STABILIZERS FOR NONDEGENERATE MATRICES OF BOUNDARY FORMAT AND STEINER BUNDLES

CARLA DIONISI

ABSTRACT. In this paper nondegenerate multidimensional matrices of boundary format in $V_0 \otimes \cdots \otimes V_p$ are investigated by their link with Steiner vector bundles on product of projective spaces. In particular it is proved that for any nondegenerate matrix A there exists an explicit action of $SL(2)$ on $V_0 \otimes \cdots \otimes V_p$ and, describing the behaviour of its stabilizer in $SL(V_0) \times \cdots \times SL(V_p)$, we prove that $Stab^0(A) \subseteq SL(2)$ and that the equality holds if and only if A belongs to a unique $SL(V_0) \times \cdots \times SL(V_p)$-orbit containing the identity matrices, according to [AO].

1. INTRODUCTION

Let V_j be a complex vector space of dimension $k_j + 1$ for $j = 0, \ldots, p$ with $k_0 = \max_i\{k_i\}$. Gelfand, Kapranov and Zelevinsky in [GKZ] proved that the dual variety of the Segre product $\mathbb{P}(V_0) \times \cdots \times \mathbb{P}(V_p)$ is a hypersurface in $(\mathbb{P}^{(k_0+1)\ldots(k_p+1)-1})^\vee$ if and only if $k_0 \leq \sum_{i=1}^{p} k_i$. The defining equation of this hypersurface is called the *hyperdeterminant* of format $(k_0+1) \times \cdots \times (k_p+1)$ and is denoted by Det. Moreover the hyperdeterminant is a homogeneous polynomial function on $V_0^* \otimes \cdots \otimes V_p^*$ so that the condition $DetA \neq 0$ is meaningful for a $(p+1)$-dimensional matrix $A \in \mathbb{P}(V_0 \otimes \cdots \otimes V_p)$ of format $(k_0+1) \times \cdots \times (k_p+1)$. The hyperdeterminant is an invariant for the natural action of $SL(V_0) \times \cdots \times SL(V_p)$

1991 *Mathematics Subject Classification.* 14F05; 15A72.

Key words and phrases. Vector bundles, multidimensional matrices, theory of invariants.

on $\mathbb{P}(V_0 \otimes \cdots \otimes V_p)$, and, in particular, if $DetA \neq 0$ then A is semistable for this action.

We denote by $Stab(A) \subset SL(V_0) \times \cdots \times SL(V_p)$ the stabilizer subgroup of A and by $Stab(A)^0$ its connected component containing the identity. The stabilizer are well known for $p \leq 1$, so that in this paper we assume $p \geq 2$.

It easy to check (see [WZ], [DO]) that the degenerate matrices fill an irreducible variety of codimension $k_0 - \sum_{i=1}^{p} k_i + 1$ and if $k_0 < \sum_{i=1}^{p} k_i$ then all matrices are degenerate. We will assume from now on that A is of **boundary format** i.e. that $k_0 = \sum_{i=1}^{p} k_i$. (A self-contained approach to hyperdeterminant of boundary format matrices can be found in [DO]).

The main result of this paper is the following:

1.1. Theorem. *Let $A \in \mathbb{P}(V_0 \otimes \cdots \otimes V_p)$ be a boundary format matrix with $DetA \neq 0$.*
Then there exists a 2-dimensional vector space U such that $SL(U)$ acts over $V_i \simeq S^{k_i}U$ and according to this action on $V_0 \otimes \cdots \otimes V_p$ we have $Stab(A)^0 \subseteq SL(U)$. Moreover $Stab(A)^0 = SL(2)$ if and only if A is an identity (defined in 2.3).

1.2. *Remark.* We emphasize that $SL(V_0) \times \ldots \times SL(V_p)$ is a "big" group, so it is quite surprising that the stabilizer found lies always in the 3-dimensional group $SL(U)$ without any dependence on p and on $\dim V_i$.

The maximal stabilizer is obtained by the "most symmetric" class of matrices corresponding to the identity matrices. Under the identifications $V_i = S^{k_i}U$ the identity is given by the natural map

$$S^{k_1}U \otimes \ldots \otimes S^{k_p}U \to S^{k_0}U$$

which is defined under the assumption $k_0 = \sum k_i$. This explains again why the condition of boundary format is so important.

Ancona and Ottaviani in [AO] prove theorem 1.1 statement for $p = 2$. We generalize their proof using the correspondence between nondegenerate boundary format matrices and vector bundles on a product of projective spaces.
Indeed, for any fixed $j \neq 0$, a $(p+1)$-dimensional matrix $A \in V_0 \otimes \cdots \otimes V_p$ of format $(k_0+1) \times \cdots \times (k_p+1)$ defines a sheaf morphism ${f_A}^{(j)}$ on the product $X = \mathbb{P}^{k_1} \times \cdots \times \widehat{\mathbb{P}^{k_j}} \times \cdots \times \mathbb{P}^{k_p}$

$$\mathcal{O}_X \otimes V_0^{\vee} \xrightarrow{{f_A}^{(j)}} \mathcal{O}_X(1, \ldots, 1) \otimes V_j; \tag{1}$$

it is easy to prove the following

1.3. Proposition. *(*[AO],[D]*) If a matrix A is of boundary format, then $DetA \neq 0$ if and only if for all $j \neq 0$ the morphism ${f_A}^{(j)}$ is surjective (so ${S^*_A}^{(j)} = Ker {f_A}^{(j)}$ is a vector bundle of rank $k_0 - k_j$).*

In the particular case $p = 2$ the (dual) vector bundle ${S_A}^{(1)}$ lives on the projective space $\mathbb{P}^n, n = k_2$ and it is a Steiner bundle as defined in [DK] (this case has been investigate in [AO]). We shall call to S_A with the name Steiner also for $p \geq 3$.
The main new technique introduced in this paper is the use of jumping hyperplanes for bundles on the product of $p-1$ projective spaces. For $p \geq 2$ there are two natural ways to introduce them that, by above correspondence, translate into two different conditions on the associated matrix and that we call weak and strong (see definition 3.1 and 3.7). They coincide when $p = 2$.

Moreover, weak and strong jumping hyperplanes are invariants of the action of $SL(V_0) \times \ldots \times SL(V_p)$ on matrices. By investigating these invariants we derive the proof of theorem 1.1 and also we obtain a characterization of a particular class of bundles called Schwarzenberger bundles (see [Sch] for the original definition in the case $p = 2$). Schwarzenberger bundles correspond exactly to such matrices A which verify the equality $Stab^0(A) = SL(2)$ in theorem 1.1, called identity matrices.

I would like to thank G. Ottaviani for many useful discussions and for his invaluable guidance.

2. Multidimensional matrices of boundary format and geometric invariant theory

In this section we recall some results proved in [AO] about multidimensional matrices from the point of view of Mumford's Geometric Invariant Theory.

2.1. Definition. [AO] A $(p+1)$-dimensional matrix of boundary format $A \in V_0 \otimes \cdots \otimes V_p$ is called **triangulable** if one of the following equivalent conditions holds:

(i) there exist bases in V_j such that $a_{i_o,\ldots,i_p} = 0$ for $i_0 > \sum_{t=1}^{p} i_t$;
(ii) there exist a vector space U of dimension 2, a subgroup $\mathbb{C}^* \subset SL(U)$ and isomorphisms $V_j \simeq S^{k_j}U$ such that if $V_0 \otimes \cdots \otimes V_p = \bigoplus_{n\in\mathbb{Z}} W_n$ is the decomposition into direct sum of eigenspaces of induced representation, then we have $A \in \bigoplus_{n\geq 0} W_n$

2.2. Definition. [AO] A $(p+1)$-dimensional matrix of boundary format $A \in V_0 \otimes \cdots \otimes V_p$ is called **diagonalizable** if one of the following equivalent conditions holds:

(i) there exist bases in V_j such that $a_{i_o,\ldots,i_p} = 0$ for $i_0 \neq \sum_{t=1}^{p} i_t$;
(ii) there exist a vector space U of dimension 2, a subgroup $\mathbb{C}^* \subset SL(U)$ and isomorphisms $V_j \simeq S^{k_j}U$ such that A is a fixed point of the induced action of $\mathbb{C}^*$

2.3. Definition. [AO] A $(p+1)$-dimensional matrix of boundary format $A \in V_0 \otimes \cdots \otimes V_p$ is an **identity** if one of the following equivalent conditions holds:

(i) there exist bases in V_j such that

$$a_{i_o,\dots,i_p} = \begin{cases} 0 & \text{for} \quad i_0 \neq \sum_{t=1}^{p} i_t \\ 1 & \text{for} \quad i_0 = \sum_{t=1}^{p} i_t \end{cases}$$

(ii) there exist a vector space U of dimension 2 and isomorphisms $V_j \simeq S^{k_j}U$ such that A belongs to the unique one dimensional $SL(U)$-invariant subspace of $S^{k_0}U \otimes \cdots \otimes S^{k_p}U$

All identity matrices fill up a distinguished $SL(V_0)\times\cdots\times SL(V_p)$-orbit in $\mathbb{P}(V_0 \otimes \ldots \otimes V_p)$.

2.4. Theorem. [AO] *Let $A \in \mathbb{P}(V_0 \otimes \ldots \otimes V_p)$ of boundary format such that $DetA \neq 0$. Then*

$$A \textit{ is triangulable} \iff A \textit{ is not stable for the action of } SL(V_0)\times\ldots\times SL(V_p)$$

2.5. Theorem. [AO] *Let $A \in \mathbb{P}(V_0 \otimes \ldots \otimes V_p)$ of boundary format such that $DetA \neq 0$. Then*

$$A \textit{ is diagonalizable} \iff \mathbb{C}^* \subset Stab(A)$$

3. Jumping hyperplanes and stabilizers

Let $p = 2$ and $S := S^1$ be the Steiner bundle on $\mathbb{P}(V_2)$ defined by a matrix $A \in V_0 \otimes V_1 \otimes V_2$ boundary format, an hyperplane $h \in \mathbb{P}(V_2^*)$ is an unstable hyperplane of S if $h^0(S^*_{|h}) \neq 0$ (see [AO]).

In particular, $\mathrm{H}^0(S^*(t))$ identifies to the space of $(k_0+1)\times 1$-column vectors v with entries in S^tV_2 such that $Av = 0$, and a hyperplane h is unstable for S if and only if there are nonzero vectors v_0 of size $(k_0+1)\times 1$ and v_1 of size $(k_1+1)\times 1$ both with constant coefficients such that

$$Av_0 = v_1 h; \tag{2}$$

the tensor $\mathcal{H} = v_0 \otimes v_1$ is called an unstable (or jumping) hyperplane for the matrix A.

For $p \geq 3$ there are at least two ways to define a jumping hyperplane. We will call them weak and strong jumping hyperplanes.

3.1. Definition. $\mathcal{H} = v_0 \otimes v_j \in V_0 \otimes V_j$ is a (j)**-weak jumping hyperplane** for A if $\exists\ v_0, w_1, \dots, w_{k_0}$ basis of V_0 such that

$$A = v_0 \otimes v_j \otimes h + \sum_{i=1}^{k_0} w_i \otimes \dots \tag{3}$$

where $h \in V_1 \otimes \cdots \otimes \widehat{V_j} \otimes \cdots \otimes V_p$ is an hyperplane for $\mathbb{P}^{k_1} \times \cdots \times \widehat{\mathbb{P}^{k_j}} \times \cdots \times \mathbb{P}^{k_p} \subset \mathbb{P}(V_1 \otimes \cdots \otimes \widehat{V_j} \otimes \cdots \otimes V_p)$.

3.2. *Remark.* The expression (3) means, as in the case $p = 2$, that $H^0(Ker f_{A|h}^{(j)}) \neq 0$.

If $\mathcal{H} = v_0 \otimes v_j$ is a (j)-weak jumping hyperplane for A then the map:

$$\begin{aligned} V_0 \otimes \cdots \otimes V_p &\to (V_0/_{<v_0>}) \otimes \cdots \otimes (V_j/_{<v_j>}) \otimes \cdots \otimes V_p \\ A &\mapsto A'_j \end{aligned}$$

gives an elementary transformation [Mar82]

3.3. *Remark.* A'_j is again of boundary format

3.4. Proposition. *If A'_j is defined as above*

$$DetA \neq 0 \Rightarrow DetA'_j \neq 0$$

Proof. If $X := \mathbb{P}^{k_1} \times \cdots \times \widehat{\mathbb{P}^{k_j}} \times \cdots \times \mathbb{P}^{k_p}$ and h is the hyperplane defined in (3.1) associated to $\mathcal{H}$, the map ${S_A}^{(j)} \to \mathcal{O}_h$ induced by a non zero section of ${S_A}^{(j)}$ is surjective (see [V2] prop.2.1).
Since $\operatorname{codim} h = 1$, then its kernel $S'^{(j)}$ is locally free sheaf [Ser65] of rank $k_0 - k_j - 1$ on X and it is the Steiner bundle associated to the matrix $A^{(j)}$ as the snake-lemma applied to the following exact diagram

show

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & S'^{(j)} & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(-1,\dots,-1)\otimes V_j^{\vee} & \xrightarrow{f_A{}^{(j)}} & \mathcal{O}_X\otimes V_0^{\vee} & \longrightarrow & S_A{}^{(j)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(-1,\dots,-1) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_h & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

i.e. $S'^{(j)} = S_{A'_j}{}^{(j)}$ and by proposition (1.3) the result follows. $\square$

3.5. *Remark.* If $W(S_{A_j})$ is the set of j-weak jumping hyperplanes of A , then the exact sequence (dual to the last column of the above diagram)

$$0 \to S_A^{(j)\vee} \to S_{A'_j}^{(j)\vee} \to \mathcal{O}_X(1,\dots,1) \to 0$$

show that $W(S_{A_j}) \subset W(S_{A'_j}) \cup \{h\}$

3.6. *Remark.* An element $g \in Stab(A)$ preserves h and it induces $\bar{g} \in SL(V_0/_{<g(v_0)>}) \times SL(V_1) \times \cdots \times SL(V_j/_{<g(v_j)>}) \times \cdots \times SL(V_p)$ such that $g \cdot A$ projects to $\bar{g} \cdot A'_j$ and the elementary transformation behaves well with respect to the action of g.

3.7. Definition. $\mathcal{H} = v_0 \otimes v_1 \otimes \cdots \otimes v_p$ is a **strong jumping hyperplane** for A if $\exists v_0, w_1, \dots, w_{k_0}$ basis of V_0 such that

$$A = v_0 \otimes v_1 \otimes \cdots \otimes v_p + \sum_{i=1}^{k_0} w_i \otimes \dots$$

3.8. *Remark.* If $\mathcal{H}$ is a strong jumping hyperplane then $\mathcal{H}$ is a (j)-weak jumping hyperplane for all $j = 1, \ldots, p$; in particular for a strong jumping hyperplane there are many elementary transformations.

3.9. *Remark.* For $p = 2$ the notations of strong jumping hyperplane and of weak jumping hyperplane coincide with each other (see [AO]).

3.10. *Example.* (*the identity*) Fixed a basis $e_0^{(j)}, \ldots, e_{k_j}^{(j)}$ in V_j for all j, the identity matrix is represented by

$$I := \sum_{\substack{i_0 = i_1 + \cdots + i_p \\ 0 \le i_j \le k_j}} e_{i_0}^{(0)} \otimes \cdots \otimes e_{i_p}^{(p)}.$$

Let $t_0, \ldots, t_{k_0}$ be any distinct complex numbers. Let w be the $(k_0 + 1) \times (k_0 + 1)$ Vandermonde matrix whose (i, j) entry is $t_j^{(i-1)}$, so acting with w over V_0, we have:

$$e_j^{(0)} = \sum_{s=0}^{k_0} \bar{e}_s^{(0)} t_s^j$$

Then substituting

$$\begin{aligned} I &= \sum_{\substack{i_0 = i_1 + \cdots + i_p \\ s = 0, \ldots k_0}} \bar{e}_{i_0}^{(0)} t_s^{i_0} \otimes e_{i_1}^{(1)} \otimes \cdots \otimes e_{i_p}^{(p)} \\ &= \sum_{s=0}^{k_0} \bar{e}_s^0 \otimes (\sum_{i_1=0}^{k_1} e_{i_1}^{(1)} t_s^{i_1}) \otimes \cdots \otimes (\sum_{i_p=0}^{k_p} e_{i_p}^{(p)} t_s^{i_p}) \end{aligned}$$

Thus, since t_i have no restrictions, I has infinitely many strong jumping hyperplane.
We call *Schwarzenberger bundle* the vector bundle associated to I (in fact in the case $p = 2$ it is exactly the same introduced by Schwarzenberger in [Sch])(see also ([AO])

3.11. Proposition. *Let A be a boundary format matrix with $Det A \neq 0$. If A has $N \geq k_0 + 3$ strong jumping hyperplanes then it is an identity.*

Proof. In the case $p = 2$ the statement is proved in [AO] (theorem 5.13) or in [V2] (theorem 3.1). Chosen V_0 and other two vector spaces among $V_1, \dots, V_p$ (say V_1 and V_2), one may perform several elementary transformations with V_0 and all the others so that we get $A' \in V'_0 \otimes V_1 \otimes V_2$ boundary format matrix with $DetA' \neq 0$ and $N' \geq k'_0 + 3$ strong jumping hyperplanes, then A' is an identity.
As in the above example, one can change the hyperplane giving the elementary transformation, so that for all N strong jumping hyperplanes we get $t_1, \dots, t_N$ distinct complex numbers and corresponding suitable basis of V_1 and V_2 :

$$\bar{e}_0^{(1)} \dots \bar{e}_{k_1}^{(1)}$$
$$\bar{e}_0^{(2)} \dots \bar{e}_{k_2}^{(2)}$$

such that the hyperplanes are given by

$$\sum_{i=0}^{k_1} \bar{e}_i^{(1)} t_j^i \quad \text{and} \quad \sum_{i=0}^{k_2} \bar{e}_i^{(2)} t_j^i \quad \text{for} j = 1, \dots N$$

Now, changing V_1 and V_2 with the pairs V_1, V_j $(j = 1, \dots p\ )$ we get

$$A := \sum_{s=0}^{k_0} \bar{e}_s^0 \otimes (\sum_{i_1=0}^{k_1} e_{i_1}^{(1)} t_s^{i_1}) \otimes \dots \otimes (\sum_{i_p=0}^{k_p} e_{i_p}^{(p)} t_s^{i_p})$$

showing that A is an identity. □

3.12. Proposition. *Let A be a boundary format matrix with $DetA \neq 0$. If A has $k_0 + 2$ strong jumping hyperplanes then it is uniquely determined by these hyperplanes.*

Proof. In the case $p = 2$ the statement is proved in [AO] (theorem 5.3). Chosen V_0 and other two vector spaces among $V_1, \dots, V_p$ (say V_1 and V_2), one may perform several elementary transformations with V_0 and all the others so that we get $A' \in V'_0 \otimes V_1 \otimes V_2$ boundary format matrix with $DetA' \neq 0$ and $N' = k'_0 + 2$ strong jumping hyperplanes, then A'

is uniquely determined. Now, changing V_1 and V_2 with the pairs V_1 and V_j $(j = 2, \dots, p)$ we detect all the 3-dimensional submatrices of A which are all uniquely determined, so also A is determined. $\square$

3.13. *Remark.* In the case $p = 2$ we know that $k_0 + 2$ jumping hyperplanes give an existence condition for the bundle S_A (it is a logarithmic bundle, see [AO]) but in the case $p \geq 3$ there is not an analog existence result.

The following proposition is well known.

3.14. Proposition. *All nondegenerate matrices of type $2 \times k \times (k+1)$ are $GL(2) \times GL(k) \times GL(k+1)$ equivalent.*

Proof. Let A, A' two such matrices. They define two exact sequences on $\mathbb{P}^1$

$$0 \to \mathcal{O}(-k) \longrightarrow \mathcal{O}^{k+1} \xrightarrow{A} \mathcal{O}(1)^k \to 0$$

$$0 \to \mathcal{O}(-k) \longrightarrow \mathcal{O}^{k+1} \xrightarrow{A'} \mathcal{O}(1)^k \to 0$$

and easily we can find the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{O}(-k) & \longrightarrow & \mathcal{O}^{k+1} & \xrightarrow{A} & \mathcal{O}(1)^k & \to & 0 \\
 & & \downarrow{\scriptstyle 1} & \searrow & \downarrow{\scriptstyle f} & & & & \\
0 & \to & \mathcal{O}(-k) & \longrightarrow & \mathcal{O}^{k+1} & \xrightarrow{A'} & \mathcal{O}(1)^k & \to & 0
\end{array}$$

$\square$

The above proposition can be reformulated as follows:

3.15. Proposition. *Every surjective morphism of vector bundles on $\mathbb{P}^1$*

$$\mathcal{O}_{\mathbb{P}^1}^{k+1} \to \mathcal{O}_{\mathbb{P}^1}(1)^k$$

is represented by an identity matrix.

Proof of theorem 1.1

Proof. We proceed by induction on k_0. If $k_0 = 2$ the theorem is true by proposition 3.15. We may suppose that $Stab(A)^0$ has dimension ≥ 1 then by theorem 2.4 the matrix A is triangulable and this implies that there exists a strong jumping hyperplane $\mathcal{H}$.
We may, also, suppose that the number of jumping hyperplanes is finite otherwise A is an identity (proposition 3.11), hence $\mathcal{H}$ is $Stab(A)^0$-invariant. Let A_1' the image of A by the elementary transformation associated to the weak jumping hyperplane $v_0 \otimes v_1$ defined by $\mathcal{H}$ (we choose $j = 1$ to have simpler notations). The matrix A_1 belongs to $V_0' \otimes V_1' \otimes V_2 \otimes \cdots \otimes V_p$ where $V_0' = V_0/_{<v_0>}$ and $V_1' = V_1/< v_1 >$, it is nondegenerate and of boundary format then, by induction, there exists a 2-dimensional vector space U such that

$$V_0' \simeq S^{k_0-1}(U), \quad V_1' \simeq S^{k_1-1}(U) \quad \text{and} \quad V_i = S^{k_i}(U) \quad \text{for all} \quad i \geq 2$$

and $Stab(A_1')^0 \subseteq SL(U)$ (by using essentially the same argument we could work in $GL(V_0) \times \cdots \times GL(V_p)$).
Since A_1' is obtained from the matrix A after the choice of two directions, any element which stabilizes A also stabilizes A_1', so $Stab(A)^0 \subseteq Stab(A')^0$. Hence $Stab(A)^0 \subseteq SL(U)$ by the inductive hypothesis.
Now, we want to show that the action is performed according to the isomorphisms $V_i \simeq S^{k_i}U$. We remark that the above considered elementary transformation gives the decomposition $V_0 = V_0' \oplus \mathbb{C}$ and $V_1 = V_1' \oplus \mathbb{C}$. Since no other morphism of $SL(U)$ in $SL(V_0) \times SL(V_1) \times SL(S^{k_2}U) \times \cdots \times SL(S^{k_p}U)$ can give $S^{k_0-1}U \otimes S^{k_1-1}U \otimes S^{k_2}U \otimes \ldots \otimes S^{k_p}U$ as an invariant summand of $V_0 \otimes V_1 \otimes S^{k_2}U \otimes S^{k_p}U$, then the inclusion

$$\mathrm{Hom}(V_0', V_1' \otimes V_2 \otimes \cdots \otimes V_p) \subset \mathrm{Hom}(V_0, V_1 \otimes V_2 \otimes \ldots V_p)$$

can be identified with the $SL(U)$-invariant inclusion

$$S^{k_0-1}U \otimes S^{k_1-1}U \otimes S^{k_2}U \otimes \ldots \otimes S^{k_p}U \subset S^{k_0}U \otimes S^{k_1}U \otimes S^{k_2}U \otimes \ldots \otimes S^{k_p}U$$

according to the natural actions; hence also $V_0 = S^{k_0}U$ and $V_1 = S^{k_1}U$.

In the case $Stab(A)^0 = SL(2)$, the action of $SL(U)$ satisfies definition 2.3, proving that A is an identity. $\square$

3.16. *Remark.* Throughout this paper we work only on nondegenerate matrices. Indeed, in the proofs we apply the induction strategy (hence the results of [AO]) and the correspondence between matrices and vector bundles described in proposition 1.3.
The characterization of the stabilizer of degenerate matrices is still an open problem.
Another interesting problem is the study of the stabilizer of general multidimensional matrices (and not necessary of boundary format).

References

[AO] V. Ancona and G. Ottaviani, *Unstable hyperplanes for Steiner bundles and multidimensional matrices*, Advances in Geometry, 1 (2001), 165-192

[D] C. Dionisi, *Multidimensional matrices and minimal resolutions of vector bundles* Ph.D. thesis, Dip. Matematica "R.Caccioppoli", Università di Napoli, 2000.

[DO] C. Dionisi and G. Ottaviani, *The Binet-Cauchy theorem for the hyperdeterminant of boundary format multdimensional matrices*, pre-print 2001, math.AG/0104281

[DK] I. Dolgacev and M. M. Kapranov, *Arrangement of hyperplanes and vector bundles on $\mathbb{P}^n$*, Duke Math. J. 71 (1993), 633-664

[GKZ] I. M. Gelfand, M. M. Kapranov and A. V. Zelevinski, *Discriminants, resultants and multidimensional determinants*, Birkhaeuser, Boston 1994

[GKZ1] I. M. Gelfand, M. M. Kapranov and A. V. Zelevinski, *Hyperdeterminants*, Adv. in Math. (1992), no. 96, 226–263

[Mar82] M. Maruyama, *Elementary tranformations in the theory of algebraic vector bundles*, Lect. Notes Math. (1982), no. 961, 241–266.

[Par] P. G. Parfenov, *Orbits and their closures in the spaces $\mathbb{C}^{k_1} \otimes \cdots \otimes \mathbb{C}^{k_r}$*, Sbornik Math. (2001), **192**, 89–112 (English transl.)

[Sch] R.L.E. Schwarzenberger, *Vector bundles on the projective plane*, Proc.London Math.Soc. **3** (1961), no. 11, 623–640.

[Ser65] J.P. Serre, *Algèbre locale, multiplicités*, Lecture Notes in Mathematics, no. 11, Springer-Verlag, 1965.

[WZ] J. Weyman and A. V. Zelevinsky, *Singularities of hyperdeterminants*, Ann. Inst. Fourier 46 (1996), 591-644.

[V1] J. Vallès, *Fibrés de Schwarzenberger et coniques de droites sauteuses*, Bull. Soc. Math. France, **128**, (2000), 433-449.

[V2] J. Vallès, *Nombre maximal d'hyperplans instables pour un fibré de Steiner*, Math. Zeitschrift 233, 507-514 (2000).

Carla Dionisi, Dipartimento di Matematica Applicata "G. Sansone", Via S. Marta, 3, I-50139, Firenze, Italy

E-mail address: dionisi@dma.unifi.it