an irreducible hypersurface in $\mathbb{P}^r(\mathbb{C})$, of equation $F(x_0, \dots, x_r) = 0$ of order $n \geq 3$, let V_{r-1} or F.
Let P be a simple point and A(P) the locus of the lines of $\mathbb{P}^r$, through P, and having with F in P multiplicity of intersection more or equal than three.
It is well known that are possible two cases:

- A(P) is the tangent hyperplane to F in P and in this case we say that P is a flex point for F;
- A(P) is quadric cone , having a double point in P, lying in the tangent hyperplane to F in P, called the *asymptotic cone* to F in P.

1.6. Definition. A simple point P is called parabolic if it is a flex point or if $A(P)$ is specialized. A simple point P is called h-parabolic for F, with $1 \le h \le r-1$, if the vertex of A(P) is a linear subspace h-dimensional of the tangent hyperplane to F in P, and $(r-1)$-parabolic if is a flex point for F.

1.7. Theorem. *The hessian* He(F) *of an hypersurface F of order n in $\mathbb{P}^r$ contains the singular points of F and then contains all the simple points of F, that for the hypersurface are flexes (if $r = 2$) or parabolic points (if $r \ge 3$).*

Proof. Let z be a simple point of F, that belongs to He(F). By lemma 1.5, $P_{z^{n-2}}(F)$ is singular in a point v. It is at least a double point, i.e. $P_{z^{n-2}}(F)$ is a quadric cone of vertex v. Hence, $P_{z^{n-2}}(F)$ has a simple point in z, so v and z are distinct and $P_{z^{n-2}}(F)$ and F have the same tangent hyperplane π. We can distinguish two cases: $r = 2$ or $r \ge 3$.
If $r = 2$, π is the line zv. This line has intersection multiplicity at least three with F in the point z. So z is a flex point for F. Vice versa if z is a flex, this line is the tangent to F in z and belongs to $P_{z^{n-2}}(F)$, then $P_{z^{n-2}}(F)$ is degenerate and $z \in$ He(F).
If $r \ge 3$, the line yz is a double line for $P_{z^{n-2}}(F) \cap \pi$ or π is a component of $P_{z^{n-2}}(F)$. So, if we exclude this case, $P_{z^{n-2}}(F) \cap \pi$ is a quadric with vertex a line at least, and the section of $P_{z^{n-2}}(F)$ with π, let $P'_{z^{n-2}}(F)$, is the locus of the lines having intersection multiplicity at least three with F in z. Then we have that $P_{z^{n-2}}(F)$ is a cone with vertex a line at least. Hence $z \in$ He(F). □

1.8. *Remark.* The hessian He(F) has in $P = z$ a multiple point if $h > 1$. But P can be a multiple point for He(F) also if it is a 1-parabolic point for F.

More precisely we have:

1.9. Theorem (Bompiani)**.** *A* 1*-parabolic point $P = z$ is multiple for* He(F) *if and only if on the vertex r of the asymptotic cone in P, $A(P)$ there exists a point $B = v$ such that:*

1. *$P_{v^1}(F)$ has in z a double point;*
2. *the osculating cone to $P_{v^1}(F)$ in z contains r with multiplicity two.*

Let F^* be the dual variety of F contained in the dual space of $\mathbb{P}^r(\mathbb{C})$.
We have $\dim F^* = r - h - 1$, with $h \in \mathbb{N}$ if and only if F is an hypersurface with h-parabolic points (see [BS1]).
Hence, if F is an hypersurface with h-parabolic points then it is made by an irreducible algebraic system $\Sigma(F)$, of dimension $r-h-1$, of h-subspace of $\mathbb{P}^r$ and in every simple point, belonging to the generic subspace of $\Sigma(F)$, F has the same tangent hyperplane.
By using theorem 1.9 Franchetta proves:

1.10. Theorem. [F1] *If V_{r-1} is an irreducible algebraic hypersurface of $\mathbb{P}^r$ with 1-parabolic points, condition necessary and sufficient for which it is a multiple component of its hessian is that every plane through the generic line r of $\Sigma := \Sigma(F)$ and not contained in the tangent hyperplane to V_{r-1} along r, intersects V_{r-1}, out of r, in a curve having with r one and only one point in common.*

And it follows:

1.11. Corollary. *If r is the generic line of Σ, we have:*

1. *the foci of the system Σ, belonging to r, coincide in a one and only one point;*
2. *the lines of Σ, cutting the line r, meet this line in the focus belonging to r.*

For the general theory of foci see [DePI].

2. Algebraic irreducible hypersurfaces of $\mathbb{P}^5$ with 1-parabolic points, multiple component of their hessian and with scheme of foci of dimension 3.

We remember:

2.1. Definition. The asymptotic lines of a variety of dimension m of $\mathbb{P}^r$ are curves such that the tangent space to the variety and the $\mathbb{P}^m$-osculating to the curve coincide.

2.2. Theorem. [[I2], lemma 21] *An algebraic irreducible hypersurface of $\mathbb{P}^r(\mathbb{C})$ with 1-parabolic points, multiple component of its hessian and such that the scheme of foci W has dimension $r-2$ is the locus of the tangents of a system of asymptotic lines of W.*

2.3. Theorem. [[I2], theorem 19] *An algebraic irreducible hypersurface of $\mathbb{P}^r(\mathbb{C})$ with 1-parabolic points, multiple component of its hessian and such that the scheme of foci W has dimension $r-2$, cannot have undetermined hessian.*

We'll prove:

2.4. Theorem. *An algebraic irreducible hypersurface of $\mathbb{P}^5$ with 1-parabolic points, contained with multiplicity in its hessian and with scheme of foci of dimension three, if exists, has degree at least sixteen.*

Proof. Let V_4^n be the considered hypersurface, of order n and let W be a 3-dimensional component of the foci locus.
By theorem 2.2 W is a variety of dimension three, with a system of asymptotic lines and V_4^n is the locus of the tangents to these lines.
So, by theorem 1.10, the multiplicity of intersection of line r of Σ with the residue intersection C^{r-1} of V_4 with a generic plane for r, in the focus belonging to r, is equal to $r-1$.
We remark that theorem 2.4 is proved if we verify that C^{r-1} passes through the focus with three linear branches, having contact of order at least four with r, i.e. intersection multiplicity at least five with the line in the focus. Then, the degree of V_4^r, if exists, is at least 16.
To verify the assert we can choose affine coordinates and we can suppose that W, variety of foci locus of Σ, is represented in a neighborhood of his generic point O, by the parametric equations:

$$x^i = c^i(u_1, u_2, u_3) \qquad (i = 1, \dots 5)$$

where O corresponds to $u_1 = u_2 = u_3 = 0$ and the c^i are defined and analytic in a neighborhood D of the origin.
We denote with the indices in height the components of the vector $\mathbf{c}$, while with an index in low the derivation respect to u_i.
We choose the parameters u_1, u_2, u_3 so that the lines $du_2 = du_3 = 0$ are the asymptotic lines of W and we suppose, furthermore, that the coordinate system of $\mathbb{C}^5$ has the origin in the point O and that the tangent space to W in O has equation $x^4 = x^5 = 0$.

We consider the matrices :

$$\begin{pmatrix} \mathbf{c_1} \\ \mathbf{c_2} \\ \mathbf{c_3} \\ \mathbf{c_{11}} \end{pmatrix} \tag{1}$$

and

$$\begin{pmatrix} \mathbf{c_1} \\ \mathbf{c_2} \\ \mathbf{c_3} \\ \mathbf{c_{111}} \end{pmatrix} \tag{2}$$

Since the lines $du_2 = du_3 = 0$ are asymptotic then these matrices have all the minors of maximum order identically zero.
By the hypotheses it follows:

$$\begin{aligned} c_1^2(0) = c_1^3(0) = c_1^4(0) = c_1^5(0) = 0 \\ c_2^1(0) = c_2^3(0) = c_2^4(0) = c_2^5(0) = 0 \\ c_3^1(0) = c_3^2(0) = c_3^4(0) = c_3^5(0) = 0 \end{aligned} \tag{3}$$

In the origin and so in its neighborhood that we can suppose coinciding with D, we have:

$$\begin{vmatrix} \mathbf{c_1} \\ \mathbf{c_2} \\ \mathbf{c_3} \end{vmatrix} \neq 0 \tag{4}$$

Let:

$$c^i(u_1, u_2, u_3) = \alpha_1^i u_1 + \alpha_2^i u_2 + \alpha_3^i u_3 + \alpha_{11}^i u_1^2 + \alpha_{22}^i u_2^2 + \alpha_{33}^i u_3^2 + \alpha_{12}^i u_1 u_2 + \dots$$

for $i = 1, \dots 5$, be the Taylor expansion of the series of c^i. Then by (3) we have:

$$\begin{aligned} \alpha_1^2 = \alpha_1^3 = \alpha_1^4 = \alpha_1^5 = 0 \\ \alpha_2^1 = \alpha_2^3 = \alpha_2^4 = \alpha_2^5 = 0 \\ \alpha_3^1 = \alpha_3^2 = \alpha_3^4 = \alpha_3^5 = 0 \end{aligned}$$

We can suppose:

$$\alpha_1^1 = \alpha_2^2 = \alpha_3^3 = 1$$

Putting equal to zero the first and the second minor of matrix (1), for $i = 4, 5$, we have:

$$\alpha^i_{11} = 0$$
$$\alpha^i_{111} = \frac{1}{3}\alpha^2_{11}\alpha^i_{12} + \frac{1}{3}\alpha^3_{11}\alpha^i_{13}$$
$$\alpha^i_{112} = \alpha^1_{11}\alpha^i_{12} + 2\alpha^2_{11}\alpha^i_{22} + \alpha^3_{11}\alpha^i_{23}$$
$$\alpha^i_{113} = \alpha^1_{11}\alpha^i_{13} + \alpha^2_{11}\alpha^i_{23} + 2\alpha^3_{11}\alpha^i_{33}$$
$$\alpha^i_{1111} = \frac{1}{2}\alpha^2_{111}\alpha^i_{12} - \frac{1}{2}\alpha^1_{11}\alpha^i_{111} - \frac{1}{2}\alpha^2_{12}\alpha^i_{111} + \frac{1}{6}\alpha^2_{11}\alpha^i_{112} + \frac{1}{2}\alpha^3_{111}\alpha^i_{13} - \frac{1}{2}\alpha^3_{13}\alpha^i_{111} + \frac{1}{6}\alpha^3_{11}\alpha^i_{113} + \frac{1}{6}\alpha^2_{11}\alpha^3_{13}\alpha^i_{12} - \frac{1}{6}\alpha^2_{11}\alpha^3_{12}\alpha^i_{13} + \frac{1}{6}\alpha^3_{11}\alpha^2_{12}\alpha^i_{13} - \frac{1}{6}\alpha^3_{11}\alpha^2_{13}\alpha^i_{12}$$

$$\ldots\ldots$$

Putting equal to zero the first and the second minor of matrix (2), for $i = 4, 5$, we have:

$$\alpha^i_{111} = 0$$
$$\alpha^i_{1111} = \frac{1}{4}\alpha^2_{111}\alpha^i_{12} + \frac{1}{4}\alpha^3_{111}\alpha^i_{13}$$
$$\alpha^i_{1112} = \alpha^1_{111}\alpha^i_{12} + 2\alpha^2_{111}\alpha^i_{22} + \alpha^3_{111}\alpha^i_{23}$$
$$\alpha^i_{1113} = \alpha^1_{111}\alpha^i_{13} + \alpha^2_{111}\alpha^i_{23} + 2\alpha^3_{111}\alpha^i_{33}$$
$$\alpha^i_{11111} = \frac{1}{10}(4\alpha^2_{1111}\alpha^i_{12} - 8\alpha^1_{11}\alpha^i_{1111} - 4\alpha^2_{12}\alpha^i_{1111} + \alpha^2_{111}\alpha^i_{112} + 4\alpha^3_{1111}\alpha^i_{13} - 4\alpha^1_{13}\alpha^i_{1111} + \alpha^3_{111}\alpha^i_{113} + 2\alpha^1_{11}\alpha^2_{111}\alpha^i_{12} + 2\alpha^1_{11}\alpha^3_{111}\alpha^i_{13} + \alpha^2_{111}\alpha^3_{13}\alpha^i_{12} - \alpha^2_{111}\alpha^3_{12}\alpha^i_{13} + \alpha^2_{12}\alpha^3_{111}\alpha^i_{13} - \alpha^1_{13}\alpha^3_{111}\alpha^i_{12})$$

$$\ldots\ldots$$

Then it follows:

$$\alpha^i_{11} = 0$$

$$\alpha^i_{111} = 0$$

$$\alpha^2_{11}\alpha^i_{12} + \alpha^3_{11}\alpha^i_{13} = 0$$

$$\ldots\ldots$$

......

The tangent t to the asymptotic line in the variable point $Q(u_1, u_2, u_3)$ of W is represented by the equations:

$$\frac{x^1 - c^1}{c_1^1} = \frac{x^2 - c^2}{c_1^2} = \frac{x^3 - c^3}{c_1^3} = \frac{x^4 - c^4}{c_1^4} = \frac{x^5 - c^5}{c_1^5}.$$

These equations represent also V_4 because this hypersurface is described by these tangents.
We study, now, the section C of V_4 with the plane π that we suppose represented by the equations:

$$x^2 = x^3 = x^5 = 0$$

The curve that we want to study is the locus of the common points to π and to the following lines:

$$x^i = c^i + \rho c_1^i, \qquad i = 1, \ldots, 5$$

We can suppose that π is represented by the equations $x^2 = x^3 = x^5 = 0$. Then it is incident to π if and only if

$$c^2 + \rho c_1^2 = c^3 + \rho c_1^3 = c^5 + \rho c_1^5 = 0$$

and if we consider the matrix $\begin{pmatrix} \mathbf{c} \\ \mathbf{c_1} \end{pmatrix}$ and we denote with Δ_{ij} its minor determined by the columns of places i, j with $1 \le i < j \le 5$, the previous conditions become:

$$\begin{cases} \Delta_{23} = 0 \\ \Delta_{25} = 0 \\ \Delta_{35} = 0 \end{cases} \tag{5}$$

The curve C has parametric equation:

$$x^1 = \frac{\Delta_{12}}{c_1^2} \quad , \quad x^4 = -\frac{\Delta_{24}}{c_1^2} \tag{6}$$

By intersecting the tangent cones to the three surfaces found out by the condition of incidence (5), we obtain the line :

$$\begin{cases} u_2 = 0 \\ u_3 = 0 \end{cases}$$

Hence the branches have the form:

$$\begin{cases} u_2 = A_2\ u_1^2 + B_2\ u_1^3 + \dots \\ u_3 = A_3\ u_1^2 + B_3\ u_1^3 + \dots \end{cases}$$

Replacing in (5) we have three branches of type:

$$(7)\qquad \begin{cases} u_2 = \ i\alpha_{11}^2 u_1^2 + \tfrac{1}{2}(1+3i)\alpha_{111}^2 u_1^3 - \tfrac{1}{2}(1+i)\alpha_{13}^2\alpha_{11}^3 u_1^3 + \dots \\ u_3 = i\alpha_{11}^3 u_1^2 + \dfrac{1}{2}(1+3i)\alpha_{111}^3 u_1^3 + \dfrac{1}{2}(1+i)\alpha_{12}^2\alpha_{11}^3 u_1^3 \\ \qquad - \dfrac{1}{2}(1+i)\alpha_{11}^2\alpha_{12}^3 u_1^3 - \dfrac{1}{2}(1+i)\alpha_{11}^3\alpha_{13}^3 u_1^3 + \dots \end{cases}$$

$$(8)\qquad \begin{cases} u_2 = \ -i\alpha_{11}^2 u_1^2 + \tfrac{1}{2}(1-3i)\alpha_{111}^2 u_1^3 - \tfrac{1}{2}(1-i)\alpha_{13}^2\alpha_{11}^3 u_1^3 + \dots \\ u_3 = \ -i\alpha_{11}^3 u_1^2 + \dfrac{1}{2}(1-3i)\alpha_{111}^3 u_1^3 + \dfrac{1}{2}(1-i)\alpha_{12}^2\alpha_{11}^3 u_1^3 \\ \qquad - \dfrac{1}{2}(1-i)\alpha_{11}^2\alpha_{12}^3 u_1^3 - \dfrac{1}{2}(1-i)\alpha_{11}^3\alpha_{13}^3 u_1^3 + \dots \end{cases}$$

$$(9)\qquad \begin{cases} u_2 = \tfrac{1}{2}\alpha_{111}^2 u_1^3 + \dots \\ u_3 = \tfrac{1}{2}\alpha_{111}^3 u_1^3 + \dots \end{cases}$$

Replacing (9) in (6) we obtain:

$$\begin{aligned} x_1 =& \frac{1}{2}u_1 + \dots \\ x_4 =& -\ \frac{\alpha_{111}^2}{2\alpha_{11}^2}[(\alpha_{33}^4\alpha_{12}^5 - \alpha_{12}^4\alpha_{33}^5)(\alpha_{13}^2\alpha_{11}^3 - \alpha_{11}^2\alpha_{13}^3) \\ & - (\alpha_{33}^4\alpha_{13}^5 - \alpha_{13}^4\alpha_{33}^5)(\alpha_{12}^2\alpha_{11}^3 - \alpha_{11}^2\alpha_{12}^3)]u_1^5 + \dots \end{aligned}$$

Replacing (7) in (6) we obtain:

$$\begin{aligned} x_1 =& \frac{(1-i)}{2}u_1 + \dots \\ x_4 =& -\ \frac{i\alpha_{11}^2}{2}[(\alpha_{33}^4\alpha_{12}^5 - \alpha_{12}^4\alpha_{33}^5)(\alpha_{13}^2\alpha_{11}^3 - \alpha_{11}^2\alpha_{13}^3) \\ & - (\alpha_{33}^4\alpha_{13}^5 - \alpha_{13}^4\alpha_{33}^5)(\alpha_{12}^2\alpha_{11}^3 - \alpha_{11}^2\alpha_{12}^3)]u_1^5 + \dots \end{aligned}$$

Replacing (8) in (6) we obtain:

$$\begin{aligned}
x_1 =& \frac{(1+i)}{2} u_1 + \dots \\
x_4 =& \frac{i\alpha_{11}^2}{2} [(\alpha_{33}^4 \alpha_{12}^5 - \alpha_{12}^4 \alpha_{33}^5)(\alpha_{13}^2 \alpha_{11}^3 - \alpha_{11}^2 \alpha_{13}^3) \\
& - (\alpha_{33}^4 \alpha_{13}^5 - \alpha_{13}^4 \alpha_{33}^5)(\alpha_{12}^2 \alpha_{11}^3 - \alpha_{11}^2 \alpha_{12}^3)] u_1^5 + \dots
\end{aligned}$$

Then we have proved that C passes through the origin with three linear branches, having contact of order at least four, with the line $x^2 = x^3 = x^4 = x^5 = 0$, so the result. □

We remark that the above calculation can be generalized without changes for every r, obtaining the proof of the following theorem:

2.5. Theorem. *An algebraic irreducible hypersurface of $\mathbb{P}^r(\mathbb{C})$, with 1 -parabolic points, multiple component of its hessian and with scheme of foci of dimension $r-2$, if exists, has degree at least $(r-1)^2$.*

A. Franchetta in [F1] conjectured that the degree is exactly $(r-1)^2$.

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