The moment map and non–periodic tilings

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joint work with Fiammetta Battaglia

Delzant theorem

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Delzant polytopes $\Longleftrightarrow$ symplectic toric manifolds

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If Φ is the moment mapping of this action we have $\Phi(M) = \Delta$

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$$\Delta \Longrightarrow M$$

explicit construction using symplectic reduction

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- $M = \frac{\Psi^{-1}(0)}{N}$, where Ψ is a moment mapping for the induced action of N on $\mathbb{C}^d$
- M inherits an action of $T^d/N \simeq T = \mathbb{R}^n/L$ from the standard action of T^d on $\mathbb{C}^d$

generalized Delzant construction

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formally, it works exactly like the Delzant construction but ...

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- the lattice L is replaced by a quasilattice Q

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- rationality is replaced by quasirationality

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- M is a $2n$–dimensional compact connected quasifold
- the torus is replaced by a quasitorus $T^d/N \simeq \mathbb{R}^n/Q$

quasifold geometry

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quasilattice

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a generalization of a lattice $L \subset \mathbb{R}^n$, a quasilattice Q is the $\mathbb{Z}$–span of a set of spanning vectors, $Y_1, \ldots, Y_d$, of $\mathbb{R}^n$

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a generalization of a torus $\mathbb{R}^n/L$, a quasitorus is the quotient $\mathbb{R}^n/Q$, Q being a quasilattice

rationality vs quasirationality

rationality vs quasirationality

given any convex polytope $\Delta \subset (\mathbb{R}^n)^*$, then there exist vectors $X_1, \ldots, X_d \in \mathbb{R}^n$ and real numbers $\lambda_1, \ldots, \lambda_d$ such that

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any given convex polytope is quasirational with respect to the quasilattice that is generated by the vectors $X_1, \ldots, X_d$

Penrose tilings

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2 examples non-periodic tilings of the plane

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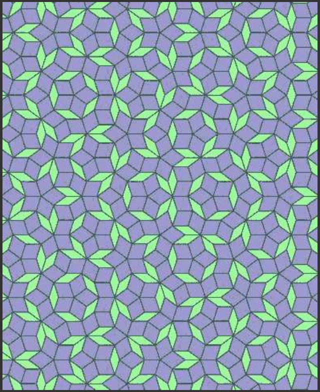


Figure: a rhombus tiling

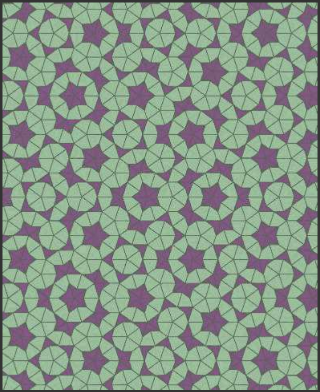


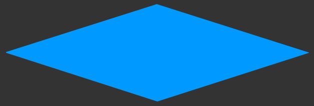
Figure: a kite and dart tiling

figures by D. Austin, reprinted courtesy of the AMS

convex Penrose tiles

convex Penrose tiles

- the thin rhombus



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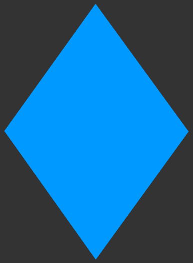

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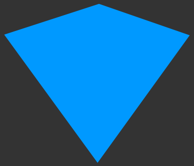

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  - the ratio of the long edge of the kite to its short edge

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find an appropriate quasilattice

choice of quasilattice for the Penrose tilings

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Let us consider the quasilattice $Q \subset \mathbb{R}^2$ generated by the vectors

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\begin{aligned}
Y_0 &= (1, 0) \\
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Y_2 &= (\cos\tfrac{4\pi}{5}, \sin\tfrac{4\pi}{5}) = \tfrac{1}{2}(-\phi, \tfrac{1}{\phi}\sqrt{2+\phi}) \\
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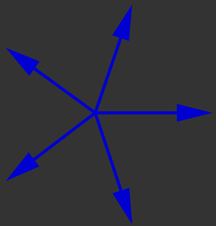

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choice of quasilattice for the Penrose tilings

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- any rhombus, thick or thin, of a given rhombus tiling is quasirational with respect to Q
- any kite of a given kite and dart tiling is quasirational with respect to Q

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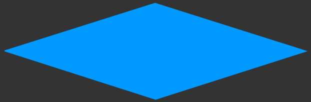

$$\Longrightarrow M = \frac{S^2_r \times S^2_r}{\Gamma}$$

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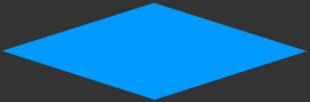

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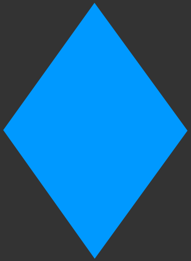

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why? what are r, R and Γ?

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thin rhombus

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thin rhombus

- symplectic reduction yields $M = \frac{S^3_r \times S^3_r}{N}$, where $N = \left\{ \exp\left(s, s + h\phi, t, t + k\phi\right) \in T^4 \,|\, s, t \in \mathbb{R}, h, k \in \mathbb{Z} \right\}$ and $r = \left(\frac{1}{2\phi} \sqrt{2 + \phi} \right)^{1/2}$

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- same, with $R = \left(\frac{1}{2}\sqrt{2+\phi}\right)^{1/2}$ instead of r

symplectic geometry of the kite and dart tiling

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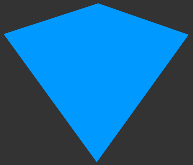

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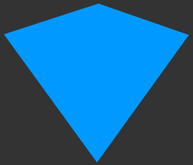
symplectic geometry of the kite and dart tiling



$$\Longrightarrow M = \frac{\left\{ (z_1,z_2,z_3,z_4)\in\mathbb{C}^4 \mid |z_1|^2+\frac{1}{\phi}|z_2|^2+|z_3|^2=\frac{\sqrt{2+\phi}}{2}, -|z_1|^2+|z_2|^2+\phi|z_4|^2=\frac{\sqrt{2+\phi}}{2\phi} \right\}}{\{ \exp(-s+\phi t, s, t, -t+\phi s)\in T^4 \mid s,t\in\mathbb{R} \}}$$

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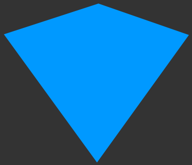

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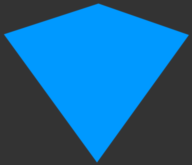
symplectic geometry of the kite and dart tiling



$$\Longrightarrow M = \frac{\left\{ (z_1,z_2,z_3,z_4)\in\mathbb{C}^4 \;\middle|\; |z_1|^2+\frac{1}{\phi}|z_2|^2+|z_3|^2=\frac{\sqrt{2+\phi}}{2}, -|z_1|^2+|z_2|^2+\phi|z_4|^2=\frac{\sqrt{2+\phi}}{2\phi} \right\}}{\{ \exp(-s+\phi t, s, t, -t+\phi s)\in T^4 \mid s,t\in\mathbb{R} \}}$$

remark

one can show that M is not the global quotient of a manifold modulo the action of a discrete group

an example of a chart

consider the open subset of $\mathbb{C}^2$ given by $\tilde{U} = \left\{ (z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + \frac{1}{\phi}|z_2|^2 < \frac{\sqrt{2+\phi}}{2}, -|z_1|^2 + |z_2|^2 < \frac{\sqrt{2+\phi}}{2\phi} \right\}$

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and the following slice of $\Psi^{-1}(0)$ that is transversal to the N–orbits

$$\begin{array}{lll} \tilde{U} & \stackrel{\tilde{\tau}}{\to} & \{(z_1, z_2, z_3, z_4) \in \Psi^{-1}(0) \mid z_3 \neq 0, z_4 \neq 0\} \\ (z_1, z_2) & \mapsto & \left(z_1, z_2, \sqrt{\frac{\sqrt{2+\phi}}{2} - |z_1|^2 - \frac{1}{\phi}|z_2|^2}, \sqrt{\frac{\sqrt{2+\phi}}{2\phi^2} + \frac{|z_1|^2 - |z_2|^2}{\phi}} \right) \end{array}$$

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$$\begin{array}{ccc} \tilde{U}/\Gamma & \xrightarrow{\tau} & U \\ [(z_1, z_2)] & \longmapsto & [\tilde{\tau}(z_1, z_2)] \end{array}$$

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Ammann tilings

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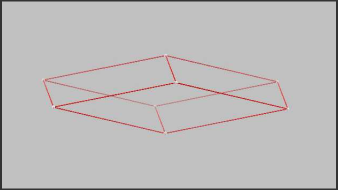
- they provide a geometrical model for the physics of certain quasicrystals
- their tiles are given by two types of rhombohedra

Ammann tiles

Ammann tiles

- the oblate rhombohedron



Ammann tiles

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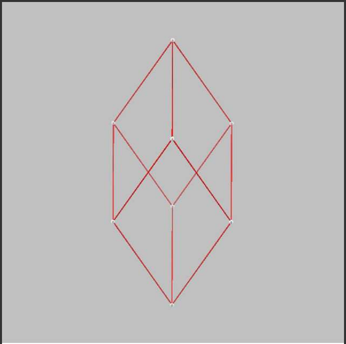

Ammann tiles

the facets of these rhombohedra are so–called golden rhombuses: the ratio of their diagonals is equal to ϕ

choice of quasilattice for the Ammann tilings

choice of quasilattice for the Ammann tilings

let us consider the quasilattice $F \subset \mathbb{R}^3$ generated by the vectors

$$
\begin{aligned}
U_1 &= \tfrac{1}{\sqrt{2}}(1, \phi - 1, \phi) \\
U_2 &= \tfrac{1}{\sqrt{2}}(\phi, 1, \phi - 1) \\
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fact

- any rhombohedron, oblate or prolate, of a given Ammann tiling is quasirational with respect to F

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- the vectors U_i have norm equal to $\sqrt{2}$

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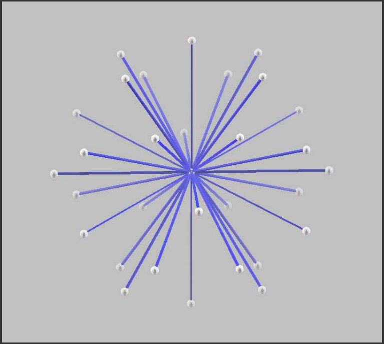

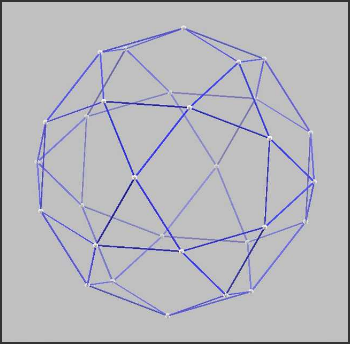

another important quasilattice in this setting

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fact

- up to a suitable rescaling, P has the property of containing all of the vertices of the Ammann tiling

the simple icosahedral lattice

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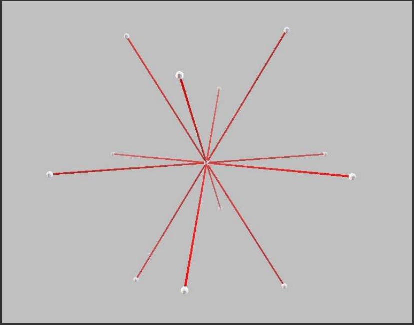

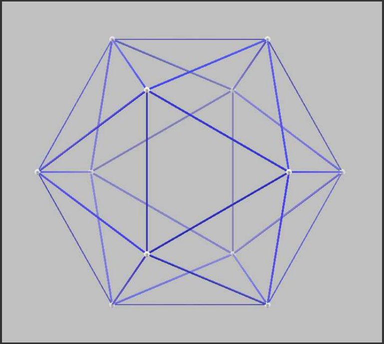

symplectic geometry of Ammann tiles

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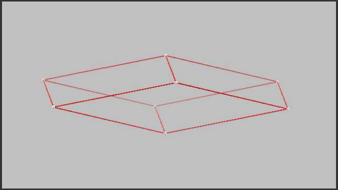
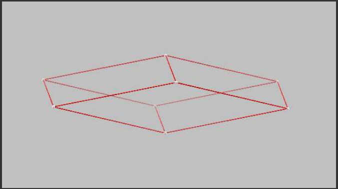


$$\Longrightarrow M = \frac{S^2_r \times S^2_r \times S^2_r}{\Gamma}$$

symplectic geometry of Ammann tiles



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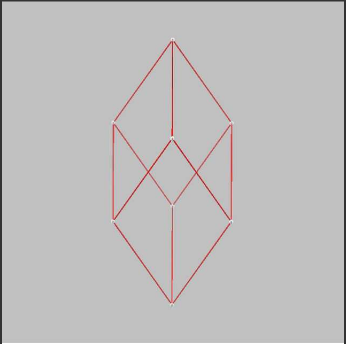

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why? what are r, R and Γ?

oblate rhombohedron

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oblate rhombohedron

- symplectic reduction yields $M = \frac{S^3_r \times S^3_r \times S^3_r}{N}$, where $N \subset T^6$ is $\{\, \exp\left(p, p+\phi h, s, s+\phi k, t, t+\phi l, p, s, t\right) \mid p, s, t \in \mathbb{R}, h, k, l \in \mathbb{Z} \,\}$ and $r = \frac{1}{\sqrt{\phi}\,\sqrt[4]{2(3-\phi)}}$

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- consider $S^1 \times S^1 \times S^1 = \{\, \exp\left(p, p, s, s, t, t\right) \in T^6 \mid p, s, t \in \mathbb{R} \,\} \subset N$

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- then $M = \frac{S^2_r \times S^2_r \times S^2_r}{\Gamma}$, with $\Gamma = \frac{N}{S^1 \times S^1 \times S^1}$

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prolate rhombohedron

why? what are r, R and Γ?

oblate rhombohedron

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prolate rhombohedron

- same, with $R = \frac{1}{\sqrt[4]{2(3-\phi)}}$ instead of r

visual aids

visual aids

- all models are built using zometool®

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- all 3D pictures are drawn using zomecad

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