Harmonics and Symplectic Geometry

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Abstract

We analyze the first 8 harmonics from the musical point of view and we remark that the 5th and 7th stand out as special. We then relate these harmonics to regular polygons, and we view these polygons from the viewpoint of symplectic geometry. Surprisingly, even in the symplectic setting, the regular pentagon and the regular heptagon, which correspond to the 5th and 7th harmonic, behave very differently from the other polygons.

Introduction

Musical harmonics are positive integer multiples of a fixed frequency, which is called the *fundamental* frequency. Harmonics can be produced in a variety of ways, with a variety of techniques and musical instruments. But perhaps one of the most interesting techniques for producing harmonics is known as *harmonic* or *overtone singing*, an ancient tradition that is still present in some countries, such as Tuva and Mongolia, and is currently being rediscovered by several contemporary musicians.

In this article we focus our attention on the first 8 harmonics of a given fundamental frequency.

In the first section we remark that two of these harmonics, the 5th and the 7th, stand out as special because their frequency is several Hertz too low to correspond to notes in our usual equal temperament musical scale. Nevertheless the musical intervals that are created between the fundamental and each of these harmonics have their distinctive melodic features that we describe. Following Laneri and Levarie–Levy, for each integer k greater than 2, we consider the regular polygon having k edges, and we remark that the distinctive melodic features of the kth harmonic relate to the symbolism of the corresponding polygon. This relationship proves to be particularly interesting in the cases $k = 5, 7$, where the corresponding polygons are the regular pentagon and the regular heptagon (see Figure 3).

In the second section we consider the regular polygons from the perspective of symplectic geometry. The Delzant Theorem, which yields, among other things, a well known correspondence between certain suitable polygons and symplectic 4–manifolds, surprisingly fails exactly for the regular pentagon and the regular heptagon. The reason for this failure is that these polygons are not rational. However, a generalization of the Delzant Theorem proven by the author in [6] allows to associate to non–rational polygons such as the these a so–called symplectic 4–*quasifold*. Quasifolds generalize manifolds and orbifolds and are wildly singular: in general they may even not be Hausdorff topological spaces.

We conclude by remarking how the peculiar features of the 5th and 7th harmonic correspond to the peculiar singularities of the symplectic quasifolds that are associated to the regular pentagon and to the regular heptagon.

1 Harmonics and Regular Polygons

We begin by recalling some facts about harmonics. We start with any given note, say for example A2 of frequency 110 Hertz. We multiply this frequency by 2 and we get 220 Hertz, which corresponds to A3, the same note one octave above; this is the 2nd harmonic. We now multiply the original frequency by the number 3 and we get 330 Hertz, which corresponds to E4, a perfect 5th two octaves above the original note; this is the 3rd harmonic. If we continue this procedure we get the following table:

harmonic	1	2	3	4	5	6	7	8
frequency	110 Hz	220 Hz	330 Hz	440 Hz	550 Hz	660 Hz	770 Hz	880 Hz
note	A2	A3	E4	A4	almost C#5	E5	almost G5	A5
interval	unison	octave	p. 5th	octave	pure major 3rd	p. 5th	harmonic 7th	octave

Of course the table could go on indefinitely, but we need to stop somewhere and 8 appears to be a rather natural stopping point; a similar discussion applies to subsequent harmonics. The same procedure could be repeated starting from any other note; naturally the notes involved will be different, but the intervals will remain the same. So it is natural to attach to each integer the corresponding musical interval.

If we look at the table, the harmonics 5 and 7 immediately stand out as being different from the others. The frequency of the 5th harmonic is lower than the equally–tempered frequency 554.37 Hertz of the note C#5, while the frequency of the 7th is lower than the equally–tempered frequency 783.99 Hertz of the note G5. Therefore the interval that corresponds to the 5th harmonic, which we refer to as the *pure major* 3*rd*, is flatter than its equally–tempered counterpart, the major 3rd. Similarly, the interval which corresponds to the 7th harmonic, which is known as the *harmonic* 7*th*, is flatter than its counterpart in equal temperament, the minor 7th.

These two harmonics also stand out because of their special melodic features. The pure major 3rd is the first truly melodic interval in the harmonic series (see Laneri [3, Chapter 6]). Perhaps one of the most striking early examples of its use is the medieval northern European practice of singing in parallel 3rds, called *gymel* or *cantus gemellus*. This interval is the heart of the *pure major triad chord*. The harmonic 7th, on the other hand, is one of the *blue notes* of blues, popularly known as *blues seven*. It is also widely used in barbershop and folk music and conveys a melancholic (blue) feeling. It can be achieved for example by vocal techniques, by bending or slide guitar techniques, or with wind instruments such the sax. Adding the 7th harmonic to the pure major triad chord yields the 7*th harmonic chord*, which blends the distinctive melodic features of the 5th and 7th harmonic. According to both Mathieu [5, pp. 318–319] and Tartini [8] this chord functions as a "fully resolved" final chord, very much unlike its equally–tempered counterpart, the minor 7th.

Let us now associate to each integer greater than 2 the corresponding regular polygon, following Laneri [3, Chapter 6] and Levarie–Levy [4, Chapters 2 and 3]. We therefore get a new table:

harmonic	1	2	3	4	5	6	7	8
polygon			triangle	square	pentagon	hexagon	heptagon	octagon
interval	unison	octave	p. 5th	octave	pure major 3rd	p. 5th	harmonic 7th	octave

Laneri and Levarie–Levy argue that each polygon symbolizes some of the characteristics of the corresponding musical interval. One of their most striking examples is the case of the 5th harmonic. If we trace the diagonals of the regular pentagon, we obtain the so–called pentagram (see Figure 1), which is known to be a symbol of *Ishtar*, the Babylonian Venus, Goddess of Love (we refer the reader to [7] for a more detailed discussion of the role of the pentagram both in mythology and symplectic geometry). The melodic character of the pure major 3rd indeed seems to correspond perfectly to this spiritual and sentimental value of the pentagon. As a matter of fact, the heptagram, the seven-pointed star that is inscribed in the heptagon (see Figure 1), is also known as the *star of Venus*. We could argue that the special character of the harmonic 7th, compared to that of a pure major 3rd, conveys a different but equally important spiritual aspect of Love. These two aspects appear to complement each other and manifest their unity in the beautiful musical expression of the harmonic 7th chord.

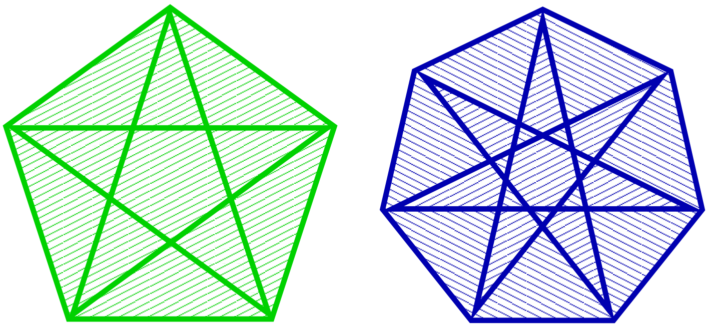

Figure 1: The Pentagram and the Heptagram

2 Regular Polygons in Symplectic Geometry

Regular polygons are special examples of simple convex dimension 2 polytopes. In symplectic geometry, a simple convex dimension n polytope that is rational and satisfies an additional integrality condition, corresponds to a compact dimension $2n$ symplectic manifold that has very special symmetry. In fact it is invariant under the effective Hamiltonian action of a dimension n torus, and the image of the corresponding moment mapping is the given polytope. This is the content of the Delzant Theorem, whose details and proof can be found in the original article [2]. If we specialize to $n = 2$ we expect the Delzant Theorem to allow us to associate to each of our six regular polygons a suitable symplectic 4–manifold. As it turns out, this is only possible for four of them: the equilateral triangle, the square, the regular hexagon and the regular octagon (see Figure 2). Any symplectic geometer will be able to show you for example that the equilateral triangle

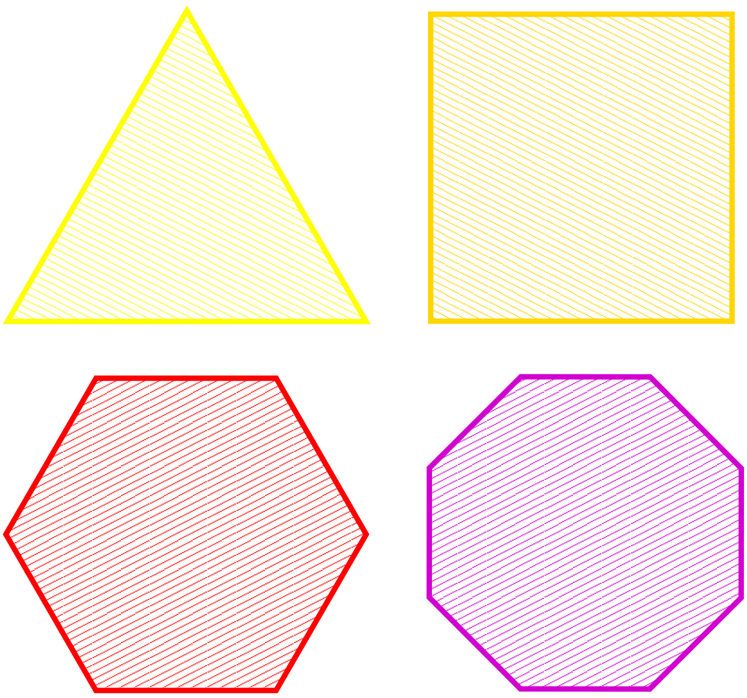

Figure 2: The Rational Regular Polygons: Triangle, Square, Hexagon and Octagon

corresponds to the Hopf quotient S^5/S^1, and that the square corresponds to $S^2 \times S^2$ (S^n denoting the standard unit sphere in $\mathbb{R}^{n+1}$). What happens to the polygons corresponding to the integers 5 and 7, the regular pentagon and the regular heptagon (Figure 3)? Why do they not correspond to a symplectic manifold? The basic issue is that neither of them is a *rational* polytope, as can be easily verified, and being rational is a crucial assumption underlying the connection between symplectic manifolds and convex polytopes. However, the

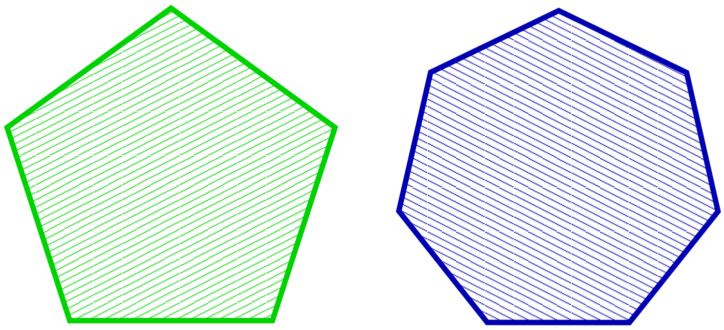

Figure 3: The Non–Rational Regular Polygons: Pentagon and Heptagon

non–rationality here can be dealt with if we allow the symplectic space to be (not mildly but) wildly singular. In fact an extension of the Delzant Theorem to general simple convex polytopes given by the author in [6] allows to associate to each such polytope a so–called symplectic *quasifold*. Quasifolds generalize both manifolds and orbifolds, and are locally modeled by open subsets of $\mathbb{R}^n$ modulo the action of a finitely generated, but typically infinite, group; in general quasifolds may not even be Hausdorff topological spaces (we refer the reader to [6] for the basic geometry of quasifolds and to the article [1] written in collaboration with F. Battaglia, for an improved version of the original definition of quasifold). In conclusion, even if there is no symplectic manifold corresponding to the regular pentagon and regular heptagon, there is a symplectic quasifold corresponding to each. The article [6] contains an explicit computation in the case of the pentagon.

Conclusion

As we have seen in the two previous sections, amongst the regular polygons with at most 8 edges, the pentagon and the heptagon play a distinctive role, whether we consider them from the point of view of music or from the point of symplectic geometry. The distinctive character and melodic features of the pure major 3rd and of the harmonic 7th reflect into the singularity of the symplectic quasifold that maps respectively to the pentagon and to the heptagon. Moreover, these musical intervals appear to stray from their tempered counterparts in much the same way as symplectic quasifolds stray from ordinary symplectic manifolds.

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