RANK 2 ACM BUNDLES WITH TRIVIAL DETERMINANT ON FANO THREEFOLDS OF GENUS 7

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Abstract. Given a smooth prime Fano threefold X of genus 7, we prove that the subset $\mathsf{M}_X^{\ell f}(2,0,4)$ of vector bundles in the moduli space $\mathsf{M}_X(2,0,4)$ of rank 2 semistable sheaves on X with $c_1 = 0$ and $c_2 = 4$ is an open dense subset of the Brill-Noether locus $W^1_{2,4}$ of rank 2 stable sheaves with degree 4 with 2 sections, defined on the homologically projectively dual curve Γ. This shows that $\mathsf{M}_X^{\ell f}(2,0,4)$ is a smooth irreducible 5-fold.

1. Introduction

In this paper we investigate the moduli spaces of rank 2 stable vector bundles on a smooth prime Fano threefold, carrying on the work taken up in [BF07] and [BF08a].

A smooth complex projective threefold X is called *Fano* if its anticanonical divisor $-K_X$ is ample. A Fano threefold X is *prime* if its Picard group is generated by the class of K_X. These varieties are classified up to deformation, see for instance [IP99, Chapter IV]. The number of deformation classes is 10, and they are characterized by the *genus*, which is the integer g such that $\deg(X) = -K_X^3 = 2\,g-2$. Recall that the genus of a prime Fano threefold take values in $\{2,\dots,10,12\}$.

Let now X be a smooth prime Fano threefold. We are interested in the Maruyama moduli scheme $\mathsf{M}_X(2,c_1,c_2)$ of semistable sheaves F on X of rank 2 with Chern classes c_1, c_2, and with $c_3 = 0$. We will be particularly interested in the subset of $\mathsf{M}_X(2,c_1,c_2)$ consisting of arithmetically Cohen-Macaulay (ACM) bundles, i.e. satisfying $\mathrm{H}^k(X,F(t)) = 0$ for all t and for $k = 1,2$. Since the rank of F is 2, we can assume $c_1 \in \{0,1\}$. We denote by $\mathsf{M}_X^{\ell f}(2,c_1,c_2)$ the subset of locally free sheaves in $\mathsf{M}_X(2,c_1,c_2)$.

The geometry of these moduli spaces has been mostly studied for $c_1 = 1$, and many results in the literature concern specific values of c_2. For instance, if one asks whether the moduli space $\mathsf{M}_X(2,1,c_2)$ is smooth and irreducible, then the answer is known (most frequently in the affirmative sense) only for low values of c_2. *Low* here means close to $m_g = \lceil (g+2)/2 \rceil$, indeed $\mathsf{M}_X(2,1,c_2)$ is empty for $c_2 < m_g$. For higher values of c_2, the space $\mathsf{M}_X(2,1,c_2)$ is known to contain a reduced component of dimension

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$2c_2 - g - 2$. We refer for more details to the papers [IM00b] (for genus 3), [IM04a], [IM07], [BF07] (for genus 7), [IM07a], [IM00a] (for genus 8), [IR05] [BF08b] (for genus 9), [AF06] (for genus 12), [BF08a] (for all genera). We stress that $\mathsf{M}_X(2,1,c_2)$ contains a component whose general element is a stable ACM bundle if and only if $m_g \leq c_2 \leq g+3$.

Much less has been said on the case of trivial determinant, i.e. when $c_1 = 0$. One sees easily that $\mathsf{M}_X^{\ell f}(2,0,c_2)$ is empty unless c_2 is even and greater than 2, so the first case to study is $\mathsf{M}_X^{\ell f}(2,0,4)$. On the other hand, a sheaf in $\mathsf{M}_X^{\ell f}(2,0,c_2)$ can be an ACM bundle if and only if $c_2 = 4$.

The study of the space $\mathsf{M}_X^{\ell f}(2,0,4)$ was first taken up by Iliev and Markushevich in [IM00b, arXiv version] for genus 3. In this case they proved that $\mathsf{M}_X^{\ell f}(2,0,4)$ has two irreducible components. Assume now $g \geq 4$. In view of [BF08a], we know that $\mathsf{M}_X^{\ell f}(2,0,4)$ contains a component of dimension 5, for all smooth prime Fano threefolds of genus g. Again a general element of this component is a stable ACM bundle.

In this paper we study the space $\mathsf{M}_X^{\ell f}(2,0,4)$, where X is a smooth prime Fano threefold of genus 7. We use the semiorthogonal decomposition of the bounded derived category $\mathbf{D^b}(X)$ obtained by Kuznetsov in [Kuz05]. More precisely, we consider the homologically projectively dual curve Γ in the sense of [Kuz06]. Recall that Γ is smooth non-hyperelliptic curve of genus 7, and that there is a natural integral functor $\mathbf{\Phi}^! : \mathbf{D^b}(X) \to \mathbf{D^b}(\Gamma)$.

Here we first prove that, given any sheaf F in $\mathsf{M}_X^{\ell f}(2,0,4)$, the sheaf $F(1)$ is mapped by $\mathbf{\Phi}^!$ to a complex concentrated in degree -1, so the shifted complex $\mathcal{F} = \mathbf{\Phi}^!(F(1))[-1]$ is in fact a locally free sheaf on Γ. The bundle $\mathcal{F}$ then turns out to belong to the Brill-Noether variety $W^1_{2,4}(\Gamma)$ of rank 2 stable bundles on Γ with degree 4 with at least 2 independent global sections. Finally, we remark that any element in $\mathsf{M}_X^{\ell f}(2,0,4)$ is an ACM bundle. This leads to our main result:

Theorem. *Let X be a smooth prime Fano threefold of genus 7 and let $\mathsf{M}_X^{\ell f}(2,0,4)$ be the subset of locally free sheaves in $\mathsf{M}_X(2,0,4)$. Then the map φ defined by:*

$$\begin{aligned} \mathsf{M}_X^{\ell f}(2,0,4) &\to W^1_{2,4}(\Gamma) \\ F &\mapsto \mathbf{\Phi}^!(F(1))[-1] \end{aligned}$$

is an open immersion. In particular, the moduli space $\mathsf{M}_X^{\ell f}(2,0,4)$ is a smooth irreducible variety of dimension 5. Any element of this space is a stable ACM bundle.

Here is the structure of our paper. In the next section we recall a few preliminary notions, while in section 3 we prove some preparatory vanishing results. Section 4 is devoted to the proof of our main theorem. Finally, in section 5 we show that, if S is a general hyperplane section surface of X, then the space $\mathsf{M}_X^{\ell f}(2,0,4)$ is embedded in $\mathsf{M}_S^{\ell f}(2,0,4)$ as a lagrangian subvariety.

2. Preliminaries

Given a smooth complex projective n-dimensional polarized variety (X, H_X) and a sheaf F on X, we write $F(t)$ for $F \otimes \mathscr{O}_X(tH_X)$. Given a pair of sheaves (F, E) on X, we will write $\mathrm{ext}^k_X(F, E)$ for the dimension of the Čech cohomology group $\mathrm{Ext}^k_X(F, E)$, and similarly $\mathrm{h}^k(X, F) = \dim \mathrm{H}^k(X, F)$. The Euler characteristic of (F, E) is defined as $\chi(F, E) = \sum_k (-1)^k \mathrm{ext}^k_X(F, E)$ and $\chi(F)$ is defined as $\chi(\mathscr{O}_X, F)$. We denote by $p(F, t)$ the Hilbert polynomial $\chi(F(t))$ of the sheaf F. The degree $\deg(L)$ of a divisor class L is defined as the degree of $L \cdot H_X^{n-1}$. The dualizing sheaf of X is denoted by ω_X.

If X is an smooth n-dimensional subvariety of $\mathbb{P}^m$, whose coordinate ring is Cohen-Macaulay, then X is said to be arithmetically Cohen-Macaulay (ACM). A locally free sheaf F on an ACM variety X is said to be an *ACM* bundle if it has no intermediate cohomology, i.e. if $\mathrm{H}^k(X, F(t)) = 0$ for all integer t and for any $0 < k < n$. The corresponding module over the coordinate ring of X is thus a maximal Cohen-Macaulay module.

Let us now recall a few well-known facts about semistable sheaves on projective varieties. We refer to the book [HL97] for a more detailed account of these notions. We recall that a torsionfree coherent sheaf F on X is (Gieseker) *semistable* if for any coherent subsheaf E, with $0 < \mathrm{rk}(E) < \mathrm{rk}(F)$, one has $p(E, t)/\mathrm{rk}(E) \leq p(F, t)/\mathrm{rk}(F)$ for $t \gg 0$. The sheaf F is called *stable* if the inequality above is always strict.

The *slope* of a sheaf F of positive rank is defined as $\mu(F) = \deg(c_1(F))/\mathrm{rk}(F)$, where $c_1(F)$ is the first Chern class of F. We recall that a torsionfree coherent sheaf F is *μ-semistable* if for any coherent subsheaf E, with $0 < \mathrm{rk}(E) < \mathrm{rk}(F)$, one has $\mu(E) < \mu(F)$. The sheaf F is called *μ-stable* if the above inequality is always strict. We recall that the *discriminant* of a sheaf F is $\Delta(F) = 2rc_2(F) - (r-1)c_1(F)^2$, where the k-th Chern class $c_k(F)$ of F lies in $\mathrm{H}^{k,k}(X)$. Bogomolov's inequality, see for instance [HL97, Theorem 3.4.1], states that if F is also μ-semistable, then we have:

$$\Delta(F) \cdot H_X^{n-2} \geq 0. \tag{2.1}$$

Recall that by Maruyama's theorem, see [Mar80], if $\dim(X) = n \geq 2$ and F is a μ-semistable sheaf of rank $r < n$, then its restriction to a general hypersurface of X is still μ-semistable.

We introduce here some notation concerning moduli spaces. We denote by $\mathsf{M}_X(r, c_1, \ldots, c_n)$ the moduli space of S-equivalence classes of rank r torsionfree semistable sheaves on X with Chern classes $c_1, \ldots, c_n$. The Chern class c_k will be denoted by an integer as soon as $\mathrm{H}^{k,k}(X)$ has dimension 1. We will drop the last values of the classes c_k when they are zero. We denote by $\mathsf{M}^{\ell f}_X(r, c_1, \ldots, c_n)$ the subset of $\mathsf{M}_X(r, c_1, \ldots, c_n)$ given by locally free sheaves.

We will work with Brill-Noether varieties of vector bundles over a smooth projective curve. We refer to [TiB91a] for some basic results. We will need also some results from [TiB91b] and [Mer01]. By definition, the *Brill-Noether variety* $W^s_{r,c}(\Gamma)$ is the scheme parameterizing rank r μ-stable bundles of degree c on Γ having at least $s+1$ independent global sections. It has expected dimension:

$$\rho(r,c,s) = 6\,r^2 - (s+1)\,(s+1-c+6\,r) + 1. \tag{2.2}$$

Recall also that the Gieseker-Petri map associated to a sheaf $\mathcal{F}$ on Γ is defined as the natural multiplication map:

$$\pi_{\mathcal{F}} : \mathrm{H}^0(\Gamma,\mathcal{F}) \otimes \mathrm{H}^0(\Gamma,\mathcal{F}^* \otimes \omega_\Gamma) \to \mathrm{H}^0(\Gamma,\mathcal{F}\otimes\mathcal{F}^*\otimes\omega_\Gamma). \tag{2.3}$$

The map $\pi_{\mathcal{F}}$ associated to a stable bundle $\mathcal{F}$ in $\mathsf{M}_\Gamma(r,d)$ is injective if and only if $[\mathcal{F}]$ is a nonsingular point of a component of $W^s_{r,d}$ of dimension $\rho(r,d,s)$. Its transpose has the form:

$$\pi_{\mathcal{F}}^\top : \mathrm{Ext}^1_\Gamma(\mathcal{F},\mathcal{F}) \to \mathrm{H}^0(\Gamma,\mathcal{F})^* \otimes \mathrm{H}^1(\Gamma,\mathcal{F}). \tag{2.4}$$

In fact, the tangent space $T_{[\mathcal{F}]}W^s_{r,d}$ is identified with the kernel of $\pi_{\mathcal{F}}^\top$.

2.1. **Prime Fano threefolds of genus 7.** We give a brief account of Mukai's description of a smooth prime Fano threefold of genus 7. For more details on the material contained in this section, we refer to [Muk88], [Muk89], [Muk95], [IM04a], [Kuz05], [IM07].

We consider thus a smooth prime Fano threefold of genus 7. which we will denote throughout the paper by X. In particular, X has Picard number 1 and the anticanonical class satisfies $-K_X = H_X$, where H_X is very ample and its class generates $\mathrm{Pic}(X)$. The divisor class H_X embeds X in $\mathbb{P}^8$ as an ACM variety. Remark that $\mathrm{H}^{k,k}(X)$ is generated by the divisor class H_X (for $k=1$), the class L_X of a line contained in X (for $k=2$), the class P_X of a closed point of X (for $k=3$). Recall that $H_X^2 = 12 L_X$. This allows to denote the Chern classes c_1,c_2,c_3 of a sheaf F on X by integers.

We recall that X is obtained as a smooth linear section of the spinor tenfold Σ^+, sitting in $\mathbb{P}^{15}$. The dual space $\check{\mathbb{P}}^{15}$ contains the dual spinor tenfold Σ^-, and the corresponding orthogonal linear section is a smooth projective canonical curve Γ of genus 7. The curve Γ can be identified with the moduli space $\mathsf{M}_X(2,1,5)$, and there exists a universal bundle $\mathscr{E}$ on $X\times\Gamma$ which makes Γ into a fine moduli space. The curve Γ is called the homologically projectively dual curve to X.

The spinor varieties $\Sigma^\pm$ can be seen as the two components of the orthogonal Grassmann variety of 4-dimensional projective subspaces contained in a smooth quadric in $\mathbb{P}^9$. We denote by $\mathcal{U}_\pm$ the restrictions to $\Sigma^\pm$ of the tautological universal subbundle. By a result of Kuznetsov, we have the following natural exact sequences on $X\times\Gamma$:

$$0 \to \mathscr{E}^* \to \mathcal{U}_- \to \mathscr{G} \to 0, \tag{2.5}$$

$$0 \to \mathscr{G} \to \mathcal{U}_+^* \to \mathscr{E} \to 0, \tag{2.6}$$

where $\mathcal{U}_-$ and $\mathcal{U}_+$ are defined on $X\times\Gamma$ using pull-backs via the projections $p : X\times\Gamma \to X$ and $q : X\times\Gamma\to\Gamma$. Here $\mathscr{G}$ is a vector bundle of rank 3 and $\mathscr{E}$ is the universal bundle mentioned above. Given a point $y\in\Gamma$ (resp. $x\in X$), and a sheaf $\mathscr{F}$ on $X\times\Gamma$, we denote by $\mathscr{F}_y$ (resp. $\mathscr{F}_x$) the restriction of $\mathscr{F}$ to $X\times\{y\}$ (resp. to $\{x\}\times\Gamma$). The Chern classes of these bundles are:

$$c_1(\mathscr{E}) = H_X + H_\Gamma, \quad c_2(\mathscr{E}) = \frac{7}{12}\,H_X\,H_\Gamma + 5\,L_X + \eta, \quad c_3(\mathscr{E}) = 0,$$

where $\eta \in \mathrm{H}^3(X) \otimes \mathrm{H}^1(\Gamma)$ satisfies $\eta^2 = 14$, and:

$$\begin{aligned} &c_1(\mathcal{U}_+) = -2H_X, && c_2(\mathcal{U}_+) = 24L_X, && c_3(\mathcal{U}_+) = -14P_X,\\ &c_1(\mathscr{G}_y) = H_X, && c_2(\mathscr{G}_y) = 7L_X, && c_3(\mathscr{G}_y) = 2P_X. \end{aligned}$$

Recall that the vector bundles $\mathcal{U}_+$ and $\mathscr{G}_y$, for any $y \in \Gamma$, are stable by [BF07, Lemma 2.5]. We will also consider the universal exact sequence:

$$0 \to \mathcal{U}_+ \to \mathscr{O}_X^{10} \to \mathcal{U}_+^* \to 0. \tag{2.7}$$

Applying the theorem of Riemann-Roch to a sheaf F on X, of (generic) rank r and with Chern classes c_1, c_2, c_3, we obtain the following formulas:

$$\begin{aligned} \chi(F) &= r + 3\,c_1 + 3\,c_1^2 - \frac{1}{2}c_2 + 2\,c_1^3 - \frac{1}{2}\,c_1\,c_2 + \frac{1}{2}\,c_3,\\ \chi(F,F) &= r^2 - \frac{1}{2}\Delta(F). \end{aligned}$$

It is well known that a general hyperplane section S of X is a smooth K3 surface (i.e., S has trivial canonical bundle and irregularity zero) of Picard number 1 (a generator is the restriction H_S of H_X to S), and sectional genus 7. We recall by [HL97, Part II, Chapter 6] that, given a stable sheaf F of rank r on S, with Chern classes c_1, c_2, the dimension at $[F]$ of the moduli space $\mathsf{M}_S(r, c_1, c_2)$ is:

$$\Delta(F) - 2\,(r^2 - 1). \tag{2.8}$$

2.2. **Derived categories.** If Y is a smooth projective variety, we denote by $\mathbf{D^b}(Y)$ its derived category, namely the derived category of complexes of sheaves on Y with bounded coherent cohomology. We refer to [GM96] and [Wei94] for definitions and notation.

Let now X be a smooth prime Fano threefold of genus 7, Γ the homologically projectively dual curve to X, and $\mathscr{E}$ the associated universal bundle defined above. The bundle $\mathscr{E}$ is defined on $X \times \Gamma$, and we denote by p and q the projections of $X \times \Gamma$ to X and Γ. As an essential tool we will use Kuznetsov's semiorthogonal decomposition of $\mathbf{D^b}(X)$, see [Kuz05]. This takes the following form:

$$\mathbf{D^b}(X) \cong \langle \mathscr{O}_X, \mathcal{U}_+^*, \mathbf{\Phi}(\mathbf{D^b}(\Gamma)) \rangle, \tag{2.9}$$

where $\mathbf{\Phi}$ is the integral functor associated to $\mathscr{E}$ defined by:

$$\mathbf{\Phi} : \mathbf{D^b}(\Gamma) \to \mathbf{D^b}(X), \qquad \mathbf{\Phi}(-) = \mathbf{R}p_*(q^*(-) \otimes \mathscr{E}). \tag{2.10}$$

Recall that the functor $\mathbf{\Phi}$ is fully faithful, and admits right and left adjoint functors $\mathbf{\Phi}^!$ and $\mathbf{\Phi}^*$ defined by:

$$\mathbf{\Phi}^! : \mathbf{D^b}(X) \to \mathbf{D^b}(\Gamma), \qquad \mathbf{\Phi}^!(-) = \mathbf{R}q_*(p^*(-) \otimes \mathscr{E}^*(\omega_\Gamma))[1], \tag{2.11}$$

$$\mathbf{\Phi}^* : \mathbf{D^b}(X) \to \mathbf{D^b}(\Gamma), \qquad \mathbf{\Phi}^*(-) = \mathbf{R}q_*(p^*(-) \otimes \mathscr{E}^*(-H_X))[3]. \tag{2.12}$$

The decomposition (2.9) provides a functorial exact triangle:

$$\mathbf{\Phi}(\mathbf{\Phi}^!(F)) \to F \to \mathbf{\Psi}(\mathbf{\Psi}^*(F)), \tag{2.13}$$

where $\mathbf{\Psi}$ is the inclusion of the subcategory $\langle \mathscr{O}_X, \mathcal{U}_+^* \rangle$ in $\mathbf{D^b}(X)$ and $\mathbf{\Psi}^*$ is the left adjoint functor to $\mathbf{\Psi}$. The k-th term of the complex $\mathbf{\Psi}(\mathbf{\Psi}^*(F))$ can be written as follows:

$$(\mathbf{\Psi}(\mathbf{\Psi}^*(F)))^k \cong \mathrm{Ext}_X^{-k}(F, \mathscr{O}_X)^* \otimes \mathscr{O}_X \oplus \mathrm{Ext}_X^{1-k}(F, \mathcal{U}_+)^* \otimes \mathcal{U}_+^*.$$

We will also use the following spectral sequences:

$$E_2^{p,q} = \mathrm{Ext}_X^p(\mathcal{H}^{-q}(a), A) \Rightarrow \mathrm{Ext}_X^{p+q}(a, A), \tag{2.14}$$

$$E_2^{p,q} = \mathrm{Ext}_X^p(B, \mathcal{H}^q(b)) \Rightarrow \mathrm{Ext}_X^{p+q}(B, b), \tag{2.15}$$

where a, b are complexes of sheaves on X, and A, B are sheaves on X. Recall that the maps in these spectral sequences are differentials:

$$d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}.$$

3. Some vanishing results

In this section, we prove some preliminary vanishing results and we prove that any locally free sheaf in $\mathsf{M}_X(2,0,4)$ is ACM. In all statements, X is a smooth prime Fano threefold of genus 7, S is a general hyperplane section surface of X, C is a general sectional curve of X and F is a locally free sheaf in $\mathsf{M}_X(2,0,4)$. We have the following exact sequences, defining respectively S and C:

$$0 \to \mathscr{O}_X(-1) \to \mathscr{O}_X \to \mathscr{O}_S \to 0, \tag{3.1}$$

$$0 \to \mathscr{O}_S(-1) \to \mathscr{O}_S \to \mathscr{O}_C \to 0. \tag{3.2}$$

Lemma 3.1. *The restrictions of $\mathcal{U}_+$ and $\mathscr{G}_y$ to S are stable vector bundles for all $y \in \Gamma$.*

Proof. We will deduce stability from Hoppe's criterion, see [Hop84, Lemma 2.6], see also [AO94, Theorem 1.2]. We have thus to show the following vanishing results:

$$\mathrm{H}^0(S, \mathscr{G}_y(-1)) = 0, \qquad \mathrm{H}^0(S, \wedge^2 \mathscr{G}_y(-1)) = 0, \tag{3.3}$$

$$\mathrm{H}^0(S, \mathcal{U}_+) = 0, \qquad \mathrm{H}^0(S, \mathcal{U}_+^*(-1)) = 0 \tag{3.4}$$

$$\mathrm{H}^0(S, \wedge^2 \mathcal{U}_+) = 0, \qquad \mathrm{H}^0(S, \wedge^2 \mathcal{U}_+^*(-1)) = 0. \tag{3.5}$$

Tensoring (3.1) by $\mathscr{G}_y(-1)$, we obtain $\mathrm{H}^0(S, \mathscr{G}_y(-1)) = 0$ since $\mathscr{G}_y$ is stable and $\mathrm{H}^1(X, \mathscr{G}_y(-2)) = 0$, see [BF07, Lemma 2.5]. Note that $\wedge^2 \mathscr{G}_y \cong \mathscr{G}_y^*(1)$. So, tensoring (3.1) by $\wedge^2 \mathscr{G}_y(-1)$, we get (3.3), since $\mathscr{G}_y$ is stable and $\mathrm{H}^1(X, \mathscr{G}_y^*(-1)) = 0$, see again [BF07, Lemma 2.5].

Applying the same argument to $\mathcal{U}_+$ and $\mathcal{U}_+^*(-1)$, we get (3.4). Finally, in view of the proof of [BF07, Lemma 2.5], in order to prove (3.5), it suffices to show $\mathrm{H}^1(X, \wedge^2 \mathcal{U}_+(-1)) = 0$ and $\mathrm{H}^1(X, \wedge^2 \mathcal{U}_+^*(-2)) = 0$. This can be checked via an easy application of Bott's theorem on the homogeneous space Σ_+. □

Lemma 3.2. *For all $y \in \Gamma$, the restrictions F_S and F_C of F to the surface S and to the curve C satisfy the following conditions:*

$$\mathrm{H}^0(S, F_S) = 0 \qquad \mathrm{H}^0(C, F_C) = 0 \tag{3.6}$$

$$\mathrm{H}^0(S, F_S \otimes \mathscr{G}_y(-t)) = 0 \qquad \mathrm{H}^0(C, F_C \otimes \mathscr{G}_y(-t)) = 0 \quad \textit{for } t \geq 1 \tag{3.7}$$

$$\mathrm{H}^0(S, F_S \otimes \mathcal{U}_+^*(-t)) = 0 \qquad \mathrm{H}^0(C, F_C \otimes \mathcal{U}_+^*(-t)) = 0 \quad \textit{for } t \geq 1 \tag{3.8}$$

Proof. Let us tensor the exact sequence (3.1) by F. Since $\mathrm{H}^0(X, F) = 0$ by stability and $\mathrm{H}^1(X, F(-1)) \cong \mathrm{H}^2(X, F)^* = 0$ by [BF08a, Lemma 4.3], we get the first vanishing in (3.6).

In the proof of Lemma [BF08a, Lemma 4.3] we have also obtained $\mathrm{H}^1(S, F_S(1)) = 0$, which implies by Serre duality $\mathrm{H}^1(S, F_S(-1)) = 0$. Then by tensoring (3.2) by F, the second vanishing in (3.6) follows.

Recall that $\mathscr{G}_y(-1)$ is isomorphic to $\wedge^2\mathscr{G}_y^*$. Then, dualizing the sequence (2.5) and restricting to $X \times \{y\}$, we obtain $\wedge^2\mathscr{G}_y^* \hookrightarrow \mathscr{O}_X^{10}$. Tensoring by F_S we get:

$$\mathrm{H}^0(S, F_S \otimes \wedge^2\mathscr{G}_y^*) \subseteq \mathrm{H}^0(S, F_S)^{10},$$

and by (3.6) we conclude that $\mathrm{H}^0(S, F_S \otimes \mathscr{G}_y(-1)) = 0$. Obviously this implies the first vanishing in (3.7) for all $t \geq 1$. The second vanishing is easily obtained replacing F_S by F_C in the above argument.

In order to prove the third part of the statement, it is enough to prove that the groups $\mathrm{H}^0(S, F_S \otimes \mathscr{E}_y(-t))$ and $\mathrm{H}^0(C, F_C \otimes \mathscr{E}_y(-t))$ are both zero. Indeed, in view of (3.7), the relations (3.8) will then easily follow making use of the exact sequence (2.6).

Notice that $\mathscr{E}_y^* \cong \mathscr{E}_y(-1)$, so from the sequence (2.5), restricted to $X \times \{y\}$, we get $\mathscr{E}_y(-1) \hookrightarrow \mathscr{O}_X^5$. Tensoring by F_S, (respectively by F_C) and using (3.6), we obtain $\mathrm{H}^0(S, F_S \otimes \mathscr{E}_y(-t)) = 0$ (respectively $\mathrm{H}^0(C, F_C \otimes \mathscr{E}_y(-t)) = 0$) for any $t \geq 1$. This completes the proof. □

Lemma 3.3. *For all $y \in \Gamma$ we have:*

$$\mathrm{Ext}^1_X(F(1), \mathscr{G}_y) = 0.$$

Proof. Let us first prove that the group $\mathrm{Ext}^1_S(F_S(1), \mathscr{G}_y) \cong \mathrm{H}^1(S, F_S \otimes \mathscr{G}_y(-1))$ vanishes. Assume the contrary, and consider the nontrivial extension of the form:

$$0 \to (\mathscr{G}_y)_S \to \widetilde{F_S} \to F_S(1) \to 0,$$

where $\widetilde{F_S}$ is a torsionfree sheaf on S with rank 5 and Chern classes $c_1(\widetilde{F_S}) = 3$, $c_2(\widetilde{F_S}) = 47$.

Notice now that $\widetilde{F_S}$ cannot be stable, since the space $\mathsf{M}_S(5, 3, 47)$ is empty by the dimension count (2.8). Then the Harder-Narasimhan filtration provides a maximal destabilizing stable quotient Q. Let K be the kernel of the projection from $\widetilde{F_S}$ onto Q. Notice that the sheaf K is reflexive by [Har80, Proposition 1.1], since $\widetilde{F_S}$ if locally free and Q is torsionfree.

Notice that the bundle $F_S(1)$ is stable by Maruyama's theorem, while $(\mathscr{G}_y)_S$ is stable by Lemma 3.1. Thus, since $\mu(K) \geq \frac{3}{5}$ and $\mathrm{rk}(K) \leq 4$, the only possible values that the pair $(\mathrm{rk}(K), c_1(K))$ can assume are $(2, 2)$ and $(3, 2)$. If the first case takes place, we have that K is a subbundle of $F(1)$ and, since K is reflexive and $F(1)$ is locally free, we have $K \cong F(1)$. This means that the extension is trivial, a contradiction.

Assume now that $\mathrm{rk}(K) = 3$ and $c_1(K) = 2$. Notice that K has to be stable since there exist no other possible destabilizing subbundles for $\tilde{F}$. Hence by the dimension count (2.8) we have $c_2(K) \geq 19$. On the other hand Q is stable with $\mathrm{rk}(Q) = 2$ and $c_1(Q) = 1$, thus by (2.8) we have $c_2(Q) \geq 5$.

But we have $47 = c_2(\tilde{F}) = c_2(K) + c_2(Q) + 24 \geq 48$, a contradiction. This proves that $\mathrm{H}^1(S, F_S \otimes \mathscr{G}_y(-1)) = 0$.

Tensoring by $F_S \otimes \mathscr{G}_y(-t)$ the exact sequence (3.2), and using the second equality in (3.7), one easily get that $\mathrm{H}^1(S, F_S \otimes \mathscr{G}_y(-t)) = 0$, for any $t \geq 1$. Tensoring now by $F \otimes \mathscr{G}_y(-t)$ the sequence (3.1), by the first vanishing in (3.7), we obtain

$$\mathrm{H}^1(X, F \otimes \mathscr{G}_y(-t-1)) \cong \mathrm{H}^1(X, F \otimes \mathscr{G}_y(-t))$$

for any $t \geq 1$. Since this groups vanish for $t \gg 0$, we conclude that $\mathrm{Ext}^1_X(F(1), \mathscr{G}_y) \cong \mathrm{H}^1(X, F \otimes \mathscr{G}_y(-1)) = 0$. □

Lemma 3.4. *For all $k \neq 3$ we have:*

$$\mathrm{Ext}^k_X(F(1), \mathcal{U}_+) = 0.$$

Proof. For $k = 0$, the statement follows from the stability of F and $\mathcal{U}_+$.

Applying the functor $\mathrm{Hom}_X(F(1), -)$ to (2.7), we obtain $\mathrm{Ext}^1_X(F(1), \mathcal{U}_+) \cong \mathrm{Hom}_X(F(1), \mathcal{U}_+^*)$, which vanishes by stability of F and $\mathcal{U}_+^*$, and $\mathrm{Ext}^2_X(F(1), \mathcal{U}_+) \cong \mathrm{Ext}^1_X(F(1), \mathcal{U}_+^*)$. It remains to prove that this last group is zero, too.

First we will prove:

$$\mathrm{Ext}^1_S(F_S(1), \mathcal{U}_+^*) \cong \mathrm{H}^1(S, F_S \otimes \mathcal{U}_+^*(-1)) = 0. \tag{3.9}$$

Assume by contradiction that there is a nontrivial extension of the form

$$0 \to (\mathcal{U}_+^*)_S \to G \to F_S(1) \to 0,$$

where G is a torsionfree sheaf on S with rank 7 and Chern classes $c_1(G) = 4$, $c_2(G) = 88$. Notice that G cannot be stable, since the space $\mathsf{M}_S(7, 4, 88)$ is empty by the dimension count (2.8).

Then the Harder-Narasimhan filtration provides a maximal destabilizing stable quotient Q. Let K be the kernel of the projection from G onto Q. Notice that the sheaf K is reflexive by [Har80, Proposition 1.1].

Recall the bundle $(\mathcal{U}_+)_S$ is stable by Lemma 3.1, while $F_S(1)$ is stable by Maruyama's theorem. So, since $\mu(K) \geq \frac{4}{7}$ and $\mathrm{rk}(K) \leq 6$, the only possible values for the pair $(\mathrm{rk}(K), c_1(K))$ are $(2, 2)$, $(3, 2)$ and $(5, 3)$.

If the first case takes place, we have that $K \cong F(1)$, hence the extension splits, a contradiction.

Assume now that $\mathrm{rk}(K) = 3$ and $c_1(K) = 2$. Notice that K has to be stable since there exist no other possible destabilizing subbundles for G. Hence by (2.8) we have $c_2(K) \geq 19$. On the other hand Q is stable with $\mathrm{rk}(Q) = 4$ and $c_1(Q) = 2$, thus by (2.8) we have $c_2(Q) \geq 22$. But we have $88 = c_2(G) = c_2(K) + c_2(Q) + 48 \geq 89$, a contradiction.

Finally assume that $\mathrm{rk}(K) = 5$ and $c_1(K) = 3$. Notice that K has to be stable since there exist no other possible destabilizing subbundles for G. Hence by (2.8) we have $c_2(K) \geq 48$. On the other hand Q is stable with $\mathrm{rk}(Q) = 2$ and $c_1(Q) = 1$, thus by (2.8) we have $c_2(Q) \geq 5$. But we have $88 = c_2(G) = c_2(K) + c_2(Q) + 36 \geq 89$, a contradiction. This proves (3.9).

Now, using (3.2) and the second equality in (3.8), one easily gets $\mathrm{H}^1(S, F_S \otimes \mathcal{U}_+^*(-t)) = 0$, for any $t \geq 1$. In turn, using (3.1) and the first vanishing in (3.8), we obtain:

$$\mathrm{H}^1(X, F \otimes \mathcal{U}_+^*(-t-1)) \cong \mathrm{H}^1(X, F \otimes \mathcal{U}_+^*(-t)).$$

for any $t \geq 1$. Since this groups vanishes for $t \gg 0$, we conclude that $\mathrm{Ext}^1_X(F(1), \mathcal{U}_+^*) \cong \mathrm{H}^1(X, F \otimes \mathcal{U}_+^*(-1)) = 0$. □

Proposition 3.5. *Let X be a smooth prime Fano threefold of genus 7. Then any locally free sheaf F in $\mathsf{M}_X(2,0,4)$ is ACM.*

Proof. We need to prove the following vanishing:

$$\mathrm{H}^k(X, F(t)) = 0,$$

for all t and for $k = 1, 2$. Notice that by Serre duality it is enough to prove only the case $k = 1$.

Fix a general hyperplane section surface S of X. By the first vanishing in (3.6) and Serre duality, we easily get that $\mathrm{H}^2(S, F_S(t)) = 0$ for all $t \geq 0$. Now, note that, by [BF08a, Lemma 4.3] and Riemann-Roch, we have $\mathrm{H}^2(X, F(-1)) = 0$. Thus, tensoring (3.1) by $F(t)$, we obtain:

$$\mathrm{H}^2(X, F(t)) = 0 \quad \text{for all } t \geq 0,$$

and by Serre duality it follows $\mathrm{H}^1(X, F(t)) = 0$ for all $t \leq -1$.

We want to prove now that $\mathrm{H}^1(X, F(t)) = 0$ for all $t \geq 0$. Fix a general sectional curve C in X and remark that by the second vanishing in (3.6) and by Serre duality we have:

$$\mathrm{h}^1(C, F_C(t)) = \mathrm{h}^0(C, F_C(-t+1)) \quad \text{for all } t \geq 1.$$

Thus, tensoring (3.2) by $F_S(t)$ and using the vanishing $\mathrm{H}^1(S, F_S) = 0$ (which holds by Riemann-Roch), we get $\mathrm{H}^1(S, F_S(t)) = 0$ for any $t \geq 1$. Finally, using again the exact sequence (3.1) tensorized by $F(t)$, since $\mathrm{H}^1(X, F) = 0$ we get $\mathrm{H}^1(X, F(t)) = 0$ for any $t \geq 1$, as we wanted. □

Remark 3.6. The previous proposition holds in fact for any smooth prime Fano threefold X of genus $g \geq 7$. Indeed the same proof works, since [BF08a, Lemma 4.3] can be applied to any locally free sheaf F in $\mathsf{M}_X(2,0,4)$ as soon as $m_g = \lceil (g+2)/2 \rceil > 4$. In turn, this takes place for all $g \geq 7$.

4. Proof of the main theorem

This section is devoted to the proof of our main theorem. Let us sketch the plan of our argument. First of all, by [BF08a, Theorem 4.10] the moduli space $\mathsf{M}_X(2,0,4)$ contains a 5-dimensional reduced irreducible component. Moreover any locally free sheaf in $\mathsf{M}_X(2,0,4)$ is stable by [BF08a, Proposition 4.16] and ACM by Proposition 3.5. Then, Lemma 4.1 will prove that, given a locally free sheaf F in $\mathsf{M}_X(2,0,4)$, the image $\varphi(F) = \mathbf{\Phi}^!(F(1))[-1]$ is a locally free sheaf on Γ, with rank 2 and degree 4. Then by Corollary (4.3) and Lemma (4.5) we will deduce that in fact $\varphi(F)$ is contained in the Brill-Noether variety $W^1_{2,4}(\Gamma)$. The fact that $\mathsf{M}^{\ell f}_X(2,0,4)$ is a smooth fivefold follows by Lemma 4.7. Moreover, it is an open dense subset of $W^1_{2,4}(\Gamma)$ by Lemma 4.8. Hence the irreducibility of $\mathsf{M}^{\ell f}_X(2,0,4)$ will follow from that of $W^1_{2,4}(\Gamma)$, which in turn is proved in [Mer01, Théorème 4], see also [Mer99]. The result of Mercat holds for any non-hyperelliptic curve, and Γ is so in view of [Muk95, Table 1]. The proof will thus be complete once we establish the lemmas of this section.

Lemma 4.1. *Let X be a smooth prime Fano threefold of genus 7 and F a locally free sheaf in $\mathsf{M}_X(2,0,4)$. Then $\mathbf{\Phi}^!(F(1))[-1]$ is a rank 2 locally free sheaf on Γ, with degree 4.*

Proof. Consider the stalk over a point $y \in \Gamma$ of the sheaf $\mathcal{H}^k(\mathbf{\Phi}^!(F(1)))$. We have:

$$\mathcal{H}^k(\mathbf{\Phi}^!(F(1)))_y \cong \operatorname{Ext}^{k+1}_X(\mathscr{E}_y, F(1)) \otimes \omega_{\Gamma,y}. \tag{4.1}$$

We would like to prove that this group vanishes for all $y \in \Gamma$ and for all $k \neq -1$. This amounts to prove that $\operatorname{Ext}^{2-k}_X(F(1), \mathscr{E}_y^*) = 0$ for $k = 0, 1, 2$. The case $k = 2$ follows immediately from the stability of F and $\mathscr{E}_y$.

Now let us apply the functor $\operatorname{Hom}_X(F(1), -)$ to the exact sequence (2.5) restricted to $X \times \{y\}$. Since $\operatorname{Hom}_X(F(1), \mathscr{O}_X) \cong \mathrm{H}^k(X, F(-1)) = 0$ for any k we have $\operatorname{Ext}^{k+1}_X(F(1), \mathscr{E}_y^*) \cong \operatorname{Ext}^k(F(1), \mathscr{G}_y)$. Hence in particular the group $\operatorname{Ext}^1_X(F(1), \mathscr{E}_y^*)$ is zero by the stability of F and $\mathscr{G}_y$ (see [BF07, Lemma 2.5]), while the group $\operatorname{Ext}^1_X(F(1), \mathscr{E}_y^*)$ vanishes by Lemma 3.3.

Finally, by Riemann-Roch we have $\chi(\mathscr{E}_y, F(1)) = 2$, so the rank of $\mathbf{\Phi}^!(F(1))$ is 2. Then we can apply the theorem of Grothendieck-Riemann-Roch to calculate $\chi(\mathbf{\Phi}^!(F(1)))$. It easily follows that $\deg(\mathbf{\Phi}^!(F(1))) = 4$. □

Notation. Let F be a sheaf in $\mathsf{M}_X^{\ell f}(2,0,4)$. We set:

$$\mathcal{F} = \mathbf{\Phi}^!(F(1))[-1].$$

We set also $A_F = \operatorname{Hom}_X(\mathcal{U}_+, F)$.

Lemma 4.2. *Let F be a sheaf in $\mathsf{M}_X^{\ell f}(2,0,4)$. Then the following relations hold:*

$$\mathcal{H}^0(\mathbf{\Phi}(\mathcal{F})) \cong A_F \otimes \mathcal{U}_+^*, \tag{4.2}$$

$$\mathcal{H}^1(\mathbf{\Phi}(\mathcal{F})) \cong F(1), \tag{4.3}$$

and A_F has dimension 2.

Proof. In order to use the decomposition (2.9), we need to compute the groups $\operatorname{Ext}^k_X(F(1), \mathscr{O}_X)$ and $\operatorname{Ext}^k_X(F(1), \mathcal{U}_+)$. Recall that $\operatorname{Ext}^k_X(F(1), \mathscr{O}_X) = 0$ for all k. On the other hand, by Lemma 3.4 we know that $\operatorname{Ext}^k_X(F(1), \mathcal{U}_+) = 0$ for all $k \neq 3$. By Riemann-Roch it follows $\operatorname{ext}^3_X(F(1), \mathcal{U}_+) = 2$. Then the exact triangle (2.13) provides thus the isomorphisms (4.2) and (4.3). □

Corollary 4.3. *The sheaf $\mathcal{F}$ has two independent global sections, and $\mathrm{H}^0(\Gamma, \mathcal{F})$ is naturally identified with A_F.*

Proof. By [Kuz05, Lemma 5.6] we have $\mathbf{\Phi}^*(\mathcal{U}_+^*) \cong \mathscr{O}_\Gamma$ and thus:

$$\mathrm{H}^0(\Gamma, \mathcal{F}) \cong \operatorname{Hom}_\Gamma(\mathscr{O}_\Gamma, \mathcal{F}) \cong \operatorname{Hom}_X(\mathcal{U}_+^*, \mathbf{\Phi}(\mathcal{F})).$$

By (4.2) it follows that $\operatorname{Hom}_X(\mathcal{U}_+^*, \mathbf{\Phi}(\mathcal{F})) \cong \operatorname{Hom}_X(\mathcal{U}_+^*, \mathcal{U}_+^* \otimes A_F) \cong A_F$, hence we have $\mathrm{h}^0(\Gamma, \mathcal{F}) = 2$. □

Lemma 4.4. *The vector bundle $\mathcal{F}$ is simple.*

Proof. We have:

$$\mathrm{Hom}_\Gamma(\mathcal{F},\mathcal{F}) \cong \mathrm{Hom}_X(\mathbf{\Phi}(\mathcal{F}), F(1)[-1]) \cong \mathrm{Hom}_X(F(1), F(1)).$$

where the last isomorphism follows immediately by the spectral sequence since (2.14), setting $A = F(1)$ and $a = \mathbf{\Phi}(\mathcal{F})$. The claim thus follows from the stability of F. □

In fact, the bundle $\mathcal{F}$ is not only simple, see the next lemma.

Lemma 4.5. *The vector bundle $\mathcal{F}$ is stable.*

Proof. Assume by contradiction that $\mathcal{F}$ is not stable. Then there exists a destabilizing exact sequence on Γ of the form:

$$0 \to \mathcal{L} \to \mathcal{F} \to \mathcal{M} \to 0, \tag{4.4}$$

where $\mathcal{L}$, $\mathcal{M}$ are line bundles, $\ell = \deg(\mathcal{L}) \geq 2$ and $m = \deg(\mathcal{M}) = 4 - \ell$.

From (4.2) it follows that for any $x \in X$, $\mathrm{h}^0(\Gamma, \mathcal{F} \otimes \mathscr{E}_x) = 10$. Then tensoring (4.4) by $\mathscr{E}_x$, we have also $\mathrm{h}^0(\Gamma, \mathcal{L} \otimes \mathscr{E}_x) \leq 10$. From Riemann-Roch it follows that $\chi(\mathcal{L} \otimes \mathscr{E}_x) = 2\,\ell \leq 10$ and thus $\ell \leq 5$.

If $\ell = 5$, we have:

$$\begin{aligned}
&\mathcal{H}^0(\mathbf{\Phi}(\mathcal{L})) \cong \mathcal{H}^0(\mathbf{\Phi}(\mathcal{F})) \cong \mathcal{H}^{-1}(\mathbf{\Phi}(\mathbf{\Phi}^!(F(1)))) \cong \mathcal{U}^* \otimes A_F,\\
&\mathcal{H}^1(\mathbf{\Phi}(\mathcal{M})) \cong \mathcal{H}^1(\mathbf{\Phi}(\mathcal{F})) \cong \mathcal{H}^0(\mathbf{\Phi}(\mathbf{\Phi}^!(F(1)))) \cong F(1),\\
&\mathcal{H}^k(\mathbf{\Phi}(\mathcal{L})) = \mathcal{H}^{k+1}(\mathbf{\Phi}(\mathcal{M})) = 0, \qquad \text{for all } k \neq 0.
\end{aligned}$$

Therefore, since the functor $\mathbf{\Phi}$ is fully faithful, we obtain:

$$\mathrm{Ext}^1_\Gamma(\mathcal{M},\mathcal{L}) \cong \mathrm{Ext}^1_X(\mathbf{\Phi}(\mathcal{M}), \mathbf{\Phi}(\mathcal{L})) \cong \mathrm{Hom}(F(1), \mathcal{U}_+^* \otimes A_F),$$

and the last group vanishes by stability of F and $\mathcal{U}_+$. This contradicts Lemma 4.4.

If $3 \leq \ell \leq 4$, we have $\mathrm{h}^0(\Gamma, \mathcal{L}) \leq 1$ by [Muk95, Table 1]. This easily implies $\mathrm{h}^0(\Gamma, \mathcal{L}) = \mathrm{h}^0(\Gamma, \mathcal{M}) = 1$. In particular the line bundle $\mathcal{M}$ is either trivial either of the form $\mathscr{O}_\Gamma(y)$, where y is a point in Γ. Applying the functor $\mathbf{\Phi}$ to (4.4) and taking cohomology we get a projection from $\mathcal{H}^1(\mathbf{\Phi}(\mathcal{F})) \cong F(1)$ to $\mathcal{H}^1(\mathbf{\Phi}(\mathcal{M}))$, hence $\mathrm{rk}(\mathcal{H}^1(\mathbf{\Phi}(\mathcal{M}))) \leq 2$. But if $\mathcal{M} \cong \mathscr{O}_\Gamma$ we have $\mathcal{H}^1(\mathbf{\Phi}(\mathcal{M})) \cong \mathcal{U}_+(1)$ which has rank 5, a contradiction. On the other hand, if $\mathcal{M} \cong \mathscr{O}_\Gamma(y)$, we can see from the exact sequence:

$$0 \to \mathscr{O}_\Gamma \to \mathcal{M} \to \mathscr{O}_y \to 0, \tag{4.5}$$

that $\mathrm{rk}(\mathcal{H}^1(\mathbf{\Phi}(\mathcal{M}))) \geq 3$, again a contradiction.

Finally, assume $\ell = 2$. Again we have $\mathrm{h}^0(\Gamma, \mathcal{L}) = \mathrm{h}^0(\Gamma, \mathcal{M}) = 1$, so the line bundle $\mathcal{M}$ is isomorphic to $\mathscr{O}_\Gamma(Z)$ where Z is an effective divisor in Γ of degree 2. We would like to prove:

$$\mathcal{H}^1(\mathbf{\Phi}(\mathcal{M})) \cong \mathcal{I}_C(1), \tag{4.6}$$

where $\mathcal{I}_C$ is the ideal sheaf of a conic $C \subset X$, so that, applying $\mathbf{\Phi}$ to (4.4) we obtain a surjection $F(1) \to \mathcal{I}_C(1)$, and F would be strictly semistable. Recall by [Kuz05, Theorem 5.3] that $\mathscr{O}_Z$ is isomorphic to $\mathbf{\Phi}^!(\mathscr{O}_C)$, for some conic $C \subset X$, and thus $\mathbf{\Phi}(\mathscr{O}_Z)$ is concentrated in degree zero. Moreover, dualizing the exact sequence (9) in [Kuz05], one gets:

$$0 \to (\mathbf{\Phi}(\mathscr{O}_Z))^*(1) \to \mathcal{U}_+(1) \to \mathcal{I}_C(1) \to 0.$$

On the other hand, applying the functor $\mathbf{\Phi}$ to the exact sequence:

$$0 \to \mathscr{O}_\Gamma \to \mathcal{M} \to \mathscr{O}_Z \to 0,$$

we obtain an exact sequence:

$$\mathbf{\Phi}(\mathscr{O}_Z) \to \mathcal{U}_+(1) \to \mathcal{H}^1(\mathbf{\Phi}(\mathcal{M})) \to 0.$$

We get thus (4.6) by the natural isomorphism $(\mathbf{\Phi}(\mathscr{O}_Z))^*(1) \cong \mathbf{\Phi}(\mathscr{O}_Z)$, provided by Grothendieck duality, see [BF07, Lemma 2.6], see also [Har66] for general reference. □

Lemma 4.6. *Let F be a sheaf in $\mathsf{M}_X^{\ell f}(2,0,4)$. Then we have:*

$$\mathrm{H}^k(X, \mathcal{U}_+ \otimes F(1)) = 0, \qquad \textit{for } k = 2, 3. \tag{4.7}$$

Proof. Recall that in Lemma 4.2 we have proved $\mathrm{H}^k(X, \mathcal{U}_+^* \otimes F) = 0$ for $k \neq 0$, so tensoring (2.7) by F we get $\mathrm{H}^k(X, \mathcal{U}_+ \otimes F) = 0$ for $k \neq 1$. In turn, tensoring (3.1) by $\mathcal{U}_+ \otimes F(1)$ and making use of stability of $(\mathcal{U}_+)_S$ and F_S we get $\mathrm{H}^2(S, \mathcal{U}_+ \otimes F(1)) = 0$. We have thus proved our statement. □

Lemma 4.7. *For any pair of sheaves F, F' in $\mathsf{M}_X^{\ell f}(2,0,4)$, we have:*

$$\mathrm{Ext}_X^2(F', F) = 0,$$
$$\mathrm{H}^1(X, \mathcal{U}_+ \otimes F(1)) = 0.$$

Proof. Recall the notation $\mathcal{F} = \mathbf{\Phi}^!(F(1))[-1]$ and set $\mathcal{F}' = \mathbf{\Phi}^!(F'(1))[-1]$. We have, for all $k \in \mathbb{Z}$:

$$\mathrm{Ext}_\Gamma^k(\mathcal{F}', \mathcal{F}) \cong \mathrm{Ext}_X^{k-1}(\mathbf{\Phi}(\mathcal{F}'), F(1)),$$

and by (2.14), we have the spectral sequence:

$$E_2^{p,q} = \mathrm{Ext}_X^p(\mathcal{H}^{-q}(\mathbf{\Phi}(\mathcal{F}')), F(1)) \Rightarrow \mathrm{Ext}_X^{p+q}(\mathbf{\Phi}(\mathcal{F}'), F(1)). \tag{4.8}$$

By Lemma 4.2, we have $\mathcal{H}^0(\mathbf{\Phi}(\mathcal{F}')) \cong A_{F'} \otimes \mathcal{U}_+^*$, and $\mathcal{H}^1(\mathbf{\Phi}(\mathcal{F}')) \cong F'(1)$. Using (4.7), the spectral sequence (4.8) becomes:

$$\begin{array}{cccc} A_{F'}^* \otimes \mathrm{Hom}_X(\mathcal{U}_+^*, F(1)) & A_{F'}^* \otimes \mathrm{Ext}_X^1(\mathcal{U}_+^*, F(1)) & 0 & 0 \\ & \searrow^{d_2^{0,0}} & \searrow^{d_2^{1,0}} & \\ \mathrm{Hom}_X(F', F) & \mathrm{Ext}_X^1(F', F) & \mathrm{Ext}_X^2(F', F) & 0 \end{array} \tag{4.9}$$

Since the map $d_2^{1,0}$ is zero, we get:

$$A_{F'}^* \otimes \mathrm{Ext}_X^1(\mathcal{U}_+^*, F(1)) \oplus \mathrm{Ext}_X^2(F', F) \cong \mathrm{Ext}_\Gamma^2(\mathcal{F}', \mathcal{F}). \tag{4.10}$$

Note that the group $\mathrm{Ext}_\Gamma^k(\mathcal{F}', \mathcal{F})$ vanishes for $k \geq 2$ since $\mathcal{F}$ and $\mathcal{F}'$ are coherent sheaves on a curve. We obtain that both groups $\mathrm{H}^1(X, \mathcal{U}_+ \otimes F(1))$ and $\mathrm{Ext}_X^2(F', F)$ are zero. This proves the lemma. □

Note that the previous Lemma holds even if we take $F' = F$. In particular, for all F in $\mathsf{M}_X^{\ell f}(2,0,4)$, we have proved:

$$\mathrm{Ext}_X^2(F, F) = 0.$$

Lemma 4.8. *Let F be a sheaf in $\mathsf{M}_X^{\ell f}(2,0,4)$ and $\mathcal{F} = \mathbf{\Phi}^!(F(1))[-1]$. Then the two tangent spaces $T_{[\mathcal{F}]} W_{2,4}^1(\Gamma)$ and $T_{[F]} \mathsf{M}_X(2,0,4)$ are naturally identified.*

Proof. Recall first that $T_{[F]}\mathsf{M}_X(2,0,4)$ is canonically identified with $\operatorname{Ext}^1_X(F,F)$. This space has dimension 5 by Lemma 4.7 and by Riemann-Roch.

On the other hand, the space $T_{[\mathcal{F}]}W^1_{2,4}(\Gamma)$ is canonically identified with the kernel of the transpose $\pi_{\mathcal{F}}^{\top}$ of the Petri map, see (2.4). Recall by Corollary 4.3 that $A_F \cong \mathrm{H}^0(\Gamma,\mathcal{F})$ and consider the natural evaluation map:

$$\mathrm{ev} : A_F \otimes \mathscr{O}_\Gamma \to \mathcal{F},$$

and note that the map $\pi_{\mathcal{F}}^{\top}$ equals $\operatorname{Ext}^1_X(\mathrm{ev},\mathcal{F})$. By definition of $\mathcal{F}$ and since $\mathbf{\Phi}^!$ is right adjoint to $\mathbf{\Phi}$, this map thus equals:

$$\operatorname{Hom}_X(\mathbf{\Phi}(\mathrm{ev}),F(1)) : \operatorname{Hom}_X(\mathbf{\Phi}(\mathcal{F}),F(1)) \to A_F^* \otimes \operatorname{Hom}_X(\mathbf{\Phi}(\mathscr{O}_\Gamma),F(1)).$$

So this map induces a map of spectral sequences from (4.9) to:

$$\begin{array}{cccc} A_F^*\otimes \operatorname{Hom}_X(\mathcal{U}_+^*,F(1)) & 0 & 0 & 0 \\ A_F^*\otimes \operatorname{Hom}_X(\mathcal{U}_+(1),F(1)) & 0 & 0 & 0 \end{array}$$

Here, the zeros in the first line are given by Lemma 4.7 and those of the second line follow from Lemma 3.4. Thus the kernel of $\operatorname{Hom}_X(\mathbf{\Phi}(\mathrm{ev}),F(1))$ is identified with $\operatorname{Ext}^1_X(F,F)$. So the two tangent spaces are naturally identified. □

Remark 4.9. The fact that the variety $W^1_{2,4}(\Gamma)$ is irreducible (and nonsingular) relies on its explicit description, obtained in [Mer01, Théorème 4], and [Mer99, Chapitre 3, Théorème A.1]. Indeed, an element $\mathcal{F}$ of $W^1_{2,4}(\Gamma)$ fits into an exact sequence:

$$0 \to \mathscr{O}_\Gamma^2 \to \mathcal{F} \to T \to 0,$$

where T is a torsion sheaf of degree 4 on Γ. Thus the space $W^1_{2,4}(\Gamma)$ is birational to a $\mathbb{P}^1$-bundle over the symmetric power $\Gamma^{(4)}$.

5. Restricting to a hyperplane section surface

Let again X be a smooth prime Fano threefold of genus 7, and consider the restriction F_S of a sheaf F in the moduli space $\mathsf{M}^{\ell f}_X(2,0,4)$ to a hyperplane section surface S of X. For general S, the sheaf F_S thus belongs to the moduli space $\mathsf{M}^{\ell f}_S(2,0,4)$. Recall by [Muk84] that the moduli space $\mathsf{M}^{\ell f}_S(2,0,4)$ is a symplectic manifold. Following an idea of Tyurin, we prove here that the restriction mapping is injective, hence that $\mathsf{M}^{\ell f}_X(2,0,4)$ is lagrangian in $\mathsf{M}^{\ell f}_S(2,0,4)$.

Proposition 5.1. *Let S be a smooth hyperplane section surface of X with $\operatorname{Pic}(S) \cong \langle H_S\rangle$. Then the restriction map:*

$$\rho : \mathsf{M}^{\ell f}_X(2,0,4) \to \mathsf{M}^{\ell f}_S(2,0,4)$$
$$F \mapsto F_S$$

is a closed embedding, and $\operatorname{Im}(\rho)$ *is a lagrangian submanifold of* $\mathsf{M}^{\ell f}_S(2,0,4)$.

Proof. The image of the restriction map ρ is a lagrangian submanifold by [Tyu04]. Thus we need only prove that ρ is well-defined and injective everywhere.

Let thus F be a sheaf in $\mathsf{M}_X^{\ell f}(2,0,4)$ and F_S be its restriction to S. First note that the sheaf F_S is a stable vector bundle. Indeed the first vanishing in (3.6) takes place for any hyperplane section surface S, and this implies stability by Hoppe's criterion since $\mathrm{Pic}(S) \cong \langle H_S \rangle$. Therefore ρ is well-defined.

In order to prove that ρ is injective, we let F' be a sheaf in $\mathsf{M}_X^{\ell f}(2,0,4)$, not isomorphic to F and we set F'_S for its restriction to S. Let us see that the existence of an isomorphism $\alpha : F_S \to F'_S$ leads to a contradiction. Tensoring (3.1) with F' provides a surjective map $F \to F'_S$. We want to prove that this map lifts to a map $\tilde{\alpha} : F \to F'$, and we note that this is the case if the obstruction group $\mathrm{Ext}^1_X(F, F'(-1))$ vanishes. But this group is dual to $\mathrm{Ext}^1_X(F', F)$, which vanishes by Lemma 4.7. Therefore we have the map $\tilde{\alpha}$, and, by stability of F and F', the map $\tilde{\alpha}$ must be an isomorphism. This is a contradiction. □

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