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MULTIDIMENSIONAL MATRICES AND MINIMAL RESOLUTIONS OF VECTOR BUNDLES

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Introduction

Cayley [Cay48] in 1848 found an explicit invariant of a given $2 \times 2 \times 2$ matrix which generalizes the usual determinant of a square matrix; for that invariant he proposed the name of hyperdeterminant. This notion was forgotten for many years; until recently Gelfand, Kapranov and Zelevinski rediscovered Cayley's results and gave a modern account of the whole subject (see the book [GKZ94] for a detailed story of this topic).

The hyperdeterminant of a multidimensional matrix $A \in V_0 \otimes \ldots \otimes V_p$ with $\dim V_i = k_i + 1$ exists if and only if $k_j \leq \sum_{i \neq j} k_i$ for all $j = 0, \ldots, p$ (theorem 1.1.2) and A is called of *boundary format* if $k_0 = \sum k_i$.

We are interested in multidimensional matrices because, in the case of boundary format and nondegenerate, they correspond to a class of vector bundles on a product of projective spaces which are called Steiner bundles (theorem 1.2.4). This class is interesting even when the base space is given by a single projective space.

Ancona and Ottaviani in [AO99] began a study of multidimensional matrices of boundary format $A \in V_0 \otimes \ldots \otimes V_p$ under the action of the reductive group $SL(V_0) \times \ldots \times SL(V_p)$ from the point of view of Mumford's Geometric Invariant Theory.

The action of $SL(V_0)\times\ldots\times SL(V_p)$ on such matrices translates to an action on the moduli space of the corresponding Steiner bundles, the invariants coincide and the stable points of both actions correspond under that translation. By investigating the properties and the invariants of the above actions, they proved that for $p\leq 2$ the stabilizer of a multidimensional matrix is contained in $SL(U)$ where U is a 2-dimensional vector space. The main result of the first chapter of the thesis is the proof that the assumption $p\leq 2$ can be dropped; it holds for every p (theorem 1.3.13).

We emphasize that $SL(V_0)\times\ldots\times SL(V_p)$ is a "big" group, so it is quite surprising that the stabilizer found lies always in the 3-dimensional group $SL(U)$ without any dependence on p and on $\dim V_i$.

The maximal stabilizer is obtained by the "most symmetric" class of matrices corresponding to the identity matrix. Under the identifications $V_i = S^{k_i}U$ the identity is given by the natural map

$$S^{k_1}U\otimes\ldots\otimes S^{k_p}U\to S^{k_0}U$$

which is defined under the assumption $k_0=\sum k_i$. This explains again why the condition of boundary format is so important.

When the base is a single projective space $\mathbb{P}(V_2)$ $(p=2)$, Steiner bundles S are defined by the following exact sequence

$$0\to V_1\otimes\mathcal{O}_{\mathbb{P}(V_2)}(-1)\to V_0\otimes\mathcal{O}_{\mathbb{P}(V_2)}\to S\to 0 \tag{1}$$

They are stable, they give smooth points in the moduli space of all stable bundles with the same rank and Chern classes, and they belong to an irreducible component of dimension $(k-1)(n-1)(n+k+1)$ where $k=k_1+1$ and $n=k_2$ ([AO94]).

This class of bundles was first studied by Schwarzenberger in 1961 [Sch61] who defined a particular class of bundles (called Schwarzenberger bundles) which corresponds, under the above correspondence, to the identity matrix. Since they have a "big" group of symmetry, their cohomological behavior is expected to be quite special.

In the section 1.4 we prove that if $\mathcal{S}$ is a Schwarzenberger bundle on $\mathbb{P}^n$

$$h^1(S^2\mathcal{S}^*)=n(n+2k-2)\quad\text{and}\quad h^2(S^2\mathcal{S}^*)\;=\binom{k-2}{2}\binom{n}{2}$$

In the second chapter of this thesis we restrict the base space to be $\mathbb{P}^2$, but we focus on a larger class of bundles $\mathcal{E}$, i.e. these bundles having free resolution

$$0\to\bigoplus_{i=1}^{k}\mathcal{O}_{\mathbb{P}^2}(-a_i)\to\bigoplus_{j=1}^{r+k}\mathcal{O}_{\mathbb{P}^2}(-b_j)\to\mathcal{E}\to 0. \tag{2}$$

(By (1), Steiner bundles are a particular case).

Since Horrocks theorem [Hor64] asserts that any torsion-free sheaf $\mathcal{F}$ on $\mathbb{P}^n$ has homological dimension at most $n-1$, thus on $\mathbb{P}^2$ any bundle has homological dimension 1, that is, has a free resolution of the type (2). Hence, the aim of the second chapter becomes to investigate stable vector bundles on the complex projective plane $\mathbb{P}^2$ by means of their minimal free resolution.

Bohnhorst and Spindler, in the paper [BS92], develop interesting techniques for the study of minimal free resolution of rank-n stable vector bundles on $\mathbb{P}^n$ having homological dimension 1. We partially adapt their results to the case of rank-r vector bundles on $\mathbb{P}^2$ with $r\geq 2$. In particular in proposition 2.2.4 we translate the condition of minimality of the resolution to a condition of admissibility of the pair (a,b), where $a=(a_1,\ldots,a_k)$ and $b=(b_1,\ldots,b_{k+r})$.

As Bohnhorst and Spindler observe, admissible pairs (a, b) define a stratification of the moduli space $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ by constructible subsets. In the section 2.2.1 (lemma 2.2.19) we estimate the codimension of such strata and we characterize the pair of the general element of the moduli space. As a corollary, a quite easy proof of the irreducibility of $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ follows (theorem 2.2.21) and some bounds about the regularity and the cohomology of its elements are found in 2.2.24 and 2.2.25.

All proofs work for rank-r vector bundles on $\mathbb{P}^2$ having strongly admissible pairs (see definition 2.2.17). The moduli space $\mathfrak{M}_{\mathbb{P}^2}(r, c_1, c_2)$ is irreducible for any r (the proofs with different techniques can be found in [Bar77a], [Ell83]), [HL93], [Pot79], [Mar78]) moreover we don't have examples of rank r- stable vector bundles on $\mathbb{P}^2$ which are not strongly admissible, so we give this conjecture:
"*the strong admissibility for the pairs of a holomorphic vector bundle $\mathcal{E}$ on the projective space is a necessary condition for the stability of $\mathcal{E}$*".

A proof of this conjecture would give as corollary also the irreducibility of $\mathfrak{M}_{\mathbb{P}^2}(r, c_1, c_2)$ for any rank r, by applying the same arguments of the proof of 2.2.20.

Finally, in the section 2.4, using the computer algebra system MACAULAY2 and an algorithm to compute admissible pair written in the Scheme dialect of Lisp, we give, in the cases $r = 2$, $c_1 = -1$ and $c_2 \leq 6$, a positive answer to the open problem of the explicit description of the filtration in constructible open subsets of moduli spaces $\mathfrak{M}_{\mathbb{P}^2}(r; c_1, c_2)$ and the results are summarized in a series of tables at the end of the thesis.

Many of the results from the second chapter are obtained in collaboration with Marco Maggesi.

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Chapter 1

Stabilizers for nondegenerate matrices of boundary format and Steiner bundles

1.1 Hyperdeterminants

1.1.1 Definition and basic properties

By an $(p+1)$-dimensional matrix we shall means an array $A = (a_{i_0,\dots,i_p})$ of complex numbers, where each index ranges over some finite set.

It is possible to define an analog of determinant of square $n \times n$ matrices also for some multidimensional matrices, called *hyperdeterminant*. Its study was initiated by Cayley [Cay48] and Schläfli [Sch52] but was largely abandoned for 150 years. Recently it is undertaken by Gelfand, Kapranov, Zelevinsky and

Weyman in [GKZ94], [GKZ92], [Wey94], [WZ96] and in this section we remind their results.

Let $p \geq 1$ be an integer and $0 \leq i_j \leq k_j$ then $A = (a_{i_0,\dots,i_p})$ is called $(p+1)$-complex matrix of format $(k_0+1)\times\cdots\times(k_p+1)$ and let $M = \mathbb{C}^{(k_0+1)\times\cdots\times(k_p+1)}$ be the spaces of these matrices. With greater generality we can fix $(p+1)$ complex vector spaces V_i for $i = 0,\dots,p$ of dimension k_i+1 and denote $M = V_0\otimes\cdots\otimes V_p$ (the context allows to avoid any ambiguity with the previous definition of M). The definition of the hyperdeterminant of A can be stated in geometric, analytic and algebraic terms ([GKZ94]).

Geometrically, consider the product $X = \mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_p}$ of several projective spaces in Segre enbedding into the projective space $\mathbb{P}^{(k_0+1)\dots(k_p+1)-1}$ (if $\mathbb{P}^{k_j}$ is the projectivization of the complex vector space V_j^* then the ambient projective space is $\mathbb{P}(V_0^*\otimes\cdots\otimes V_p^*)$). Let $X^\vee$ be the projective dual variety of X consisting of all hyperplanes in $\mathbb{P}^{(k_0+1)\dots(k_p+1)-1}$ tangent to X at some point. If $X^\vee$ is a hypersurface in $\mathbb{P}^{(k_0+1)\dots(k_p+1)-1^*}$ then its defining equation, which is a homogeneous polynomial function on $V_0^* \otimes \cdots \otimes V_p^*$ is called the *hyperdeterminant of format* $(k_0+1) \times \cdots \times (k_p+1)$ and denoted by Det. As usual, if $X^\vee$ is not an hypersurface, we set Det equal to 1, and refer to this case as *trivial*. If V_j is equipped which a basis then the element $f \in V_0 \otimes \cdots \otimes V_p$ is represented by a matrix $A = (a_{i_0,\dots i_p})$ as above, and so $DetA$ is a polynomial function of matrix entries. It is determined uniquely up to the sign by the requirement that the $Det(A)$ has integer coefficients and is irreducible over $\mathbb{Z}$.

Analytically, the hyperplane $f = 0$ belongs to $X^\vee$ if and only if f vanishes at some point of X with all first derivatives. If we consider the coordinate system

$x^{(j)} = (x_0^{(j)}, \dots, x_{k_j}^{(j)})$ on each V_j^* then $f \in V_0 \otimes \cdots \otimes V_p$ is represented after restriction on X by a multilinear form

$$F(A,x) := f(x^{(0)}, \dots, x^{(p)}) = \sum_{(i_0,\dots,i_p)} a_{i_0,\dots,i_p} x_{i_0}^{(0)} \otimes \cdots \otimes x_{i_p}^{(p)} \qquad (1.1)$$

In more invariant terms, we can think of A as an element of $V_0^* \otimes \cdots \otimes V_p^*$, or more geometrically, as a section of the vector bundle $\mathcal{O}(1,\dots,1)$ on the product of projective spaces $X = \mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_p}$. There is a natural left action of the group $G = GL(V_0) \times \cdots \times GL(V_p)$ on $V_0 \times \cdots \times V_p$ and a right action of G on M such that

$$F(Ag, x) = F(A, gx) \quad (A \in M, x \in V_0 \times \cdots \times V_p)$$

Therefore, the condition $Det(A) = 0$ means that the system of equations

$$f(x) = \frac{\partial f(x)}{\partial x_i^{(j)}} = 0 \qquad (1.2)$$

(for all i, j) has a solution $x = (x^{(0)}, \dots, x^{(p)})$ with all $x^{(j)} \neq 0$. We say that a multilinear form f (or a matrix A) satisfying this condition is *degenerate*.

Algebraically, the degeneracy of a form f can be easily characterized as follows. We denote by $\mathcal{K}(f)$ (or $\mathcal{K}(A)$) the set of points

$$x = (x^{(0)}, \dots, x^{(p)}) \in X = \mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_p}$$

such that

$$f(x^{(0)}, \dots, x^{(j-1)}, y, x^{(j+1)}, \dots, x^{(p)}) = 0$$

for every $j = 1, \dots, p$ and $y \in V_j^*$. We shall sometimes call $\mathcal{K}(A)$ the *kernel*. For a bilinear form $f(x,y)$ there is a notion of left and right kernels

$$\mathcal{K}_l(f) = \{x : f(x,y) = 0, \forall y\}, \qquad \mathcal{K}_r(f) = \{y : f(x,y) = 0, \forall x\}$$

and $\mathcal{K}(f) = \mathcal{K}_l(f) \times \mathcal{K}_r(f)$.

1.1.1 Proposition. *[GKZ94] A form f is degenerate if and only if $\mathcal{K}(f)$ is nonempty*

Proof. Computing the differential of f we see that $\mathcal{K}(f)$ is exactly the set of solution of (1.2) □

In particular, if $p = 1$ and so f is a bilinear form with a matrix A, the degeneracy of f just defined coincides with the usual notion of degeneracy and means that A is not of maximal rank. Obviously, this condition is of codimension 1 if and only if A is a square matrix, and in this case $Det(A)$ coincides with the ordinary determinant $det(A)$.

The fist natural question about hyperdeterminants is to describe all matrix for which $Det(A)$ is non-trivial, i.e. $X^\vee$ is a hypersurface, or in other words, the degeneracy of A is a codimension one condition. The matrices of such formats can be viewed as multidimensional generalizations of ordinary square matrices.

1.1.2 Theorem. *[GKZ94] The hyperdeterminant of format $(k_0+1) \times \cdots \times (k_p+1)$ exists if and only if*

$$k_j \leq \sum_{i \neq j} k_i \qquad (1.3)$$

for all j=0,..., p ((1.3) is called "polygon inequality")

Assuming that (1.3) holds, i.e. the hyperdeterminant of a matrix A is non-trivial, then the next property of Det follows at once from any of the definitions.

1.1.3 Theorem. *[GKZ94] The hyperdeterminant is relative invariant under the action of the group $GL(V_0) \times \cdots \times GL(V_p)$ (and so invariant under the action of $SL(V_0) \times \cdots \times SL(V_p)$).*

We shall identify the set of matrix (multi-)indices $I = \{(i_0, \dots, i_p) : 0 \leq i_j \leq k_j\}$ of a matrix A with the set of vertices of the product $\Delta^{k_0} \times \cdots \times \Delta^{k_p}$ of $(p+1)$ standard simplices, thus the submatrix of A correspond to the faces of $\Delta^{k_0} \times \cdots \times \Delta^{k_p}$. By a *slice in the j-direction* we mean the subset of all indices in I with the fixed j-th component, and also the corresponding submatrix of A. Two slice in the same direction are called *parallel*.

1.1.4 Corollary. *[GKZ94]*

(a) Interchanging two parallel slices leaves the hyperdeterminant invariant up to sign (which may equal 1);

(b) the hyperdeterminant is a homogeneous polynomial in the entries of each slice. The degree of homogeneity is the same for parallel slices;

(c) the hyperdeterminant does not change if we add to some slice a scalar multiple of a parallel slice;

(d) the gyperdeterminant of a matrix having two parallel slices proportional to each other is equal to 0. In particular, $Det(A) = 0$ if A has a zero slice.

1.1.5 Example. Now we will give an explicit form for the hyperdeterminant of the minimal 3-dimensional format $2 \times 2 \times 2$ (in this case the hyperdetermiant was already known to A.Cayley [Cay45]).

Let $A = (a_{ijk})$ be a matrix with $(i, j, k = 0, 1)$ then

$$\begin{aligned} Det(A) = {} & (a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{011}^2 a_{100}^2) \\ & - 2(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} + a_{000}a_{011}a_{100}a_{111} + \\ & + a_{001}a_{010}a_{101}a_{110} + a_{001}a_{011}a_{110}a_{100} + a_{010}a_{011}a_{101}a_{100}) + \\ & + 4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111}). \end{aligned} \tag{1.4}$$

Now we give a multidimensional generalization of the fact that the determinant is preserved by transposition of a matrix. For a matrix $A = (a_{i_0,\dots,i_p})$ of format $(k_0+1) \times \cdots \times (k_p+1)$ and a permutation σ of indices $0, \dots, p$ we denotes by $\sigma(A)$ the matrix of format $(k_{\sigma^{-1}(0)}+1) \times \cdots \times (k_{\sigma^{-1}(p)}+1)$, whose $(j_0, j_1, \dots, j_p)$-th entry is equal to $a_{j_{\sigma(0)},\dots,j_{\sigma(p)}}$. The following result is an immediate consequence of definitions.

1.1.6 Proposition. *[GKZ94] If A is degenerate then $\sigma(A)$ is degenerate for every permutation σ. If $Det(A)$ exists then $Det(\sigma(A))$ also exists and it is equal to $Det(A)$.*

Given A as above and $B = (b_{j_0,\dots,j_q})$ of format $(l_0+1) \times \cdots \times (l_q+1)$, if $k_p = l_0$ we may define the convolution (or product) $A * B$ of A and B as the $(p+q-1)$-dimensional matrix C of format $(k_0+1) \times \cdots \times (k_{p-1}+1)(l_1+1) \times \cdots \times (l_q+1)$ with entries

$$c_{i_0,\dots,i_{p-1},j_1,\dots,j_q} = \sum_{h=0}^{k_p} a_{i_0,\dots,i_{p-1},h} b_{h,j_1,\dots,j_q}.$$

Similarly, we can define the convolution $A *_{r,s} B$ with respect to a pair of indices r, s such that $k_r = l_s$.

1.1.7 Proposition. *[GKZ94] If A, B are degenerate then $A * B$ is also degenerate. In particular there exist polynomials $P(A,B)$ and $Q(A,B)$ in entries of A and B such that*

$$Det(A * B) = P(A,B)Det(A) + Q(A,B)Det(B)$$

Fix $p \geq 2$ and let $N(k_0, \ldots, k_p)$ be the degree of the hyperdeterminant of format $k_0 \times \cdots \times k_p)$ (if this hyperdeterminant is not defined we make the convention $N(k_0, \ldots, k_p) = 0$).

1.1.8 Theorem. *[GKZ94] The generating function for the degree $N(k_0, \ldots, k_p)$ is given by*

$$\sum_{k_0,\ldots,k_p \geq 0} N(k_0, \ldots, k_p) z_0^{k_0} \cdots z_p^{k_p} = \left(1 - \sum_{i=1}^{p} i e_i(z_0, \ldots, z_p)\right)^{-2} \tag{1.5}$$

where $e_i(z_0, \ldots, z_p)$ is the i-th elementary symmetric polynomial.

1.1.2 Hyperdeterminant of boundary format and its applications

1.1.9 Definition. A $(p+1)$-dimensional matrix $A = (a_{i_0,\ldots,i_p})$ of format $(k_0 + 1) \times \cdots \times (k_p + 1)$ is called **boundary format** if

$$k_0 = \sum_{i=1}^{p} k_i$$

(**interior format** if $k_0 < \sum_{i=1}^{p} k_i$)

In this case, the hyperdeterminant exists (by theorem 1.1.2) and its degree can be explicitly evaluate by following formula ([GKZ94]):

$$N(k_0, \ldots, k_p) = (k_1 + \cdots + k_p + 1)\binom{k_1 + \cdots + k_p}{k_1, \ldots, k_p} = \frac{(k_0+1)!}{k_1! \cdots k_p!} \tag{1.6}$$

Note that for $p = 1$ the boundary is just that of ordinary square matrices and (1.6) expresses the fact that the (ordinary) determinant of a $(k+1) \times (k+1)$ matrix has degree $k+1$.

In [GKZ94] Gelfand, Kapranov and Zelevinsky showed that if A is a boundary format matrix $Det(A)$ can be interpreted as the resultant of the system of multilinear forms, and so they gave a number of explicit formulas for $Det(A)$ similar to the classical Sylvester formula for the resultant of two binary forms. We consider p groups of variables $x = ({x_0}^{(j)}, \ldots, {x_{k_j}}^{(j)})$ for $1 \leq j \leq p$. Let $S(m_1, \ldots, m_p) := S^{m_1}V_1 \otimes \cdots \otimes S^{m_p}V_p$ denote the space of all polynomials in $x = (x^{(1)}, \ldots, x^{(p)})$ which are homogeneous of degree m_j in the variables of each group $x^{(j)}$. We shall view a matrix A as a collection of (k_0+1) multilinear forms $f_0, f_1, \ldots, f_{k_0} \in S(1, 1, \ldots, 1)$ corresponding to the slices of A in the 0-th direction:

$$f_{i_0} = \sum_{i_1,\ldots,i_p} a_{i_0,i_1,\ldots,i_p} {x_{i_1}}^{(1)} \cdots {x_{i_p}}^{(p)}. \tag{1.7}$$

1.1.10 Theorem. *[GKZ94] The hyperdeterminant $Det(A)$ of the matrix of the boundary format is equal to the resultant of the system of multilinear forms $f_0, f_1, \ldots, f_{k_0}$.In other words, A is degenerate if and only if the system of multilinear equations*

$$f_0(x) = f_1(x) = \cdots = f_{k_0}(x) = 0 \tag{1.8}$$

has non trivial solution.

Analyzing the conditions of degeneracy, theorem 1.1.10 admits following easy generalization to the case when $k_0 > k_1 + \cdots + k_p$

1.1.11 Theorem. *[GKZ94] Suppose that* $k_0 \geq k_1 + \cdots + k_p$*. Then* A *is degenerate if and only if the system (1.8) has a non-trivial solution. The subvariety of degenerate matrices has codimension* $k_0 - (k_1 + \cdots + k_p) + 1$.

Now, assuming that A is boundary format and

$$m_j = k_1 + k_2 + \cdots + k_{j-1}, \qquad j = 1, \ldots, p \tag{1.9}$$

(with the convection $m_1 = 0$, we associate to our matrix A the linear operator)

$$\partial_A : S(m_1, m_2, \ldots, m_p)^{k_0+1} \to S(1+m_1, 1+m_2, \ldots, 1+m_p)$$

given by $\partial_A(g_0, \ldots, g_{k_0}) = \sum_{i=0}^{k_0} f_i g_i$

1.1.12 Example. In the case $p = 2$, $A \in V_0 \otimes V_1 \otimes V_2$

$$\partial_A : V_0^\vee \otimes S^{k_1} V_2 \to V_1 \otimes S^{k_1+1} V_2$$

1.1.13 Remark. We remark that since $A \in V_0 \otimes \ldots \otimes V_p$ can be regarded as a map $V_0^\vee \to V_1 \otimes \cdots \otimes V_p$, taken the dual map $V_1^\vee \otimes \ldots \otimes V_p^\vee \to V_0$ (that we call also A), the theorem 1.1.10 asserts that

A is degenerate if and only if for all $i \neq 0$ there exist non zero vectors $v_i \in V_i^\vee$ such that $A(v_1 \otimes \ldots \otimes v_p) = 0$.

Moreover, the map ∂_A is obtained by tensoring A by $S^{m_1}V_1 \otimes \ldots \otimes S^{m_p}V_p$ (and by projecting on $S^{m_1+1}V_1 \otimes \ldots \otimes S^{m_p+1}V_p$ in the natural way).

1.1.14 Proposition. *[GKZ94] Each of the spaces* $S(m_1, m_2, \ldots, m_p)^{k_0+1}$ *and* $S(1+m_1, 1+m_2, \ldots, m_p)$ *has the same dimension* $N = \frac{(k_0+1)!}{k_1! \cdots k_p!}$

Proof. This follows at once from the standard fact that $dim(S^m(\mathbb{C}^{k+1})) = \binom{k+m}{k}$ $\square$

Let us choose in each of the space $S(m_1, m_2, \ldots, m_p)^{k_0+1}$ and $S(1+m_1, 1+m_2, \ldots, m_p+1)$ the basis consisting of monomials. We will denote by the same symbol ∂_A the matrix of the operator ∂_A in these bases, that by above proposition is square.

1.1.15 Lemma. *[GKZ94] The polynomial* $det(\partial_A)$ *is non-zero, and it is irreducible over* $\mathbb{Z}$

1.1.16 Theorem. *[GKZ94] We have* $Det(A) = det(\partial_A)$

1.1.3 Singularities of hyperdeterminants

Let $Y = (V_0 - \{0\}) \times \cdots \times (V_p - \{0\})$. We say that $x \in W$ is a *critical point* of a matrix $A \in M$ if (1.2) is verified for all i, j (by definition $A \in M$ is degenerate if it has at least one critical point in W). In more geometric way there is a natural projection $pr : Y \to X$ ($Y = \mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_p}$), so the coordinates $x_i^{(j)}$ of a point $y \in Y$ are the homogeneous coordinates of $pr(y) \in X$. This projection makes Y a principal fiber bundle over X with the structure group $(\mathbb{C}^*)^p$. It is clear that for every $A \in M$ the set of critical points of A in Y is a union of fibers of the projection $pr : Y \to X$. We shall say that a point $x \in X$ is a critical point of A if the fiber $pr^{-1}(x) \subset Y$ consists of critical points of A. Now we consider the incidence variety

$$Z = \{(A, x) \in M \times X : x \text{ is a critical point of} A\}$$

Then the variety $\Delta := X^\vee \subset M$ of degenerate matrices is the image $pr_1(Z)$, where pr_1 is the projection $(A, x) \to A$. This description implies at once that Δ

is irreducible(the irreducibility of Δ follows from that of Z, and Z is irreducible, since it is a vector bundle over an irreducible variety X). We know that Δ is a hypersurface in M if and only if the matrix format satisfies the polygon inequality (1.3) and the equation of Δ is the hyperdeterminant. Assuming that this holds Weyman and Zelevinsky in [WZ96] describe the irreducible components of the singular locus Δ_{sing} of the hypersurface Δ. They prove that for matrices of dimension ≥ 3 and format different from $2\times 2\times 2$, Δ_{sing} has codimension 1 in Δ and as shown in [GKZ92], this gives a complete description of the matrix formats for which the hyperdeterminant can be computed by Schläfli's methods of iterating discriminants [Sch52]. Moreover they classify the irreducible components of Δ_{sing} for all matrix format.

1.1.17 Remark. The ordinary square matrices have a natural stratification according to their rank, the set Δ is the largest closed stratum of this stratification, and the next closed stratum (that is, the set of matrices of corank ≥ 2)is exactly the singular locus Δ_{sing}.

In particular, in [WZ96] is proved that for boundary format matrices the singular locus Δ_{sing} is always an irreducible hypersurface in Δ. This is a sharp contrast with the case of interior format, in fact in this case Δ_{sing} has two irreducible components, both of codimension 1 in Δ. The origin of this difference between the interior and boundary formats lies in the fact that for boundary format the hyperdeterminant can be interpreted as the resultant of a system of multilinear forms (1.1.10).

1.1.4 Multidimensional diagonal matrices and Vandermonde matrix

1.1.18 Definition. [AO99] A $(p+1)$-dimensional matrix of boundary format $A\in V_0\otimes\cdots\otimes V_p$ is called **triangulable** if one of the following equivalent conditions holds:

i) there exist bases in V_j such that $a_{i_o,\dots,i_p}=0$ for $i_0>\sum_{t=1}^p i_t$

ii) there exist a vector space U of dimension 2, a subgroup $\mathbb{C}^*\subset SL(U)$ and isomorphisms $V_j\simeq S^{k_j}U$ such that if $V_0\otimes\cdots\otimes V_p=\bigoplus_{n\in\mathbb{Z}}W_n$ is the decomposition into direct sum of eigenspaces of induced representation, we have $A\in\bigoplus_{n\geq 0}W_n$

proof of the equivalence between i) and ii)

Let x,y be a basis of U such that $t\in\mathbb{C}^*$ acts on x and y as tx and $t^{-1}y$. Set $e_k^{(j)}:=x^k y^{k_j-k}\binom{k_j}{k}\in S^{k_j}U$ for $j>0$ and $e_k^{(0)}:=x^{k_0-k}y^k\binom{k_0}{k}\in S^{k_0}U$ so that $e_{i_0}^{(0)}\otimes\ldots\otimes e_{i_p}^{(p)}$ is a basis of $S^{k_0}U\otimes\ldots\otimes S^{k_p}U$ which diagonalizes the action of $\mathbb{C}^*$. The weight of $e_{i_0}^{(0)}\otimes\ldots\otimes e_{i_p}^{(p)}$ is $2\left(\sum_{t=1}^p i_t-i_0\right)$, hence ii) implies i). The converse is trivial.

1.1.19 Definition. [AO99] A $(p+1)$-dimensional matrix of boundary format $A\in V_0\otimes\cdots\otimes V_p$ is called **diagonalizable** if one of the following equivalent conditions holds:

i) there exist bases in V_j such that $a_{i_o,\dots,i_p}=0$ for $i_0\neq\sum_{t=1}^p i_t$

ii) there exist a vector space U of dimension 2, a subgroup $\mathbb{C}^*\subset SL(U)$ and isomorphisms $V_j\simeq S^{k_j}U$ such that A is a fixed point of the induced action of $\mathbb{C}^*$

1.1.20 Definition. [AO99] A $(p+1)$-dimensional matrix of boundary format $A \in V_0 \otimes \cdots \otimes V_p$ is an **identity** if one of the following equivalent conditions holds:

i) there exist bases in V_j such that

$$a_{i_o,\ldots,i_p} = \begin{cases} 0 & \text{for} \quad i_0 \neq \sum_{t=1}^{p} i_t \\ 1 & \text{for} \quad i_0 = \sum_{t=1}^{p} i_t \end{cases}$$

ii) there exist a vector space U of dimension 2 and isomorphisms $V_j \simeq S^{k_j}U$ such that A belongs to the unique one dimensional $SL(U)$-invariant subspace of $S^{k_0}U \otimes \cdots \otimes S^{k_p}U$

The equivalence between i) and ii) follows easily from the following remark: the matrix A satisfies the condition ii) if and only if it corresponds to the natural multiplication map $S^{k_1}U \otimes \ldots \otimes S^{k_p}U \to S^{k_0}U$ (after a suitable isomorphism $U \simeq U^*$ has been fixed).

The above definitions agrees with some in [WZ96] where a $(p+1)$-complex matrix $A = (a_{i_0,\ldots,i_p})$ of format $(k_0+1) \times \cdots \times (k_p+1)$ is called *diagonal* if $a_{i_0,\ldots,i_p} = 0$ unless $i = (i_0,\ldots,i_p)$ is a diagonal multi-index for $k = (k_0,\ldots,k_p)$ i.e. if i and $(k-i)$ satisfy the "polygon" inequality (1.3).

We shall see that a generic diagonal matrix A is nondegenerate as proved in [WZ96]. This is a consequence of the following statement: there exists an extreme monomial

$$\prod_{i=(i_0,\ldots,i_p)} a_i^{d(i,k)} \tag{1.10}$$

appearing in $Det(A)$ such that $d(i,k) > 0$ if and only if i is diagonal for k. (Recall that a monomial is *extreme* if it corresponds to a vertex of the Newton

polytope of $Det(A)$). To construct such a monomial we recall that in [GKZ94] and [GKZ92] is shown that any coefficient of the degree $N(k_0,\ldots,k_p)$ in the formula (1.6) has a combinatorial expression as a sum of positive summands; furthermore, a combinatorial argument shows that $N(k_0,\ldots,k_p) > 0$ exactly when $k = (k_0,\ldots,k_p)$ is diagonal.

Now consider the square root of the generating function (1.6), i.e., the series

$$\sum_{k_0,\ldots,k_p \geq 0} M(k_0,\ldots,k_p) z_0^{k_0} \cdots z_p^{k_p} = \left(1 - \sum_{i=1}^{p} ie_i(z_0,\ldots,z_p)\right)^{-1} \tag{1.11}$$

For any two non-negative integer vectors $i = (i_0,\ldots,i_p), k = (k_0,\ldots,k_p)$ we set

$$d(i,k) = M(i_0,\ldots,i_p) M(k_0-i_0,\ldots,k_p-i_p) \tag{1.12}$$

1.1.21 Theorem. *[WZ96] For every interior or boundary matrix format, the monomial (1.10) with the exponent given by (1.12) is an extreme monomial in $Det(A)$ appearing with the coefficient ± 1. The exponent $d(i,k)$ is positive if and only if i is diagonal for k.*

This theorem proves that the diagonal matrices are nondegenerate and since the hyperdeterminant was defined only up to the sign, then it gives a natural choice of the sign, by requiring that the monomial given by (1.10),(1.12) occurs in $Det(A)$ with coefficient 1.

1.1.22 Remark. If the matrix format is boundary, then the determinantal formula $Det(A)$ given by (1.1.16), implies that the hyperdeterminant of diagonal matrix is just the monomial (1.10). This is no longer true for interior format. Using computer algebra system MACAULAY, Weyman and Zelevinsky found that the hyperdeterminant of a diagonal $3 \times 3 \times 3$ matrix is given by

$$\begin{aligned} Det(A) =&(a_{000}a_{222})^8(a_{110}a_{101}a_{011}a_{112}a_{121}a_{211})^2(a_{000}^2a_{111}^4a_{222}^2+ \\ &+ 8a_{000}a_{111}^2a_{222}(a_{000}a_{112}a_{121}a_{211} + +a_{222}a_{110}a_{101}a_{011}))+ \\ &+ 16(a_{000}a_{112}a_{121}a_{211})^2 + 16(a_{222}a_{110}a_{101}a_{011})^2+ \\ &- 32a_{000}a_{110}a_{101}a_{011}a_{112}a_{121}a_{211}a_{222}) \end{aligned} \tag{1.13}$$

Here the diagonal monomial is the only one occurring with coefficient 1. There are two other extreme monomials occurring in (1.13): both of them have coefficient 16 and not contain the variable a_{111}. It follows that there exists a nondegenerate diagonal matrix having $a_{111} = 0$.

Now we assume that the matrix format is boundary.We shall construct another special class of matrices analogs of the classical Vandermonde matrix. Let $\Lambda = (\lambda_{i,j})_{0\le i\le k_0, 1\le j\le p}$ be a $(k_0 + 1) \times (p)$ complex matrix. We define **Vandermonde-type** matrix $A = A(\Lambda)$ of format $(k_0 + 1) \times \cdots \times (k_p + 1)$ by the formula

$$a_{i_0,i_1,\ldots,i_p} = \lambda_{i,1}^{i_1}\lambda_{i,2}^{i_2}\cdots\lambda_{i,p}^{i_p} \tag{1.14}$$

If $p = 1, k_0 = k_1 = k$ then $A = A(\Lambda)$ is the usual $(k+1) \times (k+1)$ Vandermonde matrix

$$\begin{pmatrix} 1 & \lambda_0 & \lambda_0^2 & \ldots & \lambda_0^k \\ 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_k & \lambda_k^2 & \ldots & \lambda_k^k \end{pmatrix}$$

1.1.23 Proposition. *The matrix $A(\Lambda)$ is nondegenerate if and only if for each $j = 2, \ldots, p$ the numbers $\lambda_{0,j}, \lambda_{1,j}, \ldots, \lambda_{k_0,j}$ are mutually distinct.*

1.2 Geometric invariant theory and nondegenerate matrices of boundary format

We consider the natural action of $SL(V_0) \times \ldots \times SL(V_p)$ on $\mathbb{P}(V_0 \otimes \ldots \otimes V_p)$. We may suppose $p \ge 2$. The definitions of triangulable, diagonalizable and identity boundary format matrix, given in the last section, apply to elements of $\mathbb{P}(V_0 \otimes \ldots \otimes V_p)$ as well. In particular all identity matrices fill a distinguished orbit in $\mathbb{P}(V_0 \otimes \ldots \otimes V_p)$. Moreover, we know that the hyperdeterminant of multidimensional matrices is a homogeneous polynomial function over $V_0 \otimes \ldots \otimes V_p$ then the condition $Det\ A \ne 0$ is meaningful for $A \in \mathbb{P}(V_0 \otimes \ldots \otimes V_p)$. The function Det is $SL(V_0) \times \ldots \times SL(V_p)$-invariant (see theorem (1.1.3)), in particular if $Det\ A \ne 0$ then A is semistable for the action of $SL(V_0) \times \ldots \times SL(V_p)$. We denote by $Stab\ (A) \subset SL(V_0) \times \ldots \times SL(V_p)$ the stabilizer subgroup of A and by $Stab\ (A)^0$ its connected component containing the identity.

We remember that if $e_0^{(j)}, \ldots, e_{k_j}^{(j)}$ is a basis in V_j so that every $A \in V_0 \otimes \cdots \otimes V_p$ has coordinate form

$$A = \sum a_{i_0,\ldots,i_p} e_{i_0}^{(0)} \otimes \cdots \otimes e_{i_p}^{(p)} \tag{1.15}$$

and if $x_0^{(j)}, \ldots, x_{k_j}^{(j)}$ are the coordinates in V_j, then A has the following different descriptions:

1. A multilinear form

$$A = \sum_{(i_0,\ldots,i_p)} a_{i_0,\ldots,i_p} x_{i_0}^{(0)} \cdots x_{i_p}^{(p)}$$

2. For any fixed $j \ne 0$ an ordinary matrix $M_A{}^{(j)} = (m_{i_j i_0})$ of size $(k_j + 1) \times$

(k_0+1) whose entries are multilinear forms

$$m_{i_j i_0} = \sum_{(i_1,\ldots\hat{i_j}\ldots,i_p)} a_{i_0,\ldots,i_p} x^{(1)}_{i_1} \otimes \cdots \otimes x^{\hat{(j)}}_{i_j} \otimes \cdots \otimes x^{(p)}_{i_p} \tag{1.16}$$

3. For any fixed $j \neq 0$ a sheaf morphism $f_A{}^{(j)}$ on the product $X = \mathbb{P}^{k_1} \times \cdots \times \widehat{\mathbb{P}^{k_j}} \times \cdots \times \mathbb{P}^{k_p}$

$$\mathcal{O}_X \otimes V_0^\vee \to \mathcal{O}_X(1,\ldots,1) \otimes V_j \tag{1.17}$$

1.2.1 Theorem. *Let A be a matrix of boundary format, let $f^{(1)}_A$ be a surjective map and $S^* := ker f^{(1)}_A$ a vector bundle then*

$$h^0(S^*(k_1, k_1+k_2, \ldots, k_1+\ldots+k_{p-1}) = 0 \tag{1.18}$$

Proof. We have that S^* is a vector bundle on $X = \mathbb{P}^{k_2} \times \cdots \times \mathbb{P}^{k_p}$ of rank equal to $dimX = k_0 - k_1$ and $detS^* = \mathcal{O}_X(-k_1-1,\ldots,-k_1-1)$, hence

$$S^*(k_1, k_1+k_2, \ldots, k_1+\ldots+k_{p-1}) \simeq \wedge^{k_0-k_1-1} S(-1, k_2-1, \ldots, k_2+k_3+\cdots+k_{p-1}-1)$$

The result follows from the (k_0-k_1-1)-power of the sequence dual to (1.17):

$$\begin{aligned}
&0 \to S^{k_0-k_1-1}V_1^\vee \otimes \mathcal{O}(-k_2-\cdots-k_p, , -k_3-\cdots-k_p, \ldots, -k_p) \to \ldots \\
&\ldots \to V_0 \otimes S^{(k_0-k_1-2)}V_1^\vee \otimes \mathcal{O}(-k_2-\cdots-k_p-1, -k_3-\cdots-k_p-1, \ldots, -k_p-1) \to \ldots \\
&\ldots \to \wedge^{k_0-k_1-1}V_0^\vee \otimes \mathcal{O}(-1, k_2-1, \ldots, k_2+k_3+\cdots+k_{p-1}-1) \to \ldots \\
&\ldots \to \wedge^{k_0-k_1-1} S(-1, k_2-1, \ldots, k_2+\cdots+k_{p-1}-1) \to 0
\end{aligned} \tag{1.19}$$

□

1.2.2 Theorem. *The following properties are equivalent*

i) The system $f_0(x) = \cdots = f_{k_0}(x)$ has only trivial solution on

ii) f_A is surjective and $S^* = Kerf^{(1)}_A$ is a vector bundle on $X = \mathbb{P}^{k_2} \times \cdots \times \mathbb{P}^{k_p}$ of rank $k_0 - k_1$

iii) $det(\partial_A) \neq 0$.

Proof.

i) $\Longrightarrow$ ii) It follows obviously by definition of f_i.

ii) $\Longrightarrow$ iii) By proposition 1.1.14 ∂_A is a map between finite vector spaces having same dimension $\frac{(k_0+1)!}{k_1!k_2!\ldots k_p!}$. If f_A is surjective the kernel of ∂_A is equal to $\mathrm{H}^0(S^*(k_1, k_1+k_2, \ldots, k_1+\cdots+k_{p-1}))$ which vanishes by the above lemma, then ∂_A is an injective map and $det(\partial_A) \neq 0$.

iii) $\Longrightarrow$ i) If the system has non trivial solution then there exist nonzero vectors $v_i \in V_i^\vee$, $i = 1, \ldots p$ such that $A(v_1 \otimes \cdots \otimes v_p) = 0$ and the nonzero tensor $v_1^{\otimes(m_1+1)} \otimes v_2^{\otimes(m_2+1)} \otimes \cdots \otimes v_p{}^{\otimes(m_p+1)}$ maps to zero, hence $det(\partial_A) = 0$. □

1.2.3 Remark. Above theorem suggests to give as definition of hyperdeterminant of a matrix A the usual determinant of the square matrix ∂_A.

More in general, given an opportune order on the integers k_i in the similar way of (1.9), theorem 1.1.10 easily translates into:

1.2.4 Theorem. *The following properties are equivalent*

i) $DetA \neq 0$;

ii) for all $j \neq 0$ the matrix ${M_A}^{(j)}$ has constant rank $k_j + 1$ on

$X = \mathbb{P}^{k_1} \times \ldots \widehat{\mathbb{P}^{k_j}} \times \cdots \times \mathbb{P}^{k_p}$;

iii) for all $j \neq 0$ the morphism ${f_A}^{(j)}$ is surjective so that ${S^*}_A{}^{(j)} = Ker {f_A}^{(j)}$ is a vector bundle of rank $k_0 - k_j$

In the particular case $p = 2$ the (dual) vector bundle ${S_A}^{(1)}$ lives on the projective space $\mathbb{P}^n, n = k_2$ and is a Steiner bundle as defined in [DK93] (this case has been investigate in [AO99]). We can keep to S_A the name Steiner also for $p \geq 3$.

Moreover, the action of $SL(V_0) \times \cdots \times SL(V_p)$ on A translates for all j to an action on the corresponding bundle ${S^*}_A{}^{(j)}$ in two steps: first the action of $SL(V_0) \times SL(V_j)$ leaves the bundle in the same isomorphism class, then $SL(V_1) \times \cdots \times \widehat{SL(V_j)} \times \cdots \times SL(V_p)$ acts on the classes, i.e. on the moduli space of Steiner bundles. It follows that the invariants of the matrices for the action of $SL(V_0) \times \cdots \times SL(V_p)$ coincide with the invariants of the action of $SL(V_1) \times \cdots \times \widehat{SL(V_j)} \times \cdots \times SL(V_p)$ on the moduli space of the corresponding bundles and the stable points of both actions correspond to each other. In particular, investigating the properties and the invariant of both the above action, Ancona and Ottaviani in [AO99], proved the following main results.

1.2.5 Theorem. *[AO99] Let $A \in \mathbb{P}(V_0 \otimes \ldots \otimes V_p)$ of boundary format such that Det $A \neq 0$. Then*

$$A \text{ is triangulable} \iff A \text{ is not stable for the action of } SL(V_0) \times \ldots \times SL(V_p)$$

1.2.6 Theorem. *[AO99] Let $A \in \mathbb{P}(V_0 \otimes \ldots \otimes V_p)$ of boundary format such that*

Det $A \neq 0$. Then

$$A \text{ is diagonalizable} \iff \mathbb{C}^* \subset Stab(A)$$

In the case $p = 2$ in the following proposition we give an alternative simple proof of this result.

1.2.7 Proposition. *Let $A \in \mathbb{P}(V_0 \otimes V_1 \otimes V_2)$ of boundary format such that Det $A \neq 0$. If $\mathbb{C}^* \subseteq GL(V_0) \times GL(V_1) \times GL(V_2)$ stabilizes A then A is diagonalizable.*

Proof. For all i=0,1,2 we choose a basis of V_i such that

$$a_{i0j} = 0 \quad \text{if} \quad i \neq j \quad \text{and} \quad \forall i \geq k_2 + 1$$

and we put

$$\begin{aligned} \alpha_0 \leq \cdots \leq \ & \alpha_{k_0} \\ \beta_0 \geq \cdots \geq \ & \beta_{k_1} \\ \gamma_0 \geq \cdots \geq \ & \gamma_{k_2} \end{aligned}$$

the weights of the action of $\mathbb{C}^*$ respectively on V_0,V_1 and V_2, hence we have a region on left-up of the matrix M_A corresponding to maximal weight.

Claim: $\alpha_0 < \alpha_1$ and $\beta_0 > \beta_1$.

Assume, as a contradiction, that there exist two integers $s \geq 1$ and $t \geq 1$ such that

$$\begin{aligned} \alpha_0 = \cdots = \ & \alpha_s \\ \beta_0 = \cdots = \ & \beta_t. \end{aligned} \tag{1.20}$$

Since $\mathbb{C}^* \subset Stab(A)$, then (1.20) implies also that $\gamma_0 = \cdots = \gamma_s$ and in the left-up minor of M_A of order $(t+1) \times (s+1)$ we have only the variables $x_0, \ldots, x_s$

where $x_0, \ldots, x_{k_2}$ is the coordinate system of $\mathbb{P}(V_2)$. Moreover, in the right-up minor of order $(t+1) \times (k_0 - s)$ we have only the variables $x_{s+1}, \ldots, x_{k_2}$, the left down minor of order $(k_1 - t) \times (s+1)$ is the zero matrix and in the last minor, that we call A', we have all the variables x_i.

If $s < t$ the left-up minor has rank less than $(t+1)$, hence, the first $(t+1)$-rows of M_A are linearly dependent by putting $x_i = 0$ for $i \geq s+1$. This contradicts, by theorem 1.2.4, the hypothesis $DetA \neq 0$.

If $s = t$, since $s, t \leq 1$, by calculating the zero of the determinant of the left-up minor, we get a point of $\mathbb{P}(V_2)$ where the first s rows are dependent as in the above case.

If $s > t$ the transposed matrix of A' gives on $\mathbb{P}^{k_2}$ the sheaf morphism

$$\mathcal{O}_{\mathbb{P}^{k_2}}^{k_1 - t} \to \mathcal{O}_{\mathbb{P}^{k_2}}^{k_0 - s}(1)$$

whose rank drops on a subvariety of codimension $\leq k_0 - s - k_1 + t + 1 \leq k_2 = \dim \mathbb{P}(V_2)$, then we get, as contradiction, that A is degenerate. So, (1.20) cannot occur.

Now, we suppose that $k_1 = 1$ then for all $i = 0, .., k_2 + 1$ $m_{1,i} = a_{i,1,i-1} x_{i-1}$ i.e. A is diagonal. In fact, by comparing:

the weights of $m_{1,i}$ with the weight of $m_{0,k}$ we get $a_{i1i} = 0$;

the weights of $m_{1,i}$ we get $a_{ikj} = 0 \quad \forall j \geq i+2$;

the weights of $m_{1,i}$ with the weights of $m_{0,t}$, we get $a_{i1(i+1)} = 0$ if $t > i$ and $a_{i1j} = 0$ $\forall j \leq i-2$ if $t < i$.

Also in the case $k_1 > 1$, the same comparisons gives that $m_{j0} = 0$ for all $j > 0$. The minor obtained by cutting-out the first rows and the first column has

maximal rank (since A is nondegenerate), is boundary format and it is stabilized by $\mathbb{C}^*$, then, by induction on k_1, it is diagonal. Hence, also A is diagonal. $\square$

1.2.8 Theorem. *[AO99] Let $A \in \mathbb{P}(V_0 \otimes V_1 \otimes V_2)$ of boundary format such that $Det\ A \neq 0$. Then there exists a 2-dimensional vector space U such that $SL(U)$ acts over $V_i \simeq S^{k_i} U$ and according to this action on $V_0 \otimes V_1 \otimes V_2$ we have $Stab\ (A)^0 \subset SL(U)$. Moreover the following cases are possible*

$$Stab\ (A)^0 \simeq \begin{cases} 0 \\ \mathbb{C} \\ \mathbb{C}^* \\ SL(2) \quad \textit{(this case occurs if and only if } A \textit{ is an identity)} \end{cases}$$

1.2.9 Remark. When A is an identity then $Stab\ (A) \simeq SL(2)$.

In the next section we extend the first part of this result to the case $p \geq 2$ and using jumping hyperplane we characterize Schwarzenberger bundles on product of projective spaces.

1.3 Jumping hyperplanes, Steiner bundles and stabilizers

Let $p = 2$ and $S := S^1$ be the Steiner bundle on $\mathbb{P}(V_2)$ defined by a matrix $A \in V_0 \otimes V_1 \otimes V_2$ boundary format, an hyperplane $h \in \mathbb{P}(V_2^*)$ is an instable hyperplane of S if $h^0(S^*_{|h}) \neq 0$ (see [AO99]).

In particular, $\mathrm{H}^0(S^*(t))$ identifies the spaces of $(k_0+1)\times 1$-column vectors v with entries in S^tV_2 such that $Av=0$, and an hyperplane h is instable for S if and only if there are nonzero vectors v_0 of size $(k_0+1)\times 1$ and $(k_1+1)\times 1$ both with constant coefficients such that

$$Av_0 = v_1 h. \tag{1.21}$$

(the tensor $\mathcal{H} = v_0 \otimes v_1$ is called instable (or jumping) hyperplane for the matrix A)

For $p \geq 3$ there are at least two ways to define a jumping hyperplane. We will them weak and strongly jumping hyperplanes.

1.3.1 Definition. $\mathcal{H} = v_0 \otimes v_j \in V_0 \otimes V_j$ is a (j)-**weak jumping hyperplane** for A if $\exists$ $v_0, w_1, \dots, w_{k_0}$ basis of V_0 such that

$$A = v_0 \otimes v_j \otimes h + \sum_{i=1}^{k_0} w_i \otimes \dots \tag{1.22}$$

where $h \in V_1 \otimes \dots \otimes \widehat{V_j} \otimes \dots \otimes V_p$ is an hyperplane for $\mathbb{P}^{k_1} \times \dots \times \widehat{\mathbb{P}^{k_j}} \times \dots \times \mathbb{P}^{k_p} \subset \mathbb{P}(V_1 \otimes \dots \otimes \widehat{V_j} \otimes \dots \otimes V_p)$.

1.3.2 Remark. The expression (1.22) means, as in the case $p=2$, that $H^0(Kerf_{A_{|h}}^{(j)}) \neq 0$.

If $\mathcal{H} = v_0 \otimes v_j$ is a (j)-weak jumping hyperplane for A then the map:

$$V_0 \otimes \dots \otimes V_p \to (V_0/_{<v_0>}) \otimes \dots \otimes (V_j/_{<v_j>}) \otimes \dots \otimes V_p$$
$$A \mapsto A'_j$$

gives an elementary transformation [Mar82]

1.3.3 Remark. A'_j is again of boundary format

1.3.4 Theorem. *If A'_j is defined as above*

$$DetA \neq 0 \Rightarrow DetA'_j \neq 0$$

Proof. If $X := \mathbb{P}^{k_1} \times \dots \times \widehat{\mathbb{P}^{k_j}} \times \dots \times \mathbb{P}^{k_p}$ and h is the hyperplane defined in (1.3.1) associated to $\mathcal{H}$, the map $\mathcal{O}_h \to {S_A}^{*(j)}_{|h}$ induces a surjective map $S_A^{*(j)} \to \mathcal{O}_h$ and an exact sequence:

$$0 \to S'^{(j)} \to {S_A}^{(j)} \to \mathcal{O}_h \to 0$$

where by definitions $S'^{(j)} = {S_{A'_j}}^{(j)}$. Since $\operatorname{codim} h = 1$ then it is locally free [Ser65] and by theorem (1.2.4) the result follows. □

1.3.5 Remark. An element $g \in SL(V_0) \times SL(V_j)$ preserves h and it induces $\bar{g} \in SL(V_0/_{<g(v_0)>}) \times SL(V_j/_{<g(v_j)>})$ such that $g \cdot A$ projects to $\bar{g}A'_j$ and the elementary transformation is well behaved with respect the action of g.

1.3.6 Definition. $\mathcal{H} = v_0 \otimes v_1 \otimes \dots \otimes v_p$ is a **strong jumping hyperplane** for A if $\exists v_0, w_1, \dots, w_{k_0}$ basis of V_0 such that

$$A = v_0 \otimes v_1 \otimes \dots \otimes v_p + \sum_{i=1}^{k_0} w_i \otimes \dots$$

1.3.7 Remark. If $\mathcal{H}$ is a strong jumping hyperplane then $\mathcal{H}$ is a (j)-weak jumping hyperplane for all $j = 1, \dots, p$.
In particular for a strong jumping hyperplane there are many elementary transformations.

1.3.8 Remark. For $p=2$ the notation of strong jumping hyperplane coincides to weak jumping hyperplane (see [AO99])

1.3.9 Example. (*the identity*) The identity matrix is represented by

$$I := \sum_{\substack{i_0=i_1+\cdots+i_p \\ 0\le i_j\le k_j}} e^{(0)}_{i_0}\otimes\cdots\otimes e^{(p)}_{i_p}$$

Let $t_0,\ldots,t_{k_0}$ be any distinct complex numbers. Let w be the $(k_0+1)\times(k_0+1)$ Vandermonde matrix whose (i,j) entry is $t_j^{(i-1)}$, so acting with w over V_0 we have:

$$e^{(0)}_j = \sum_{s=0}^{k_0} \bar{e}^{(0)}_s t^j_s$$

Then substituting

$$\begin{aligned} I &= \sum_{\substack{i_0=i_1+\cdots+i_p \\ s=0,\ldots k_0}} \bar{e}^{(0)}_{i_0} t_s^{i_0}\otimes\cdots\otimes e^{(p)}_{i_p} \\ &= \sum_{s=0}^{k_0}\bar{e}^0_s\otimes(\sum_{i_1=0}^{k_1} e^{(1)}_{i_1}t_s^{i_1})\otimes\cdots\otimes(\sum_{i_p=0}^{k_p} e^p_{i_p}t_s^{i_p}) \end{aligned}$$

Thus, since t_i have no restrictions, I has infinitely many strong jumping hyperplane.

We call Schwarzenberger bundle the vector bundle associated to I (in fact in the case $p=2$ it is exactly the same introduced by Schwarzenberger in [Sch61])(see also ([AO99])

1.3.10 Proposition. *Let A be boundary format matrix with $DetA\neq 0$. If A has $N\geq k_0+3$ strong jumping hyperplanes then it is an identity.*

Proof. In the case $p=2$ the statement is prove in [AO99] (theorem 5.13). Chosen V_0 and other two vector spaces among $V_1,\ldots,V_p$ (say V_1 and V_2), one may perform several elementary transformations whit V_0 and all the others so

that we get $A'\in V'_0\otimes V_1\otimes V_2$ boundary format matrix with $DetA'\neq 0$ and $N'\geq k'_0+3$ strong jumping hyperplanes, then A' is an identity.

(Hence in $\mathbb{P}(V_1)$ and $\mathbb{P}(V_2)$ there are rational normal curves because $S_{A'}$ is a Schwarzenberger bundle).

As in the above example, one can change the hyperplane giving the elementary transformation, so that for all N strong jumping hyperplanes we get $t_1,\ldots,t_N$ distinct complex numbers and corresponding suitable basis of V_1 and V_2 :

$$\begin{gathered} \bar{e}^{(1)}_0\ldots\bar{e}^{(1)}_{k_1} \\ \bar{e}^{(2)}_0\ldots\bar{e}^{(2)}_{k_2} \end{gathered}$$

such that the hyperplanes are given by

$$\sum_{i=0}^{k_1}\bar{e}^{(1)}_i t^i_j \quad\text{and}\quad \sum_{i=0}^{k_2}\bar{e}^{(2)}_i t^i_j \quad \text{for} j=1,\ldots N$$

Now, changing V_1 and V_2 with the pairs V_1, V_j $(j=1,\ldots p\ )$ we get

$$A := \sum_{s=0}^{k_0}\bar{e}^0_s\otimes(\sum_{i_1=0}^{k_1} e^{(1)}_{i_1}t_s^{i_1})\otimes\cdots\otimes(\sum_{i_p=0}^{k_p} e^p_{i_p}t_s^{i_p})$$

showing that A is an identity. □

1.3.11 Proposition. *Let A be boundary format matrix with $DetA\neq 0$. If A has k_0+2 strong jumping hyperplanes then it is uniquely determined by these hyperplanes.*

Proof. In the case $p=2$ the statement is prove in [AO99] (theorem 5.3). Chosen V_0 and other two vector spaces among $V_1,\ldots,V_p$ (say V_1 and V_2), one may perform several elementary transformations whit V_0 and all the others so that we get $A'\in V'_0\otimes V_1\otimes V_2$ boundary format matrix with $DetA'\neq 0$ and $N'=k'_0+2$

strong jumping hyperplanes, then A' is uniquely determined. Now, changing V_1 and V_2 with the pairs V_1 and V_j $(j = 2, \dots, p)$ we detect all the 3-dimensional submatrix of A which are all uniquely determined, then also A is. $\square$

1.3.12 Remark. In the case $p = 2$ we know that $k_0 + 2$ jumping hyperplanes give an existence condition for the bundle S_A (it is a logarithmic bundle, see [AO99]) but in the case $p \geq 3$ there is not analog existence result. Moreover, by definitions, there exists a unique matrix A with $k_0 + 1$ strong jumping hyperplanes given.

1.3.13 Theorem. *Let A be boundary format matrix with $DetA \neq 0$. Then there exists a 2-dimensional vector space U such that $SL(U)$ acts over $V_i \simeq S^{k_i}U$ and according to this action on $V_0 \otimes \cdots \otimes V_p$ we have $Stab(A)^0 \subseteq SL(U)$.*

Proof. We proceed by induction on k_0. If $k_0 = 2$ then $p = 2$ and $V_0 \otimes V_1 \otimes V_2 \simeq \mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and a Steiner bundle on the line $\mathbb{P}^1$ is Schwarzenberger $(A = I)$, then the result follows. We may suppose that $Stab(A)^0$ has dimension ≥ 1 then by theorem 1.2.5 the matrix A is triangulable and this implies that there exists a strong jumping hyperplane $\mathcal{H}$.

We may, also, suppose that the number of jumping hyperplanes is finite otherwise A is an identity (proposition 1.3.10), hence $\mathcal{H}$ is $Stab(A)^0$- invariant. Let A'_1 the image of A by the elementary transformation associated to the weak jumping hyperplane $v_0 \otimes v_1$ defined by $\mathcal{H}$ (we choose $j = 1$ to have simpler notations). The matrix A_1 belongs to $V'_0 \otimes V'_1 \otimes V_2 \otimes \cdots \otimes V_p$ where $V'_0 = V_0/_{<v_0>}$ and $V'_1 = V_1/< v_1 >$, it is nondegenerate and of boundary format then, by induction,

there exists a 2-dimensional vector space U such that

$$V'_0 \simeq S^{k_0-1}(U), \quad V'_1 \simeq S^{k_1-1}(U) \quad \text{and} \quad V_i = S^{k_i}(U) \quad \text{for all} \quad i \geq 2$$

and $Stab(A'_1)^0 \subseteq SL(U)$. (By using essentially the same argument we could work in $GL(V_0 \times \cdots \times GL(V_p))$

Since A'_1 is obtained from the matrix A after the choice of two directions, any element which stabilizes A also stabilizes A'_1, so $Stab(A)^0 \subseteq Stab(A')^0$.

Moreover, we remark that the above considered elementary transformation gives the decomposition $V_0 = V'_0 \oplus \mathbb{C}$ and $V_1 = V'_1 \oplus \mathbb{C}$.

Since no other morphism of $SL(U)$ in $SL(V_0) \times SL(V_1) \times SL(S^{k_2}U) \times \cdots \times SL(S^{k_p}U)$ can give $S^{k_0-1}U \otimes S^{k_1-1}U \otimes S^{k_2}U \otimes \ldots \otimes S^{k_p}U$ as an invariant summand of $V_0 \otimes V_1 \otimes S^{k_2}U \otimes S^{k_p}U$, then the inclusion

$$\mathrm{Hom}(V'_0, V'_1 \otimes V_2 \otimes \cdots \otimes V_p) \subset \mathrm{Hom}(V_0, V_1 \otimes V_2 \otimes \ldots V_p)$$

identifies to the $SL(U)$-invariant inclusion

$$S^{k_0-1}U \otimes S^{k_1-1}U \otimes S^{k_2}U \otimes \ldots \otimes S^{k_p}U \subset S^{k_0}U \otimes S^{k_1}U \otimes S^{k_2}U \otimes \ldots \otimes S^{k_p}U$$

according to the natural actions, hence also $V_0 = S^{k_0}U$ and $V_1 = S^{k_1}U$. $\square$

1.4 Cohomology of Schwarzenberger bundles

In [AO94] it was proved that a Steiner bundles S on $\mathbb{P}^n$ defined by the sequences

$$0 \to I \otimes \mathcal{O}(-1) \to W \otimes \mathcal{O} \to S \to 0$$

where I and W are vector spaces of dimension k and $k+n$ respectively, are stable, give smooth points in the moduli space of all stable bundles with the same rank and Chern classes and they belong to an irreducible component of dimension $(k-1)(n-1)(n+k+1)$.
Moreover, in [AO99], it was proved that for a Steiner bundle S as above, $h^0(S^*(t)) = 0$ if and only if $t \le k-1$.

In the class of Steiner bundle we want to give a cohomological characterization of Schwarzenberger bundles, hence, we calculate the dimension of the cohomology of their second symmetric power.

Throughout this section $\mathbb{K}$ denotes an algebraically closed field of characteristic zero. U denotes a 2-dimensional $\mathbb{K}$ vector space ($U =< s,t >$), $S_n = S^n U$ its n-th symmetric power ($S_n =< s^n, s^{n-1}t, \dots, t^n >$) and $\mathbb{P}^n = \mathbb{P}(S_n)$.
First, we prove the existance of a special Steiner bundles (or Schwarzenberger bundle) $\mathcal{S}$ on $\mathbb{P}^n$ defined by the following exact sequence

$$0 \to \mathcal{S}^* \to S^\vee_{n+k-1} \otimes \mathcal{O} \xrightarrow{B} S^\vee_{k-1} \otimes \mathcal{O}(1) \to 0 \tag{1.23}$$

where

$$B = \begin{pmatrix} s^n & s^{n-1}t & \cdots & st^{n-1} & t^n & & \\ & \ddots & \ddots & & \ddots & \ddots & \\ & & s^n & s^{n-1}t & \cdots & st^{n-1} & t^n \end{pmatrix}$$

and second we calculate its cohomology.

There is a natural exact sequence of GL(U)-equivariant maps for any $k \ge 2, n \ge 1$ (Clebsch-Gordan sequence):

$$0 \to \overset{2}{\wedge} U \otimes S_{k-2} \otimes S_{n-1} \xrightarrow{\beta} S_{k-1} \otimes S_n \xrightarrow{\mu} S_{k+n-1} \to 0 \tag{1.24}$$

where μ is the multiplication map and β is defined by $(s \wedge t) \otimes f \otimes g \to (sf \otimes tg - tf \otimes sg)$.

1.4.1 Lemma. *Let B, C be vector spaces of dimension k and $2n(k-1)$. A linear map $B \otimes S_n^\vee \xrightarrow{b} C$ induces a sheaf homomorphism $B \otimes \Omega^1(1) \xrightarrow{\tilde{b}} C \otimes \mathcal{O}$*

Proof. By the dual Euler sequence:

$$0 \to \Omega^1(1) \to \mathcal{O} \otimes S_n^\vee \to \mathcal{O}(1) \to 0$$

we obtain

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \Omega^2(2) \arrow[r] & \mathcal{O} \otimes \overset{2}{\wedge} S_n^\vee \arrow[rr] \arrow[dr] & & \mathcal{O}(1) \otimes S_n^\vee \arrow[r] & \mathcal{O}(2) \arrow[r] & 0 \\
& & & \Omega^1(2) \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd} \tag{1.25}$$

i.e the following two exact sequences:

$$0 \to \Omega^2(2) \to \mathcal{O} \otimes \overset{2}{\wedge} S_n^\vee \xrightarrow{f} \Omega^1(2) \to 0 \tag{1.26}$$

$$0 \to \Omega^1(2) \xrightarrow{g} \mathcal{O}(1) \otimes S_n^\vee \to \mathcal{O}(2) \to 0 \tag{1.27}$$

Then $\tilde{b}$ is defined by the composition:

$$0 \to B \otimes \Omega^1(1) \xrightarrow{g \otimes id_{\mathcal{O}(-1)}} B \otimes S_n^\vee \otimes \mathcal{O} \xrightarrow{b \otimes id_{\mathcal{O}}} C \otimes \mathcal{O} \tag{1.28}$$

i.e $\tilde{b} = (b \otimes id_{\mathcal{O}}) \circ (g \otimes id_{\mathcal{O}(-1)})$ where $g \otimes id_{\mathcal{O}(-1)}$ is an injective map. $\square$

1.4.2 Remark. It is true also the conversely of the previous lemma. In fact, given the homomorphism $\tilde{b}$ the linear map b can be defined as $b = \mathrm{H}^0(\tilde{b}^\vee)^\vee$ where $\tilde{b}^\vee : C^\vee \otimes \mathcal{O} \to B^\vee \otimes T_{\mathbb{P}^n}(-1)$. By Euler sequence we get:
$0 \to \mathrm{H}^0(\mathcal{O} \otimes S_n) \xrightarrow{\sim} \mathrm{H}^0(T_{\mathbb{P}^n}(-1)) \to 0$
$\Rightarrow \mathrm{H}^0(\tilde{b}^\vee) : \mathrm{H}^0(C^\vee \otimes \mathcal{O}) \to \mathrm{H}^0(B^\vee \otimes T_{\mathbb{P}^n}(-1))$
with : $\mathrm{H}^0(C^\vee \otimes \mathcal{O}) \simeq C^\vee \otimes \mathbb{C}$
$\mathrm{H}^0(B^\vee \otimes T(-1)) \simeq B^\vee \otimes \mathbb{C} \otimes V_n$
Then dualizing it is induced the map:
$b : B \otimes S_n \to C$

Now, if we denote

$$B := S_{k-1}^\vee \quad \text{and} \quad C := \overset{2}{\wedge} U^\vee \otimes S_{k-2}^\vee \otimes S_{n-1}^\vee \tag{1.29}$$

then, by previous lemma we can define the morphism

$$\tilde{b} : S_{k-1}^\vee \otimes \Omega^1(1) \to \overset{2}{\wedge} U^\vee \otimes S_{k-2}^\vee \otimes S_{n-1}^\vee \otimes \mathcal{O}$$

where $b = \beta^\vee$ and β is defined in (1.24).

1.4.3 Lemma. *$\tilde{b}$ is a surjective map.*

Proof. We consider the following exact commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & S_{k-1}^\vee \otimes \Omega^1(1) \arrow[r, "\tilde{b}"] \arrow[d, "f"] & \overset{2}{\wedge} U^\vee \otimes S_{k-2}^\vee \otimes S_{n-1}^\vee \otimes \mathcal{O} \arrow[d, equal] & \\
0 \arrow[r] & S_{k+n-1}^\vee \otimes \mathcal{O} \arrow[r, "\mu^\vee"] \arrow[d, "\varphi"] & S_{k-1}^\vee \otimes S_n^\vee \otimes \mathcal{O} \arrow[r, "\beta^\vee"] \arrow[d, "g"] & \overset{2}{\wedge} U^\vee \otimes S_{k-2}^\vee \otimes S_{n-1}^\vee \otimes \mathcal{O} \arrow[r] & 0 \\
 & S_{k-1}^\vee \otimes \mathcal{O}(1) \arrow[r, equal] & S_{k-1}^\vee \otimes \mathcal{O}(1) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd} \tag{1.30}$$

where $\varphi := g \circ \mu^\vee$ and $N = Ker\varphi \cong Ker\tilde{b}$
In fact, if $x \in Ker\tilde{b} \Rightarrow \tilde{b}(x) = \beta^\vee(f(x)) = 0 \Rightarrow f(x) \in Ker\beta^\vee = Im\mu^\vee$
$\Rightarrow \exists a \in S_{k+n-1}^\vee \otimes \mathcal{O}$ such that $f(x) = \mu^\vee(a) \in Imf = Kerg$
$g(\mu^\vee(a)) = 0 \Rightarrow a \in Ker\varphi$.
then we can define the morphism

$\Phi : Ker\tilde{b} \to Ker\varphi$ defined by

$x \mapsto a$

Claim: Φ is an isomorphism.
Obviously Φ is surjective by definition. Moreover, since f and $\mu^\vee$ are injective maps, if $x \neq x'$ and $\Phi(x) \neq \Phi(x')$ then $f(x) = \mu^\vee(\Phi(x)) \neq \mu^\vee(\Phi(x')) = f(x')$ (absurde) then Φ is also injective.
Now, by Snake lemma applied to exact diagram (1.30) it sufficient to prove that

φ is surjective, but this follows immediately observing that the map

$$\varphi^{\vee} : S_{k-1} \otimes \mathcal{O}(-1) \xrightarrow{g^{\vee}} S_{k-1} \otimes S_n \otimes \mathcal{O} \xrightarrow{\mu} S_{k+n-1} \otimes \mathcal{O} \tag{1.31}$$

is represented by the $k \times (k+n)$ matrix:

$$\begin{pmatrix} s^n & s^{n-1}t & \cdots & st^{n-1} & t^n & & & & \\ & \ddots & \ddots & & \ddots & \ddots & & & \\ & & & s^n & s^{n-1}t & \cdots & st^{n-1} & t^n \end{pmatrix}$$

which has maximal rank.

$\square$

1.4.4 Remark. The bundle N in the previous lemma is exactly the Schwarzenberger bundle $\mathcal{S}^*$ that we wanted.

Now, by diagram (1.30) and identifications (1.29) the Schwarzenberger bundle $\mathcal{S}^*$ verifies:

$$0 \to \mathcal{S}^* \to B \otimes \Omega^1(1) \to C \otimes \mathcal{O} \to 0 \tag{1.32}$$

By using the same technique of [Dio98], we perform its second symmetric and its alternating power and we get

$$\begin{tikzcd} 0 \arrow[r] & S^2\mathcal{S}^* \arrow[r] & \tilde{A} \arrow[rr] \arrow[dr] & & B \otimes C \otimes \Omega^1(1) \arrow[r] & \overset{2}{\wedge} C \otimes \mathcal{O} \arrow[r] & 0 \\ & & & M' \arrow[ur] \arrow[dr] & & & \\ & & 0 \arrow[ur] & & 0 & & \end{tikzcd} \tag{1.33}$$

where $\tilde{A} := S^2(B \otimes \Omega^1(1)) = (S^2B \otimes S^2(\Omega^1(1))) \oplus (\overset{2}{\wedge} B \otimes \Omega^2(2))$

and

$$\begin{tikzcd} 0 \arrow[r] & \overset{2}{\wedge} \mathcal{S}^* \arrow[r] & \overline{A} \arrow[rr] \arrow[dr] & & B \otimes C \otimes \Omega^1(1) \arrow[r] & \mathcal{O} \otimes S^2C \arrow[r] & 0 \\ & & & M \arrow[ur] \arrow[dr] & & & \\ & & 0 \arrow[ur] & & 0 & & \end{tikzcd} \tag{1.34}$$

where $\overline{A} := \overset{2}{\wedge} (B \otimes \Omega^1(1)) = (\overset{2}{\wedge} B \otimes S^2(\Omega^1(1))) \oplus (S^2B \otimes \Omega^2(2))$

Diagram (1.33) gives the following two exact sequences:

$$O \to \mathrm{H}^0(M') \to \mathrm{H}^1(S^2\mathcal{S}^*) \to \mathrm{H}^1(\tilde{A}) \to \mathrm{H}^1(M') \to \mathrm{H}^2(S^2(\mathcal{S}^*)) \to \mathrm{H}^2(\tilde{A}) \to \cdots \tag{1.35}$$

$$\begin{array}{ccccccccc} O \to \mathrm{H}^0(M') \to & B \otimes C \otimes \mathrm{H}^0(\Omega^1(1)) & \to \overset{2}{\wedge} C \to \mathrm{H}^1(M') \to & B \otimes C \otimes \mathrm{H}^1(\Omega^1(1)) & \to \cdots \\ & \| & & \| & \\ & 0 & & 0 & \end{array} \tag{1.36}$$

Sequence (1.36) implies: $\mathrm{H}^0(M') = 0$ and $\mathrm{H}^1(M') \cong \overset{2}{\wedge} C$

Then, by using the two formulas:

$\mathrm{H}^1(\tilde{A}) = (S^2B \otimes \mathrm{H}^1(S^2\Omega^1(1))) \oplus (\overset{2}{\wedge} B \otimes \mathrm{H}^1(\Omega^2(2)) = S^2B \otimes \overset{2}{\wedge} S_n^{\vee}$

and: $\mathrm{H}^2(\tilde{A}) = (S^2B \otimes \mathrm{H}^2(S^2\Omega^1(1))) \oplus (\overset{2}{\wedge} B \otimes \mathrm{H}^2(\Omega^2(2)) = 0$

sequence (1.35) becomes:

$$0 \to \mathrm{H}^1(S^2\mathcal{S}^*) \to \mathrm{H}^1(\tilde{A}) \to \mathrm{H}^1(M') \to \mathrm{H}^2(S^2(\mathcal{S}^*)) \to 0$$

i.e.

$$0 \to \mathrm{H}^1(S^2\mathcal{S}^*) \to S^2B \otimes \overset{2}{\wedge} S_n^\vee \xrightarrow{\tilde{\Phi}} \overset{2}{\wedge} C \to \mathrm{H}^2(S^2\mathcal{S}^*) \to 0$$

$$\Longrightarrow \qquad \mathrm{H}^2(S^2\mathcal{S}^*) \cong \mathrm{Coker}(\tilde{\Phi}) = (\mathrm{Ker}(\tilde{\Phi}^\vee))^\vee$$

Then:

$$\mathrm{H}^2(S^2\mathcal{S}^*)^\vee = \mathrm{Ker}\left[\overset{2}{\wedge}(S_{k-2} \otimes S_{n-1}) \xrightarrow{\tilde{\Phi}^\vee} S^2(S_{k-1}) \otimes \overset{2}{\wedge} S_n\right]$$

Moreover, diagram (1.34) gives the following two exact sequences:

$$O \to \mathrm{H}^0(M) \to \mathrm{H}^1(\overset{2}{\wedge}\mathcal{S}^*) \to \mathrm{H}^1(\overline{A}) \to \mathrm{H}^1(M) \to \mathrm{H}^2(\overset{2}{\wedge}\mathcal{S}^*) \to \mathrm{H}^2(\overline{A}) \to \cdots \tag{1.37}$$

$$\begin{array}{ccccccccccc} O \to \mathrm{H}^0(M) \to B \otimes C \otimes & \mathrm{H}^0(\Omega^1(1)) & \to S^2C \otimes & \mathrm{H}^0(\mathcal{O}) & \to \mathrm{H}^1(M) \to 0 \to \cdots \text{ and,} \\ & \| & & \| & \\ & 0 & & S^2C & \end{array} \tag{1.38}$$

from sequence (1.38), we get

$\mathrm{H}^0(M) = 0$ and $\mathrm{H}^1(M) \simeq S^2C$

Then, since :

$\mathrm{H}^1(\overline{A}) = (\mathrm{H}^1(S^2(\Omega^1(1)) \otimes \overset{2}{\wedge} B) \oplus (S^2B \otimes \mathrm{H}^1(\Omega^2(2))) = \overset{2}{\wedge} B \otimes \overset{2}{\wedge} S_n^\vee$

and $\mathrm{H}^2(\overline{A}) = 0$ sequence (1.37) becomes:

$$\begin{array}{l} O \to \mathrm{H}^0(M) \to \mathrm{H}^1(\overset{2}{\wedge}\mathcal{S}^*) \to \mathrm{H}^1(\overline{A}) \to \mathrm{H}^1(M) \to \mathrm{H}^2(\overset{2}{\wedge}\mathcal{S}^*) \to 0 \\ \quad \| \\ \quad 0 \end{array}$$

i.e. $0 \to \mathrm{H}^1(\overset{2}{\wedge}\mathcal{S}^*) \to \overset{2}{\wedge} B \otimes \overset{2}{\wedge} S_n^\vee \xrightarrow{\overline{\Phi}} S^2C \to \mathrm{H}^2(\overset{2}{\wedge}\mathcal{S}^*) \to 0$

$$\Longrightarrow \qquad \mathrm{H}^2(\overset{2}{\wedge}\mathcal{S}^*) \cong \mathrm{Coker}(\overline{\Phi}) = (\mathrm{Ker}(\overline{\Phi}^\vee))^\vee$$

Then we obtain :

$$(\mathrm{H}^2(\overset{2}{\wedge}\mathcal{S}^*))^\vee = \mathrm{Ker}\left[S^2(S_{k-2} \otimes S_{n-1}) \xrightarrow{\overline{\Phi}^\vee} \overset{2}{\wedge} S_{k-1} \otimes \overset{2}{\wedge} S_n\right]$$

Same reasonings of [Dio98] can be repeated in this case and we get that there are two injective SL(2)-equivariant morphisms

$$\tilde{\varepsilon} \ : \overset{2}{\wedge} S_{k-3} \otimes S^2S_{n-2} \to \overset{2}{\wedge}(S_{k-2} \otimes S_{n-1})$$

$$\bar{\varepsilon} \ : \ S^2S_{k-3} \otimes S^2S_{n-2} \to S^2(S_{k-2} \otimes S_{n-1})$$

such that: $Im\tilde{\varepsilon} \subset \mathrm{Ker}\ \tilde{\Phi}^\vee$ and $Im\bar{\varepsilon} \subset \mathrm{Ker}\ \bar{\Phi}^\vee$

Moreover, analogously to [OT94](page 202) we get

$$H^2(\mathcal{S}^* \otimes \mathcal{S}^*) \cong Ker(\Phi^\vee)^\vee$$

where

$$\Phi^\vee : S_{k-2}^{\otimes 2} \otimes S_{n-1}^{\otimes 2} \to S_{k-1}^{\otimes 2} \otimes \overset{2}{\wedge} S_n$$

and there is an isomorphism of SL(2)-representations

$$\varepsilon : S_{k-3}^\vee \otimes S_{k-3}^\vee \otimes S^2S_{n-2}^\vee \to Ker(\Phi^\vee)$$

1.4.5 Theorem. *For any special Steiner bundle $\mathcal{S}^*$*

$$\mathrm{H}^2(S^2\mathcal{S}^*) \simeq \bigwedge^2 (S_{k-3})^\vee \otimes S^2(S_{n-2})^\vee$$
$$and \qquad \mathrm{H}^2(\bigwedge^2 \mathcal{S}^*) \simeq S^2(S_{k-3})^\vee \otimes \bigwedge^2 (S_{n-2})^\vee$$

Proof. We can consider the following diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{H}^2(\bigwedge^2 \mathcal{S}^*)^\vee \arrow[r] \arrow[d] & S^2(S_{k-2}\otimes S_{n-1}) \arrow[r, "\Phi^\vee"] \arrow[d] & \bigwedge^2 S_{k-1}\otimes \bigwedge^2 S_n \arrow[r] \arrow[d] & \mathrm{H}^1(\bigwedge^2 \mathcal{S}^*)^\vee \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{H}^2(\mathcal{S}^*\otimes \mathcal{S}^*)^\vee \arrow[r] \arrow[d] & S_{k-2}^{\otimes 2}\otimes S_{n-1}^{\otimes 2} \arrow[r, "\Phi^\vee"] \arrow[d] & S_{k-1}^{\otimes 2}\otimes \bigwedge^2 S_n \arrow[r] \arrow[d] & \mathrm{H}^1(\mathcal{S}^*\otimes \mathcal{S}^*)^\vee \arrow[d] & \\
0 \arrow[r] & \mathrm{H}^2(S^2\mathcal{S}^*)^\vee \arrow[r] \arrow[d] & \bigwedge^2 (S_{k-2}\otimes S_{n-1}) \arrow[r, "\tilde{\Phi}^\vee"] \arrow[d] & S^2S_{k-1}\otimes \bigwedge^2 S_n \arrow[r] \arrow[d] & \mathrm{H}^1(S^2\mathcal{S}^*)^\vee \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

Then, we can consider the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S^2S_{k-3}\otimes S^2S_{n-2} \arrow[r] \arrow[d, "\bar{\varepsilon}"] & S_{k-3}^{\otimes 2}\otimes S^2S_{n-2} \arrow[r] \arrow[d, "\varepsilon"] & \bigwedge^2 S_{k-3}\otimes S^2S_{n-2} \arrow[r] \arrow[d, "\tilde{\varepsilon}"] & 0 \\
0 \arrow[r] & \mathrm{H}^2(\bigwedge^2 \mathcal{S}^*)^\vee \arrow[r] & \mathrm{H}^2(\mathcal{S}^*\otimes \mathcal{S}^*)^\vee \arrow[r] \arrow[d] & H^2(S^2\mathcal{S}^*)^\vee \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

and by the Snake-Lemma there is the exact sequence :

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\ \bar{\varepsilon} \arrow[r] \arrow[d, equal] & \mathrm{Ker}\ \varepsilon \arrow[r] \arrow[d, equal] & \mathrm{Ker}\ \tilde{\varepsilon} \arrow[r] \arrow[d, equal] & \mathrm{Coker}\ \bar{\varepsilon} \arrow[r] & \mathrm{Coker}\ \varepsilon \arrow[r] & \mathrm{Coker}\ \tilde{\varepsilon} \arrow[r] \arrow[d, equal] & 0 \\
 & 0 & 0 & 0 & & & 0 &
\end{tikzcd}$$

$\Rightarrow$ Coker $\bar{\varepsilon} = 0 \Rightarrow \bar{\varepsilon}$ is an isomorphism $\Rightarrow \tilde{\varepsilon}$ is an isomorphism.

Thus:

$$\mathrm{H}^2(S^2\mathcal{S}^*)^\vee \cong \bigwedge^2 (S_{k-3}) \otimes S^2(S_{n-2})$$
$$\text{and} \qquad \mathrm{H}^2(\bigwedge^2 \mathcal{S}^*)^\vee \cong S^2(S_{k-3}) \otimes \bigwedge^2 (S_{n-2})$$

as we wanted.

□

By above results we get:

$$h^2(S^2\mathcal{S}^*) = \binom{k-2}{2}\binom{n}{2} \qquad \text{and} \qquad h^2(\bigwedge^2 \mathcal{S}^*) = \binom{k-1}{2}\binom{n-1}{2} \tag{1.39}$$

Now, by performing second symmetric power of first column of diagram (1.30) we obtain:

$$0 \to S^2\mathcal{S}^* \to S^2S^\vee_{n+k-1}\otimes \mathcal{O} \to S^\vee_{n+k-1}\otimes S^\vee_{k-1}\otimes \mathcal{O}(1) \to \bigwedge^2 S^\vee_{k-1}\otimes \mathcal{O}(2) \to 0$$

then the Euler-Poincaré characteristic of $S^2\mathcal{S}^*$ is

$$\chi(S^2\mathcal{S}^*) = \binom{n+k+1}{2} + \binom{k}{2}\binom{n+2}{2} - k(n+k)(n+1)$$

Moreover, since for Steiner bundles $h^0(\mathcal{S}^*) = 0$ (see [AO99] proposition 3.4), then $\mathrm{H}^0(S^2\mathcal{S}^*) = 0$ and after calculation we get

1.4.6 Theorem. *Any Schwarzenberger bundle $\mathcal{S}^*$ verifies:*

$$h^1(S^2\mathcal{S}^*) = h^2(S^2\mathcal{S}^*) - \chi(S^2\mathcal{S}^*) = n(n+2k-2)$$

Chapter 2

Minimal resolution of general stable vector bundles on $\mathbb{P}^2$

2.1 Generalities and notations

Let us write a free resolution $F_\bullet \to \mathcal{E} \to 0$ of a vector bundle $\mathcal{E}$ in the form

$$\cdots \to \bigoplus_{q\in\mathbb{Z}} B_{1,q} \otimes_{\mathbb{C}} \mathcal{O}(-q) \to \bigoplus_{q\in\mathbb{Z}} B_{0,q} \otimes_{\mathbb{C}} \mathcal{O}(-q) \to \mathcal{E} \to 0, \qquad (2.1)$$

where the complex vector spaces $B_{p,q}$ are zero but a finite number. Then, we denote by $b_{p,q}(\mathcal{E}) := \dim \operatorname{Tor}_p(\mathcal{E}, \mathbb{C})_q$ the Betti numbers of $\mathcal{E}$ and with $b_{p,q}(F_\bullet) := \dim B_{p,q}$ the Betti numbers of the resolution (2.1). We recall that, if $F_\bullet$ is minimal, then $b_{p,q}(F_\bullet) = b_{p,q}(\mathcal{E})$ for all p, q.

A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is said m-regular if $\mathrm{H}^q(\mathbb{P}^n, \mathcal{F}(m-q)) = 0$ for all $q > 0$.

2.1.1 Theorem (Castelnuovo-Mumford). *[Mum66] An m-regular sheaf is*

$(m+1)$-*regular.*

The *regularity* of a sheaf is defined as the minimum integer r for which $\mathcal{F}$ is r-regular. By the previous theorem and the semicontinuity of cohomology groups, the regularity is a upper semicontinuous function on the related moduli space.

2.1.2 Theorem ([Gre89]). *For all coherent sheaves $\mathcal{F}$ on $\mathbb{P}^n$,*

$$\operatorname{reg}(\mathcal{F}) = \max\{q-p \quad | \quad b_{p,q}(\mathcal{F}) \neq 0\}. \tag{2.2}$$

Another parameter that measures the "complexity" of a resolution is its *length*: we say that $0 \to \mathcal{A}_n \to \cdots \to \mathcal{A}_0 \to \mathcal{F} \to 0$ is a resolution of $\mathcal{F}$ of length n. The homological dimension $\operatorname{hd}(\mathcal{F})$ of a sheaf $\mathcal{F}$ is the length of its minimal free resolution. One fundamental tool to control the homological dimension is the following result that can be regarded as a refined version of the Hilbert syzygy theorem:

2.1.3 Theorem (Horrocks [Hor64]). *A torsion-free sheaf $\mathcal{F}$ on $\mathbb{P}^n$ has homological dimension at most $n-1$.*

2.2 Admissible pairs and resolutions

Let $\mathcal{E}$ be a rank r vector bundle on $\mathbb{P}^2$. By Horrocks theorem [Hor64], $\mathcal{E}$ has homological dimension at most 1, that is, it has a free resolution of the form

$$0 \to \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^2}(-a_i) \xrightarrow{\Phi} \bigoplus_{j=1}^{r+k} \mathcal{O}_{\mathbb{P}^2}(-b_j) \to \mathcal{E} \to 0. \tag{2.3}$$

We do not assume that the resolution (2.3) is minimal. Vector bundles of homological dimension 0, i.e., splitting vector bundles, are a special case in which either $k=0$ or the resolution (2.3) is not minimal; however, they are of marginal interest to us since they are not stable.

We suppose that the two sequences a_i and b_i are indexed in nondecreasing order

$$\begin{aligned} &a_1 \le a_2 \le \cdots \le a_k, \\ &b_1 \le b_2 \le \cdots \le b_k \le \cdots \le b_{r+k}. \end{aligned} \tag{2.4}$$

We call $(a,b) = ((a_1,\ldots,a_k),(b_1,\ldots,b_{r+k}))$ the *associated pair* to the resolution (2.3). If the resolution (2.3) is minimal, we call (a,b) the pair associated to the bundle $\mathcal{E}$. Notice that the associated pair and the Betti numbers of a resolution encode exactly the same information; in particular $\max(a_k-1, b_{r+k})$ is the Castelnuovo-Mumford regularity of $\mathcal{E}$.

The Chern classes c_1, c_2 of $\mathcal{E}$ are determined by a_i and b_j with the formulas

$$c_1 = \sum_{i=1}^{k} a_i - \sum_{i=1}^{k+r} b_i, \tag{2.5}$$

$$2c_2 - c_1^2 = \sum_{i=1}^{k} a_i^2 - \sum_{i=1}^{k+r} b_i^2. \tag{2.6}$$

2.2.1 Definition. The pair (a,b) is said to be *admissible* if

$$a_i > b_{2+i} \qquad \text{for all } i = 1,\ldots,k \tag{2.7}$$

For brevity we say that the resolution (2.3) is admissible if the associated pair (a,b) is.

2.2.2 Example. By Euler sequence

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus 3} \to T_{\mathbb{P}^2} \to 0,$$

the tangent bundle of $\mathbb{P}^2$ has associated pair $((0),(-1,-1,-1))$ which is admissible.

2.2.3 Example. The resolution

$$0 \to \mathcal{O} \to \mathcal{O}(-1) \oplus \mathcal{O}(1)^{\oplus 3} \to T_{\mathbb{P}^2} \oplus \mathcal{O}(-1) \to 0,$$

is not admissible. Remark that $T_{\mathbb{P}^2} \oplus \mathcal{O}(-1)$ is not a stable bundle.

More generally, we can consider the associated pair (a,b) to any vector bundle of homological dimension 1 on $\mathbb{P}^n$ with $n \geq 2$. In that case, we say that (a,b) is admissible if $a_i > b_{n+i}$ for $i = 1,\dots,k$, as in [BS92].

2.2.4 Proposition. *If $r \geq 2$, resolution (2.3) is minimal if and only if it is admissible and every constant entry of the matrix $(\phi_{i,j})$ is zero.*

Proof. Obviously, if the pair is admissible and every constant entry of the matrix $(\phi_{i,j})$ is zero then (2.3) is minimal. Conversely, for $r = 2$ the statement was proved by Bohnhorst and Spindler ([BS92] proposition 2.3). Now suppose that $r > 2$ and (2.3) is minimal. Since $\mathcal{E}(b_{r+k})$ is globally generated, Bertini's theorem ensures that a generic map $f\colon \mathcal{O}(-b_{r+k}) \to \mathcal{E}$ is injective. Then in the following commutative diagram columns and rows are exact and $\mathcal{E}''$ is locally free:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathcal{O}(-b_{r+k}) \arrow[r, "Id"] \arrow[d] & \mathcal{O}(-b_{r+k}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \oplus_{i=1}^{k}\mathcal{O}(-a_i) \arrow[r] \arrow[d, "Id"] & \oplus_{i=1}^{r+k}\mathcal{O}(-b_i) \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \oplus_{i=1}^{k}\mathcal{O}(-a_i) \arrow[r] \arrow[d] & \oplus_{i=1}^{r+k-1}\mathcal{O}(-b_i) \arrow[r] \arrow[d] & \mathcal{E}'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.8}$$

The minimality of the middle row yields the minimality of the last row. Using induction on r, we may assume that the last row is admissible. Then the middle row is also admissible. □

2.2.5 Theorem. *Let $r = 2$ and suppose that the resolution (2.3) is admissible. Let $c_1 = \sum a_i - \sum b_j$ be the first Chern class and μ be the slope of $\mathcal{E}$. Then $\mathcal{E}$ is semistable (respectively stable) if and only if*

$$b_1 \geq -\mu \qquad (\textit{resp. } b_1 > -\mu). \tag{2.9}$$

Proof. This is theorem 2.7 of [BS92] in the case of rank-2 vector bundles on $\mathbb{P}^2$. □

For the case of higher rank, we have no chances to extend the above arithmetical characterization since stable and unstable vector bundles may have the same associated pair.

2.2.6 Proposition. *If resolution (2.3) is admissible (in particular if it is minimal) and $\mathcal{E}$ is semistable (resp. stable) then the associated pair (a,b) verifies $b_1 \geq -\mu$ (resp. $b_1 > -\mu$).*

Proof. If $\mathcal{E}$ is semistable (resp. stable), then

$$\mathrm{H}^0(\mathcal{E}(m)) = 0 \qquad \forall m < -\mu(\mathcal{E}) \quad (\text{resp. } \forall m \leq -\mu(\mathcal{E})) \tag{2.10}$$

but, from the exact sequence

$$0 \to \bigoplus_{i=1}^{k} \mathcal{O}(-a_i + b_1) \to \bigoplus_{j=1}^{r+k} \mathcal{O}(-b_j + b_1) \to \mathcal{E}(b_1) \to 0, \tag{2.11}$$

we have $\mathrm{H}^0(\mathcal{E}(b_1)) \neq 0$ then $b_1 > -\mu(\mathcal{E})$ (resp. $b_1 \geq -\mu(\mathcal{E})$). □

2.2.7 Lemma. *If resolution (2.3) is admissible and $\mathcal{E}$ is semistable, then*

$$b_{r+k} < a_k.$$

Proof. If $b_{r+k} \geq a_k$ then $\mathcal{E}$ split as $\mathcal{E} = \mathcal{O}(-b_{k+r}) \oplus \mathcal{E}''$ and by admissibility

$$\begin{aligned}\sum_{i=1}^{k} a_i - \sum_{j=1}^{r+k-1} b_j &= -\sum_{i=1}^{2} b_i + \sum_{i=1}^{k}(a_i - b_{n+i}) - \sum_{i=k+3}^{r+k-1} b_i \\ &\geq -2b_{r+k} + k - (r-3)b_{r+k} \\ &> (1-r)b_{r+k}\end{aligned} \tag{2.12}$$

then we have $\mu(\mathcal{E}'') > -b_{r+k} = \mu(\mathcal{O}(-b_{r+k}))$ which contradicts the semistability of $\mathcal{E}$. □

We denote by $\mathfrak{I}$ the set of all admissible pairs (a,b) associated to rank r-vector bundles on $\mathbb{P}^2$ with Chern classes c_1, c_2 satisfying the conditions $b_1 > -\mu = (\sum a_i - \sum b_j)/r$ and $b_{r+k} < a_k$. Proposition 2.2.6 shows that the set $\mathfrak{I}$ contains the set of all possible associated pair to a stable vector bundle in $\mathfrak{M}_{\mathbb{P}^2}(r, c_1, c_2)$ and coincides exactly with it for $r = 2$. Then

$$\mathfrak{M}_{\mathbb{P}^2}(r, c_1, c_2) = \coprod_{(a,b)\in\mathfrak{I}} \mathfrak{M}(a,b) \tag{2.13}$$

where $\mathfrak{M}(a,b)$ will be the subset (possibly empty) of $\mathfrak{M}_{\mathbb{P}^2}(r, c_1, c_2)$ of vector bundles with associated pair (a,b).

2.2.8 Remark. Let us fix a pair $(a,b) \in \mathfrak{I}$ and define $F_0 = \oplus_{j=1}^{k+r}\mathcal{O}(-b_j), F_1 = \oplus_{i=1}^{k}\mathcal{O}(-a_i)$. We consider the linear subspace V of $Hom(F_1, F_0)$ consisting of those homomorphisms $\phi : F_1 \to F_0$ such that every constant entry of the matrix

$(\phi_{i,j})$ is zero and The set $X = \{\phi \in V | \phi \text{ is injective}, coker\phi \text{ is locally free}\}$ is a Zariski open subspace of V. The cokernel of the universal homomorphism

$$\Phi : pr_2^* F_1 \to pr_2^* F_0 \qquad on \quad X \times \mathbb{P}^2$$

induces a morphism

$$\tau : X \to \mathfrak{M}$$

By definition $\mathfrak{M}(a,b) = \tau(X)$. Especially $\mathfrak{M}(a,b)$ is a constructible subset of $\mathfrak{M}$.

The following result was stated and proved by Bohnhorst and Spindler [BS92] for rank-n vector bundles on $\mathbb{P}^n$ with homological dimension 1, but their proof works on $\mathbb{P}^2$ for vector bundles of any rank without modifications.

2.2.9 Theorem. *For all $(a,b) \in \mathfrak{I}$, the closed set $\overline{\mathfrak{M}(a,b)}$ is an irreducible algebraic subset of $\mathfrak{M}_{\mathbb{P}^2}(r, c_1, c_2)$ of dimension:*

$$\begin{aligned}\dim \overline{\mathfrak{M}(a,b)} &= \dim \operatorname{Hom}(F_1, F_0) + \dim \operatorname{Hom}(F_0, F_1) \\ &\quad - \dim \operatorname{End}(F_1) - \dim \operatorname{End}(F_0) + 1 - \#\{(i,j) : a_i = b_j\},\end{aligned} \tag{2.14}$$

where $F_0 = \oplus_{j=1}^{k+r}\mathcal{O}(-b_j)$, $F_1 = \oplus_{i=1}^{k}\mathcal{O}(-a_i)$.

We introduce a partial order on $\mathfrak{I}$, namely

$$\begin{aligned}(\tilde{a}, \tilde{b}) \preceq (a,b) \qquad \Longleftrightarrow \qquad \exists \quad \alpha_1, \cdots, \alpha_t \in \mathbb{Z} \quad &\text{such that up to order} \\ \tilde{a} &= (a_1, \cdots, a_k, \alpha_1, \cdots, \alpha_t) \\ and \quad \tilde{b} &= (b_1, \cdots, b_{k+n}, \alpha_1, \cdots, \alpha_t)\end{aligned}$$

2.2.10 Lemma. *If $(a,b), (\tilde{a}, \tilde{b}) \in \mathfrak{I}$ then*

$$(\tilde{a}, \tilde{b}) \preceq (a,b) \Longrightarrow \mathfrak{M}^1(\tilde{a}, \tilde{b}) \subset \overline{\mathfrak{M}^1(a,b)}$$

In the following lemma we find an upper bound on the regularity of semistable vector bundles on $\mathbb{P}^2$ of rank 2. A lower bound is given in corollary 2.2.24.

2.2.11 Theorem. *A normalized semistable rank 2 bundle $\mathcal{E}$ on $\mathbb{P}^2$ is c_2-regular.*

Proof. For brevity's sake, we set $\xi_i := a_i - b_{i+2}$ and $t_i := b_{i+2} - b_2$. Obviously, $\xi_i \geq 1$ and $t_i \geq 0$. We rewrite (2.5) as

$$\sum_{i=1}^{k} \xi_i = \sum_{i=1}^{2} b_i + c_1 \tag{2.15}$$

and by (2.6), using inequalities (2.4), (2.7), (2.9) we get

$$\begin{aligned} \sum_{i=1}^{2} b_i^2 + 2c_2 - c_1^2 = \sum_{i=1}^{k}(a_i^2 - b_{i+2}^2) &= \sum_{i=1}^{k} \xi_i(2b_2 + 2t_i + \xi_i) \\ &\geq 2b_2 \sum_{i=1}^{k} \xi_i + \sum_{i=1}^{k}(2t_i + \xi_i) \\ &= (2b_2+1)(\sum_{i=1}^{2} b_i + c_1) + 2\sum_{i=1}^{k} t_i. \end{aligned} \tag{2.16}$$

If we suppose that $b_2 + \sum_{i=1}^{k} t_i \geq c_2 + 1$, then

$$\sum_{i=1}^{2} b_i^2 + 2b_2 - c_1^2 - (2b_2+1)(\sum_{i=1}^{2} b_i + c_1) \geq 2.$$

Since the left side is non increasing with respect to b_1, we may restrict ourselves to the case $b_1 = -c_1$. But $\sum_{i=2}^{2} b_i^2 + 2b_2 - (2b_2+1)(\sum_{i=2}^{2} b_i) \geq 2$ it easily seen to be impossible. Then $\sum_{i=1}^{k} t_i$ must be at most $c_2 - b_2$ and in particular

$$b_{k+2} = b_2 + t_k \leq b_2 + \sum t_i \leq c_2. \tag{2.17}$$

Now, we must show that $a_k \leq c_2 + 1$. We rewrite (2.5) as $\sum_{i=1}^{k-1} \xi_i = \sum_{i=1}^{2} b_i +$

$b_{k+2} - a_k + c_1$ and by (2.6)

$$\begin{aligned} \sum_{i=1}^{2} + b_i^2 + b_{k+2}^2 - a_k^2 + 2c_2 - c_1^2 = \sum_{i=1}^{k-1}(a_i^2 - b_{i+2}^2) = \sum_{i=1}^{k-1} \xi_i(2b_2 + 2t_i + \xi_i) \geq \\ \geq 2b_2 \sum_{i=1}^{k-1} \xi_i + \sum_{i=1}^{k-1} \xi_i \geq (2b_2+1)(\sum_{i=1}^{2} b_i + b_{k+2} - a_k + c_1) \end{aligned} \tag{2.18}$$

that can be put in the form

$$\begin{aligned} \sum_{i=1}^{2} b_i^2 - (2b_2+1)(\sum_{i=1}^{2} b_i + c_1) + 2c_2 - c_1^2 \geq \\ \geq (a_k - b_{k+2})(a_k + b_{k+2} - 2b_2 - 1). \end{aligned} \tag{2.19}$$

Suppose that $a_k \geq c_2 + 2$. By (2.17) we have $a_k - b_{k+2} \geq c_2 + 2 - c_2 = 2$ and we observe also that $a_k + b_{k+2} - 2b_2 - 1 \geq c_2 - b_2 + 1$. Substituting and simplifying, (2.19) becomes

$$\sum_{i=1}^{2} b_i^2 - (2b_2+1)(\sum_{i=1}^{2} b_i + c_1) - c_1^2 \geq 2 + c_1^2 \tag{2.20}$$

As before, we can restrict ourselves to the case $b_1 = -c_1$ obtaining

$$\sum_{i=2}^{2} b_i^2 - (2b_2+1)\sum_{i=2}^{2} b_i \geq 2$$

that do not have solution for b_i positive. Then $a_k \leq c_2 + 1$. □

2.2.12 Remark. The above theorem is sharp in the sense that the admissible pairs $((c_2+1),(0,1,c_2))$ and $((c_2+1),(1,1,c_2))$ are associated to rank 2 semistable bundles of Chern classes c_1, c_2 that have regularity c_2.

2.2.13 Remark. It is also possible to prove that a semistable rank 2 bundle on $\mathbb{P}^2$ is c_2 regular if $c_1 = 0$ and (c_2+1)-regular if $c_1 = -1$ using the bounds on dimension of cohomology groups proved by Elencwajg and Forster (proposition 2.18 [EF80]) and Grauert-Mülich theorem.

2.2.14 Remark. From (2.15) and the thesis of previous theorem, the value k in (2.3) is bounded by:

$$k \leq \sum_{i=1}^{k} \xi_i = \sum_{i=1}^{2} b_i + c_1 \leq 2c_2 + c_1 \tag{2.21}$$

Hence, for fixed Chern classes c_1, c_2, there are only a finite number of admissible pair of rank 2 vector bundles and we can write an algorithm to enumerate such pairs restricting the search to a finite domain and to list them with the help of a computer.

2.2.15 Example. Associated admissible pairs for some Chern classes

1. Each bundle $\mathcal{E} \in \mathfrak{M}_{\mathbb{P}^2}(2; -1, 2)$ has a resolution of the form:

$$0 \to \mathcal{O}(-3) \to \mathcal{O}(-1) \oplus \mathcal{O}(-2)^{\oplus 2} \to \mathcal{E} \to 0$$

   and $((3), (1, 1, 2))$ is the associated pair. This is the only admissible pair in this case. In the section 2.3 we will describe explicitly this moduli space.

2. Admissible pairs associated to $\mathcal{E} \in \mathfrak{M}_{\mathbb{P}^2}(2; -1, 5)$:

```
   a_i          b_j            Codim.
=======================================
  (3 4)        (2 2 2 2)      |  0
  (3 3 4)      (2 2 2 2 3)    |  2
  (4 4)        (1 2 3 3)      |  2
  (3 5)        (1 2 2 4)      |  4
  (6)          (1 1 5)        |  6
```


3. Admissible pairs associated to $\mathfrak{M}_{\mathbb{P}^2}(2; -1, 9)$

```
   a_i          b_j            Codim.
=======================================
  (4 4 4 4 4)  (3 3 3 3 3 3 3) |  0
  (4 4 5)      (2 3 3 3 3)     |  1
  (4 4 4 5)    (2 3 3 3 3 4)   |  4
  (5 5)        (2 2 3 4)       |  4
  (5 5 5)      (1 3 4 4 4)     |  6
  (4 5 5)      (2 2 3 4 4)     |  6
  (4 4 6)      (2 2 3 3 5)     |  6
  (4 7)        (2 2 2 6)       |  8
  (3 6)        (2 2 2 4)       |  8
  (6)          (1 3 3)         |  8
  (3 5 6)      (2 2 2 4 5)     |  9
  (3 4 7)      (2 2 2 3 6)     |  9
  (5 6)        (1 3 3 5)       |  9
  (4 6)        (1 3 3 4)       |  9
  (3 3 8)      (2 2 2 2 7)     |  10
  (4 4 7)      (1 3 3 3 6)     |  10
  (4 5 6)      (1 3 3 4 5)     |  10
  (6 6)        (1 2 5 5)       |  10
  (4 8)        (1 2 3 7)       |  11
  (5 7)        (1 2 4 6)       |  11
  (3 9)        (1 2 2 8)       |  12
  (10)         (1 1 9)         |  14
```

2.2.1 Natural pairs and general vector bundles

We say that $(a,b) = ((a_1,\dots,a_k),(b_1,\dots,b_{r+k}))$ is a *natural pair* if it is admissible and

$$b_{r+k} < a_1, \qquad a_k \le b_1 + 2. \tag{2.22}$$

The above inequalities imply $a_i \neq b_j$ for all i and j.

We observe that natural pairs are parametrized by three integers s, k, α such that

$$k \ge 1 \qquad \text{and} \qquad -k+1 \le \alpha \le k+r \tag{2.23}$$

as follows: the pair $(a,b)_{s,k,\alpha}$ corresponding to the triple (s,k,α) is the pair associated to a resolution of the form

$$0 \to \mathcal{O}(-s-1)^k \to \mathcal{O}(-s)^\alpha \oplus \mathcal{O}(-s+1)^{r+k-\alpha} \to \mathcal{E} \to 0 \tag{2.24}$$

if $\alpha \ge 0$, or of the form

$$0 \to \mathcal{O}(-s-1)^{k+\alpha} \oplus \mathcal{O}(-s)^{-\alpha} \to \mathcal{O}(-s+1)^{r+k} \to \mathcal{E} \to 0 \tag{2.25}$$

if $\alpha < 0$. We have excluded the case $\alpha = -k$ so that s is the regularity of the pair, i.e. $s = \max(a_k - 1, b_{r+k})$.

Through this section we are going to show that resolutions of general vector bundles have natural pairs.

2.2.16 Theorem. *If $r = 2$, one has $\operatorname{codim} \overline{\mathfrak{M}(a,b)} = 0$ if and only if (a,b) is a natural pair.*

As a remarkable consequence we will derive a quite simple proof of the irreducibility of moduli spaces of stable vector bundles on $\mathbb{P}^2$ and we will compute the regularity and the cohomology of their general elements.

We recall that, since $\dim \operatorname{Ext}^2(\mathcal{F},\mathcal{F}) = 0$ for any stable vector bundle $\mathcal{F}$ on $\mathbb{P}^2$, the relative moduli $\mathfrak{M}_{\mathbb{P}^2}(r,c_1,c_2)$ space is smooth of dimension

$$\dim \operatorname{Ext}^1(\mathcal{F},\mathcal{F}) = 2rc_2 - (r-1)c_1^2 - r^2 + 1. \tag{2.26}$$

Let us consider the function $A(t) := h^2(\mathcal{O}(t))$ and the finite differences of first and second order $(\Delta_u A)(t) := A(t+u) - A(t)$ and $(\Delta_v \Delta_u A)(t) := (\Delta_u A)(t+v) - (\Delta_u A)(t)$.

2.2.17 Definition. The pair (a,b) is said to be *strongly admissible* if

$$a_i > b_{r+i} \qquad \text{for all } i = 1,\dots,k \tag{2.27}$$

For $r = 2$ strongly admissibility and admissibilility coincide.

2.2.18 Example. The map Φ in (2.3) can be expressed by a $(r+k)\times k$ matrix of forms $(\phi_{i,j})$ of degree $\deg(\phi_{i,j}) = (b_i - a_j)$. If (a,b) is a strongly admissible pair and $\omega_0 \dots \omega_r$ are linear forms in general position on $\mathbb{P}^n$, then the $(r+k)\times k$ matrix

$$(\phi_{ij}) := \begin{bmatrix} \omega_0^{a_1-b_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & & \omega_0^{a_k-b_k} \\ \omega_r^{a_1-b_{r+1}} & & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \dots & \omega_r^{a_k-b_{r+k}} \end{bmatrix} \tag{2.28}$$

defines a minimal free resolution and $\mathcal{E} := \operatorname{coker}\Phi$ is a vector bundle with associated pair (a,b).

2.2.19 Lemma. *If a strongly admissible pair (a,b) is associated to a stable vector bundle $\mathcal{E}$ on $\mathbb{P}^2$, Then*

$$\begin{aligned}\operatorname{codim}\overline{\mathfrak{M}(a,b)} = &\sum_{i=1}^{r} h^1(\mathcal{E}(b_i)) + \#\{(i,j) : a_i = b_j\} \\ &+ \sum_{i,j=1}^{r}(\Delta_{b_{i+r}-a_i}\Delta_{b_{j+r}-a_j}A)(a_i - b_{j+r})\end{aligned} \tag{2.29}$$

Proof. Let

$$0 \to F_1 \to F_0 \to \mathcal{E} \to 0 \tag{2.30}$$

be the minimal resolution of $\mathcal{E}$ where

$$F_0 = \bigoplus_{j=1}^{k+r} \mathcal{O}(-b_j), \qquad F_1 = \bigoplus_{i=1}^{k} \mathcal{O}(-a_i). \tag{2.31}$$

The stability of $\mathcal{E}$ ensures the vanishing $\dim(\operatorname{Ext}^2(\mathcal{E},\mathcal{E})) = h^2(\mathcal{E}^* \otimes \mathcal{E}) = 0$ so that $h^2({F_0}^* \otimes \mathcal{E}) = h^2({F_1}^* \otimes \mathcal{E})$. Then from (2.30) we easily find the following data:

$$\begin{aligned}h^0({F_0}^* \otimes \mathcal{E}) &= h^0({F_0}^* \otimes F_0) - h^0({F_0}^* \otimes F_1),\\ h^0({F_1}^* \otimes \mathcal{E}) &= h^0({F_1}^* \otimes F_0) - h^0({F_1}^* \otimes F_1),\\ \dim(\operatorname{Ext}^1(\mathcal{E},\mathcal{E})) &= h^1(\mathcal{E}^* \otimes \mathcal{E}) = \\ &= h^1({F_0}^* \otimes \mathcal{E}) - h^1({F_1}^* \otimes \mathcal{E}) + \\ &\quad + h^0({F_1}^* \otimes \mathcal{E}) - h^0({F_0}^* \otimes \mathcal{E}) + 1\end{aligned} \tag{2.32}$$

and from (2.14) we have

$$\begin{aligned}\operatorname{codim}\overline{\mathfrak{M}(a,b)} &= \dim(\operatorname{Ext}^1(\mathcal{E},\mathcal{E})) - \dim\overline{\mathfrak{M}(a,b)} = \\ &= h^1({F_0}^* \otimes \mathcal{E}) - h^1({F_1}^* \otimes \mathcal{E}) + \#\{(i,j) : a_i = b_j\}\end{aligned} \tag{2.33}$$

Now, by splitting F_0 as $\mathcal{O}(-b_1) \oplus \cdots \oplus \mathcal{O}(-b_{r-1}) \oplus \tilde{F_0}$ with $\tilde{F_0} := \oplus_{i=r+1}^{k+r}\mathcal{O}(-b_i)$, the above formula becomes

$$\begin{aligned}\operatorname{codim}\overline{\mathfrak{M}(a,b)} = &\sum_{i=1}^{r} h^1(\mathcal{E}(b_i)) + \#\{(i,j) : a_i = b_j\} \\ &+ h^1(\tilde{F_0}^* \otimes \mathcal{E}) - h^1({F_1}^* \otimes \mathcal{E}).\end{aligned} \tag{2.34}$$

Since $h^2(\tilde{F_0}^* \otimes F_0) = h^2(\tilde{F_0}^* \otimes \tilde{F_0})$ and $h^2({F_1}^* \otimes F_0) = h^2({F_1}^* \otimes \tilde{F_0})$ so

$$\begin{aligned}h^1(\tilde{F_0}^* \otimes \mathcal{E}) - h^1({F_1}^* \otimes \mathcal{E}) = \quad & h^2(\tilde{F_0}^* \otimes F_1) - h^2(\tilde{F_0}^* \otimes F_0) \\ & - h^2({F_1}^* \otimes F_1) + h^2({F_1}^* \otimes F_0) = \\ & = h^2(\tilde{F_0}^* \otimes F_1) - h^2(\tilde{F_0}^* \otimes \tilde{F_0}) \\ & - h^2({F_1}^* \otimes F_1) + h^2({F_1}^* \otimes \tilde{F_0}) = \\ & = \sum_{i,j=1}^{r}\Big[h^2(\mathcal{O}(b_{i+r} - a_j)) - h^2(\mathcal{O}(b_{i+r} - b_{j+r})) \\ & \quad - h^2(\mathcal{O}(a_i - a_j)) + h^2(\mathcal{O}(a_i - b_{j+r}))\Big]\end{aligned} \tag{2.35}$$

Finally the equation (2.34) becomes

$$\begin{aligned}\operatorname{codim}\overline{\mathfrak{M}(a,b)} = &\sum_{i=1}^{r} h^1(\mathcal{E}(b_i)) + \#\{(i,j) : a_i = b_j\} \\ &+ \sum_{i,j=1}^{r}(\Delta_{b_{i+r}-a_i}\Delta_{b_{j+r}-a_j}A)(a_i - b_{j+r})\end{aligned} \tag{2.36}$$

□

Proof of theorem 2.2.16. It can be verified by direct computation from theorem 2.2.9 that, if $\mathcal{E}$ has natural pair, then the codimension of $\overline{\mathfrak{M}(a,b)}$ is zero.

Conversely, let u, v be two non-negative integers. Since all finite difference $(\Delta_u A)(t) := A(t+u) - A(t)$ are non decreasing functions of t, then

$$(\Delta_v \Delta_u A)(t) \geq 0 \tag{2.37}$$

and by the previous lemma

$$\operatorname{codim} \overline{\mathfrak{M}(a,b)} \geq \sum_{i=1}^{2} h^1(\mathcal{E}(b_i)) + \#\{(i,j) : a_i = b_j\}. \tag{2.38}$$

If $\operatorname{codim} \overline{\mathfrak{M}(a,b)} = 0$, we have $a_k \leq b_1 + 2$ and $\#\{(i,j) : a_i = b_j\} = 0$, since $h^1(\mathcal{E}(b_1)) = 0$ implies $h^2(F_1(b_1)) = 0$. This forces (a,b) to be a natural pair. $\square$

2.2.20 Proposition. *Let $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ be nonempty and*

$$s := \max\{\rho \in \mathbb{Z} : 2\rho^2 + 2c_1\rho - 2\rho \leq 2c_2 - c_1^2 + c_1 - 1\}, \tag{2.39}$$

or, equivalently,

$$s := \min\{\rho \in \mathbb{Z} : 2\rho^2 + 2c_1\rho + 2\rho \geq 2c_2 - c_1^2 - c_1\}. \tag{2.39bis}$$

If α and k are defined by

$$\begin{aligned} \alpha &:= 2c_2 - c_1^2 + 2 - 2s^2 - 2c_1 s, \\ k &:= (2s + c_1 - 2 + |\alpha|)/2, \end{aligned} \tag{2.40}$$

then $(a,b)_{s,k,\alpha}$ is the only natural pair of $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$.

Proof. This is a verification; we outline the main steps of the computation. In the first place, one must ensure that the natural pair $(a,b)_{s,k,\alpha}$ is actually associated to vector bundles in $\mathfrak{M}_{\mathbb{P}^2}(r, c_1, c_2)$. This amount to show that, according

with equations (2.5) and (2.6), the pair $(a,b)_{s,k,\alpha}$ has the appropriate Chern classes and that conditions (2.23) hold.

From theorem 2.2.16, any pair (a,b) such that $\dim \overline{\mathfrak{M}(a,b)} = 0$ is a natural pair of the form $(a,b)_{s,k,\alpha}$. From resolutions (2.24) and (2.25) with $r = 2$ we find that α, k must satisfy (2.40). Then it remains to verify that s is uniquely determined from c_1, c_2 and satisfy (2.39). By substitution, the inequalities $-k < \alpha \leq k+2$ turn into

$$2s^2 + 2c_1 s - c_1 - 2s + 1 \leq 2c_2 - c_1^2 \leq 2s^2 + 2c_1 s + c_1 + 2s. \tag{2.41}$$

Since the intervals $[2s^2 + 2c_1 s - c_1 - 2s + 1, \quad 2s^2 + 2c_1 s + c_1 + 2s]$ are disjoint for s varying in $\mathbb{Z}$, then equations (2.39) and (2.39bis) give the only suitable value for s. $\square$

2.2.21 Theorem. *Moduli spaces of stable rank 2 vector bundles on $\mathbb{P}^2$ are irreducible.*

Proof. Moduli spaces of stable rank 2 vector bundles on $\mathbb{P}^2$ are smooth. By the previous proposition they can have only one connected component. $\square$

2.2.22 Corollary. *The general element of $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ has natural cohomology.*

The above corollary justify the terminology *"natural pair"*. A different proof for it, working also for higher rank, can be found in [HL93], by using sophisticated techniques of stacks theory.

2.2.23 Example. If $r = 2$ and $c_1 = -1$, in the following table we put the natural pairs for $c_2 \leq 15$.

c2	a_i	b_j
1	(2)	(1 1 1)
2	(3)	(1 1 2)
3	(3 3)	(1 2 2 2)
4	(3 3 3)	(2 2 2 2 2)
5	(3 4)	(2 2 2 2)
6	(4 4)	(2 2 2 3)
7	(4 4 4)	(2 2 3 3 3)
8	(4 4 4 4)	(2 3 3 3 3 3)
9	(4 4 4 4 4)	(3 3 3 3 3 3 3)
10	(4 4 4 5)	(3 3 3 3 3 3)
11	(4 5 5)	(3 3 3 3 3)
12	(5 5 5)	(3 3 3 3 4)
13	(5 5 5 5)	(3 3 3 4 4 4)
14	(5 5 5 5 5)	(3 3 4 4 4 4 4)
15	(5 5 5 5 5 5)	(3 4 4 4 4 4 4 4)

Now, we are going to give some inequalities on the regularity and the cohomology of stable vector bundles using proposition 2.2.20. In particular, for rank 2 vector bundles, the next two corollaries give respectively a refined version of corollary 5.4 in [Bru80] and proposition 7.1 in [Har78].

2.2.24 Corollary. *A general vector bundle $\mathcal{E}$ in $\mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ has regularity*

$\operatorname{reg}(\mathcal{E}) = s$, where s is given by (2.39).

2.2.25 Corollary. *Let $[\mathcal{E}]$ be a vector bundle in $\mathfrak{M} = \mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ and s defined by (2.39). Then $\mathrm{H}^0(\mathcal{E}(t)) \neq 0$ if*

$$\begin{aligned} &t \geq s && \textit{when} \quad 2s^2 + 2c_1 s + 2s = 2c_2 - c_1^2 - c_1, \\ &t \geq s - 1 && \textit{otherwise.} \end{aligned}$$

The above inequality is sharp, in the sense that, if $\mathcal{E}$ is general, it gives a necessary and sufficient condition.

Proof. Let $((a_1, \ldots, a_k), (b_1, \ldots, b_{k+2}))$ be the admissible pair associated to a vector bundle $\mathcal{E}$ in $\mathfrak{M}$. Then one has $\mathrm{H}^0(\mathcal{E}(t)) \neq 0$ if and only if $t - b_1 \geq 0$. By semicontinuity of cohomology groups and theorem 2.2.21, it is enough to restrict ourselves to the case where $\mathcal{E}$ is general. So, by (2.24) and (2.25) one has $\mathrm{H}^0(\mathcal{E}(t)) \neq 0$ if and only if

$$\begin{aligned} &t \geq s && \text{when} \quad \alpha = k + 2 \\ &t \geq s - 1 && \text{otherwise} \end{aligned}$$

and the condition $\alpha = k + 2$ is equivalent to $2s^2 + 2c_1 s + 2s = 2c_2 - c_1^2 - c_1$ by (2.40). □

2.2.26 Remark. It is known that the moduli space $\mathfrak{M} = \mathfrak{M}_{\mathbb{P}^2}(r, c_1, c_2)$ is irreducible for any rank r (various proofs of this result can be found in [Bar77a], [Ell83]), [HL93], [Pot79], [Mar78]). If (a, b) is a strongly admissible natural pair and there exists a rank r -vector bundle $\mathcal{E} \in \mathfrak{M}(a, b)$ then $\overline{\mathfrak{M}}(a, b) = 0$ hence, by irreducibility of $\mathfrak{M}$, as in the proof of proposition 2.2.20, we get that (a, b) is uniquely determinated, i.e. $(a,b)=(a,b)_{s,k,\alpha}$ where

$$s := \max\{\rho \in \mathbb{Z} : r\rho^2 + 2c_1\rho - r\rho \leq 2c_2 - c_1^2 + c_1 - 1\}, \tag{2.42}$$

and

$$\begin{aligned} \alpha &:= 2c_2 - c_1^2 + r - rs^2 - 2c_1 s, \\ k &:= (rs + c_1 - r + |\alpha|)/2. \end{aligned} \tag{2.43}$$

Some remarks about strong admissibility

Let $r=3$, $a=(2,2,4,4,4)$ and $b=(1,1,1,1,3,3,3,3)$.

Coherent sheaves on $\mathbb{P}^2$ of type (a,b) form an irreducible family and the general sheaf in the family is a vector bundle because (a,b) is admissible. Moreover, the family contains sheaves of the form $I_P \oplus I_Q \oplus I_Z$ where P,Q are point in $\mathbb{P}^2$ and Z is a general set of six point of $\mathbb{P}^2$. These sheaves are semistable because they are ideal sheaves and because direct sums of semistable sheaves of same slope are semistable [OSS80]. By openness of semistability, the general sheaf of type (a,b) is a semistable vector bundle, with (a,b) admissible but not strongly admissible (by Hoppe criterion it is not stable).

So far we did not succeed in finding stable vector bundles with admissible pairs not strongly admissible. This fact and the above remark suggest to formulate following conjecture:

"*If $\mathcal{E}$ is a stable rank r-vector bundle on the projective space then its minimal resolution is strongly admissible*".

A proof of this conjecture would give as corollary also the irreducibility of $\mathfrak{M}_{\mathbb{P}^2}(r,c_1,c_2)$ by applyng the same arguments of the proof of 2.2.20.

2.3 The sheaves $\mathcal{F}(r,f_1,f_2)$

Now we turn our attention on a specific example. We want to give an explicit description of the moduli space of rank-2 stable vector bundles on $\mathbb{P}^2$ with Chern classes $c_1=-1$, $c_2=2$ in the spirit of the previous sections. With this aim we are going to introduce the class of sheaves $\mathcal{F}(r,f_1,f_2)$.

2.3.1 Lemma. *Any stable rank-2 vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ with Chern classes $c_1=-1$, $c_2=2$ is presented by a minimal resolution of the form*

$$0 \to \mathcal{O}(-3) \to \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus 2} \to \mathcal{E} \to 0. \tag{2.44}$$

Proof. After few tries, with the aid of theorem 2.2.11, one sees that $((3),(1,1,2))$ is the only admissible pair with the prescribed rank and Chern classes. ☐

Let U be a complex vector space of dimension 3 and let $\mathbb{P}^2=\mathbb{P}(U)$ be the projective space equipped with the $\mathrm{SL}_{\mathbb{C}}(3)=\mathrm{SL}_{\mathbb{C}}(U)$ action. An element $(r,f_1,f_2)\in \mathrm{H}^0(\mathcal{O}(1)\oplus\mathcal{O}(2)^{\oplus 2})$ defines a morphism $\mathcal{O}(-3)\to\mathcal{O}(-2)\oplus\mathcal{O}(-1)^{\oplus 2}$ of sheaves on $\mathbb{P}^2$. We suppose $r\neq 0$ to avoid uninteresting pathologies and we denote by $\mathcal{F}(r,f_1,f_2)$ the sheaf defined by the following exact sequence:

$$0 \to \mathcal{O}(-3) \xrightarrow{(r,f_1,f_2)} \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus 2} \to \mathcal{F}(r,f_1,f_2) \to 0. \tag{2.45}$$

The rank 2 sheaf $\mathcal{F}(r,f_1,f_2)$ has Chern classes $c_1(\mathcal{F})=-1$ and $c_2(\mathcal{F})=2$. In this section we are going to study some properties of the sheaves $\mathcal{F}(r,f_1,f_2)$.

The matrix

$$B = \begin{bmatrix} \lambda & 0 & 0 \\ \omega_1 & \alpha_{1,1} & \alpha_{1,2} \\ \omega_2 & \alpha_{2,1} & \alpha_{2,2} \end{bmatrix}, \tag{2.46}$$

where λ, $\alpha_{i,j}$ are constants and ω_1, ω_2 linear forms on $\mathbb{P}^2$, defines an isomorphism of exact sequences (of sheaves)

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}(-3) \arrow[r, "(r,f_1,f_2)"'] \arrow[d, "Id"] & \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus 2} \arrow[r] \arrow[d, "B"] & \mathcal{F}(r,f_1,f_2) \arrow[r] \arrow[d, "\phi_B"] & 0 \\ 0 \arrow[r] & \mathcal{O}(-3) \arrow[r, "(s,g_1,g_2)"'] & \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{\oplus 2} \arrow[r] & \mathcal{F}(s,g_1,g_2) \arrow[r] & 0 \end{tikzcd} \tag{2.47}$$

and, in particular, a morphism ϕ_B between $\mathcal{F}(r, f_1, f_2)$ and $\mathcal{F}(s, g_1, g_2)$ where s, g_1, g_2 satisfy:

$$^t(s, g_1, g_2) = B.^t(r, f_1, f_2). \tag{2.48}$$

Conversely, every morphism between sheaves $\phi\colon \mathcal{F}(r, f_1, f_2) \to \mathcal{F}(s, g_1, g_2)$ is induced by a morphism of corresponding exact sequences i.e. $\phi = \phi_B$ for a suitable matrix B.

2.3.2 Lemma. *Two sheaves $\mathcal{F}(r, f_1, f_2)$, $\mathcal{F}(s, g_1, g_2)$ are isomorphic if and only if (r, f_1, f_2) and (s, g_1, g_2) satisfy*

$$\begin{aligned} s &= \lambda r \\ g_1 &= \omega_1 r + \alpha_{1,1} f_1 + \alpha_{1,2} f_2 \\ g_2 &= \omega_2 r + \alpha_{2,1} f_1 + \alpha_{2,2} f_2 \end{aligned} \tag{2.49}$$

where λ, $\alpha_{i,j}$ are constants and ω_i linear forms on $\mathbb{P}^2$ such that

$$\lambda \det(\alpha_{i,j}) \neq 0. \tag{2.50}$$

Proof. The map ϕ_B is an isomorphism if and only if B has maximal rank. □

Let $\mathcal{F}$ be the sheaf $\mathcal{F} = \mathcal{F}(r, f_1, f_2)$. We denote by $V(f)$ the zeros set of a form f on $\mathbb{P}^2$, by $\Sigma(r)$ the net of degenerated conics containing $V(r)$ and by

$\Sigma(r, f_1, f_2)$ the linear system of conics of the form $V(\omega r + \alpha_1 f_1 + \alpha_2 f_2)$ where ω is a linear form and α_1, α_2 are scalars. By the previous lemma, $V(r)$ and $\Sigma(r, f_1, f_2)$ are intrinsically determined by $\mathcal{F}$. Note that the pencil generated by $V(f_1)$ and $V(f_2)$ is not uniquely individuated, but its dimension is an invariant canonically associated to $\mathcal{F}$.

2.3.3 Lemma. *Let us fix a system of homogeneous coordinates x_0, x_1, x_2 on $\mathbb{P}^2$. Up to the action of* $\mathrm{SL}_{\mathbb{C}}(3)$ *every sheaf $\mathcal{F}(r, f_1, f_2)$ (with r not identically zero) is isomorphic to one of the following sheaves:*

(i) $\mathcal{F}(x_0, x_1^2, x_2^2)$,

(ii) $\mathcal{F}(x_0, x_1x_2, x_2^2)$,

(iii) $\mathcal{F}(x_0, x_1x_2, x_1x_2)$,

(iv) $\mathcal{F}(x_0, x_1^2, x_1^2)$,

(v) $\mathcal{F}(x_0, x_1x_2, 0)$,

(vi) $\mathcal{F}(x_0, x_1^2, 0)$,

(vii) $\mathcal{F}(x_0, 0, 0)$.

Proof. We can choose an homogeneous coordinate system such that $r = x_0$. The linear system $\Sigma(r, f_1, f_2)$ is given by the conics $V(\beta_0 x_0^2 + \beta_1 x_0 x_1 + \beta_2 x_0 x_2 + \alpha_1 f_1 + \alpha_2 f_2)$ depending on β and α. Without loss of generality, we may assume that the monomials x_0^2, x_0x_1, x_0x_2 do not appear in f_1 e f_2 i.e. $V(f_i)$ are conics of rank 2 with a singularity in $(1, 0, 0)$. We are going to study the sheaf $\mathcal{F}(r, f_1, f_2)$ in dependence of the dimension $d(r, f_1, f_2)$ of $\Sigma(r, f_1, f_2)$.

If $d(r,f_1,f_2)=5$, the dimension of the linear system is maximal and the conics x_0^2, x_0x_1, x_0x_2, f_1, f_2 are in general position. In this case the pencil of conics generated by $V(f_1)$ and $V(f_2)$ contains two (distinct or not) rank 1 conics, i.e. double line through $(1,0,0)$. If the two conics are distinct, we can assume, up to change of coordinates, that they are defined by the equations $x_1^2=0$, $x_2^2=0$. This corresponds to the case (i) of the lemma. When the two conics coincide, $\Sigma(r,f_1,f_2)$ has a base point on the line $V(r)$ (case (ii)). If $d(r,f_1,f_2)=4$, either f_1 and f_2 are coincident (cases (iii) and (iv)) or distinct (cases (v) and (vi)). By looking at the base locus of $\Sigma(r,f_1,f_2)$ we can distinguish case (iii) from case (iv) and case (v) from (vi). Finally, if $d(r,f_1,f_2)=3$ both conics $V(f_1)$, $V(f_2)$ are in $\Sigma(r)$ and we get case (vii). □

2.3.4 Remark. In the classification of the last lemma we observe that

- case (i) is the unique locally free sheaf and, in particular, it is stable (Hoppe's criterion),
- case (ii) is stable because one can show that its dual is the bundle $T\mathbb{P}^2(-1)$,
- cases (iii) and (iv) are not stable, in fact, if $V(f_1)$, $V(f_2)$ coincide, we may assume $f_1=f_2$ and give explicitly a destabilizing subsheaf $\mathcal{J}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-3) & \xrightarrow[(r,f_1,f_1)]{} & \mathcal{O}(-2)\oplus\mathcal{O}(-1)^{\oplus 2} & \longrightarrow & \mathcal{F}(r,f_1,f_2) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle Id} & & \uparrow\begin{bmatrix}1&0\\0&1\\0&1\end{bmatrix} & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{O}(-3) & \xrightarrow[(r,f_1)]{} & \mathcal{O}(-2)\oplus\mathcal{O}(-1) & \longrightarrow & \mathcal{J} & \longrightarrow & 0 \end{array} \tag{2.51}$$

- case (v) and (vi) are not stable and now a destabilizer subsheaf is the following $\mathcal{J}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-3) & \xrightarrow[(r,f_1,0)]{} & \mathcal{O}(-2)\oplus\mathcal{O}(-1)^{\oplus 2} & \longrightarrow & \mathcal{F}(r,f_1,f_2) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle Id} & & \uparrow\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix} & & \uparrow & & \\ 0 & \longrightarrow & \mathcal{O}(-3) & \xrightarrow[(r,f_1)]{} & \mathcal{O}(-2)\oplus\mathcal{O}(-1) & \longrightarrow & \mathcal{J} & \longrightarrow & 0 \end{array} \tag{2.52}$$

- case (vii) is not torsion free sheaf.

As a consequence, we get the following

2.3.5 Lemma. *$\mathcal{F}(r,f_1,f_2)$ is a stable sheaf if and only if $\Sigma(r,f_1,f_2)$ has dimension 5.*

2.3.1 Jumping lines of $\mathcal{F}(x_0,x_1^2,x_2^2)$

In this section we are going to study the jumping lines of first and second kind of a locally free sheaf $\mathcal{F}=\mathcal{F}(r,f_1,f_2)$. After lemma 2.3.3, it is enough for us to consider the case of $\mathcal{F}=\mathcal{F}(x_0,x_1^2,x_2^2)$.

Jumping lines of second kind were introduced by Hulek in [Hul79] where it is also proved that they are a powerful tool for the classification of rank-2 vector bundles on $\mathbb{P}^2$ with odd first Chern class.

2.3.6 Proposition. *Let $\mathcal{F}$ be the sheaf $\mathcal{F}(x_0,x_1^2,x_2^2)$.*

1. *The line $\{x_0=0\}$ is the unique jumping line of $\mathcal{F}$.*

2. The jumping lines of second kind of $\mathcal{F}$ are only the lines through the points $(0,1,0)$, $(0,0,1)$.

Proof. The canonical isomorphism of 2-dimensional vector spaces $V \simeq V^* \otimes \det V$ give the isomorphism of vector bundles $\mathcal{F} \simeq \mathcal{F}^*(-1)$. Then $\mathcal{F}$ has the following injective free resolution:

$$0 \to \mathcal{F} \to \mathcal{O} \oplus \mathcal{O}(1)^2 \xrightarrow{\phi} \mathcal{O}(2) \to 0 \tag{2.53}$$

where ϕ is given by the matrix (x_0, x_1^2, x_2^2). For a jumping line ℓ, the map:

$$\mathrm{H}^0(\phi_\ell(-1)) : \mathrm{H}^0(\mathcal{O}_\ell \oplus \mathcal{O}(-1)^2) \to \mathcal{O}(1)$$

is not injective and this is true if and only if ℓ is the line $\{x_0 = 0\}$. Moreover, ℓ is a jumping line of second kind if and only if the map:

$$\mathrm{H}^0(\phi_{\ell^2}) : \mathrm{H}^0(\mathcal{O}_{\ell^2}(1) \oplus \mathcal{O}_{\ell^2}^2) \to \mathrm{H}^0(\mathcal{O}_{\ell^2}(2))$$

has not maximal rank ad by calculating this is verified is and only if ℓ is a line through $(0,1,0)$ or $(0,0,1)$. $\square$

2.3.2 The moduli space $\mathfrak{M}_{\mathbf{Sheaf},\mathbb{P}^2}(-1,2)$

Let $\mathfrak{M}_{\mathrm{Sheaf},\mathbb{P}^2}(-1,2)$ be the moduli space of stable rank 2 sheaves on $\mathbb{P}^2$ with Chern classes $c_1 = -1$, $c_2 = 2$ and let $\mathfrak{M}^\circ \subset \mathfrak{M}_{\mathrm{Sheaf},\mathbb{P}^2}(-1,2)$ be the open set which appears in a minimal resolution same as (2.45). Since the first Chern class is odd, semistable sheaves are stable.

Let M be the incidence variety:

$$M := \left\{ \big((P_1,P_2),\ell\big) \in S^2\mathbb{P}^2 \times \mathbb{P}^{2^*} \quad | \quad P_1, P_2 \in \ell; \quad \ell \subset \mathbb{P}^2 \right\}. \tag{2.54}$$

where $S^2\mathbb{P}^2$ is the space parameterizing all pairs of points of $\mathbb{P}^2$. If $\mathbb{P}^5$ denotes the space of non zero symmetric matrices 3×3, since $S^2\mathbb{P}^{2^*}$ is isomorphic to the space of degenerated (non zero) conics on $\mathbb{P}^2$,

$$M \simeq \left\{ (x,A) \in \mathbb{P}^2 \times \mathbb{P}^5 : \quad \det A = 0, \quad Ax = 0 \right\}. \tag{2.55}$$

2.3.7 Theorem. *The varieties $\mathfrak{M}^\circ$ and M are isomorphic.*

Proof. By lemma 2.3.5 every point of $\mathfrak{M}^\circ$ is a class of sheaves of type $\mathcal{F}(r, f_1, f_2)$ where the linear system $\Sigma(r, f_1, f_2)$ has dimension 5. The pencil of conics generated by $V(f_1)$ and $V(f_2)$ contains two conics (possibly coincident) tangent to $V(r)$ in the points P_1, P_2. Then we get the map

$$\begin{array}{cccc} \phi : & \mathfrak{M}^\circ & \to & M \\ & [\mathcal{F}(r,f_1,f_2)] & \mapsto & (P_1,P_2,V(r)) \end{array}$$

The map ϕ is bijective since, given a conic g_1 (not divisible by r) tangent at $V(r)$ in P_1 (or P_2), every other conic tangent at $V(r)$ in P_1 (or P_2) is $V(\omega r + g_1)$ with an appropriate ω. $\square$

2.3.8 Corollary. *The moduli space of vector bundles $\mathfrak{M}_{\mathbb{P}^2}(-1,2)$ is isomorphic to $S^2\mathbb{P}^2 \setminus \Delta$ where Δ is the diagonal of $S^2\mathbb{P}^2$.*

Proof. By lemma 2.3.1, the space $\mathfrak{M}_{\mathbb{P}^2}(-1,2)$ consists precisely of locally free sheaf of type $\mathcal{F}(r, f_1, f_2)$ that is elements of $\mathfrak{M}^\circ$ corresponding to points (P_1, P_2, r) of M where P_1 and P_2 are distinct. $\square$

2.3.9 Remark. The same description of $\mathfrak{M}_{\mathbb{P}^2}(-1,2)$ obtained with a different approach can be found in [OSS80] page 344.

2.3.10 Remark. With the the isomorphism of the previous theorem, toric sheaves in $\mathfrak{M}^\circ$ admit the following characterization. The action of the torus $T^2 := \mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{P}^2$, which in coordinates is given by:

$$\begin{array}{cccc} \alpha: & T^2 \times \mathbb{P}^2 & \to & \mathbb{P}^2 \\ & ((t_1,t_2),(x_0,x_1,x_2)) & \mapsto & (x_0, t_1x_1, t_2x_2) \end{array}$$

has fixed points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and fixed lines $x_0 = 0$, $x_1 = 0$, $x_2 = 0$. The induced action on M is given by

$$\alpha_{t_1,t_2}(P_1, P_2, r)) = (\alpha_{t_1,t_2}(P_1), \alpha_{t_1,t_2}(P_2), \alpha_{t_1,t_2}(r)). \tag{2.56}$$

A point $(P_1, P_2, r) \in M$ corresponds at a toric sheaf $[\mathcal{F}] \in \mathfrak{M}^\circ$ if and only if P_1 and P_2 are fixed points and r is a fixed line. In conclusion we have 3 toric bundles presented by

$$\mathcal{F}(x_i, x_j^2, x_k^2) \qquad i,\ j,\ k \text{ distinct}$$

and 6 toric (not locally free) sheaves presented by

$$\mathcal{F}(x_i, x_jx_k, x_k^2) \qquad i,\ j,\ k \text{ distinct.}$$

2.4 Monads of stable rank-2 bundles on $\mathbb{P}^2$

For stable rank-2 vector bundles on $\mathbb{P}^2$ with fixed Chern classes c_1, c_2, theorem 2.2.5 and equations (2.5), (2.6) give sufficient conditions to find relative admissible pairs. Remark 2.2.14 ensures the finiteness of $\mathfrak{I}$ and permit to write an algorithm to enumerate such pairs by restricting the search to a finite domain. The sets $\mathfrak{M}(a,b)$ for all pairs (a,b) varying in $\mathfrak{I}$ give a partition of the moduli space $\mathfrak{M} := \mathfrak{M}_{\mathbb{P}^2}(2, c_1, c_2)$ in constructible subsets. The structure of this partition is in general not known. In fact we ignore which numerical conditions must be fulfilled by two admissible pairs (a,b) and (a',b') for $\mathfrak{M}(a,b)$ to be in the closure of $\mathfrak{M}(a',b')$.
Theorem 2.2.10 provides some information in this regard and in the previous section we found the admissible pair of general bundle (i.e., the pair of the open dense subset of the moduli space), but, even in the simpler cases, these results are too weak to complete the analysis of $\mathfrak{M}$ (see examples 2.2.15).

Even if the general problem still remains open, some more help in this investigation comes from the following application of Beilinson theorem (see [OSS80]).

2.4.1 Proposition. *A normalized r-bundle $\mathcal{E}$ over $\mathbb{P}^2$ is the cohomology of a monad*

$$0 \to H \otimes \mathcal{O}(-1) \xrightarrow{A} K \otimes \mathcal{O} \xrightarrow{B} L \otimes \mathcal{O}(1) \to 0 \tag{2.57}$$

where $H = \mathrm{H}^1(\mathbb{P}^2, \mathcal{E}(-2))$, $K = \mathrm{H}^1(\mathbb{P}^2, \mathcal{E} \otimes \Omega^1)$ *and* $L = \mathrm{H}^1(\mathbb{P}^2, \mathcal{E}(-1))$.

Hence, for fixed Chern classes, different admissible pairs correspond to different maps (or matrices) A and B of the "same" monad. This allows us to construct vector bundles on $\mathbb{P}^2$ starting from a suitable surjective maps $B\colon K\otimes\mathcal{O}\to L\otimes\mathcal{O}(1)$ with the following steps:

(i) find a generic element A of the algebraic set of maps

$$\{T: H\otimes\mathcal{O}(-1)\to L\otimes\mathcal{O}\mid BT=0\}\ ;$$

(ii) test if A is injective (otherwise choose a new B);

(iii) take the cohomology of the monad (2.57).

We denote by $\mathcal{E}(B)$ a vector bundle obtained with this method.

For $c_1=-1$ and some values of c_2 we can describe the structure of the moduli space completely by writing explicitly the matrices A and B of some deformation of vector bundles.

For now, we don't have a unique method to write all the possible deformations. In the next section we give an explicit description of the moduli space in the case resumed in the following figure. (We restrict our attention at the case $c_1=-1$ since for odd first Chern class the notions of semistability and stability coincide).

All matrices are found using computer algebra system MACAULAY2 and the algorithm to compute admissible pairs and their "codimension" is written in the Scheme dialect of lisp and available in [Mag99].

c_2	a	b	codim
2	(3)	(1 1 2)	0
3	(3 3)	(1 2 2 2)	0
	(4)	(1 1 3)	2
4	(3 3 3)	(2 2 2 2 2)	0
	(4)	(1 2 2)	1
	(3 4)	(1 2 2 3)	2
	(5)	(1 1 4)	4
5	(3 4)	(2 2 2 2)	0
	(3 3 4)	(2 2 2 2 3)	2
	(4 4)	(1 2 3 3)	2
	(3 5)	(1 2 2 4)	4
	(6)	(1 1 5)	6
6	(4 4)	(2 2 2 3)	0
	(3 4 4)	(2 2 2 3 3)	2
	(4 4 4)	(1 3 3 3 3)	3
	(3 3 5)	(2 2 2 2 4)	4
	(5)	(1 2 3)	4
	(4 5)	(1 2 3 4)	5
	(3 6)	(1 2 2 5)	6
	(7)	(1 1 6)	8

Admissible pair of $\mathfrak{M}_{\mathbb{P}^2}(2;-1,c_2)$

The cases $c_2=2,3$ are obvious.

2.4.1 $\mathfrak{M}_{\mathbb{P}^2}(2;-1,4)$

A vector bundle $\mathcal{E}$ in $\mathfrak{M}_{\mathbb{P}^2}(2;-1,4)$ is the cohomology of a monad

$$0\to\mathcal{O}(-1)^3\xrightarrow{A}\mathcal{O}^9\xrightarrow{B}\mathcal{O}(1)^4\to 0$$

Let B^1 be the matrix

$$B^1 = \begin{pmatrix} x_0 & x_1 & x_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_0 & x_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0 & x_1 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_0 & x_1 & x_2 \end{pmatrix}.$$

With the aid of a computer algebra system one easily find that the vector bundle $\mathcal{E}^1 = \mathcal{E}(B)$ belongs to to the stratum $\mathfrak{M}_1 := \mathfrak{M}\big((5),(1,1,4)\big)$.

Analogously, the matrix

$$B^2_\xi = \begin{pmatrix} x_0 & x_1 & x_2 & \xi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_0 & x_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0 & x_1 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_0 & x_1 & x_2 \end{pmatrix}$$

defines an element $\mathcal{E}(B^2_\xi)$ in $\mathfrak{M}_2 := \mathfrak{M}\big((3,4),(1,2,2,3)\big)$ for any sufficiently generic linear form ξ. By choosing $\xi = \varepsilon(x_0+x_1+x_2)$ we find a family of vector bundles $\mathcal{E}^2_\varepsilon := \mathcal{E}(B^2_{\varepsilon(x_0+x_1+x_2)})$ for a generic $\varepsilon \in \mathbb{C}^*$. Since for $\varepsilon = 0$ we have $B^2_0 = B^1$, then $\mathcal{E}^2_\varepsilon$ is a family of vector bundles having limit in $\mathfrak{M}^1$, that is, $\mathfrak{M}_1 \subset \overline{\mathfrak{M}}_2$.

One can show that $\mathfrak{M}_1 \subset \overline{\mathfrak{M}}_3$ similarly by taking the limit of the family $\mathcal{E}^3_\varepsilon := \mathcal{E}(B^3_{\varepsilon(x_0+x_1+x_2)})$ of vector bundles in $\mathfrak{M}_3 := \mathfrak{M}\big((4),(1,2,2)\big)$ defined by

$$B^3_\xi = \begin{pmatrix} x_0 & x_1 & x_2 & \xi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_0 & x_1 & x_2 & \xi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0 & x_1 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_0 & x_1 & x_2 \end{pmatrix}.$$

Moreover, by lemma 2.2.10, we have $\mathfrak{M}_2 \subset \overline{\mathfrak{M}}_3$ and all the previous subset are on the boundary of $\mathfrak{M}\big((3,3,3),(2,2,2,2,2)\big)$ which is the stratum of the general element.

2.4.2 $\mathfrak{M}_{\mathbb{P}^2}(2;-1,5)$

For the moduli space $\mathfrak{M}_{\mathbb{P}^2}(2;-1,5)$ the monad (2.57) take the form

$$0 \to \mathcal{O}(-1)^4 \xrightarrow{A} \mathcal{O}^{11} \xrightarrow{B} \mathcal{O}(1)^5 \to 0$$

We are going to consider the following family of matrices depending on two linear forms ξ, η on $\mathbb{P}^2$:

$$B^1_{\xi,\eta} = \begin{pmatrix} x_0 & x_1 & x_2 & \xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_0 & x_1 & x_2 & \eta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0 & x_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_0 & x_1 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0 & x_1 & x_2 \end{pmatrix}$$

$$B^2_{\xi,\eta} = \begin{pmatrix} x_0 & \xi & x_0 & 0 & 0 & x_1 & x_2 & 0 & 0 & 0 & 0 \\ \eta & x_0 & 0 & x_0 & 0 & 0 & x_1 & x_2 & 0 & 0 & 0 \\ 0 & 0 & x_0 & 0 & x_0 & 0 & 0 & x_1 & x_2 & 0 & 0 \\ 0 & 0 & 0 & x_0 & 0 & x_0 & 0 & 0 & x_1 & x_2 & 0 \\ 0 & 0 & 0 & 0 & x_0 & 0 & x_0 & 0 & 0 & x_1 & x_2 \end{pmatrix}$$

Then one may easly verify with a computer algebra system that when ξ and

η are null $\mathcal{E}(B^1_{0,0})$ and $\mathcal{E}(B^1_{0,0})$ belong to $\mathfrak{M}\big((6),(1,1,5)\big)$. For generic ξ one has

$$\begin{aligned} &\mathcal{E}(B^1_{\xi,0}) \quad \text{belongs to} \quad \mathfrak{M}\big((3,5),(1,2,2,4)\big),\\ &\mathcal{E}(B^1_{\xi,\xi}) \quad \text{belongs to} \quad \mathfrak{M}\big((4,4),(1,2,3,3)\big),\\ &\mathcal{E}(B^2_{\xi,\xi}) \quad \text{belongs to} \quad \mathfrak{M}\big((3,3,4),(2,2,2,2,3)\big). \end{aligned}$$

It follows, by taking the limit for $\varepsilon \to 0$ in the matrices $B^1_{\varepsilon\xi,0}$, $B^1_{\xi,\varepsilon\xi}$ and $B^2_{\varepsilon\xi,\varepsilon\xi}$ that

$$\begin{aligned} \mathfrak{M}\big((6),(1,1,5)\big) &\subset \overline{\mathfrak{M}}\big((3,5),(1,2,2,4)\big),\\ \mathfrak{M}\big((3,5),(1,2,2,4)\big) &\subset \overline{\mathfrak{M}}\big((4,4),(1,2,3,3)\big),\\ \mathfrak{M}\big((3,5),(1,2,2,4)\big) &\subset \overline{\mathfrak{M}}\big((3,3,4),(2,2,2,2,3)\big). \end{aligned}$$

Moreover, all the strata are in the closure of the 0-codimensional stratum $\mathfrak{M}\big((3,4),(2,2,2,2)\big)$ while the closures of $\mathfrak{M}\big((4,4),(1,2,3,3)\big)$ and $\mathfrak{M}\big((3,3,4),(2,2,2,2,3)\big)$ are disjoint since they have the same dimension. This complete the description of the strata in $\mathfrak{M}_{\mathbb{P}^2}(2;-1,5)$.

2.4.3 $\mathfrak{M}_{\mathbb{P}^2}(2;-1,6)$

Let us suppose that $\mathcal{E}$ be a vector bundle in $\mathfrak{M}_{\mathbb{P}^2}(2;-1,6)$ and apply prop 2.4.1 to both $\mathcal{E}$ and $\mathcal{E}(1)$. Then one obtain $\mathcal{E}$ and $\mathcal{E}(1)$ respectively as cohomology of monads

$$0 \to \mathcal{O}(-1)^5 \xrightarrow{A} \mathcal{O}^{13} \xrightarrow{B} \mathcal{O}(1)^6 \to 0, \tag{2.58}$$

$$0 \to \mathcal{O}(-1)^6 \xrightarrow{A} \mathcal{O}^{13} \xrightarrow{B} \mathcal{O}(1)^5 \to 0. \tag{2.59}$$

This give us more freedom to describe $\mathcal{E}$ since upon twist of $\mathcal{E}$ we can start the construction of the previous sections from a matrix B that fits in (2.58) or in (2.59). For brevity we denote the eight strata according to the following list

$$\begin{aligned} \mathfrak{M}_1 &= \mathfrak{M}\big((7),(1,1,6)\big), & \mathfrak{M}_5 &= \mathfrak{M}\big((3,3,5),(2,2,2,2,4)\big),\\ \mathfrak{M}_2 &= \mathfrak{M}\big((3,6),(1,2,2,5)\big), & \mathfrak{M}_6 &= \mathfrak{M}\big((4,4,4),(1,3,3,3,3)\big),\\ \mathfrak{M}_3 &= \mathfrak{M}\big((4,5),(1,2,3,4)\big), & \mathfrak{M}_7 &= \mathfrak{M}\big((3,4,4),(2,2,2,3,3)\big),\\ \mathfrak{M}_4 &= \mathfrak{M}\big((5),(1,2,3)\big), & \mathfrak{M}_8 &= \mathfrak{M}\big((4,4),(2,2,2,3)\big). \end{aligned}$$

We begin by using (2.59). Let $\xi = (\xi_1,\xi_2,\xi_3,\xi_4,\xi_5)$ and $\eta = (\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)$ be vectors of linear form on $\mathbb{P}^2$ and consider the matrix

$$B^1_{\xi,\eta} = \begin{pmatrix} x_0 & 0 & \xi_1 & 0 & 0 & 0 & \eta_1 & x_1 & x_2 & 0 & 0 & 0 & 0\\ 0 & x_0 & 0 & \xi_2 & 0 & 0 & 0 & \eta_2 & x_1 & x_2 & 0 & 0 & 0\\ 0 & 0 & x_0 & 0 & \xi_3 & 0 & 0 & 0 & \eta_3 & x_1 & x_2 & 0 & 0\\ 0 & 0 & 0 & x_0 & 0 & \xi_4 & 0 & 0 & 0 & \eta_4 & x_1 & x_2 & 0\\ 0 & 0 & 0 & 0 & x_0 & 0 & \xi_5 & 0 & 0 & 0 & \eta_5 & x_1 & x_2 \end{pmatrix}$$

As in the previous sections, one may play with various possibilities for ξ and η with a computer algebra program. We call $\mathcal{E}(B_{\xi,\eta})$ the resulting vector bundles in $\mathfrak{M}_{\mathbb{P}^2}(2;-1,6)$. It turns out that for generic forms α_i and β_j with $i,j = 1\dots5$ one has

- for $\xi = 0$ and $\eta = 0$ the vector bundle $\mathcal{E}(B_{0,0})$ belongs to $\mathfrak{M}_1$;
- $\mathcal{E}(B_{(0\dots0,\alpha_5),0})$ and $\mathcal{E}(B_{0,(\beta_1,0\dots0)})$ belong to $\mathfrak{M}_2$;

- $\mathcal{E}(B_{(\alpha_1,\alpha_2,0,0,0),(\beta_1,\beta_2,0,0,0)})$ belongs to $\mathfrak{M}_3$;
- $\mathcal{E}(B_{(\alpha_1,\alpha_2,\alpha_3,0,\alpha_5),0})$ belongs to $\mathfrak{M}_4$;
- $\mathcal{E}(B_{(0,0,0,\alpha_1,\alpha_2),(0,0,0,\beta_1,\beta_2)})$ belongs to $\mathfrak{M}_5$.
- $\mathcal{E}(B_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4,0),0})$ and $\mathcal{E}(B_{(\alpha_1,\alpha_2,\alpha_3,0,\alpha_5),(\beta_1,\beta_2,0,0,0)})$ belong to $\mathfrak{M}_6$;
- $\mathcal{E}(B_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4,0),(\beta_1,\beta_2,0,0,0)})$ and $\mathcal{E}(B_{(0,\alpha_2,0,\alpha_4,\alpha_5),(0,0,0,\beta_4,\beta_5)})$ belong to $\mathfrak{M}_7$.

Moreover for general linear forms ω the matrix

$$B^2 = \begin{pmatrix} x_0 & 0 & 0 & 0 & 0 & \omega & 0 & x_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & x_0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & x_2 & 0 & 0 & 0 \\ 0 & 0 & x_0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & x_2 & 0 & 0 \\ 0 & 0 & 0 & x_0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & x_2 & 0 \\ 0 & 0 & 0 & 0 & x_0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & x_1 & x_2 \end{pmatrix}$$

gives the vector bundle $\mathcal{E}(B^2)$ belongs to $\mathfrak{M}_5$. Then

- $\mathfrak{M}_1$ is contained in the closure of any other constructible subsets;
- $\mathfrak{M}_2$ is contained in $\overline{\mathfrak{M}}_3$, $\overline{\mathfrak{M}}_5$, $\overline{\mathfrak{M}}_6$, $\overline{\mathfrak{M}}_7$ and $\overline{\mathfrak{M}}_8$;
- $\mathfrak{M}_3 \subset \overline{\mathfrak{M}}_6$, $\mathfrak{M}_4 \subset \overline{\mathfrak{M}}_6$, $\mathfrak{M}_6 \subset \overline{\mathfrak{M}}_7$ and $\mathfrak{M}_5 \subset \overline{\mathfrak{M}}_7$
- $\mathfrak{M}_4$ and $\mathfrak{M}_5$ so they are unrelated each other.

Now, let us look a the monad (2.58). If

$$B^2_{\xi,\eta} = \begin{pmatrix} x_0 & x_1 & x_2 & \eta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_0 & x_1 & x_2 & \xi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0 & x_1 & x_2 & \xi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_0 & x_1 & x_2 & \xi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0 & x_1 & x_2 & \xi & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_0 & x_1 & x_2 \end{pmatrix}.$$

and

$$B^3_{\xi,\eta} = \begin{pmatrix} x_0 & 0 & \eta & 0 & 0 & 0 & x_1 & x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_0 & 0 & \eta & 0 & 0 & 0 & x_1 & x_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_0 & 0 & \xi & 0 & 0 & 0 & x_1 & x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_0 & 0 & \xi & 0 & 0 & 0 & x_1 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_0 & 0 & \xi & 0 & 0 & 0 & x_1 & x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_0 & 0 & \xi & 0 & 0 & 0 & x_1 & x_2 \end{pmatrix}$$

then, for a generic linear form ω on $\mathbb{P}^2$ and $\varepsilon \in \mathbb{C}^*$ one has

$$[\mathcal{E}(B^2_{\omega,0})] \in \mathfrak{M}_2, \qquad [\mathcal{E}(B^2_{\omega,\varepsilon\omega})] \in \mathfrak{M}_4,$$
$$[\mathcal{E}(B^3_{\omega,0})] \in \mathfrak{M}_3, \qquad [\mathcal{E}(B^3_{\omega,\varepsilon\omega})] \in \mathfrak{M}_7.$$

It follows $\mathfrak{M}_2 \subset \overline{\mathfrak{M}}_4$ and $\mathfrak{M}_3 \subset \overline{\mathfrak{M}}_7$. Also remember that $\mathfrak{M}_3 \subset \overline{\mathfrak{M}}_4$ by 2.2.10.

In general, by semicontinuity of the the dimension of the cohomology groups h^q, if X and Y are constructible subsets of $\mathfrak{M}_{\mathbb{P}^2}(c_1,c_2)$ and $Y \in \overline{X}$, then $h^q(\mathcal{E}(t)) \geq h^q(\mathcal{F}(t))$ for all $t \in \mathbb{Z}$, $\mathcal{E} \in Y$, $\mathcal{F} \in X$.

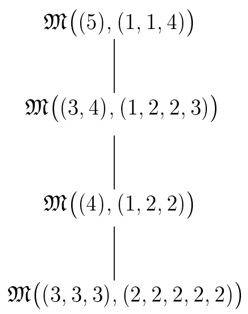
If $\mathcal{E} \in \mathfrak{M}_6$ then $h^1(\mathcal{E}(4)) = 3$ and $h^1(\mathcal{E}(5)) = 0$, while if $\mathcal{F} \in \mathfrak{M}_5$ then $h^1(\mathcal{F}(4)) = 2$ and $h^1(\mathcal{E}(5)) = 1$ hence $\mathfrak{M}_6$ and $\mathfrak{M}_5$ are not related.

So far, it remains unclear if $\mathfrak{M}_3$ is in the closure of $\mathfrak{M}^5$ or not.

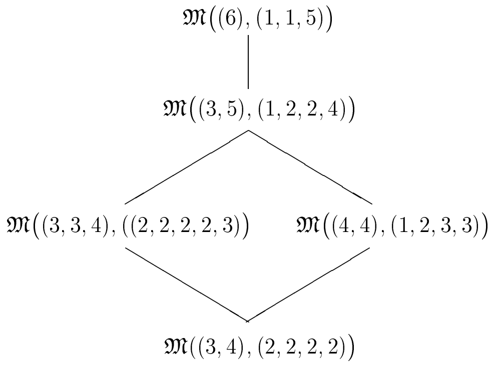
2.4.4 Tables

The results of the above section can be resumed in the following pictures

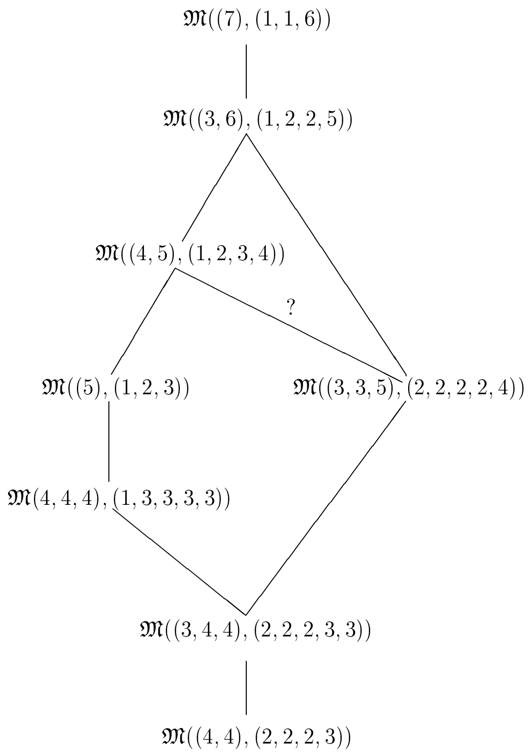
Filtration of $\mathfrak{M}_{\mathbb{P}^2}(2;-1,4)$



Filtration of $\mathfrak{M}_{\mathbb{P}^2}(2;-1,5)$



Filtration of $\mathfrak{M}_{\mathbb{P}^2}(2;-1,6)$



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