VECTOR BUNDLES ON FANO THREEFOLDS OF GENUS 7 AND BRILL-NOETHER LOCI

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Abstract. Given a smooth prime Fano threefold X of genus 7 we consider its homologically projectively dual curve Γ and the natural integral functor $\mathbf{\Phi}^! : \mathbf{D^b}(X) \to \mathbf{D^b}(\Gamma)$.

We prove that, for $d \geq 6$, $\mathbf{\Phi}^!$ gives a birational map from a component of the moduli scheme $\mathsf{M}_X(2, 1, d)$ of rank 2 stable sheaves on X with $c_1 = 1$, $c_2 = d$ to a generically smooth $(2\,d - 9)$-dimensional component of the Brill-Noether variety $W^{2\,d-11}_{d-5,5\,d-24}$ of stable vector bundles on Γ of rank $d - 5$ and degree $5\,d - 24$ with at least $2\,d - 10$ sections.

This map turns out to be an isomorphism for $d = 6$, and the moduli space $\mathsf{M}_X(2, 1, 6)$ is fine. For general X, this moduli space is a smooth irreducible threefold.

1. Introduction

Let X be a smooth complex projective variety of dimension 3, with Picard number one, and assume that the anticanonical divisor K_X is ample. Then X is called a Fano threefold, and one defines the index i_X of X as the greatest integer i such that $-K_X/i$ lies in $\mathrm{Pic}(X)$. We are interested in the Maruyama moduli scheme $\mathsf{M}_X(2, c_1, c_2)$ of semistable sheaves of rank 2 and Chern classes c_1, c_2, and with $c_3 = 0$, defined on a Fano threefold X.

The maximum value of i_X is 4, and in this case X must be isomorphic to $\mathbb{P}^3$. The study of the moduli space $\mathsf{M}_{\mathbb{P}^3}(2, c_1, c_2)$ was pioneered by Barth in [8], and pursued later by several authors. Roughly speaking, the main questions concern rationality, irreducibility and smoothness of these moduli spaces; many of them are still open. Among the main tools to study the problem, we recall monads and Beilinson's theorem, see [9], [11] and [63]. The next case is $i_X = 3$. Then X has to be isomorphic to a quadric hypersurface. This case was considered by Ein and Sols (see [17]) and later by Ottaviani and Szurek, see [65].

In the case $i_X = 2$, there are 5 deformation classes of Fano threefolds as it results from Iskovskikh's classification, see [40]. Perhaps the most studied among them is the cubic hypersurface V_3 in $\mathbb{P}^4$. The geometry of these threefolds is deeply linked to the properties of the families of curves they contain. A cornerstone in this sense is the paper [14] of Clemens and Griffiths on V_3. For a survey of results about moduli spaces of vector bundles on V_3 we refer to [10]. In particular we mention [16], [49] and [6].

In the case $i_X = 1$, we say that X is a *prime* Fano threefold. Then, one defines the genus of X as the integer $g = -K_X^3/2 + 1$. The genus satisfies $2 \leq g \leq 12$, $g \neq 11$, and there are 10 deformation classes of prime Fano threefolds, characterized by value of g. The birational geometry of prime Fano threefolds has been extensively studied as well, see [40]. The geometry of the moduli spaces of rank 2 vector bundles on X has been more recently investigated by several authors, for instance in the papers [35] (for genus 3), [36], [37] (for genus 7), [33], [34] (for genus 8), [38] (for genus 9), [4] (for genus 12). Among the main tools we mention the Abel-Jacobi map and Serre's correspondence between rank 2 vector bundles and curves contained in X.

The purpose of the present paper is to investigate the properties of the moduli spaces of rank 2 bundles on a smooth prime Fano threefold X, making use of homological methods. We first observe that (under a mild generality assumption on X), given any integer $d \geq g/2 + 1$, the moduli space $\mathsf{M}_X(2, 1, d)$ contains a generically smooth component $\mathsf{M}(d)$ of dimension $2\,d - g - 2$, such that its general element F is a stable locally free sheaf with $\mathrm{H}^1(X, F(-1)) = 0$, see Theorem 3.7. The

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condition $\mathrm{H}^1(X, F(-1)) = 0$ implies $\mathrm{H}^k(X, F(-1)) = 0$ for all k for F in $\mathsf{M}_X(2, 1, d)$, and we think of this vanishing as an analogue of the instanton condition on $\mathbb{P}^3$.

Then, we focus on genus 7, where an analogue of Beilinson's theorem is provided by the semiorthogonal decomposition of the bounded derived category $\mathbf{D^b}(X)$ obtained by Kuznetsov in [44]. We use this decomposition to study the component $\mathsf{M}(d)$. More precisely, we consider the homologically projectively dual curve Γ in the sense of [45], and the corresponding integral functor $\mathbf{\Phi}^! : \mathbf{D^b}(X) \to \mathbf{D^b}(\Gamma)$. Making use of the canonical resolution of a general element of $\mathsf{M}(d)$, we show that $\mathbf{\Phi}^!$ gives a birational map φ from $\mathsf{M}(d)$ to a component of $W^{2\,d-11}_{d-5,5\,d-24}$ (Theorem 5.9), where we denote by $W^s_{r,c}$ the Brill-Noether variety of stable vector bundles on Γ of rank r and degree c with at least $s+1$ independent global sections.

We prove that the map φ is in fact an isomorphism in the case $d = 6$. In particular the moduli space $\mathsf{M}_X(2, 1, 6)$ is fine and isomorphic to a connected threefold (Theorem 5.11, part A). If X is general enough, the moduli space $\mathsf{M}_X(2, 1, 6)$ is actually smooth and irreducible (Theorem 5.11, part B). We also exhibit an involution of $\mathsf{M}_X(2, 1, 6)$ which interchanges the set of sheaves which are not globally generated with the one of those which are not locally free. Finally we show that, if S is a general hyperplane section surface, the space $\mathsf{M}_X(2, 1, 6)$ embeds as a Lagrangian subvariety of $\mathsf{M}_S(2, 1, 6)$ with respect to the Mukai form, away from finitely many double points (Theorem 5.19).

The paper is organized as follows. In Section 2 we review the geometry of Fano threefolds X of genus 7 and the structure of their derived category. In Section 3 we construct (under mild generality assumptions) a generically smooth component $\mathsf{M}(d)$ of $\mathsf{M}_Y(2, 1, d)$, over a smooth prime Fano threefold Y (of any genus), and we recall some basic facts concerning bundles with minimal c_2. Then again we work on genus 7: in Section 4, we prove that the functor $\mathbf{\Phi}^!$ provides an isomorphism between the Hilbert scheme $\mathscr{H}^0_3(X)$ and the symmetric cube $\Gamma^{(3)}$, (see Theorem 4.5). Section 5 contains our main results.

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2. Preliminaries

We introduce here some basic material. Throughout the paper we work over the field of complex numbers, and X will denote a smooth connected complex projective n-dimensional variety, while H_X will denote an ample divisor class on X, so (X, H_X) will be called a polarized manifold.

2.1. **Notation and preliminary results.** Given a polarized manifold (X, H_X), and a coherent sheaf F on X, we write $F(t)$ for $F \otimes \mathscr{O}_X(tH_X)$. Given a subscheme Z of X, we write F_Z for $F \otimes \mathscr{O}_Z$ and we denote by $\mathcal{I}_{Z,X}$ the ideal sheaf of Z in X, and by $N_{Z,X}$ its normal sheaf. We will frequently drop the second subscript. Given a pair of coherent sheaves (F, E) on X, we will write $\mathrm{ext}^k_X(F, E)$ for the dimension of the vector space $\mathrm{Ext}^k_X(F, E)$, and similarly $\mathrm{h}^k(X, F) = \dim \mathrm{H}^k(X, F)$. The Euler characteristic of (F, E) is defined as $\chi(F, E) = \sum_k (-1)^k \mathrm{ext}^k_X(F, E)$ and $\chi(F)$ is defined as $\chi(\mathscr{O}_X, F)$. We denote by $p(F, t)$ the Hilbert polynomial $\chi(F(t))$ of the sheaf F. The canonical bundle of X is denoted by ω_X. We define also the natural evaluation map:

$$e_{E,F} : \mathrm{Hom}_X(E, F) \otimes E \to F.$$

Assuming that H_X is very ample, we have that X is embedded in $\mathbb{P}^m$, and we say that X is *ACM* (for *arithmetically Cohen-Macaulay*) if its coordinate ring is Cohen-Macaulay. If X is ACM and $n \geq 1$, a locally free sheaf F on X is called *ACM* if it has no intermediate cohomology, i.e. if $\mathrm{H}^k(X, F(t)) = 0$ for all integer t and for any $0 < k < n$. This is equivalent to $\oplus_{t\in\mathbb{Z}} \mathrm{H}^0(X, F(t))$ being a Cohen-Macaulay module over the coordinate ring of X.

2.1.1. *Torsion-free, reflexive and locally free sheaves.* For any integer $k \geq 0$, the k-th Chern class $c_k(F)$ takes values in $\mathrm{H}^{k,k}(X)$ and is defined for a vector bundle of finite rank F on X, and in fact for any element F of Grothendieck group of all coherent sheaves on X (for more details see e.g. [28, Appendix A]). Depending on the context, one can also think of $c_1(F)$ as a divisor class in $\mathrm{Pic}(X)$. Given coherent sheaves F_1, F_2 on X, we will write $c_1(F_1) \geq c_1(F_2)$ if $c_1(F_1) - c_1(F_2)$ is effective. In the sequel, the Chern classes will be denoted by integers as soon as $\mathrm{H}^{k,k}(X)$ has dimension 1 and the choice of a generator is clear.

We denote by $F^* = \mathcal{H}om_X(F, \mathscr{O}_X)$ the dual of a coherent sheaf F on X. Recall that a coherent sheaf F on X is *reflexive* if the natural map $F \to F^{**}$ of F to its double dual is an isomorphism. We recall here some basic facts on reflexive sheaves, which will be useful in the sequel, and we refer to [29] for more details. Any locally free sheaf is reflexive, and any reflexive sheaf is torsion-free. A coherent sheaf F on X is reflexive if and only if F is the kernel of a surjective map $E \to G$, with G torsion-free and E locally free, see [29, Proposition 1.1]. Moreover, by [29, Proposition 1.9], any reflexive rank-1 sheaf is invertible (recall that X is smooth and irreducible). Finally, we will often use a straightforward generalization of [29, Proposition 2.6] which implies that the third Chern class $c_3(F)$ of a rank 2 reflexive sheaf F on a smooth projective threefold satisfies $c_3(F) \geq 0$, and vanishes if and only if F is locally free. We conclude this section with the following easy and useful lemma.

Lemma 2.1. *Let F be a vector bundle on X and $\mathcal{F}$ be a torsion-free sheaf such that $c_1(F) \geq c_1(\mathcal{F})$ and $\mathrm{rk}(F) = \mathrm{rk}(\mathcal{F})$. Then an injective map $F \to \mathcal{F}$ is an isomorphism.*

Proof. Consider an injective map $F \to \mathcal{F}$. Its determinant $\mathscr{O}_X(c_1(F)) \to \mathscr{O}_X(c_1(\mathcal{F}))$ is non-zero so $c_1(F) \leq c_1(\mathcal{F})$, hence $c_1(F) = c_1(\mathcal{F})$. Therefore, we have the exact sequence:

$$0 \to F \to \mathcal{F} \to T \to 0,$$

where the quotient T is a torsion sheaf with $c_1(T) = 0$. Assume $T \neq 0$. Note that $\mathscr{E}xt^k_X(T, F) \cong \mathscr{E}xt^k_X(T, \mathscr{O}_X) \otimes F = 0$ for $k = 0, 1$, because T is supported in codimension at least 2, since $c_1(T) = 0$. Then setting $A = T$ and $B = \mathscr{O}_X$ in the spectral sequence (2.6) (see below) we get $\mathrm{Ext}^1_X(T, F) = 0$. This implies $\mathcal{F} \cong F \oplus T$, which is a contradiction because $\mathcal{F}$ is torsion-free. ☐

2.1.2. *Semistable sheaves and their moduli spaces.* We refer to the book [32] for a detailed account of all the notions introduced here. We recall that a torsion-free coherent sheaf F on X is *Gieseker-semistable* with respect to H_X (shortly, *semistable*) if for any coherent subsheaf E, with $\mathrm{rk}(E) < \mathrm{rk}(F)$, one has $p(E, t)/\mathrm{rk}(E) \leq p(F, t)/\mathrm{rk}(F)$ for $t \gg 0$. The sheaf F is called *stable* if the inequality above is strict for all E and $t \gg 0$. It is straightforward to see that a sheaf F is semistable (respectively stable) if and only if for any torsion-free quotient Q, with $\mathrm{rk}(Q) < \mathrm{rk}(F)$, one has $p(Q, t)/\mathrm{rk}(Q) \geq p(F, t)/\mathrm{rk}(F)$ (respectively $p(Q, t)/\mathrm{rk}(Q) > p(F, t)/\mathrm{rk}(F)$) for $t \gg 0$.

The *slope* of a sheaf F of positive rank is defined as $\mu(F) = \deg(c_1(F) \cdot H_X^{n-1})/\mathrm{rk}(F)$. We recall that a torsion-free coherent sheaf F is *μ-semistable* if for any coherent subsheaf E, with $\mathrm{rk}(E) < \mathrm{rk}(F)$, one has $\mu(E) \leq \mu(F)$. The sheaf F is called *μ-stable* if the above inequality is strict for all E. We recall that the *discriminant* of a sheaf F is:

$$\Delta(F) = 2rc_2(F) - (r-1)c_1(F)^2. \tag{2.1}$$

Bogomolov's inequality, see e.g. [32, Theorem 3.4.1], states that if F is μ-semistable, then we have:

$$\Delta(F) \cdot H_X^{n-2} \geq 0. \tag{2.2}$$

Recall that by Maruyama's theorem, see [51], if $\dim(X) = n \geq 2$ and F is a μ-semistable sheaf of rank $r < n$, then its restriction to a general divisor in $|\mathscr{O}_X(H_X)|$ is still μ-semistable.

We introduce here some notation concerning moduli spaces. Once fixed the polarization H_X, we denote by $\mathsf{M}_X(r, c_1, \ldots, c_n)$ the moduli space of S-equivalence classes of rank-r sheaves which are H_X-semistable and have Chern classes $c_1, \ldots, c_n$. We will drop the last values of the classes c_k when they are zero. We denote by $\mathsf{M}^{\mathsf{s}}{}_X(r, c_1, \ldots, c_n)$ the subset of stable sheaves of $\mathsf{M}_X(r, c_1, \ldots, c_n)$. The point of $\mathsf{M}^{\mathsf{s}}{}_X(r, c_1, \ldots, c_n)$ represented by a sheaf F will be denoted again by F, with a slight abuse of notation. We denote by $\mathscr{H}^g_d(X)$ the union of components of the *Hilbert scheme* of closed subschemes Z of X with Hilbert polynomial $p(\mathscr{O}_Z, t) = dt + 1 - g$, containing integral curves of degree d and arithmetic genus g.

We use the following terminology: any claim referring to a *general* element in a given parameter space P, will mean that the claim holds for all elements of P, except possibly for those who lie in a countable union of Zariski closed subsets of P.

2.1.3. *Homological algebra.* As a basic tool, we will use the bounded derived category of coherent sheaves. Namely, given X as above, we will consider the derived category $\mathbf{D^b}(X)$ of complexes of sheaves on X with bounded coherent cohomology. For definitions and notation we refer to [20] and [31]. In particular we write $[j]$ for the j-th shift to the right in the derived category.

Recall that a full triangulated subcategory $\mathcal{A}$ of $\mathbf{D^b}(X)$ is called left or right admissible if the inclusion $i_{\mathcal{A}} : \mathcal{A} \hookrightarrow \mathbf{D^b}(X)$ has a left or right adjoint, which will be denoted as usual by $i^*_{\mathcal{A}}$ and

$i^!_{\mathcal{A}}$ ($\mathcal{A}$ is called admissible if it is so in both ways). Assuming $\mathcal{A}$ admissible, we have $\mathbf{D^b}(X) = \langle \mathcal{A}, {}^\perp\mathcal{A}\rangle = \langle \mathcal{A}^\perp, \mathcal{A}\rangle$, $\mathcal{A}^\perp$ is left admissible and ${}^\perp\mathcal{A}$ is right admissible. In this situation, the left and right mutations through $\mathcal{A}$ are defined respectively as $i_{\mathcal{A}^\perp} i^*_{\mathcal{A}^\perp}$ and $i_{{}^\perp\mathcal{A}} i^!_{{}^\perp\mathcal{A}}$. An object A of $\mathbf{D^b}(X)$ is *exceptional* if $\mathrm{Hom}_{\mathbf{D^b}(X)}(A, A[i])$ is $\mathbb{C}$ if $i = 0$ and 0 if $i \neq 0$.

If $\mathcal{A}$ is generated by an exceptional object A, and B is an object of $\mathbf{D^b}(X)$, the left and right mutations of B through A are defined, respectively, by the triangles:

$$\mathrm{L}_A B[-1] \to \mathrm{Hom}(A, B) \otimes A \to B \to \mathrm{L}_A B,$$
$$\mathrm{R}_A B \to A \to \mathrm{Hom}(A, B)^* \otimes B \to \mathrm{R}_A B[1].$$

We refer to [12, 21] for more details.

Let Z be a local complete intersection subvariety of X. In view of the Fundamental Local Isomorphism (see [27, Proposition III.7.2]), we have the natural isomorphisms:

$$\mathscr{E}xt^k_X(\mathscr{O}_Z, \mathscr{O}_Z) \cong \mathscr{E}xt^{k-1}_X(\mathcal{I}_Z, \mathscr{O}_Z) \cong \bigwedge^k N_Z, \tag{2.3}$$

$$\mathscr{T}or^X_k(\mathscr{O}_Z, \mathscr{O}_Z) \cong \mathscr{T}or^X_{k-1}(\mathcal{I}_Z, \mathscr{O}_Z) \cong \bigwedge^k N^*_Z. \tag{2.4}$$

We will use some well-known spectral sequences, most frequently the following ones:

$$E_2^{p,q} = \mathrm{Ext}^p_X(\mathcal{H}^{-q}(a), A) \Longrightarrow \mathrm{Ext}^{p+q}_X(a, A), \tag{2.5}$$

$$E_2^{p,q} = \mathrm{H}^p(X, \mathscr{E}xt^q_X(A, B)) \Longrightarrow \mathrm{Ext}^{p+q}_X(A, B). \tag{2.6}$$

where a, b are complexes of sheaves on X, and A, B are sheaves on X. Recall that the maps in the E_2 term of these spectral sequences are differentials of the form:

$$d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}.$$

2.1.4. *Brill-Noether loci for vector bundles on a smooth projective curve.* We recall here some basic results in Brill-Noether theory, for definitions and notations we refer for instance to [71]. Let Γ be a smooth connected complex projective curve of genus g. The Brill-Noether locus $W^s_{r,c} \subset \mathsf{M}^{\mathsf{s}}{}_{\Gamma}(r, c)$ is defined to be the subvariety consisting of rank r stable bundles of degree c on Γ having at least $s+1$ independent global sections. The expected dimension of this variety is:

$$\rho(r, c, s) = r^2(g-1) - (s+1)\,(s+1-c+r(g-1)) + 1.$$

Consider a stable rank r vector bundle $\mathcal{F}$ on Γ, with $\mathcal{F}$ in $\mathsf{M}^{\mathsf{s}}{}_{\Gamma}(r, c)$. We define the Gieseker-Petri map as the natural linear application:

$$\pi_{\mathcal{F}} : \mathrm{H}^0(\Gamma, \mathcal{F}) \otimes \mathrm{H}^0(\Gamma, \mathcal{F}^* \otimes \omega_\Gamma) \to \mathrm{H}^0(\Gamma, \mathcal{F} \otimes \mathcal{F}^* \otimes \omega_\Gamma).$$

The map $\pi_{\mathcal{F}}$ is injective if and only if $\mathcal{F}$ is a non-singular point of a component of $W^s_{r,d}$ of dimension $\rho(r, d, s)$. We will use more frequently in the sequel the transpose of the Petri map, that reads:

$$\pi^\top_{\mathcal{F}} : \mathrm{Ext}^1_\Gamma(\mathcal{F}, \mathcal{F}) \to \mathrm{H}^0(\Gamma, \mathcal{F})^* \otimes \mathrm{H}^1(\Gamma, \mathcal{F}).$$

In fact the tangent space to $W^s_{r,d}$ at the point $\mathcal{F}$ can be interpreted as the kernel of $\pi^\top_{\mathcal{F}}$, while the space containing obstructions at $\mathcal{F}$ is identified with the cokernel of $\pi^\top_{\mathcal{F}}$.

2.1.5. *Smooth prime Fano threefolds.* Assume now $\dim(X) = 3$. Recall that X is called *Fano* if its anticanonical divisor class $-K_X$ is ample. A Fano threefold X is said to be *prime* if its Picard group is generated by the class of K_X. These varieties are classified up to deformation, see for instance [40, Chapter IV]. The number of deformation classes is 10, and they are characterized by the *genus*, which is the integer g such that $\deg(X) = -K_X^3 = 2\,g - 2$. Recall that the genera of prime Fano threefolds take values in $\{2, 3, \ldots, 9, 10, 12\}$. If $-K_X$ is very ample, we say that X is *non-hyperelliptic*. In this case we have $g \geq 3$.

If X is a prime Fano threefold of genus g, the Hilbert scheme $\mathscr{H}^0_1(X)$ of lines contained in X is a scheme of dimension 1. It is known by [39] that the normal bundle of a line $L \subset X$ splits either as $\mathscr{O}_L \oplus \mathscr{O}_L(-1)$ or as $\mathscr{O}_L(1) \oplus \mathscr{O}_L(-2)$. The Hilbert scheme $\mathscr{H}^0_1(X)$ contains a component which is non-reduced at any point if and only if the normal bundle of a general line L in that component splits as $\mathscr{O}_L(1) \oplus \mathscr{O}_L(-2)$. In this case, the threefold X is said to be *exotic* (see [66]). On the other hand, we say that X is *ordinary* if it contains a line L with normal bundle $\mathscr{O}_L \oplus \mathscr{O}_L(-1)$, equivalently if $\mathscr{H}^0_1(X)$ has a generically smooth component. Recall that, if X is general enough, $\mathscr{H}^0_1(X)$ is in fact a smooth irreducible curve (see [40, Theorem 4.2.7], and the references therein).

Let us recall that a non-hyperelliptic prime Fano threefold X is exotic if and only if it contains infinitely many non-reduced conics (see [13]). For $g \geq 9$, the results of [23] and [66] imply that X is non-exotic unless $g = 12$ and X is the Mukai-Umemura threefold, see [62]. In fact, the only other known examples of exotic prime Fano threefolds besides Mukai-Umemura's case are those containing a cone. For instance if X is the Fermat quartic threefold in $\mathbb{P}^4$ ($g = 3$), then $\mathscr{H}_1^0(X)$ is a curve with 40 irreducible components, each of multiplicity 2 (see [72]). We do not know if there exist exotic prime Fano threefolds of genus 7. In view of a result of Iliev-Markushevich (restated in Proposition 4.1 further on), this amounts to ask whether there are non-tetragonal smooth curves Γ of genus 7 admitting infinitely many divisors $\mathcal{L}$ of type g_5^1 such that $K_\Gamma - 2\mathcal{L}$ is effective (see Remark 4.2).

Remark that the cohomology groups $\mathrm{H}^{k,k}(X)$ of a prime Fano threefold X are generated by the divisor class H_X (for $k = 1$), the class L_X of a line contained in X (for $k = 2$), the class P_X of a closed point of X (for $k = 3$). Hence we will denote the Chern classes of a sheaf on X by the integral multiple of the corresponding generator. Recall that, if X has genus g, we have $H_X^2 = (2g - 2)L_X$. Given a smooth curve $C \subset X$ of degree d and genus p_a, we have:

$$c_1(\mathscr{O}_C) = 0, \qquad c_2(\mathscr{O}_C) = -d, \qquad c_3(\mathscr{O}_C) = 2 - 2p_a - d. \tag{2.7}$$

Applying the theorem of Riemann-Roch to a sheaf F on X, of (generic) rank r and with Chern classes c_1, c_2, c_3, we obtain the following formulas:

$$\chi(F) = r + \frac{11+g}{6}\,c_1 + \frac{g-1}{2}\,c_1^2 - \frac{1}{2}c_2 + \frac{g-1}{3}\,c_1^3 - \frac{1}{2}\,c_1\,c_2 + \frac{1}{2}\,c_3, \tag{2.8}$$

$$\chi(F,F) = r^2 - \frac{1}{2}\Delta(F), \tag{2.9}$$

and, in case $r = 2$ and $g = 7$, formula (2.8) becomes:

$$\chi(F) = 2 + 3\,c_1 + 3\,c_1^2 - \frac{1}{2}c_2 + 2\,c_1^3 - \frac{1}{2}\,c_1\,c_2 + \frac{1}{2}\,c_3. \tag{2.10}$$

Recall that if $T \neq 0$ is a torsion sheaf supported in codimension $p > 0$, then $c_k(T) = 0$ for $k < p$, while $(-1)^{p-1}c_p(T)$ is the class of the scheme-theoretic support of T in $\mathrm{H}^{p,p}(X)$ (see e.g. [19]). Moreover since $\chi(T(t))$ is positive for $t \gg 0$, looking at the dominant term of $\chi(T(t))$, we see that $(-1)^{p-1}c_p(T) > 0$.

Recall also that a smooth projective surface S is a *K3 surface* if it has trivial canonical bundle and irregularity zero. A general hyperplane section of a non-hyperelliptic prime Fano threefold of genus g is a K3 surface S whose Picard group is generated by the restriction H_S of H_X to S, and whose (sectional) genus equals g. We consider stability with respect to H_S. Given a stable sheaf F of rank r on a K3 surface S with Chern classes c_1, c_2, the dimension at F of the moduli space $\mathsf{M}_S(r, c_1, c_2)$ is:

$$\Delta(F) - 2\,(r^2 - 1). \tag{2.11}$$

For this equality we refer for instance to [32, Part II, Chapter 6].

Remark 2.2. Assume that X is a prime Fano threefold, and let L be a line contained in X, with $N_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(-1)$. Then we have:

$$\operatorname{ext}^1_X(\mathscr{O}_L, \mathscr{O}_L) = 1, \quad \operatorname{ext}^2_X(\mathscr{O}_L, \mathscr{O}_L) = 0.$$

One can easily check this statement, using (2.3) and (2.6).

2.2. **Geometry of Fano threefolds of genus 7.** We recall here the construction of a Fano threefold of genus 7 as a linear section of the spinor 10-fold, outlined by Mukai in [56], [57]. See also [58], [59], [36].

Let V be a 10-dimensional $\mathbb{C}$-vector space, equipped with a non-degenerate quadratic form. The algebraic group $\mathsf{Spin}(V)$ corresponds to a Dynkin diagram of type D_5. It admits two 16-dimensional irreducible representations S^+ and S^-, called the half-spin representations, having maximal weight respectively $\lambda_+ = \lambda_4$ and $\lambda_- = \lambda_5$, where the 4th and 5th nodes of D_5 are connected only to the unique trivalent node. These representations are naturally dual to each other.

The corresponding roots $\alpha_+ = \alpha_4$ and $\alpha_- = \alpha_5$ give rise to the Hermitian symmetric spaces Σ^+ and Σ^-, defined by $\Sigma^\pm = \mathsf{Spin}(10)/\mathsf{P}(\alpha_\pm)$, where $\mathsf{P}(\alpha_+)$ and $\mathsf{P}(\alpha_-)$ are the parabolic subgroups associated respectively to α_+ and α_-. These can be seen as the connected components of the orthogonal Grassmann variety $\mathbb{G}_Q(\mathbb{P}^4, \mathbb{P}(V))$ of 4-dimensional isotropic linear subspaces $\mathbb{P}^4$ contained in the smooth quadric hypersurface Q in $\mathbb{P}^9 = \mathbb{P}(V)$ corresponding to the quadratic form on V. Note that

$\mathscr{O}_{\mathbb{G}_Q(\mathbb{P}^4,\mathbb{P}(V))}(1)|_{\Sigma_\pm} = \mathscr{O}_{\Sigma_\pm}(2H)$. We denote by $\mathcal{U}_\pm$ the restriction of the tautological subbundle on $\mathbb{G}_Q(\mathbb{P}^4,\mathbb{P}(V))$ to $\Sigma^\pm$. We have thus the universal exact sequence:

$$0 \to \mathcal{U}_\pm \to V \otimes \mathscr{O}_{\Sigma^\pm} \to \mathcal{U}_\pm^* \to 0. \tag{2.12}$$

Under the duality on V given by Q, the bundle $\mathcal{U}_\pm$ is isomorphic to $\mathcal{U}_\pm^\perp$. The hyperplane divisors $H_{\Sigma^\pm}$ provide natural equivariant embeddings of $\Sigma^\pm$ into $\mathbb{P}(\mathsf{S}^\pm)$. Given a subvariety $Y \subset \Sigma^\pm$, we denote by H_Y the restriction of $H_{\Sigma^\pm}$ to Y.

Now choose a 9-dimensional vector subspace A of S^+, and consider its (7-dimensional) orthogonal space $B = A^\perp \subset \mathsf{S}^-$ under the duality $(\mathsf{S}^+)^* \cong \mathsf{S}^-$. We define:

$$X = \Sigma^+ \cap \mathbb{P}(A) \subset \mathbb{P}(\mathsf{S}^+), \tag{2.13}$$

$$\Gamma = \Sigma^- \cap \mathbb{P}(B) \subset \mathbb{P}(\mathsf{S}^-). \tag{2.14}$$

If the subspace A is general enough, then X is smooth and it turns out that it is a prime Fano threefold of genus 7, in particular we have $K_X = -H_X$, $H_X^3 = 12$. Further, any such prime Fano threefold of genus 7 is obtained in this way. In turn, the curve Γ is a smooth canonical curve of genus 7, called the *homologically projective dual curve* of X. By [59, Table 1], we know that the curve Γ is not hyperelliptic, nor trigonal nor tetragonal and $W^2_{1,6}$ is empty. Moreover, a general curve of genus 7 is of this kind.

2.2.1. *Semiorthogonal decomposition of the derived category of X.* Here we briefly sketch the construction due to Kuznetsov [44], of the semiorthogonal decomposition of $\mathbf{D^b}(X)$. The first ingredient of this decomposition is the restriction to X of the vector bundle $\mathcal{U}_+$ (still denoted by $\mathcal{U}_+$): this is an exceptional bundle, and we have the exceptional pair:

$$(\mathscr{O}_X, \mathcal{U}_+^*).$$

Then, we consider the product variety $X \times \Gamma$, together with the two projections $p\colon X \times \Gamma \to X$, $q\colon X \times \Gamma \to \Gamma$. The symmetric form on V provides the following natural exact sequence on $X \times \Gamma \subset \Sigma^+ \times \Sigma^-$:

$$0 \to \mathscr{E}^* \to \mathcal{U}_- \to \mathcal{U}_+^* \xrightarrow{\alpha} \mathscr{E} \to 0 \tag{2.15}$$

(here $\mathcal{U}_\pm$ denotes also the pull-back of $\mathcal{U}_\pm$ to $X \times \Gamma$). It turns out that $\mathscr{E}$ is a locally free sheaf on $X \times \Gamma$ with the following invariants:

$$\begin{aligned} c_1(\mathscr{E}) &= H_X + H_\Gamma, \\ c_2(\mathscr{E}) &= \frac{7}{12}\, H_X\, H_\Gamma + 5\,L + \eta, \end{aligned} \tag{2.16}$$

where η sits in $\mathrm{H}^3(X,\mathbb{C}) \otimes \mathrm{H}^1(\Gamma,\mathbb{C})$ and satisfies $\eta^2 = 14$. In view of the results of [61], [60], [44] and [36], the vector bundle $\mathscr{E}$ is a universal object for moduli functors on X and Γ in the sense specified as follows.

Theorem 2.3 (Mukai, Iliev-Markushevich, Kuznetsov)**.** *Let $A \subset \mathsf{S}^+$ be chosen so that X defined by* (2.13) *is a smooth threefold, and define Γ and $\mathscr{E}$ as in* (2.14), (2.15). *Then:*

i) the curve Γ is isomorphic to $\mathsf{M}_X(2,1,5)$,
ii) the manifold X is isomorphic to the Brill-Noether locus of stable bundles $\mathcal{E}$ on Γ with $\mathrm{rk}(\mathcal{E}) = 2$, $\det(\mathcal{E}) \cong H_\Gamma$, $\mathrm{h}^0(\Gamma,\mathcal{E}) = 5$,
iii) the bundle $\mathscr{E}$ universally represents both moduli problems (i) *and* (ii)*,*
iv) for all $y \in \Gamma$, the sheaf $\mathscr{E}_y$ is a globally generated ACM vector bundle.

Given points $x \in X$, $y \in \Gamma$, and given a vector bundle $\mathscr{F}$ on $X \times \Gamma$, we denote by $\mathscr{F}_y$ (resp. $\mathscr{F}_x$) the bundle over X (resp. over Γ) obtained restricting $\mathscr{F}$ to $X \times \{y\}$ (resp. to $\{x\} \times \Gamma$). We still denote by $\mathcal{U}_+$ (resp. $\mathcal{U}_-$) the restriction of $\mathcal{U}_\pm$ to X (resp. to Γ). The vector bundles $\mathcal{U}_+$ and $\mathcal{U}_-$ have rank 5. We have $c_1(\mathcal{U}_-) = -2H_\Gamma$ and:

$$c_1(\mathcal{U}_+) = -2, \qquad c_2(\mathcal{U}_+) = 24, \qquad c_3(\mathcal{U}_+) = -14.$$

We define the following exact functors:

$$
\begin{aligned}
&\mathbf{\Phi} : \mathbf{D^b}(\Gamma) \to \mathbf{D^b}(X), && \mathbf{\Phi}(-) = \mathbf{R}p_*(q^*(-) \otimes \mathscr{E}),\\
&\mathbf{\Phi}^! : \mathbf{D^b}(X) \to \mathbf{D^b}(\Gamma), && \mathbf{\Phi}^!(-) = \mathbf{R}q_*(p^*(-) \otimes \mathscr{E}^*(H_\Gamma))[1],\\
&\mathbf{\Phi}^* : \mathbf{D^b}(X) \to \mathbf{D^b}(\Gamma), && \mathbf{\Phi}^*(-) = \mathbf{R}q_*(p^*(-) \otimes \mathscr{E}^*(-H_X))[3].
\end{aligned} \tag{2.17}
$$

We recall that $\mathbf{\Phi}$ is fully faithful, $\mathbf{\Phi}^*$ is left adjoint to $\mathbf{\Phi}$, and $\mathbf{\Phi}^!$ is right adjoint to $\mathbf{\Phi}$. The main result of [44] provides the following semiorthogonal decomposition:

$$\mathbf{D^b}(X) \cong \langle \mathscr{O}_X, \mathcal{U}_+^*, \mathbf{\Phi}(\mathbf{D^b}(\Gamma))\rangle. \tag{2.18}$$

This decomposition will be used to write a canonical resolution of a given sheaf over X. In view of [21], given a sheaf F over X, the decomposition (2.18) provides a functorial exact triangle:

$$\mathbf{\Phi}(\mathbf{\Phi}^!(F)) \to F \to \mathbf{\Psi}(\mathbf{\Psi}^*(F)), \tag{2.19}$$

where $\mathbf{\Psi}$ is the inclusion of the subcategory $\langle \mathscr{O}_X, \mathcal{U}_+^*\rangle$ in $\mathbf{D^b}(X)$ and $\mathbf{\Psi}^*$ is the left adjoint functor to $\mathbf{\Psi}$. The k-th term of the complex $\mathbf{\Psi}(\mathbf{\Psi}^*(F))$ can be written as follows:

$$(\mathbf{\Psi}(\mathbf{\Psi}^*(F)))^k \cong \operatorname{Ext}_X^{-k}(F, \mathscr{O}_X)^* \otimes \mathscr{O}_X \oplus \operatorname{Ext}_X^{1-k}(F, \mathcal{U}_+)^* \otimes \mathcal{U}_+^*. \tag{2.20}$$

Remark 2.4. Given a sheaf F on X, one can describe more explicitly the map $F \to \mathbf{\Psi}(\mathbf{\Psi}^*(F))$. We do this here, in order to show that the complex $\mathbf{\Psi}(\mathbf{\Psi}^*(F))$ is *minimal*, i.e. the only non-zero maps in the complex are from copies of $\mathscr{O}_X$ to copies of $\mathcal{U}_+^*$. In other words, for any k, the differential d^k from the $(k-1)$-st term to the k-th term is strictly triangular.

We consider the product $X \times X$ and the projections q_1 and q_2 onto the two factors. Let $\Delta : X \to X \times X$ be the diagonal embedding. We denote by $\mathbf{K}$ the following complex on $X \times X$:

$$\mathcal{U}_+ \boxtimes \mathcal{U}_+ \to \mathscr{O}_X \boxtimes \mathscr{O}_X \to \Delta_*(\mathscr{O}_X)$$

obtained restricting the standard resolution of the diagonal on the Grassmannian, see [41]. Now denote by $\mathbf{U}$ the complex $\mathscr{O}_X \boxtimes \mathscr{O}_X[3] \to \mathcal{U}_+^* \boxtimes \mathcal{U}_+^*[3]$. Since $(\Delta_*(\mathscr{O}_X))^* \cong \Delta_*(\mathscr{O}_X(1))[-3]$, dualizing $\mathbf{K}$ and shifting by 3, we get that $\mathbf{K}^*[3]$ is quasi-isomorphic to $\Delta_*(\mathscr{O}_X(1)) \to \mathbf{U}$. This quasi-isomorphism can be rewritten as a distinguished triangle

$$\mathbf{K}^*(-1,0)[3] \to \Delta_*(\mathscr{O}_X) \to \mathbf{U}(-1,0).$$

Then, given a sheaf F on X, the complex $\mathbf{\Psi}(\mathbf{\Psi}^*(F))$ is given by $\mathbf{R}q_{2*}(q_1^*(F(-1)) \otimes \mathbf{U})$, and the map $F \to \mathbf{\Psi}(\mathbf{\Psi}^*(F))$ is induced by $\Delta_*(\mathscr{O}_X) \to \mathbf{U}(-1,0)$. Finally, we recall the vanishing $\operatorname{Ext}_X^k(\mathcal{U}_+^*, \mathscr{O}_X) = \operatorname{Ext}_X^k(\mathscr{O}_X, \mathcal{U}_+^*) = 0$ for any $k > 0$.

Having this in mind, one can easily prove the minimality statement, indeed [41, Lemma 1.6] applies, and we can use [1, Lemma 3.2] to deduce that the differentials between the graded pieces of $\mathbf{R}q_{2*}(q_1^*(F(-1)) \otimes \mathcal{U}_+^* \boxtimes \mathcal{U}_+^*)$ are zero, as well as the differentials between the graded pieces of $\mathbf{R}q_{2*}(q_1^*(F(-1)))$.

Remark 2.5. Given an object F of $\mathbf{D^b}(X)$, we have an exact triangle:

$$F \to \mathbf{\Psi}(\mathbf{\Psi}^*(F)) \to \mathbf{\Phi}(\mathbf{\Phi}^!(F))[1],$$

so we may think of $\mathbf{\Phi}(\mathbf{\Phi}^!(-))[1]$ as the *right mutation functor* $R_{\langle \mathscr{O}_X, \mathcal{U}_+^*\rangle}$ with respect to the subcategory $\langle \mathscr{O}_X, \mathcal{U}_+^*\rangle$ of $\mathbf{D^b}(X)$, see [21].

2.2.2. *Some lemmas on universal bundles.* We close this section with some lemmas regarding the image via the integral functors defined above of some natural sheaves on X and Γ. These results will be needed further on. From the sequence (2.15) we obtain:

$$0 \to \mathscr{E}^* \to \mathcal{U}_- \to \mathscr{G} \to 0, \tag{2.21}$$

$$0 \to \mathscr{G} \to \mathcal{U}_+^* \to \mathscr{E} \to 0, \tag{2.22}$$

where $\mathscr{G}$ is a rank 3 vector bundle with $c_1(\mathscr{G}) = H_X - H_\Gamma$.

Lemma 2.6. *The vector bundles $\mathcal{U}_+$ and $\mathscr{G}_y$, for any $y \in \Gamma$, are stable and ACM. Moreover, we have* $\mathrm{H}^0(\Gamma, \mathscr{G}_x) = 0$ *for any $x \in X$.*

Proof. Let us prove first that $\mathrm{H}^0(\Gamma, \mathscr{G}_x) = 0$ for any $x \in X$. Notice that $\mathscr{G}_x \cong \wedge^2 \mathscr{G}_x^*(-1)$, because $\mathscr{G}_x$ has rank 3 and $c_1(\mathscr{G}_x) \cong -H_\Gamma$. Let us dualize (2.21) and restrict it to $\{x\} \times \Gamma$. We obtain an inclusion:

$$\wedge^2 \mathscr{G}_x^*(-1) \hookrightarrow \wedge^2 \mathcal{U}_-^*(-1).$$

Then we have $\mathrm{H}^0(\Gamma, \mathscr{G}_x) \subset \mathrm{H}^0(\Gamma, \wedge^2 \mathcal{U}_-^*(-1))$. So it suffices to show that the latter space is 0. To prove this, one can tensor by $\wedge^2 \mathcal{U}_-^*(-1)$ the Koszul complex:

$$0 \to \wedge^9 A \otimes \mathscr{O}_{\Sigma_-}(-9) \to \cdots \to A \otimes \mathscr{O}_{\Sigma_-}(-1) \to \mathscr{O}_{\Sigma_-} \to \mathscr{O}_\Gamma \to 0,$$

and the conclusion follows applying Borel-Bott-Weil theorem (see e.g. [74]) on Σ_- to the homogeneous vector bundles $\wedge^2 \mathcal{U}_-^*(-t)$, for $t = 1, \ldots, 10$.

Let us now turn to $\mathcal{U}_+$. Consider the Koszul complex:

$$0 \to \wedge^7 B \otimes \mathscr{O}_{\Sigma_+}(-7) \to \cdots \to B \otimes \mathscr{O}_{\Sigma_+}(-1) \to \mathscr{O}_{\Sigma_+} \to \mathscr{O}_X \to 0,$$

and tensor it with $\mathcal{U}_+$. Applying Borel-Bott-Weil theorem on Σ_+ we obtain that, for any t, the homogeneous vector bundles $\mathcal{U}_+(t)$ on Σ_+ have natural cohomology and more precisely we get:

$$\mathrm{H}^k(\Sigma_+, \mathcal{U}_+(-t)) = 0, \qquad \text{for} \qquad \begin{cases} \text{all } k \text{ and } t = 0, \ldots, 7, \\ k \neq 0 \text{ and } t < 0, \\ k \neq 10 \text{ and } t > 7. \end{cases}$$

Then it easily follows that $\mathcal{U}_+$ is an ACM bundle on X and

$$\mathrm{H}^k(X, \mathcal{U}_+) = 0, \quad \text{for all } k.$$

Applying the same argument to $\wedge^2 \mathcal{U}_+$, we obtain the following:

$$\mathrm{H}^k(X, \wedge^2 \mathcal{U}_+) = 0, \quad \text{for } k \neq 1, \text{ and} \quad \mathrm{h}^1(X, \wedge^2 \mathcal{U}_+) = 1.$$

In particular, Serre duality implies:

$$\mathrm{H}^0(X, \wedge^4 \mathcal{U}_+(1)) = 0, \qquad \mathrm{H}^0(X, \wedge^3 \mathcal{U}_+(1)) = 0. \tag{2.23}$$

By Hoppe's criterion, see [2, Theorem 1.2] and [30, Lemma 2.6], this proves stability of $\mathcal{U}_+$.

Recall that the dual of an ACM vector bundle is also ACM. Therefore, the dual bundles of $\mathcal{U}_+$ and $\mathscr{E}_y$ are ACM by (iv) of Theorem 2.3. This easily implies, by (2.21) and (2.22), that the bundle $\mathscr{G}_y$ is ACM. To prove that $\mathscr{G}_y$ is stable, by Hoppe's criterion it is enough to show that the groups $\mathrm{H}^0(X, \mathscr{G}_y^*)$, $\mathrm{H}^0(X, \mathscr{G}_y(-1))$ both vanish. We consider the restriction to $X \times \{y\}$ of (2.22). Since $\mathcal{U}_+^*(-1) \cong \wedge^4 \mathcal{U}_+(1)$, we obtain the latter vanishing by (2.23). Dualizing the same exact sequence and using $\mathrm{H}^1(X, \mathscr{E}_y^*) = 0$ (recall that $\mathscr{E}_y^*$ is ACM) we get the former. □

Lemma 2.7. *Given an object $\mathcal{F}$ in $\mathbf{D^b}(\Gamma)$ and an object F in $\mathbf{D^b}(X)$ we have the following functorial isomorphisms:*

$$\mathbf{R}\mathcal{H}om_X(\mathbf{\Phi}(\mathcal{F}), \mathscr{O}_X) \cong \mathbf{\Phi}(\mathbf{R}\mathcal{H}om_\Gamma(\mathcal{F}, \mathscr{O}_\Gamma)) \otimes \mathscr{O}_X(-1)[1], \tag{2.24}$$

$$\mathbf{R}\mathcal{H}om_\Gamma(\mathbf{\Phi}^!(F), \mathscr{O}_\Gamma) \cong \mathbf{\Phi}^!(\mathbf{R}\mathcal{H}om_X(F, \mathscr{O}_X)) \otimes \omega_\Gamma^*[1]. \tag{2.25}$$

Proof. By Grothendieck duality, (see [27, Chapter III], or [15]), given a complex $\mathscr{K}$ on $X \times \Gamma$, we have:

$$\mathbf{R}\mathcal{H}om_X(\mathbf{R}p_*(\mathscr{K}), \mathscr{O}_X) \cong \mathbf{R}p_*(\omega_\Gamma \otimes \mathbf{R}\mathcal{H}om_{X \times \Gamma}(\mathscr{K}, \mathscr{O}_{X \times \Gamma}))[1], \tag{2.26}$$

$$\mathbf{R}\mathcal{H}om_\Gamma(\mathbf{R}q_*(\mathscr{K}), \mathscr{O}_\Gamma) \cong \mathbf{R}q_*(\omega_X \otimes \mathbf{R}\mathcal{H}om_{X \times \Gamma}(\mathscr{K}, \mathscr{O}_{X \times \Gamma}))[3], \tag{2.27}$$

and the isomorphisms are functorial. Recall that $\omega_X \cong \mathscr{O}_X(-1)$ and $\omega_\Gamma \cong \mathscr{O}_\Gamma(H_\Gamma)$. So by (2.16) we have $\mathscr{E}^* \otimes \omega_\Gamma \cong \mathscr{E} \otimes \mathscr{O}_X(-1)$. Then, setting $\mathscr{K} = q^*(\mathcal{F}) \otimes \mathscr{E}$ in (2.26), we get (2.24). Setting $\mathscr{K} = p^*(F) \otimes \mathscr{E}^* \otimes \omega_\Gamma[1]$ in (2.27), we obtain (2.25). □

Remark 2.8. Given an object F in $\mathbf{D^b}(X)$, whose class in the Grothendieck group has rank r and Chern classes c_1, c_2, c_3, let us assume that $\mathbf{\Phi}^!(F)$ is concentrated in degree k, i.e. $\mathcal{H}^j(\mathbf{\Phi}^!(F)) = 0$ for $j \neq k$, so that $\mathbf{\Phi}^!(F)[k]$ is the coherent sheaf $\mathbf{R}^k p_*(q^*(F) \otimes \mathscr{E})$. The sheaf F is sometimes called a WIT sheaf in this case, and by base change, since $\mathscr{E}$ is locally free one sees that $\mathbf{R}^k p_*(q^*(F) \otimes \mathscr{E})$ is locally free (see for instance [54]). Similarly, WIT sheaves over Γ give rise to locally free sheaves on X.

Note also that the rank of $\mathbf{\Phi}^!(F)[k]$ is the dimension of the vector space $\mathcal{H}^k(\mathbf{\Phi}^!(F))_y$ for general $y \in \Gamma$, and for WIT sheaves it coincides with $(-1)^{k+1}\chi(\mathscr{E}_y, F)$, by definition of $\mathbf{\Phi}^!$. Moreover, in this case applying Riemann-Roch formula we get:

$$\operatorname{rk}(\mathbf{\Phi}^!(F)[k]) = (-1)^k \left(-c_1 + c_1c_2 - 4c_1^3 - c_3\right). \tag{2.28}$$

On the other hand, Grothendieck-Riemann-Roch formula gives:

$$\deg(\mathbf{\Phi}^!(F)[k]) = (-1)^k \left(-6c_1 + 6c_1^2 - c_2 - 24c_1^2 + 6c_1c_2 - 6c_3\right). \tag{2.29}$$

Lemma 2.9. *The following relations hold on Γ, for each point $y \in \Gamma$:*

$$\mathbf{\Phi}^*(\mathscr{O}_X) \cong \mathcal{U}_-, \qquad \mathbf{\Phi}^*(\mathcal{U}_+^*) \cong \mathscr{O}_\Gamma, \qquad \mathbf{\Phi}^*(\mathscr{E}_y) \cong \mathscr{O}_y, \tag{2.30}$$

$$\mathbf{\Phi}^!(\mathscr{O}_X) = 0, \qquad \mathbf{\Phi}^!(\mathcal{U}_+^*) = 0, \qquad \mathbf{\Phi}^!(\mathscr{E}_y) \cong \mathscr{O}_y, \tag{2.31}$$

and on X:

$$\mathcal{H}^0(\mathbf{\Phi}(\mathscr{O}_\Gamma)) \cong \mathcal{U}_+^*, \qquad \mathcal{H}^1(\mathbf{\Phi}(\mathscr{O}_\Gamma)) \cong \mathcal{U}_+(1), \tag{2.32}$$

$$\mathcal{H}^k(\mathbf{\Phi}(\mathscr{O}_\Gamma)) = 0, \qquad \text{for } k \neq 0, 1,$$

$$\mathbf{\Phi}(\mathscr{O}_y) \cong \mathscr{E}_y. \tag{2.33}$$

Proof. The isomorphism (2.33) follows immediately from the definition of $\mathbf{\Phi}$. Since the functor $\mathbf{\Phi}$ is fully faithful we easily obtain also the relations $\mathbf{\Phi}^*(\mathscr{E}_y) \cong \mathbf{\Phi}^!(\mathscr{E}_y) \cong \mathscr{O}_y$. It is clear that $\mathbf{\Phi}^!(\mathscr{O}_X) = \mathbf{\Phi}^!(\mathcal{U}_+^*) = 0$.

The isomorphism $\mathbf{\Phi}^*(\mathcal{U}_+^*) \cong \mathscr{O}_\Gamma$ is proved in [44, Lemma 5.6]. Twisting (2.21) by $\mathscr{O}_{X\times\Gamma}(-H_X)$ and taking $\mathbf{R}q_*$, we get $\mathbf{\Phi}^*(\mathscr{O}_X) \cong \mathcal{U}_-$. Indeed, we have $\mathrm{H}^k(X, \mathscr{G}_y(-H_X)) = 0$ for any integer k, since the vanishing for $k = 1, 2$ follows from the fact that $\mathscr{G}_y$ is ACM (by Lemma 2.6), and the vanishing for $k = 0, 3$ follows from the fact that $\mathscr{G}_y$ is stable (again by Lemma 2.6).

It remains to compute $\mathbf{\Phi}(\mathscr{O}_\Gamma)$. Let $S : F \mapsto F \otimes \mathscr{O}_X(-1)[3]$ be the Serre functor of $\mathbf{D^b}(X)$. Now we replace the semiorthogonal decomposition (2.18) by:

$$\langle \mathcal{U}_+^*, \mathbf{\Phi}(\mathbf{D^b}(\Gamma)), \mathscr{O}_X(1) \rangle, \tag{2.34}$$

and we note that right-mutating $\mathcal{U}_+^*$ through $\mathbf{\Phi}(\mathbf{D^b}(\Gamma))$ is equivalent to left-mutating $S^{-1}(\mathcal{U}_+^*)$ through $\mathscr{O}_X(1)$. Now we clearly have $S^{-1}(\mathcal{U}_+^*) \cong \mathcal{U}_+^*(1)[-3]$ and its left mutation through $\mathscr{O}_X(1)$ is $\mathcal{U}_+(1)[-2]$ by (2.12). Hence the mutation triangle:

$$R_{\mathbf{\Phi}(\mathbf{D^b}(\Gamma))}\mathcal{U}_+^* \to \mathcal{U}_+^* \to \mathbf{\Phi}(\mathbf{\Phi}^*(\mathcal{U}_+^*))$$

becomes, using also (2.30), the exact triangle:

$$\mathcal{U}_+(1)[-2] \to \mathcal{U}_+^* \to \mathbf{\Phi}(\mathscr{O}_\Gamma)$$

and taking cohomology we prove (2.32). □

The following corollary of Lemma (2.7) has been pointed out by the referee. We set τ for the functor $\mathcal{F} \mapsto \mathbf{R}\mathcal{H}om_\Gamma(\mathcal{F}, \omega_\Gamma)$ defined on $\mathbf{D^b}(\Gamma)$.

Corollary 2.10. *Set T for the functor $F \mapsto \mathbf{\Phi}(\mathbf{\Phi}^!(\mathbf{R}\mathcal{H}om_X(F, \mathscr{O}_X)))[1]$. Then T is an autoequivalence of ${}^\perp\langle \mathscr{O}_X, \mathcal{U}_+^* \rangle$. Moreover, we have $\mathbf{\Phi}^! \circ T = \tau \circ \mathbf{\Phi}^!$.*

Proof. By Remark 2.5 one has $\mathbf{\Phi}(\mathbf{\Phi}^!(F^*))[1] = R_{\langle \mathscr{O}_X, \mathcal{U}_+^* \rangle}(F^*)$. It is easy to see that the right mutation functor $R_{\langle \mathscr{O}_X, \mathcal{U}_+^* \rangle}$ is an equivalence of $\langle \mathscr{O}_X, \mathcal{U}_+^* \rangle^\perp$ onto ${}^\perp\langle \mathscr{O}_X, \mathcal{U}_+^* \rangle$. Further, by sending F to $\mathbf{R}\mathcal{H}om_X(F, \mathscr{O}_X)$, we clearly get an equivalence of ${}^\perp\langle \mathscr{O}_X, \mathcal{U}_+^* \rangle$ onto $\langle \mathcal{U}_+, \mathscr{O}_X \rangle^\perp = \langle \mathscr{O}_X, \mathcal{U}_+^* \rangle^\perp$. It follows that T is an autoequivalence of ${}^\perp\langle \mathscr{O}_X, \mathcal{U}_+^* \rangle$. To prove $\mathbf{\Phi}^! \circ T = \tau \circ \mathbf{\Phi}^!$, it suffices to use (2.25). □

Denoting by $L_{\langle \mathcal{U}_+, \mathscr{O}_X \rangle}$ the left mutation with respect to $\langle \mathcal{U}_+, \mathscr{O}_X \rangle$ of $\mathbf{D^b}(X)$, we have:

$$R_{\langle \mathscr{O}_X, \mathcal{U}_+^* \rangle}(\mathbf{R}\mathcal{H}om_X(F, \mathscr{O}_X)) \cong \mathbf{R}\mathcal{H}om_X(L_{\langle \mathcal{U}_+, \mathscr{O}_X \rangle}(F), \mathscr{O}_X). \tag{2.35}$$

Remark 2.11. Let S be a general hyperplane section of Y. Note that if F is an ACM bundle on Y, then the restriction F_S is ACM too. For instance, if Y is a prime Fano threefold of genus 7, for all $y \in \mathsf{M}_Y(2, 1, 5)$ the restriction to S of the bundle $\mathscr{E}_y$ is ACM, too.

3. Rank 2 stable sheaves on prime Fano threefolds

Throughout this section, Y will denote a smooth non-hyperelliptic complex prime Fano threefold. In this section we present some results concerning rank 2 stable sheaves F with $c_1(F) = 1$ on Y. We will first analyze the cases of minimal c_2 (see the next subsection) and then look for bundles with higher c_2.

3.1. **Rank 2 stable sheaves with $c_1 = 1$ and minimal c_2.** We provide a lower bound on $c_2(F)$ for the existence of F, namely $\mathsf{M}_Y(2,1,c_2)$ is non-empty if and only if $c_2(F) \geq m_g = \lceil \frac{g+2}{2} \rceil$. Then we give some properties of F in the cases $c_2 = m_g$ and $c_2 = m_g + 1$, see Proposition 3.4. This description is deeply inspired on the analysis of the case $g = 8$ pursued by Iliev and Manivel in [33]. Finally, we restate a result concerning non-emptyness and generic smoothness of this space (Theorem 3.5).

Lemma 3.1. *Let Y be a smooth non-hyperelliptic Fano threefold of genus g, and let F be a rank 2 stable sheaf on Y with $c_1(F) = H_Y$. Then we have:*

$$c_2(F) \geq \frac{g+2}{2}. \tag{3.1}$$

Proof. Let $S \subset Y$ be a general hyperplane section surface. Since Y is non-hyperelliptic, by Moishezon's theorem [53], we have $\mathrm{Pic}(S) \cong \mathbb{Z} = \langle H_S \rangle$. Consider the restriction $F_S = F \otimes \mathscr{O}_S$ and notice that the sheaf F_S is still torsion-free, since S is general. Moreover it is semistable by Maruyama's theorem ([52]), hence stable since $c_1(F_S) = H_S$ and $\mathrm{Pic}(S) = \langle H_S \rangle$. Since S is a K3 surface, the dimension of the moduli space $\mathsf{M}_S(2,1,c_2(F_S))$ can be computed by (2.11) and (2.1) and it is $4\,c_2(F_S) - 2\,g - 4$. So this number has to be non-negative, and we obtain (3.1). □

In view of the previous lemma we define:

$$m_g = \left\lceil \frac{g+2}{2} \right\rceil. \tag{3.2}$$

Lemma 3.2. *Let C be a curve in $\mathscr{H}^0_d(Y)$, with $d < m_g$. Then C is Cohen-Macaulay and we have $\mathrm{H}^k(Y, \mathcal{I}_C) = 0$ for all k.*

Proof. First observe that the curve C has no isolated or embedded points. Indeed, the purely 1-dimensional piece $\tilde{C}$ of C is a curve of degree d and arithmetic genus ℓ, where ℓ is the length of the zero-dimensional piece of C. In order to see that, for $\ell > 0$, this leads to a contradiction, one notes that since $\mathrm{H}^0(Y, \mathcal{I}_{\tilde{C}}) = 0$, we have $\mathrm{h}^2(Y, \mathcal{I}_{\tilde{C}}) \geq \chi(\mathcal{I}_{\tilde{C}}) = \ell$. Thus we would have a non-zero element of $\mathrm{Ext}^1_Y(\mathcal{I}_{\tilde{C}}(1), \mathscr{O}_Y)$, corresponding to a rank 2 sheaf F with $c_1(F) = 1$, $c_2(F) = d$. It is easy to see that the sheaf F would be stable. Indeed, assuming that there exists a destabilizing torsion-free subsheaf K, then it is easy to check that $\mathrm{rk}(K) = 1$ and $c_1(K) = 1$ and we have the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & \mathscr{O}_Y \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d] & \mathcal{I}_{\tilde{C}}(1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}_Y \arrow[r] & S \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where T has rank 0 and $c_1(T) = 0$. This implies that T is supported at a subvariety $Z \subset Y$ of dimension less than or equal to 1. It follows that $\mathrm{Ext}^1_Y(T, \mathscr{O}_Y) \cong \mathrm{H}^2(Z, T(-1))^* = 0$. Note that, by the above diagram, the element in $\mathrm{Ext}^1_Y(\mathcal{I}_{\tilde{C}}(1), \mathscr{O}_Y)$ corresponding to F (i.e. to the middle row) is the image of the element in $\mathrm{Ext}^1_Y(T, \mathscr{O}_Y)$ corresponding to S (the bottom row). But we have seen $\mathrm{Ext}^1_Y(T, \mathscr{O}_Y) = 0$, so the middle row of the above diagram splits, a contradiction.

Hence the sheaf F is stable, thus contradicting Lemma 3.1. The above argument implies that the group $\mathrm{H}^2(Y, \mathcal{I}_C)$ vanishes. Note that this implies the statement by Riemann-Roch. □

Lemma 3.3. *Let S be a K3 surface of Picard number 1 and sectional genus g, and let $m = m_g$ be defined by (3.2). Then for any $k \geq 1$ and for any $\ell \leq m + k - 2$, S contains no zero-dimensional subscheme Z of length ℓ with $\mathrm{h}^1(S, \mathcal{I}_Z(1)) = k$. Moreover if $g \geq 4$, then for any $k \geq 2$ and any $\ell \leq m + k - 1$, S contains no zero-dimensional subscheme Z of length ℓ with $\mathrm{h}^1(S, \mathcal{I}_Z(1)) = k$.*

Proof. We split the induction argument in two steps.

Step 1. *For any $\ell \leq m + k - 2$, there are no subschemes Z of S of length ℓ with $\mathrm{h}^1(S, \mathcal{I}_Z(1)) = k$, $\forall k \geq 1$.*

We prove our statement by induction on $k \geq 1$. Note that any scheme Z of length 1 has obviously $\mathrm{h}^1(S, \mathcal{I}_Z(1)) = 0$. Assume first $k = 1$, and consider a subscheme $Z \subset S$ of length $\ell \geq 2$ and with $\mathrm{h}^1(S, \mathcal{I}_Z(1)) = 1$. By Serre duality we have $\mathrm{Ext}^1_S(\mathcal{I}_Z(1), \mathscr{O}_S) \cong \mathrm{H}^1(S, \mathcal{I}_Z(1))^*$. Let F be the sheaf on S defined by the non-trivial extension:

$$0 \to \mathscr{O}_S \to F \to \mathcal{I}_Z(1) \to 0.$$

Notice that $c_1(F) = 1$, $c_2(F) = \ell$. We want to prove now that F is stable, for then the dimension (2.11) must be non-negative, and this implies that $\ell \geq m$. To do it, assume that Q is a destabilizing quotient of F, hence Q is torsion-free of rank 1, and $c_1(Q) \leq 0$. If the composition of $\mathscr{O}_S \hookrightarrow F$ and $F \to Q$ is non-zero then it is necessarily an isomorphism by Lemma 2.1, hence F would contain $\mathscr{O}_S$ as a direct summand. But this is impossible since the extension is non-trivial. This implies that Q is a quotient of $\mathcal{I}_Z(1)$. The kernel of the surjection $\mathcal{I}_Z(1) \to Q$ is torsion (both sheaves have rank 1) hence zero since $\mathcal{I}_Z(1)$ is torsion-free, so $\mathcal{I}_Z(1) \cong Q$. This is impossible since $c_1(Q) \leq 0$ and $c_1(\mathcal{I}_Z(1)) = 1$.

Now, assuming the claim for $k \geq 1$, we shall prove it for $k + 1$, namely we shall prove that a subscheme Z of length $\ell \leq m + k - 1$ with $\mathrm{h}^1(S, \mathcal{I}_Z(1)) = k + 1$ cannot exist. Indeed, given such Z, we choose a chain of subschemes $Z_1 \subset \cdots \subset Z_\ell = Z$ with Z_j of length j. This corresponds to a chain of twisted ideals $\mathcal{I}_{Z_\ell}(1) \subset \cdots \subset \mathcal{I}_{Z_1}$, with associated exact sequences of the form

$$0 \to \mathcal{I}_{Z_{j+1}}(1) \to \mathcal{I}_{Z_j}(1) \to \mathscr{O}_x \to 0,$$

for a point $x \in S$. Note that $\mathrm{h}^1(S, \mathcal{I}_{Z_j}(1))$ equals $\mathrm{h}^1(S, \mathcal{I}_{Z_{j+1}}(1)) + \varepsilon_j$, with $\varepsilon_j \in \{0, 1\}$. There must be some $j < \ell$ such that $\varepsilon_j = 1$, for $\mathrm{h}^1(S, \mathcal{I}_{Z_1}(1)) = 0$. Let j_0 be the greatest such j, and observe that $\mathrm{h}^1(S, \mathcal{I}_{Z_{j_0}}(1)) = k$. Then by induction hypothesis $j_0 = \mathrm{len}(Z_{j_0}) \geq m + k - 1$, hence $l \geq j_0 + 1 \geq m + k$.

Step 2. *We assume now that $g \geq 4$ and we prove that there are no subschemes of S of length $m + k - 1$ and $\mathrm{h}^1(S, \mathcal{I}_Z(1)) = k$, for any $k \geq 2$.*

We prove the statement by induction on $k \geq 2$. Assume first $k = 2$. Suppose that Z is a subscheme of length $m + 1$ and $\mathrm{h}^1(S, \mathcal{I}_Z(1)) = 2$. Let F be the rank 3 sheaf associated to Z by the non-trivial extension:

$$0 \to \mathscr{O}_S \otimes \mathrm{H}^1(S, \mathcal{I}_Z(1)) \to F \to \mathcal{I}_Z(1) \to 0.$$

Note that $\mathrm{rk}(F) = 3$, and $c_1(F) = 1$, $c_2(F) = c_2(\mathcal{I}_Z(1)) = m + 1$. We prove now that F is stable. Let Q be a destabilizing quotient of F. We may assume that Q is semistable. This implies that $1 \leq \mathrm{rk}(Q) \leq 2$ and $c_1(Q) \leq 0$.

If $\mathrm{rk}(Q) = 1$, then we conclude as in Step 1. Then we may assume that $\mathrm{rk}(Q) = 2$. Consider the kernel K of the projection $F \to Q$. We have $\mathrm{rk}(K) = 1$ and $c_1(K) \geq 1$. The map $K \to F$ then gives an injective map $K \to \mathcal{I}_Z(1)$, so that $K \cong \mathcal{I}_{Z'}(1)$, for some subscheme Z' of S containing Z. In particular we have $c_1(K) = 1$ and $c_1(Q) = 0$. We have thus the following diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & \mathscr{O}_S^2 \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d] & \mathcal{I}_Z(1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{O}_S^2 \arrow[r] & Q \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Here T has rank 0 and $c_1(T) = 0$ hence $c_2(T) \leq 0$. Note that, since Q is semistable, Bogomolov inequality (2.2) gives $c_2(Q) \geq 0$. But $c_2(Q) = c_2(T) \leq 0$ hence $c_2(Q) = 0$. Then $c_2(T) = 0$ and so

$T = 0$, and this implies that Q is isomorphic to $\mathscr{O}_S^2$. We conclude that F should contain $\mathscr{O}_S^2$ as a direct summand, which is not the case by the choice of the extension giving F.

We have thus proved that F is stable. But by (2.11), the dimension of the moduli space $\mathsf{M}_S(3, 1, c_2(F))$ equals $6\, c_2(F) - 4\, g - 12$ and for $g \geq 4$ this dimension is negative, a contradiction.

Finally by induction on $k \geq 2$ one easily proves, as in Step 1, that there are no subschemes of S of length ℓ and $\mathrm{h}^1(S, \mathcal{I}_Z(1)) = k$, with $\ell \leq m + k - 1$.

□

The following proposition is inspired on the approach of Iliev and Manivel, see [33].

Proposition 3.4. *Let Y be a smooth non-hyperelliptic Fano threefold of genus g and set $m = m_g$. Let F be a rank 2 stable sheaf on Y, with $c_1(F) = 1$, $c_2(F) = c_2 \in \{m, m+1\}$, $c_3(F) = c_3 \geq 0$. When $c_2 = m + 1$, we assume also $g \geq 4$. Then:*

i) $\mathrm{H}^k(Y, F(-1)) = 0$, for all $k \in \mathbb{Z}$, and $\mathrm{H}^j(Y, F) = 0$, for all $j \neq 0$;
ii) if $c_2 = m$, then F is locally free. If moreover $g \geq 4$, then F is globally generated and ACM;
iii) if $c_2 = m + 1$, then F is either locally free, or there exists an exact sequence:

$$0 \to F \to E \to \mathscr{O}_L \to 0, \tag{3.3}$$

where E is a rank 2 vector bundle with $c_1(E) = 1$, $c_2(E) = m$ and L is a line contained in Y.

Proof. Note that since Y is non-hyperelliptic, we have $g \geq 3$.

Step 1. *We prove* (i) *for $j = 2, 3$ and $k = 0, 3$. In the meantime we show* $\mathrm{H}^0(Y, F) \neq 0$.

The sheaf F is stable, hence torsion-free. Assume that $\mathrm{H}^2(Y, F) \neq 0$. Then any non-trivial element of $\mathrm{H}^2(Y, F)^* \cong \mathrm{Ext}^1_Y(F, \mathscr{O}_Y(-1))$ provides an extension of the form:

$$0 \to \mathscr{O}_Y(-1) \to \tilde{F} \to F \to 0,$$

where $\tilde{F}$ is a rank 3 sheaf which is easily seen to be semistable. This sheaf satisfies $c_1(\tilde{F}) = 0$ and $c_2(\tilde{F}) = c_2 - (2g - 2)$. Since either $c_2 = m$, or $c_2 = m + 1$ and $g \geq 4$, it follows that $c_2(\tilde{F}) < 0$, which contradicts Bogomolov's inequality (2.2). We have proved $\mathrm{H}^2(Y, F) = 0$.

By (2.8) we compute $\chi(F) = 3 + g - c_2 + \frac{1}{2} c_3$. Then $\mathrm{h}^0(Y, F) > 0$, i.e. there exists a non-zero global section of F.

By stability we have $\mathrm{H}^0(Y, F(-1)) = 0$. Moreover by Serre duality and stability we have $\mathrm{H}^3(Y, F(-1)) = \mathrm{H}^3(Y, F) = 0$. Indeed, $\mathrm{H}^3(Y, F(-1))$ is dual to $\mathrm{Hom}_Y(F, \mathscr{O}_Y)$. This group is zero, for the image of a nontrivial map $F \to \mathscr{O}_Y$ would be a destabilizing quotient of F. Similarly we prove that $\mathrm{H}^3(Y, F) = 0$.

Step 2. *Take double duals.* Setting $E = F^{**}$, we consider the double dual exact sequence:

$$0 \to F \to E \to T \to 0, \tag{3.4}$$

where T is a torsion sheaf supported in codimension at least 2, so $c_1(T) = 0$ and $c_2(T) \leq 0$. The sheaf E has rank 2, $c_1(E) = 1$ and $c_2(E) \leq c_2(F)$.

Let us show that E is stable. Assuming the contrary, we let K be a destabilizing subsheaf of E, and we note that K must have rank 1 and $c_1(K) \geq 1$. Let K' be the pull-back of K to F. The support of the image K'' of K in T is contained in the support of T, which has codimension at least 2, so that $c_1(K'') = 0$. Thus $c_1(K') \geq 1$ and K' destabilizes F, a contradiction.

By Step 1, we have: $\mathrm{H}^k(Y, E(-1)) = 0$, for $k = 0, 3$ and $\mathrm{H}^j(Y, E) = 0$ for $j = 2, 3$. Moreover since $\mathrm{h}^0(Y, F) > 0$, clearly we have $\mathrm{h}^0(Y, E) > 0$.

Step 3. *Show that E is locally free, satisfying* $\mathrm{H}^k(Y, E(-1)) = 0$, $\forall k$ *and* $\mathrm{H}^j(Y, E) = 0$ *for* $j = 2, 3$.

Recall that the sheaf E is reflexive, so its singularity locus has codimension at least 3, hence $E_S = E \otimes \mathscr{O}_S$ is locally free for a general hyperplane section S of Y. Moreover since E is stable, by Maruyama's theorem the sheaf E_S is μ-semistable, hence stable, for a general S. Fix a hyperplane section S, such that E_S is locally free and stable, and consider the exact sequence:

$$0 \to E(-1) \to E \to E_S \to 0. \tag{3.5}$$

Since $\mathrm{h}^0(Y, E(-1)) = 0$, it follows that $\mathrm{h}^0(S, E_S) \geq \mathrm{h}^0(Y, E) > 0$. Let Z be the zero locus of a general section of E_S. Note that Z has dimension zero and length $c_2(E) \geq m$ (Lemma 3.1), and recall the exact sequence:

$$0 \to \mathscr{O}_S \to E_S \to \mathcal{I}_Z(1) \to 0. \tag{3.6}$$

By Serre duality and stability we have $\mathrm{H}^2(S,E_S)^* \cong \mathrm{H}^0(S,E_S^*) = 0$, so the induced map $\mathrm{H}^1(S,\mathcal{I}_Z(1)) \to \mathrm{H}^2(S,\mathscr{O}_S)$ is surjective. By Lemma 3.3, since the subscheme Z is zero-dimensional of length either m, or $m+1$ (and in this case $g \geq 4$), then we must have $\mathrm{h}^1(S,\mathcal{I}_Z(1)) = 1$. Hence from (3.6), using $\mathrm{H}^1(S,\mathscr{O}_S) = 0$, $\mathrm{h}^2(S,\mathscr{O}_S) = 1$ and $\mathrm{H}^2(S,E_S) = 0$, we get $\mathrm{H}^1(S,E_S) = 0$. Taking global sections of (3.5), since $\mathrm{H}^2(Y,E) = 0$, we obtain $\mathrm{H}^2(Y,E(-1)) = 0$. So:

$$\chi(E(-1)) = -\,\mathrm{h}^1(Y,E(-1)) \leq 0. \tag{3.7}$$

On the other hand, formula (2.8) yields:

$$\chi(E(-1)) = c_3(E(-1))/2,$$

hence $c_3(E(-1)) \leq 0$. But $E(-1)$ is reflexive, so a direct generalization of [29, Proposition 2.6] gives:

$$c_3(E(-1)) = \mathrm{h}^0(Y, \mathscr{E}xt^1_Y(E(-1),\omega_Y)) \geq 0,$$

and $c_3(E(-1)) = 0$ if and only if $E(-1)$ is locally free. We deduce $c_3(E(-1)) = 0$, so $E(-1)$ is locally free (and also E is). Now from (3.7) we obtain $\mathrm{H}^1(Y,E(-1)) = 0$. Note that, using (3.5), this implies also $\mathrm{H}^1(Y,E) = 0$. Statement (i) thus holds whenever $F \cong E$.

Step 4. *Assume that $c_2 = m$ and prove that F is locally free.*

By the previous step it is enough to prove that $F \cong E$. Note that since E is stable, by Lemma 3.1 we have $c_2(E) \geq m = c_2$ and so we get $c_2(T) = c_2(E) - c_2 \geq 0$. On the other hand $c_2(T) \leq 0$ since the support of T has codimension ≥ 2. Hence $c_2(T)$ vanishes. Thus the sheaf T is supported on a subscheme of codimension 3.

Now, since $c_1(T) = c_2(T) = 0$, by (3.4) we have $c_3(T) = c_3(E) - c_3 \geq 0$. By Step 3 we know that $c_3(E) = 0$, hence the assumption $c_3 \geq 0$ forces $c_3(T) = 0$. We have thus proved that $T = 0$, so the sheaf F is isomorphic to E, hence it is locally free. Moreover, since $F \cong E$, by the previous step we have the vanishings $\mathrm{H}^2(Y,F(-1)) = 0$, $\mathrm{H}^1(Y,F(-1)) = 0$, and $\mathrm{H}^1(Y,F) = 0$. Of course, this completes the proof of (i) in the case $c_2 = m$.

Step 5. *Assume $c_2 = m$ and $g \geq 4$ and show that F is globally generated and ACM.*

Following [33, Proposition 5.4] one reduces to show that for any point $x \in Y$ and for a general surface S' through x, the vector bundle $F_{S'} = F \otimes \mathscr{O}_{S'}$ is globally generated. Clearly it is enough to prove that $\mathcal{I}_Z(1)$ is globally generated, where we denote again by Z the zero locus of a general global section of $F_{S'}$. This amounts to show that Z is cut out scheme-theoretically by its linear span, in other word that Z cannot be contained in a subscheme $Z' \subset S'$ such that $\mathrm{len}(Z') = m+1$ and $\mathrm{h}^1(S,\mathcal{I}_Z(1)) = 2$. But no such subscheme exists by Lemma 3.3, as soon as $g \geq 4$.

Finally F is ACM, by Griffith's theorem, [68, Theorem 5.52], since F is globally generated. This completes the proof of (ii).

Step 6. *Assume $c_2 = m+1$ and $g \geq 4$, and prove* (iii).

We have to show (3.3) in case F is not locally free, and still (i) has to be shown in this case. We consider the exact sequence (3.4), introduced in Step 2. Recall that E is locally free by Step 3, so $c_3(E) = 0$. Therefore, assuming $c_2(E) = m+1$ we get $c_2(T) = 0$ hence $c_3(T) \geq 0$, but $c_3(T) = -c_3(F) \leq 0$ so $T = 0$ hence $F \cong E$ contradicting that F is not locally free. Then, we must have $c_2(E) = m$ and $c_2(T) = -1$. By statement (ii), E is also globally generated. Therefore the 1-dimensional piece of the support of T is a line $L \subset Y$. Since the hyperplane section S chosen at Step 3 is general, we may also assume that $L \cap S = x$, for a point $x \in Y$.

A general global section of F_S (respectively, of E_S) vanishes on a subscheme $Z' \subset S$ (respectively, $Z \subset S$). The scheme Z has length m and satisfies $\mathrm{h}^1(S,\mathcal{I}_Z(1)) = 1$, and we have:

$$0 \to \mathcal{I}_{Z'}(1) \to \mathcal{I}_Z(1) \to \mathscr{O}_x \to 0. \tag{3.8}$$

Since the sheaf E is globally generated, $\mathcal{I}_Z(1)$ is too, hence Z is cut sheaf-theoretically by hyperplanes. Then the map $\mathrm{H}^0(S,\mathcal{I}_{Z'}(1)) \to \mathrm{H}^0(S,\mathcal{I}_Z(1))$ induced by (3.8) is not an isomorphism. We obtain $\mathrm{h}^1(S,\mathcal{I}_{Z'}(1)) = 1$, which easily implies $\mathrm{H}^1(S,F_S) = 0$. Taking global sections of the exact sequence

$$0 \to F(-1) \to F \to F_S \to 0, \tag{3.9}$$

and recalling that $\mathrm{H}^2(Y,F) = 0$, we get $\mathrm{H}^2(Y,F(-1)) = 0$. We can now follow the argument of Step 3, to prove that $c_3(F) = 0$, and $\mathrm{H}^1(Y,F(-1)) = 0$. Indeed, (2.8) gives $\chi(F(-1)) = c_3(F)/2$ and this value is non-negative by assumption. But we have proved $\mathrm{H}^0(Y,F(-1)) = \mathrm{H}^2(Y,F(-1)) =$

0 so $\chi(F(-1)) \leq 0$ hence $c_3(F) = 0$ and we deduce from $\chi(F(-1)) = 0$ that $\mathrm{H}^1(Y, F(-1)) = \mathrm{H}^3(Y, F(-1)) = 0$. By (3.9) we conclude that $\mathrm{H}^1(Y, F) = 0$. This completes the proof of (i) for $c_2 = m + 1$.

Now we mimic a remark of Druel, see [16]. Namely, since $\mathrm{H}^1(Y, F(-1)) = 0$, we have $\mathrm{H}^0(Y, T(-1)) = 0$. It follows that T is a Cohen-Macaulay curve and by a Hilbert polynomial computation we obtain $T \cong \mathscr{O}_L$ for a given line $L \subset Y$. This concludes the proof of (iii).

□

Finally, we reproduce here, for the reader's convenience, a result summarizing the information we have on the moduli space $\mathsf{M}_Y(2, 1, m_g)$, taken from [13, Theorem 3.2]. For a more accurate description of such space, we refer to [13] and the references therein.

Theorem 3.5. *Let Y be a smooth non-hyperelliptic prime Fano threefold of genus g. Then the space $\mathsf{M}_Y(2, 1, m_g)$ is not empty and any sheaf F satisfies:*

$$F \otimes \mathscr{O}_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(1), \qquad \textit{for some line } L \subset Y. \tag{3.10}$$

Assume further that Y is ordinary if $g = 3$ and that Y is contained in a smooth quadric if $g = 4$. Then the space $\mathsf{M}_Y(2, 1, m_g)$ contains a sheaf F satisfying:

$$\mathrm{Ext}^2_Y(F, F) = 0.$$

The space $\mathsf{M}_Y(2, 1, m_g)$ is equidimensional of dimension 0 if g is even and 1 if g is odd.

Recall that the non-emptyness of the space $\mathsf{M}_Y(2, 1, m_g)$ is derived from a case by case analysis, going back to [47] for $g = 3$, [48] for $g = 4, 5$, [24] for $g = 6$, [36], [37], [44], for $g = 7$, [25], [26], [57] for $g = 8$, [38] for $g = 9$, [57] for $g = 10$, [43] (see also [67], [18]) for $g = 12$.

3.2. **A good component of the moduli space $\mathsf{M}_Y(2, 1, d)$.** Recall that Y denotes a non-hyperelliptic smooth prime Fano threefold. We will construct a good (in the sense specified by Theorem 3.7) component of the space $\mathsf{M}_Y(2, 1, d)$, and for this we will need to assume that Y is ordinary. In particular we will assume that the Hilbert scheme $\mathscr{H}^0_1(Y)$ has a generically smooth component. In case $g = 4$ we will have to assume that Y is contained in a smooth quadric in $\mathbb{P}^5$, due to the restriction appearing in the previous theorem.

We start with the next proposition, which attributes a particular emphasis to the condition $\mathrm{H}^1(Y, F(-1)) = 0$. We think of it as an evidence that this condition is analogous to the vanishing $\mathrm{H}^1(\mathbb{P}^3, E(-2)) = 0$ required for E in $\mathsf{M}_{\mathbb{P}^3}(2, 0, d)$ to be an instanton bundle, see [5]. Note that the condition $\mathrm{H}^1(Y, F(-1)) = 0$ holds for any F in $\mathsf{M}_Y(2, 1, d)$, with $d = m_g$ and $d = m_g + 1$ (see Proposition 3.4).

Proposition 3.6. *Let Y be a smooth prime Fano threefold and d an integer. If a sheaf $F \in \mathsf{M}_Y(2, 1, d)$ satisfies $\mathrm{H}^1(Y, F(-1)) = 0$, then we have the following vanishings:*

$$\mathrm{H}^k(Y, F(-1)) = 0, \quad \textit{for any } k, \tag{3.11}$$

$$\mathrm{H}^1(Y, F(-t)) = 0, \quad \textit{for any } t \geq 1. \tag{3.12}$$

Proof. First let us prove (3.11). By stability and Serre duality, we have $\mathrm{H}^0(Y, F(-1)) = \mathrm{H}^3(Y, F(-1)) = 0$. By (2.8) it is easy to compute that $\chi(F(-1)) = 0$, and this implies the vanishing for $k = 2$.

In order to prove (3.12), let us take a general hyperplane section S of Y. Then we have the restriction exact sequence, for any integer t,

$$0 \to F(-1-t) \to F(-t) \to F_S(-t) \to 0. \tag{3.13}$$

Note that the sheaf F_S is semistable, by Maruyama's theorem. This implies $\mathrm{H}^0(Y, F_S(-t)) = 0$ for any $t \geq 1$. Then, taking cohomology of (3.13), we deduce that $\mathrm{H}^1(Y, F(-t)) = 0$, for any $t \geq 1$. □

Now, we construct inductively a component of $\mathsf{M}_Y(2, 1, d)$, for all $d \geq m_g$. This component is generically smooth of the expected dimension and its general element F is locally free and satisfies $\mathrm{H}^1(Y, F(-1)) = 0$.

Theorem 3.7. *Let Y be a smooth ordinary prime Fano threefold of genus g, and if $g = 4$ we further assume that Y is contained in a smooth quadric in $\mathbb{P}^5$, and we let $m = m_g$. Then we can choose a line $L \subset Y$ with $N_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(-1)$ such that, for any integer $d \geq m$, there exists a rank 2 stable locally free sheaf F with $c_1(F) = 1$, $c_2(F) = d$, and satisfying:*

$$\mathrm{Ext}^2_Y(F, F) = 0, \tag{3.14}$$

$$\mathrm{H}^1(Y, F(-1)) = 0, \tag{3.15}$$

$$\mathrm{H}^0(L, F_L(-2)) = 0. \tag{3.16}$$

The sheaf F belongs to a generically smooth component of $\mathsf{M}_Y(2, 1, d)$ of dimension:

$$2d - g - 2.$$

Proof. We work by induction on $d \geq m$. For $d = m$, it is enough to choose L and F according to Theorem 3.5. Indeed, the only property that we need to check in this case is (3.16), but this is clear since we may assume that (3.10) holds for a line L such that $N_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(-1)$.

Now we work out the induction process, and we divide it into several steps.

Step 1. *Construct a sheaf F_d in $\mathsf{M}_Y(2, 1, d)$ starting with a sheaf F_{d-1} in $\mathsf{M}_Y(2, 1, d-1)$ and a line $L \subset X$.*

We can choose in the inductively defined generically smooth component of $\mathsf{M}_Y(2, 1, d-1)$ a rank 2 locally free sheaf F_{d-1} satisfying $\mathrm{Ext}^2_Y(F_{d-1}, F_{d-1}) = 0$, $\mathrm{H}^1(Y, F_{d-1}(-1)) = 0$, and $\mathrm{H}^0(L, F_{d-1}(-2)) = 0$. From the last vanishing it easily follows that $F_{d-1} \otimes \mathscr{O}_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(1)$. Therefore there exists a (unique up to a non-zero scalar) surjective morphism $F_{d-1} \otimes \mathscr{O}_L \to \mathscr{O}_L$. Then we get a projection σ as the composition of surjective morphisms: $F_{d-1} \to F_{d-1} \otimes \mathscr{O}_L \to \mathscr{O}_L$. We denote by F_d the kernel of σ and we have the exact sequence:

$$0 \to F_d \to F_{d-1} \xrightarrow{\sigma} \mathscr{O}_L \to 0. \tag{3.17}$$

Step 2. *Prove that F_d lies in $\mathsf{M}_Y(2, 1, d)$ and satisfies* (3.14), (3.15), (3.16), *and:*

$$\mathrm{H}^2(Y, F_d(t)) = -t - 1 \quad \textit{for} \quad t \ll 0. \tag{3.18}$$

It is easy to see, using (2.7) to compute the Chern classes of $\mathscr{O}_L$, that the sheaf F_d fitting in (3.17) is a rank 2 (non-reflexive) torsion-free sheaf with $c_1(F_d) = 1$, $c_2(F_d) = d$, $c_3(F_d) = 0$. Moreover F_d is stable, because the slope of a destabilizing subsheaf (necessarily of rank 1) of F_d would a positive integer, hence this sheaf would destabilize also F_{d-1}, which is slope-stable.

We have $\mathrm{H}^1(Y, F_d(-1)) = 0$ since $\mathrm{H}^1(Y, F_{d-1}(-1)) = 0$ by induction and $\mathrm{H}^0(Y, \mathscr{O}_L(-1)) = 0$. So (3.15) holds. In order to prove (3.14), let us apply the functor $\mathrm{Hom}_Y(F_d, -)$ to (3.17). This gives the exact sequence:

$$\mathrm{Ext}^1_Y(F_d, \mathscr{O}_L) \to \mathrm{Ext}^2_Y(F_d, F_d) \to \mathrm{Ext}^2_Y(F_d, F_{d-1}).$$

We will prove that both the first and the last terms of the above sequence vanish. To prove the vanishing of the latter, apply $\mathrm{Hom}_Y(-, F_{d-1})$ to the exact sequence (3.17). We get the exact sequence:

$$\mathrm{Ext}^2_Y(F_{d-1}, F_{d-1}) \to \mathrm{Ext}^2_Y(F_d, F_{d-1}) \to \mathrm{Ext}^3_Y(\mathscr{O}_L, F_{d-1}).$$

By induction, we have $\mathrm{Ext}^2_Y(F_{d-1}, F_{d-1}) = 0$. Serre duality yields, since $F^*_{d-1} \cong F_{d-1}(-1)$,

$$\mathrm{Ext}^3_Y(\mathscr{O}_L, F_{d-1})^* \cong \mathrm{H}^0(Y, \mathscr{O}_L \otimes F^*_{d-1}(-1)) \cong \mathrm{H}^0(L, F_{d-1}(-2x)) = 0.$$

Therefore we obtain $\mathrm{Ext}^2_Y(F_d, F_{d-1}) = 0$. To show the vanishing of the group $\mathrm{Ext}^1_Y(F_d, \mathscr{O}_L)$, we apply the functor $\mathrm{Hom}_Y(-, \mathscr{O}_L)$ to (3.17). We are left with the exact sequence:

$$\mathrm{Ext}^1_Y(F_{d-1}, \mathscr{O}_L) \to \mathrm{Ext}^1_Y(F_d, \mathscr{O}_L) \to \mathrm{Ext}^2_Y(\mathscr{O}_L, \mathscr{O}_L).$$

The rightmost term vanishes by Remark 2.2. By Serre duality on L we get $\mathrm{Ext}^1_Y(F_{d-1}, \mathscr{O}_L) \cong \mathrm{H}^1(L, F^*_{d-1}) \cong \mathrm{H}^0(L, F_{d-1}(-2))^*$. But this group vanishes by induction. We have thus established (3.14). Note that, since clearly $\hom_Y(F_d, F_d) = 1$ and $\mathrm{Ext}^3_Y(F_d, F_d) = \mathrm{Hom}_Y(F_d, F_d(-1))^* = 0$ by stability, then by (2.9) and (2.1) we compute $\chi(F_d, F_d) = 3 + g - 2d$, which implies

$$\mathrm{ext}^1_Y(F_d, F_d) = 2\,d - g - 2. \tag{3.19}$$

Now let us prove property (3.16). Tensoring (3.17) by $\mathscr{O}_L$ we get the exact sequence of coherent sheaves on L:

$$0 \to \mathscr{T}or_1^Y(\mathscr{O}_L, \mathscr{O}_L) \to F_d \otimes \mathscr{O}_L \to F_{d-1} \otimes \mathscr{O}_L \to \mathscr{O}_L \to 0. \tag{3.20}$$

By (2.4) we know that $\mathscr{T}or_1^Y(\mathscr{O}_L,\mathscr{O}_L) \cong N_L^* \cong \mathscr{O}_L \oplus \mathscr{O}_L(1)$, by the choice of L. Now we twist (3.20) by $\mathscr{O}_L(-2)$ and take global sections. By induction $\mathrm{H}^0(L, F_{d-1}(-2)) = 0$, so our claim (3.16) follows easily.

Finally we prove (3.18). Note that since F_{d-1} is locally free, using Serre's vanishing theorem [28, III, Theorem 5.2] and Serre duality we get $\mathrm{h}^i(Y, F_{d-1}(t)) = 0$ for $i = 1, 2$ and $t \ll 0$. Hence (3.17) provides $\mathrm{h}^2(Y, F_d(t)) = \mathrm{h}^1(Y, \mathscr{O}_L(t))$, for $t \ll 0$. Now for any $t < 0$ we have $\mathrm{h}^1(Y, \mathscr{O}_L(t)) = -\chi(\mathscr{O}_L(t)) = -t-1$, and this conclude the proof of Step 2.

Step 3. *Define and study an open subset U of $\mathsf{M}_Y(2,1,d)$ containing F_d.*

We define now the subset of the moduli space $\mathsf{M}_Y(2,1,d)$, of sheaves F satisfying (3.14), (3.15), (3.16), and also:

$$\mathrm{H}^2(Y, F(t)) \leq -t-1 \quad \text{for} \quad t \ll 0. \tag{3.21}$$

All these properties are open by semicontinuity (see [28, Theorem 12.8] and [7, Satz 3 (i)]), so our subset is open. We have proved that F_d lies in this subset, and we let U be the irreducible component of this subset that contains F_d.

Now we want to prove that any element $F \in U$ is either locally free or it fits in a sequence of the form

$$0 \to F \to F^{**} \to \mathscr{O}_L \to 0, \quad \text{for some } L \in \mathscr{H}_1^0(Y). \tag{3.22}$$

Assume then that $F \in U$ is not locally free, and consider the double dual exact sequence:

$$0 \to F \to F^{**} \to T \to 0.$$

Clearly $T \neq 0$ is a torsion sheaf whose support W has dimension at most 1. We want to prove $T \cong \mathscr{O}_L$ for some line L. It is easy to see that F^{**} is stable. This implies $\mathrm{H}^0(Y, F^{**}(-1)) = 0$ which in turn gives $\mathrm{H}^0(Y, T(-1)) = 0$, since $\mathrm{H}^1(X, F(-1)) = 0$. So $W = \mathrm{supp}(T)$ contains no isolated or embedded points, i.e. it is a Cohen-Macaulay curve.

In order to prove that W must have degree 1, by (2.7) it is enough to show $c_2(T) \geq -1$. Note first that for any $t < 0$, we have $\chi(T(t)) = -\mathrm{h}^1(Y, T(t))$. Recall that, by [29, Remark 2.5.1], we have $\mathrm{H}^1(Y, F^{**}(t)) = 0$ for all $t \ll 0$. Thus, tensoring (3.17) by $\mathscr{O}_Y(t)$ and taking cohomology, we obtain $\mathrm{h}^1(Y, T(t)) \leq \mathrm{h}^2(Y, F(t))$ for all $t \ll 0$. By (3.21), it follows $\chi(T(t)) \geq t+1$ for all $t \ll 0$. On the other hand, for any integer t, we can compute:

$$c_1(T(t)) = 0, \quad c_2(T(t)) = c_2(T) = d - c_2(F^{**}), \quad c_3(T(t)) = c_3(F^{**}) - (2t+1)c_2(T),$$

hence by Riemann-Roch:

$$\chi(T(t)) = -c_2(T)(t+1) + \frac{c_3(F^{**})}{2}.$$

Summing up we have, for all $t \ll 0$:

$$-c_2(T)(t+1) + \frac{c_3(F^{**})}{2} \geq t+1.$$

This implies that $c_2(T) \geq -1 + \frac{c_3(F^{**})}{2(t+1)}$ for all $t \ll 0$, which implies $c_2(T) \geq -1$.

We have thus proved that T is of the form $\mathscr{O}_L(a)$, for some $L \in \mathscr{H}_1^0(Y)$, and for some integer a. Then $c_3(T) = c_3(F^{**}) = 1 + 2\,a$, so $a \geq 0$, see [29, Proposition 2.6]. On the other hand we have seen $\mathrm{H}^0(Y, T(-1)) = 0$, so $a \leq 0$. So $T \cong \mathscr{O}_L$.

Step 4. *Flatly deform F_d in U to a stable vector bundle F.*

In order to conclude the proof we need now to prove that U contains locally free sheaves, and so it is possible to deform F_d to a locally free sheaf. Assume by contradiction that all the sheaves $F \in U$ fit in a sequence of the form (3.22). Up to possibly shrinking U, since all our sheaves are stable, there exists the universal family $\mathscr{F}$ of sheaves on $U \times Y$ (see [69, Theorem 1.2.1 part (4)]). The corresponding double dual sequence reads:

$$0 \to \mathscr{F} \to \mathscr{F}^{**} \to \mathscr{T} \to 0, \tag{3.23}$$

on $U \times Y$. Clearly, for any $F \in U$, the sequence (3.23) restricts to the sequence (3.22) on $\{F\} \times Y$. Then the sheaf $\mathscr{T}$ gives a family of lines in Y, while $\mathscr{F}^{**}$ gives a family of sheaves in $\mathsf{M}_Y(2,1,d-1)$. This defines a map $\iota : U \to \mathsf{M}_Y(2,1,d-1) \times \mathscr{H}_1^0(Y)$, which is injective. Indeed for any $F \in U$ and $\iota(F) = (F^{**}, L)$, tensoring the exact sequence (3.22) by $\mathscr{O}_L(-2)$ and taking global sections we see that, since $F \in U$ satisfies (3.16), then we also have $\mathrm{H}^0(L, F^{**}(-2)) = 0$. This is equivalent to the condition $F^{**} \otimes \mathscr{O}_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(1)$, which implies that $\hom_Y(F^{**}, \mathscr{O}_L) = 1$. Hence the surjective map $F^{**} \to \mathscr{O}_L$ is unique up to a non-zero scalar, and its kernel is determined (up to isomorphism) by F^{**} and L and must be isomorphic to F.

The image of ι is irreducible (for U is) and lies in the product of a M with an irreducible component of $\mathscr{H}_1^0(Y)$, because M is the component of $\mathsf{M}_Y(2,1,d-1)$ containing $F_{d-1} \cong F_d^{**}$. The dimension of M equals $\mathrm{ext}^1_Y(F_{d-1}, F_{d-1}) = 2\,d - g - 4$ and so

$$\dim U \leq 2\,d - g - 3 < 2\,d - g - 2 = \dim U,$$

where the last equality follows from (3.19). This gives a contradiction and we conclude that U must contain locally free sheaves. $\square$

By Theorem 3.7 we can now give the following definition.

Definition 3.8. Choose a component $\mathscr{H}$ of $\mathscr{H}_1^0(Y)$ containing a line L such that $N_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(-1)$. Choose a component $\mathsf{M}(m_g)$ of the moduli space $\mathsf{M}_Y(2,1,m_g)$ containing a sheaf F satisfying the properties listed in Theorem 3.5, with respect to some line contained in $\mathscr{H}$. Then, for each $d \geq m_g+1$, we recursively define $\mathsf{N}(d)$ as the set of non-reflexive sheaves fitting as kernel in an exact sequence of the form (3.17), with $F_{d-1} \in \mathsf{M}(d-1)$, and $\mathsf{M}(d)$ as the component of the moduli scheme $\mathsf{M}_Y(2,1,d)$ containing $\mathsf{N}(d)$. In view of Theorem 3.7 the component $\mathsf{M}(d)$ is generically smooth of dimension $2d - g - 2$ and contains $\mathsf{N}(d)$ as an irreducible divisor.

Note that the previous definition may depend on several choices. Indeed, there might exist several different components $\mathsf{N}(d)$ and the results we prove hold for each of them.

4. Rational cubics on Fano threefolds of genus 7

Let X be a smooth prime Fano threefold of genus 7, and let Γ be its homologically projectively dual curve. For $1 \leq d \leq 4$, the subset of $\mathscr{H}_d^0(X)$ containing rational normal curves is described by the results of [37]. It is known to have dimension d, and to be isomorphic to $W^1_{1,5}$ for $d = 1$, isomorphic to $\Gamma^{(2)}$ for $d = 2$, and birational to $\Gamma^{(3)}$ for $d = 3$. The isomorphism $\mathscr{H}_2^0(X) \cong \Gamma^{(2)}$ was also proved by Kuznetsov, making use of the semiorthogonal decomposition of $\mathbf{D^b}(X)$.

Here we first rephrase in our framework the results concerning lines contained in X. Then, we make more precise the result on cubics, showing that the Hilbert scheme $\mathscr{H}_3^0(X)$ is in fact isomorphic to the symmetric cube $\Gamma^{(3)}$.

4.1. **Lines on a Fano threefold of genus 7.**

The fact that the Hilbert scheme of lines contained in X is isomorphic to the Brill-Noether locus $W^1_{1,5}$ is due to Iliev-Markushevich, [37]. This reformulation will be used further on.

Proposition 4.1 (Iliev-Markushevich)**.** *Let X be a smooth prime Fano threefold of genus 7. Then we have the following isomorphisms:*

$$\mathscr{H}_1^0(X) \to W^1_{1,5}, \qquad L \mapsto \mathbf{\Phi}^!(\mathscr{O}_L)[-1], \tag{4.1}$$

$$\mathscr{H}_1^0(X) \to W^2_{1,7}, \qquad L \mapsto \mathbf{\Phi}^!(\mathscr{O}_L(-1)). \tag{4.2}$$

Proof. Let $L \subset X$ be a line contained in X. For any $y \in \Gamma$, we have $\mathscr{E}_y^* \otimes \mathscr{O}_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(-1)$, so $\mathrm{h}^0(X, \mathscr{E}_y^* \otimes \mathscr{O}_L) = 1$. Indeed $\mathscr{E}_y$ is globally generated for each y and has degree 1 on L. So $\mathcal{H}^k(\mathbf{\Phi}^!(\mathscr{O}_L))_y = 0$ for all $y \in \Gamma$ and all $k \neq -1$. Hence $\mathbf{\Phi}^!(\mathscr{O}_L)[-1]$ is a line bundle on Γ, by (2.28) and (2.7), and has degree 5 by Grothendieck-Riemann-Roch (2.29). Now observe that, using (2.32) and the spectral sequence (2.5), we get:

$$\begin{aligned}\mathrm{H}^0(\Gamma, \mathbf{\Phi}^!(\mathscr{O}_L)[-1]) &\cong \mathrm{Hom}_X(\mathbf{\Phi}(\mathscr{O}_\Gamma), \mathscr{O}_L[-1]) \cong \\ &\cong \mathrm{Hom}_X(\mathcal{H}^1(\mathbf{\Phi}(\mathscr{O}_\Gamma)), \mathscr{O}_L) \cong \\ &\cong \mathrm{H}^0(L, \mathcal{U}_+^*(-1)).\end{aligned}$$

Since $\mathcal{U}_+^*$ is globally generated and $c_1(\mathcal{U}_+^*) = 2$, one sees easily that $\mathcal{U}_+^* \otimes \mathscr{O}_L \cong \mathscr{O}_L^3 \oplus \mathscr{O}_L(1)^2$, hence the space $\mathrm{H}^0(L, \mathcal{U}_+^*(-1))$ must have dimension 2. So $\mathbf{\Phi}^!(\mathscr{O}_L)[-1]$ lies in $W^1_{1,5}$, and (4.1) is well-defined.

Let us now define the inverse of (4.1). For any $\mathcal{L}$ in $W^1_{1,5}$, we set $\mathcal{L} \mapsto \mathcal{H}^1(\mathbf{\Phi}(\mathcal{L}))$, and we will see that this map is clearly an inverse of (4.1), as soon as it is well-defined. Hence we have to prove that $\mathcal{H}^1(\mathbf{\Phi}(\mathcal{L}))$ is a sheaf of the form $\mathscr{O}_L$, for a line $L \subset X$. Let us accomplish this task.

Since the curve Γ is not tetragonal, it is easy to see that $\mathcal{L}$ is globally generated, i.e. we have:

$$0 \to \mathcal{L}^* \to \mathscr{O}_\Gamma^2 \to \mathcal{L} \to 0. \tag{4.3}$$

Moreover, a general global section of $\mathcal{L}$ vanishes at 5 distinct points $y_1, \ldots, y_5 \in \Gamma$, hence it gives:

$$0 \to \mathscr{O}_\Gamma \to \mathcal{L} \to \bigoplus_{i=1,\ldots,5} \mathscr{O}_{y_i} \to 0. \tag{4.4}$$

Set $\mathscr{K}_\mathcal{L} = \mathcal{H}^0(\mathbf{\Phi}(\mathcal{L}))$ and $\mathscr{F}_\mathcal{L} = \mathcal{H}^1(\mathbf{\Phi}(\mathcal{L}))$. By (2.32) and (2.33), applying $\mathbf{\Phi}$ to this sequence gives:

$$0 \to \mathcal{U}_+^* \to \mathscr{K}_\mathcal{L} \to \bigoplus_{i=1,\ldots,5} \mathscr{E}_{y_i} \xrightarrow{\beta} \mathcal{U}_+(1) \to \mathscr{F}_\mathcal{L} \to 0. \tag{4.5}$$

Since, for any $i = 1, \ldots, 5$, the bundle $\mathscr{E}_{y_i} \cong \mathbf{\Phi}(\mathscr{O}_{y_i})$ lies in ${}^\perp\mathcal{U}_+^*$, then we get, using (2.32) and the fact that $\mathbf{\Phi}$ is fully faithful, we get:

$$\begin{aligned} \mathrm{Hom}_X(\mathscr{E}_{y_i}, \mathcal{U}_+(1)) \cong \mathrm{Hom}_X(\mathscr{E}_{y_i}, \mathbf{\Phi}(\mathscr{O}_\Gamma)[1]) \cong \mathrm{Hom}_X(\mathbf{\Phi}(\mathscr{O}_{y_i}), \mathbf{\Phi}(\mathscr{O}_\Gamma)[1]) \cong \\ \cong \mathrm{Hom}_\Gamma(\mathscr{O}_{y_i}, \mathscr{O}_\Gamma[1]) \cong \mathrm{Ext}^1_\Gamma(\mathscr{O}_{y_i}, \mathscr{O}_\Gamma) \end{aligned}$$

Note that, since $\mathcal{L}$ is torsion free, then the restriction of the extension (4.4) onto each $\mathscr{O}_{y_i}$ is nontrivial, hence the restriction of β on each $\mathscr{E}_{y_i}$ is nontrivial. This proves in particular that $\beta \neq 0$, thus we consider the non-zero sheaf $\mathrm{Im}(\beta)$. By stability of the $\mathscr{E}_{y_i}$'s and of $\mathcal{U}_+(1)$, the sheaf $\mathrm{Im}(\beta)$ can only have slope either $3/5$, or $1/2$. If the latter case takes place, we also know that $\mathrm{Im}(\beta)$ is isomorphic either to $\mathscr{E}_{y_i}$, or $\mathscr{E}_{y_i} \oplus \mathscr{E}_{y_j}$, for some $i \neq j \in \{1, \ldots, 5\}$. But we have seen that this is impossible.

Hence we must have $\mu(\mathrm{Im}(\beta)) = 3/5$. So β is generically surjective, $\mathscr{K}_\mathcal{L}$ is a torsion-free sheaf of rank 10 and $\mathscr{F}_\mathcal{L}$ is a torsion sheaf. For a general point $x \in X$, we have using Serre duality $\mathrm{h}^0(\Gamma, \mathcal{L}^* \otimes \mathscr{E}_x) = \mathrm{h}^1(\Gamma, \mathcal{L} \otimes \mathscr{E}_x) = \mathrm{rk}(\mathscr{F}_\mathcal{L}) = 0$, then we conclude $\mathcal{H}^0(\mathbf{\Phi}(\mathcal{L}^*)) = 0$ (recall that $\mathcal{H}^0(\mathbf{\Phi}(\mathcal{L}^*)) = 0$ is torsion-free). Then, applying $\mathbf{\Phi}$ to (4.3) we get an injective map $\iota : (\mathcal{U}_+^*)^2 \subset \mathscr{K}_\mathcal{L}$, which is an isomorphism by Lemma 2.1. We compute now by (4.5) that $\mathscr{F}_\mathcal{L}$ has Chern classes $(0, -1, 1)$, and again by (4.5) we see $\mathrm{H}^0(X, \mathscr{F}_\mathcal{L}(-1)) = 0$. This suffices to deduce $\mathscr{F}_\mathcal{L} \cong \mathscr{O}_L$.

Let us look at (4.2). Consider $\mathcal{P} = \mathcal{L}^* \otimes \omega_\Gamma$, which is a line bundle of degree 7. By Serre duality we get $\mathrm{h}^0(\Gamma, \mathcal{P}) = \mathrm{h}^1(\Gamma, \mathcal{L}) = 3$, hence $\mathcal{P}$ lies in $W^2_{1,7}$. Applying (2.25), since $\mathbf{R}\mathcal{H}om_X(\mathscr{O}_L, \mathscr{O}_X)[2] \cong \mathscr{O}_L(-1)$, we obtain the functorial isomorphism:

$$\mathbf{\Phi}^!(\mathscr{O}_L(-1)) \cong \mathcal{L}^* \otimes \omega_\Gamma.$$

Therefore, (4.2) is also well-defined. We have then a commutative diagram:

$$\begin{tikzcd} \mathscr{H}_1^0(X) \arrow[r] \arrow[d, equal] & W^1_{1,5} \arrow[d, "\mathcal{L} \mapsto \mathcal{L}^* \otimes \omega_\Gamma"] & \mathscr{O}_L \arrow[r, mapsto] \arrow[d, equal] & \mathbf{\Phi}^!(\mathscr{O}_L[-1]) \arrow[d] \\ \mathscr{H}_1^0(X) \arrow[r] & W^2_{1,7} & \mathscr{O}_L \arrow[r, mapsto] & \mathbf{\Phi}^!(\mathscr{O}_L(-1)). \end{tikzcd}$$

Then (4.2) is an isomorphism, and we are done. □

A few comments are in order. First, observe that $\mathrm{Ext}^k_X(\mathscr{O}_L, \mathscr{O}_X) = 0$ for all k, while $\mathrm{Ext}^k_X(\mathscr{O}_L, \mathcal{U}_+) = 0$ for $k \neq 3$, and $\mathrm{Ext}^3_X(\mathscr{O}_L, \mathcal{U}_+) = A_L^*$, where A_L denotes $\mathrm{H}^0(\Gamma, \mathbf{\Phi}^!(\mathscr{O}_L)[-1])$. Making use of the exact triangle (2.19), this gives the isomorphisms:

$$\mathcal{H}^k(\mathbf{\Phi}(\mathbf{\Phi}^!(\mathscr{O}_L))) \cong \begin{cases} \mathscr{O}_L & \text{for } k = 0, \\ A_L \otimes \mathcal{U}_+^* & \text{for } k = -1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.6}$$

Note that (4.5) gives the resolution of $\mathscr{O}_L$:

$$0 \to \mathcal{U}_+^* \to \bigoplus_{i=1,\ldots,5} \mathscr{E}_{y_i} \xrightarrow{\beta} \mathcal{U}_+(1) \to \mathscr{O}_L \to 0.$$

Notice also that, given a line $L \subset X$, taking $\mathcal{P} = \mathbf{\Phi}^!(\mathscr{O}_L(-1))^* \otimes \omega_\Gamma$, we have $\mathcal{P}$ in $W^2_{1,7}$. Using (2.19), we can write the canonical resolution of $\mathscr{O}_L(-1)$, that reads:

$$0 \to \mathscr{O}_X \to (\mathcal{U}_+^*)^3 \xrightarrow{\varsigma} \mathbf{\Phi}(\mathcal{P}) \to \mathscr{O}_L(-1) \to 0. \tag{4.7}$$

Note that, considering the evaluation map $e_\mathcal{P} = e_{\mathscr{O}_\Gamma, \mathcal{P}} : \mathscr{O}_\Gamma^3 \to \mathcal{P}$, the map ς above is just $\mathcal{H}^0(\mathbf{\Phi}(e_\mathcal{P}))$. This gives an explicit description of the inverse of (4.2) as $\mathcal{P} \mapsto \mathrm{cok}(\mathcal{H}^0(\mathbf{\Phi}(e_\mathcal{P})))$.

Remark 4.2. In view of the isomorphism $\mathscr{H}^0_1(X) \cong W^1_{1,5}$, we note that the threefold X is exotic if and only if $W^1_{1,5}$ has a component which is non-reduced at any point. It is well-known (see e.g. [3, Proposition 4.2]) that $\mathcal{L}$ is a singular point of $W^1_{1,5}$ if and only if the Petri map:

$$\pi_{\mathcal{L}} : \mathrm{H}^0(\Gamma, \mathcal{L}) \otimes \mathrm{H}^0(\Gamma, \mathcal{L}^* \otimes \omega_\Gamma) \to \mathrm{H}^0(\Gamma, \omega_\Gamma).$$

is not injective. We have seen that any line bundle $\mathcal{L}$ in $W^1_{1,5}$ is globally generated. Therefore $\ker(\pi_{\mathcal{L}})$ is isomorphic to $\mathrm{H}^0(\Gamma, \mathcal{L}^* \otimes \mathcal{L}^* \otimes \omega_\Gamma)$.

This proves that the threefold X is exotic if and only if Γ admits infinitely many line bundles $\mathcal{L}$ in $W^1_{1,5}$ such that $\mathcal{L}^* \otimes \mathcal{L}^* \otimes \omega_\Gamma$ is effective.

4.2. **Conics on a Fano threefold of genus 7.** Kuznetsov's result on conics contained in X, see [44] asserts that, if $C \subset X$ is a connected Cohen-Macaulay curve of arithmetic genus 0 (a conic), then $\mathbf{\Phi}^!(\mathscr{O}_C)$ is the structure sheaf of a length-2 subscheme of Γ, and we have the following canonical resolution:

$$0 \to \mathscr{O}_X \to \mathcal{U}_+^* \to \mathbf{\Phi}(\mathbf{\Phi}^!(\mathscr{O}_C)) \to \mathscr{O}_C \to 0. \tag{4.8}$$

Let us look at what happens for reducible conics.

Lemma 4.3. *Let $M, N \subset X$ be distinct lines and set $\mathcal{M} = \mathbf{\Phi}^!(\mathscr{O}_M)[-1]$, $\mathcal{P} = \mathbf{\Phi}^!(\mathscr{O}_N(-1))$. Then:*

$$N \cap M \neq \emptyset \quad \Leftrightarrow \quad \mathrm{H}^0(\Gamma, \mathcal{M}^* \otimes \mathcal{P}) \neq 0 \quad \Leftrightarrow \quad \mathrm{h}^0(\Gamma, \mathcal{M}^* \otimes \mathcal{P}) = 1.$$

In this case, setting $C = M \cup N$, we have $\mathbf{\Phi}^!(\mathscr{O}_C) \cong \mathcal{P}/\mathcal{M}$.

Proof. Recall that $\mathcal{M} \in W^1_{1,5}$ and $\mathcal{P} \in W^2_{1,7}$. First note that $\mathrm{H}^0(\Gamma, \mathcal{M}^* \otimes \mathcal{P}) \neq 0$ is equivalent to $\mathrm{h}^0(\Gamma, \mathcal{M}^* \otimes \mathcal{P}) = 1$, since $\deg(\mathcal{M}^* \otimes \mathcal{P}) = 2$ so a pencil of sections of $\mathcal{M}^* \otimes \mathcal{P}$ would turn Γ into a hyperelliptic curve, which is not the case.

If $M \cap N \neq \emptyset$ then M and N meet at a single point, and we have the exact sequence:

$$0 \to \mathscr{O}_N(-1) \to \mathscr{O}_C \to \mathscr{O}_M \to 0. \tag{4.9}$$

Applying $\mathbf{\Phi}^!$ to the sequence above, by Proposition 4.1 we have:

$$0 \to \mathcal{M} \to \mathcal{P} \to \mathbf{\Phi}^!(\mathscr{O}_C) \to 0$$

We deduce that $\mathrm{H}^0(\Gamma, \mathcal{M}^* \otimes \mathcal{P}) \neq 0$, and $\mathbf{\Phi}^!(\mathscr{O}_C) \cong \mathcal{P}/\mathcal{M}$.

Conversely, assume $\mathrm{H}^0(\Gamma, \mathcal{M}^* \otimes \mathcal{P}) \neq 0$. We get a commutative exact diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_\Gamma^2 \arrow[r] \arrow[d] & \mathscr{O}_\Gamma^3 \arrow[r] \arrow[d] & \mathscr{O}_\Gamma \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{M} \arrow[r] & \mathcal{P} \arrow[r] & \mathscr{O}_Z \arrow[r] & 0,
\end{tikzcd}$$

where $Z \subset \Gamma$ has length 2 and the vertical maps are natural (surjective) evaluations. Applying $\mathbf{\Phi}$ to this diagram and taking cohomology, using (4.8) and (4.7), we get:

$$\begin{tikzcd}
0 \arrow[r] & (\mathcal{U}_+^*)^2 \arrow[r] \arrow[d, equal] & (\mathcal{U}_+^*)^3 \arrow[r] \arrow[d] & \mathcal{U}_+^* \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & (\mathcal{U}_+^*)^2 \arrow[r] \arrow[d] & \mathbf{\Phi}(\mathcal{P}) \arrow[r] \arrow[d] & \mathbf{\Phi}(\mathscr{O}_Z) \arrow[r] \arrow[d] & \mathscr{O}_L \arrow[r] \arrow[d, equal] & 0 \\
& 0 \arrow[r] & \mathscr{O}_N(-1) \arrow[r] & \mathscr{O}_C \arrow[r] & \mathscr{O}_L \arrow[r] & 0
\end{tikzcd}$$

This gives back (4.9) with C a conic in X, so $M \cap N \neq \emptyset$. ☐

Our next goal is to investigate the Hilbert scheme $\mathscr{H}^0_3(X)$. We will need the following lemma.

Lemma 4.4. *Let C be any Cohen-Macaulay curve of degree $d \geq 3$ and arithmetic genus p_a contained in X. Then $\mathbf{\Phi}^!(\mathscr{O}_C)$ is a vector bundle on Γ of rank $d - 2 + 2p_a$ and degree $7\,d - 12 + 12\,p_a$.*

Proof. The following argument is inspired on the proof of [44, Lemma 5.1]. We have to prove that, for each $y \in \Gamma$, the group $\mathrm{H}^0(X, \mathscr{E}_y^* \otimes \mathscr{O}_C)$ vanishes. By (2.22), it is enough to prove:

$$\mathrm{H}^0(C, \mathcal{U}_+) = 0. \tag{4.10}$$

Assume the contrary, and consider a non-zero global section u in $\mathrm{H}^0(C, \mathcal{U}_+)$. Let U be the 1-dimensional subspace spanned by u. By (2.12), we have $U \subset \mathrm{H}^0(C, \mathcal{U}_+) \subset \mathrm{H}^0(C, \mathscr{O}_X \otimes V) \cong V$.

Set $V' = U^{\perp}/U$. Then the orthogonal Grassmann variety $\mathbb{G}_Q(\mathbb{P}^3, \mathbb{P}(V')) \subset \Sigma_+$ is a quadric and clearly the curve C is contained in $X' := \mathbb{G}_Q(\mathbb{P}^3, \mathbb{P}(V')) \cap X$. Recall that X is a linear section of Σ^+, then X must contain either a 2-dimensional quadric or a plane. But this is impossible by Lefschetz theorem.

This proves that $\mathbf{\Phi}^!(\mathscr{O}_C)$ is a vector bundle on Γ. By (2.28) and (2.7), we conclude that $\mathrm{rk}(\mathbf{\Phi}^!(\mathscr{O}_C)) = d - 2 + 2p_a$. Finally, by (2.29), we compute the degree of $\mathbf{\Phi}^!(\mathscr{O}_C)$. □

4.3. **Hilbert scheme of rational cubics on X and the symmetric cube of Γ.** We are now in position to prove the following result.

Theorem 4.5. *Let X be a smooth prime Fano threefold of genus 7, and Γ be its homologically projectively dual curve. The map $\psi : \mathscr{O}_C \mapsto (\mathbf{\Phi}^!(\mathscr{O}_C))^* \otimes \omega_\Gamma$ gives an isomorphism between $\mathscr{H}_3^0(X)$ and $\Gamma^{(3)}$. In particular $\mathscr{H}_3^0(X)$ is a smooth irreducible threefold.*

Proof. Let us start by observing that, by Riemann-Roch, the correspondence $\tau : \mathcal{L} \mapsto \mathcal{L}^* \otimes \omega_\Gamma$ provides an isomorphism between $W^3_{1,9}$ and $W^0_{1,3}$. Moreover, $\mathrm{h}^1(\Gamma, \mathcal{L}) = \mathrm{h}^0(\Gamma, \tau(\mathcal{L})) = 1$ if and only if $\mathrm{h}^1(\Gamma, \mathcal{L}) \neq 0$ since the curve Γ is not trigonal (see [59, Table 1]). We deduce that $W^0_{1,3} \cong \Gamma^{(3)}$.

Let us now look at the map ψ, and check that it is well-defined. Let C be a curve corresponding to an element of $\mathscr{H}_3^0(X)$. Recall that by Lemma 3.2 the curve C is Cohen-Macaulay. We have seen in Lemma 4.4 that the sheaf $\mathcal{L} = \mathbf{\Phi}^!(\mathscr{O}_C)$ is a line bundle of degree 9 on Γ. We have to prove that $\mathcal{L}$ lies in $W^3_{1,9}$, equivalently that $\mathrm{H}^1(\Gamma, \mathcal{L}) \neq 0$. Note that:

$$\mathrm{H}^1(\Gamma, \mathcal{L}) \cong \mathrm{Ext}^1_\Gamma(\mathscr{O}_\Gamma, \mathbf{\Phi}^!(\mathscr{O}_C)) \cong \mathrm{Ext}^1_X(\mathbf{\Phi}(\mathscr{O}_\Gamma), \mathscr{O}_C).$$

Recall that $\mathcal{H}^0(\mathbf{\Phi}(\mathscr{O}_\Gamma)) \cong \mathcal{U}_+^*$ and $\mathcal{H}^q(\mathbf{\Phi}(\mathscr{O}_\Gamma))$ for all $q \neq 0, 1$ (see (2.32)). In view of the vanishing (4.10), we have $\mathrm{Hom}_X(\mathcal{U}_+^*, \mathscr{O}_C) = 0$ so $\mathrm{ext}^1_X(\mathcal{U}_+^*, \mathscr{O}_C) = 1$ by Riemann-Roch. Therefore, by the spectral sequence (2.5), we get:

$$\mathrm{Ext}^p(\mathcal{H}^{-q}(\mathbf{\Phi}(\mathscr{O}_\Gamma), \mathscr{O}_C) \Rightarrow \mathrm{H}^1(\Gamma, \mathcal{L}) \neq 0.$$

We have now proved that $\mathcal{L}$ lies in $W^3_{1,9}$, hence we have the morphism $\psi : \mathscr{O}_C \mapsto (\mathbf{\Phi}^!(\mathscr{O}_C))^* \otimes \omega_\Gamma$ from $\mathscr{H}_3^0(X)$ to $\Gamma^{(3)}$ as $\psi = \tau \circ \varphi$.

Now, we would like to construct an inverse map $\vartheta : W^3_{1,9} \to \mathscr{H}_3^0(X)$ of φ. We consider a line bundle $\mathcal{L}$ in $W^3_{1,9}$, and the natural evaluation map of sections of $\mathcal{L}$:

$$e_\mathcal{L} := e_{\mathscr{O}_\Gamma, \mathcal{L}} : \mathrm{H}^0(\Gamma, \mathcal{L}) \otimes \mathscr{O}_\Gamma \to \mathcal{L}.$$

Applying the functor $\mathbf{\Phi}$ to this map and taking cohomology, we get a map:

$$\mathcal{H}^0(\mathbf{\Phi}(e_\mathcal{L})) : \mathrm{H}^0(\Gamma, \mathcal{L}) \otimes \mathcal{U}_+^* \to \mathbf{\Phi}(\mathcal{L}).$$

We would like to show that the cokernel of this map is the structure sheaf of a rational curve of degree 3 on X. This will define an inverse map ϑ to φ. So let $Z = \{y_1, y_2, y_3\}$ be three, non necessarily distinct points of Γ. We have an exact sequence:

$$0 \to \mathscr{O}_\Gamma \to \mathscr{O}_\Gamma(Z) \to \mathscr{O}_Z \to 0.$$

We set $\mathscr{F}_Z = \mathcal{H}^1(\mathbf{\Phi}(\mathscr{O}_\Gamma(Z)))$. Applying $\mathbf{\Phi}$ to the sequence above, by (2.32) and (2.33), we get:

$$0 \to \mathcal{U}_+^* \to \mathcal{H}^0(\mathbf{\Phi}(\mathscr{O}_\Gamma(Z))) \to \bigoplus_{i=1,2,3} \mathscr{E}_{y_i} \xrightarrow{\xi} \mathcal{U}_+(1) \to \mathscr{F}_Z \to 0.$$

Reasoning as in the proof of Proposition 4.1, one can prove that $\mathrm{Im}(\xi)$ has rank 5 and first Chern class 3, i.e. $\mu(\mathrm{Im}(\xi)) = 3/5$. So ξ is generically surjective, and $\ker(\xi)$ is reflexive (by [29, Proposition 1.1]) with rank 1 and $c_1(\ker(\xi)) = 0$, i.e. $\ker(\xi) \cong \mathscr{O}_X$. We have thus:

$$0 \to \mathscr{O}_X \to \bigoplus_{i=1,2,3} \mathscr{E}_{y_i} \xrightarrow{\xi} \mathcal{U}_+(1) \to \mathscr{F}_Z \to 0.$$

It follows that $\mathscr{F}_Z$ is a sheaf of rank 0 with Chern classes $(0, -3, 1)$. Now, twisting by $\mathscr{O}_X(-1)$ the sequence above and taking cohomology, we get $\mathrm{H}^0(X, \mathscr{F}_Z(-1)) = 0$. It follows that $\mathscr{F}_Z$ is torsion-free over a Cohen-Macaulay curve C_Z of arithmetic genus 0 and degree 3 in X.

Dualizing the sequence above we have:

$$0 \to \mathcal{U}_+^*(-1) \to \bigoplus_{i=1,2,3} \mathscr{E}^*_{y_i} \to \mathscr{O}_X \to \mathscr{E}xt^2_X(\mathscr{F}_Z, \mathscr{O}_X) \to 0.$$

So we can define our map ϑ as $\vartheta(Z) = \mathbf{R}\mathcal{H}om_X(\mathscr{F}_Z, \mathscr{O}_X[-2])$. Applying Lemma 2.7, we see that ϑ and φ are inverse to each other. $\square$

5. Vector bundles on Fano threefolds of genus 7

In this section, we assume that X is a smooth prime Fano threefold of genus 7, and we let Γ be its homologically projectively dual curve. We set up a birational correspondence between the component $\mathsf{M}(d)$ of Definition 3.8 and a component of the Brill-Noether variety $W^{2\,d-11}_{d-5,5\,d-24}$. This correspondence will turn out to be an isomorphism for $d = 6$. In this section, we will need to assume that X is ordinary for $d \geq 7$, in order to ensure the existence of the well-behaved component $\mathsf{M}(d)$, constructed in Theorem 3.7.

5.1. **Vanishing results.** In order to setup the correspondence mentioned above, we will have to prove that various cohomology groups are zero. This is the purpose of the next series of lemmas.

Lemma 5.1. *Let $d \geq 6$ and let F be a sheaf in $\mathsf{M}_X(2,1,d)$ such that:*

$$\mathrm{H}^1(X, F(-1)) = 0. \tag{5.1}$$

Then we have:

$$\mathrm{Ext}^k_X(F, \mathscr{O}_X) = 0, \qquad \textit{for any } k \in \mathbb{Z}, \tag{5.2}$$

$$\mathrm{Ext}^k_X(F, \mathcal{U}_+) = 0, \qquad \textit{for any } k \neq 2, \tag{5.3}$$

$$\mathrm{ext}^2_X(F, \mathcal{U}_+) = 2d - 10. \tag{5.4}$$

Proof. By Serre duality and (5.1), the vanishing (5.2) follows from Proposition 3.6.

Moreover, for $k = 0, 3$ we have $\mathrm{Ext}^k_X(F, \mathcal{U}_+) = 0$ by stability of the sheaves $\mathcal{U}_+$ and F. Since $\mathrm{Ext}^1_X(F, \mathscr{O}_X) = 0$, by applying the functor $\mathrm{Hom}_X(F, -)$ to the sequence (2.12) we get $\mathrm{Ext}^1_X(F, \mathcal{U}_+) \cong \mathrm{Hom}_X(F, \mathcal{U}_+^*)$, which vanishes by stability of F and $\mathcal{U}_+$. This proves (5.3). Finally, by the Riemann-Roch formula (2.10) we obtain the last equality. $\square$

Lemma 5.2. *Let $d \geq 6$ and let F be a sheaf in $\mathsf{M}_X(2,1,d)$ satisfying (5.1). Then, for all $y \in \Gamma$:*

$$\mathrm{Ext}^k_X(\mathscr{E}_y, F) = 0, \qquad \textit{for all } k \neq 1, \tag{5.5}$$

$$\mathrm{Ext}^j_X(F, \mathscr{E}_y) = 0, \qquad \textit{for all } j \geq 2. \tag{5.6}$$

Moreover, if F is locally free, then (5.6) holds for $j = 0$ as well.

Proof. Let us prove the first vanishing. Notice that for $k < 0$ and for $k > 3$ the claim is obvious since X has dimension 3. For $k = 3$, the claim amounts to $\mathrm{Hom}_X(F, \mathscr{E}_y(-1)) = \mathrm{Hom}_X(F, \mathscr{E}_y^*) = 0$, which follows from stability of F and $\mathscr{E}_y$.

For $k = 0$, we have to show that $\mathrm{Hom}_X(\mathscr{E}_y, F) = 0$. Assume the contrary, and let f be a non-trivial map $f : \mathscr{E}_y \to F$. Note that the image I of f must have rank 2, for if it had rank 1 it would destabilize either F (if $c_1(I) \geq 1$), or $\mathscr{E}_y$ (if $c_1(I) \leq 0$). So f is injective, and so an isomorphism by Lemma 2.1. But we have $c_2(\mathscr{E}_y) = 5$ and $c_2(F) = d \geq 6$, a contradiction.

For $k = 2$, let us show that $\mathrm{Ext}^1_X(F, \mathscr{E}_y^*) = 0$. Applying the functor $\mathrm{Hom}_X(F, -)$ to the restriction of (2.21) to $X \times \{y\}$, we get:

$$\mathrm{Hom}_X(F, \mathscr{G}_y) \to \mathrm{Ext}^1_X(F, \mathscr{E}_y^*) \to \mathrm{Ext}^1_X(F, \mathscr{O}_X) \otimes (\mathcal{U}_-)_y.$$

It is easy to see that the term on the left hand side vanishes by virtue of stability of $\mathscr{G}_y$ and F (see Lemma 2.6). On the other hand, the rightmost term vanishes by (5.2). We have thus proved (5.5).

Let us now turn to (5.6). For $j = 3$, one easily sees that this group vanishes by stability of F and $\mathscr{E}_y$. In order to prove (5.6) for $j = 2$, we apply the functor $\mathrm{Hom}_X(F, -)$ to (2.22), and by stability of $\mathscr{G}_y$, we are reduced to show that $\mathrm{Ext}^2_X(F, \mathcal{U}_+^*) = 0$. But this follows easily applying $\mathrm{Hom}_X(F, -)$ to (2.12).

Finally, if F is locally free, then a nonzero map $f \in \mathrm{Hom}_X(F, \mathscr{E}_y)$ must be injective by stability of F and $\mathscr{E}_y$. Hence, by Lemma 2.1, the map f is an isomorphism, but this is impossible because $c_2(F) \neq c_2(\mathscr{E}_y)$. $\square$

Lemma 5.3. *Let $d \geq 7$ and let F be a sheaf in $\mathsf{M}_X(2,1,d)$ satisfying (5.1). Then we have:*

$$\mathrm{Hom}_X(\mathcal{U}_+^*, F) = 0.$$

Proof. Fix a point y in Γ. Applying the functor $\mathrm{Hom}_X(-,F)$ to the exact sequence (2.22) we obtain an exact sequence:

$$0 \to \mathrm{Hom}_X(\mathscr{E}_y, F) \to \mathrm{Hom}_X(\mathcal{U}_+^*, F) \to \mathrm{Hom}_X(\mathscr{G}_y, F).$$

By Lemma 5.2 we know that the leftmost term vanishes. Assume that the rightmost does not, and consider a non-zero map $f : \mathscr{G}_y \to F$. Set $F' = \mathrm{Im}(f)$ and note that, by stability of the sheaves F and $\mathscr{G}_y$, we must have $\mathrm{rk}(F') = 2$ and $c_1(F') = 1$. We have thus an exact sequence:

$$0 \to F' \to F \to T \to 0, \tag{5.7}$$

where T is a torsion sheaf, with $\dim(\mathrm{supp}(T)) \leq 1$. Note that F' is stable, since any destabilizing subsheaf would destabilize also F. By [29, Propositions 1.1 and 1.9], the sheaf $\ker(f)$ must be a line bundle of degree zero. This means that $\ker(f) \cong \mathscr{O}_X$, and we have an exact sequence:

$$0 \to \mathscr{O}_X \to \mathscr{G}_y \to F' \to 0.$$

So F' satisfies $c_2(F') = 7$, $c_3(F') = 2$. Hence by (5.7) it follows that $d = c_2(F) = 7 + c_2(T) \leq 7$, since $c_2(T)$ is non-positive. Hence we have $d = 7$ and in this case we have $c_2(T) = 0$, and $c_3(T) = -2$. But this is a contradiction since $c_3(T)$ must be non-negative. We have thus proved our claim. □

5.2. **Canonical resolution of a bundle in $\mathsf{M}_X(2,1,d)$.** We will show here that a sheaf F in $\mathsf{M}_X(2,1,d)$ which satisfies $\mathrm{H}^1(X, F(-1)) = 0$ admits a canonical resolution having two terms, see the formula (5.9) below. Recall that, if the threefold X is ordinary, such a sheaf exists for all $d \geq 6$ by Theorem 3.7.

Proposition 5.4. *Let $d \geq 6$ and let F be a sheaf in $\mathsf{M}_X(2,1,d)$ such that $\mathrm{H}^1(X, F(-1)) = 0$. Then $\mathcal{F} = \mathbf{\Phi}^!(F)$ is a simple vector bundle on Γ, with:*

$$\mathrm{rk}(\mathcal{F}) = d - 5, \qquad \deg(\mathcal{F}) = 5\,d - 24. \tag{5.8}$$

Moreover, F admits the following canonical resolution:

$$0 \to \mathrm{Ext}^2_X(F, \mathcal{U}_+)^* \otimes \mathcal{U}_+^* \xrightarrow{\zeta_F} \mathbf{\Phi}(\mathcal{F}) \to F \to 0, \tag{5.9}$$

and $\mathbf{\Phi}(\mathcal{F})$ is a simple vector bundle.

Proof. Consider the stalk over a point $y \in \Gamma$ of the sheaf $\mathcal{H}^k(\mathbf{\Phi}^!(F))$. We have:

$$\mathcal{H}^k(\mathbf{\Phi}^!(F))_y \cong \mathrm{Ext}_X^{k+1}(\mathscr{E}_y, F) \otimes \omega_{\Gamma,y}.$$

Hence by Lemma 5.2 it follows that this group vanishes for all $y \in \Gamma$ and for all $k \neq 0$, so $\mathbf{\Phi}^!(F)$ can be identified with a coherent sheaf $\mathcal{F}$ on Γ. By Remark 2.8 the sheaf $\mathcal{F}$ is locally free over Γ, with rank $d-5$, and $\deg(\mathcal{F}) = 5\,d - 24$.

Let us exhibit the resolution (5.9). Note that, by formula (2.20) and Lemma 5.1 we get that the complex $(\mathbf{\Psi}(\mathbf{\Psi}^*(F)))$ is concentrated in degree -1 and isomorphic to $\mathrm{Ext}^2_X(F, \mathcal{U}_+)^* \otimes \mathcal{U}_+^*$. Hence the exact triangle (2.19) provides the resolution (5.9) for F.

This resolution also proves that $\mathcal{H}^i(\mathbf{\Phi}(\mathcal{F})) = 0$, for all $i \neq 0$, which means (by the definition (2.17) of $\mathbf{\Phi}$) that $\mathbf{R}^i p_*(q^*(\mathcal{F} \otimes \mathscr{E})) = 0$ for all $i \neq 0$. Then by [22, Corollaries 7.9.9 and 7.9.10], we conclude that $\mathbf{\Phi}(\mathcal{F}) = \mathbf{R}^0 p_*(q^*(\mathcal{F}) \otimes \mathscr{E})$ is a locally free sheaf on X.

Let us now prove that $\mathcal{F}$ is a simple bundle. If $d = 6$, then $\mathcal{F}$ is a line bundle, hence it is obviously simple. For $d \geq 7$ we want to prove that the group $\mathrm{Hom}_\Gamma(\mathcal{F}, \mathcal{F}) \cong \mathrm{Hom}_X(\mathbf{\Phi}(\mathcal{F}), F)$ is 1-dimensional. Applying the functor $\mathrm{Hom}_X(-, F)$ to the sequence (5.9) we obtain:

$$\mathrm{Hom}_X(\mathbf{\Phi}(\mathbf{\Phi}^!(F)), F) \cong \mathrm{Hom}_X(F, F),$$

since the term $\mathrm{Hom}_X(\mathcal{U}_+^*, F)$ vanishes by Lemma 5.3. Hence $\mathcal{F} = \mathbf{\Phi}^!(F)$ is simple, for F is. Since the functor $\mathbf{\Phi}$ is fully faithful, it follows that also the vector bundle $\mathbf{\Phi}(\mathcal{F})$ is simple. □

Lemma 5.5. *Let $d \geq 7$ and let F be as in the previous proposition, and set $\mathcal{F} = \mathbf{\Phi}^!(F)$, $A_F = \mathrm{Ext}^2_X(F, \mathcal{U}_+)^*$. Then we have the natural isomorphism:*

$$A_F \cong \mathrm{H}^0(\Gamma, \mathcal{F}) \cong \mathrm{Hom}_X(\mathcal{U}_+^*, \mathbf{\Phi}(\mathcal{F})).$$

In particular $\mathrm{h}^0(\Gamma, \mathcal{F}) = 2d - 10$.

Proof. By Lemma 5.3 we know that $\mathrm{Hom}_X(\mathcal{U}_+^*, F) = 0$. Therefore, applying the functor $\mathrm{Hom}_X(\mathcal{U}_+^*, -)$ to the resolution (5.9) we obtain:

$$A_F = \mathrm{Ext}^2_X(F,\mathcal{U}_+)^* \cong \mathrm{Hom}_X(\mathcal{U}_+^*, \mathbf{\Phi}(\mathcal{F})).$$

By adjunction and (2.30) we have the isomorphisms:

$$\mathrm{Hom}_X(\mathcal{U}_+^*, \mathbf{\Phi}(\mathcal{F})) \cong \mathrm{Hom}_\Gamma(\mathbf{\Phi}^*(\mathcal{U}_+^*), \mathcal{F}) \cong \mathrm{H}^0(\Gamma, \mathcal{F}).$$

The last statement follows by (5.4). □

Lemma 5.6. *Let $d \geq 6$ and let F be a locally free sheaf in $\mathsf{M}_X(2,1,d)$ with $\mathrm{H}^1(X, F(-1)) = 0$. Set $\mathcal{F} = \mathbf{\Phi}^!(F)$. Then $\mathcal{F}$ is globally generated and we have the exact sequence:*

$$0 \to \mathcal{F}^* \to \mathrm{Ext}^2_X(F,\mathcal{U}_+)^* \otimes \mathscr{O}_\Gamma \to \mathcal{F} \to 0. \tag{5.10}$$

Proof. Consider the complex $\mathbf{\Phi}^*(F)$. Let us compute the stalk over the point $y \in \Gamma$ of the sheaf $\mathcal{H}^k(\mathbf{\Phi}^*(F))$. We obtain:

$$\mathcal{H}^{-k}(\mathbf{\Phi}^*(F))_y \cong \mathrm{Ext}^k_X(F, \mathscr{E}_y)^*.$$

But the above vector space vanishes for any $k \neq 1$ in view of Lemma 5.2, for F is locally free. So $\mathbf{\Phi}^*(F)[-1]$ is a locally free sheaf. Moreover, applying (2.25) we have $(\mathbf{\Phi}^!(F))^* \cong \mathbf{R}\mathcal{H}om_\Gamma(\mathcal{F}, \mathscr{O}_\Gamma) \cong \mathbf{\Phi}^!(F^*) \otimes \omega_\Gamma^*[1]$. Using the definition of $\mathbf{\Phi}^!$ and the isomorphism $F^* \cong F(-H_X)$, we have $\mathbf{\Phi}^!(F^*) \otimes \omega_\Gamma^*[1] \cong \mathbf{R}q_*(p^*(F) \otimes \mathscr{E}^*(H_\Gamma - H_X))[1] \otimes \mathscr{O}_\Gamma(-H_\Gamma)[1]$. By definition of $\mathbf{\Phi}^*$ we have $\mathbf{\Phi}^*(F)[-1] = \mathbf{R}q_*(p^*(F) \otimes \mathscr{E}^*(-H_X))[2]$, and so we get:

$$\mathbf{\Phi}^*(F)[-1] \cong (\mathbf{\Phi}^!(F))^*.$$

Remark that, for any sheaf $\mathcal{P}$ on the curve Γ, since the functor $\mathbf{\Phi}$ is fully faithful, we have:

$$\mathbf{\Phi}^*(\mathbf{\Phi}(\mathcal{P})) \cong \mathcal{P}.$$

Thus, applying the functor $\mathbf{\Phi}^*$ to (5.9), we obtain, in view of (2.30), the exact sequence (5.10). Hence, the sheaf $\mathcal{F}$ is globally generated. □

In order to set up our correspondence between $\mathsf{M}(d)$ and $W^{2\,d-11}_{d-5,5\,d-24}$, we have to prove that, for a general F in $\mathsf{M}(d)$, the vector bundle $\mathbf{\Phi}^!(F)$ over Γ is stable. This is done in the next lemma.

Lemma 5.7. *For each integer $d \geq 6$, there exists a Zariski dense open subset $\Omega(d) \subset \mathsf{M}(d)$, such that each point F_d of $\Omega(d)$ satisfies $\mathrm{H}^1(X, F_d(-1)) = 0$, and $\mathcal{F}_d = \mathbf{\Phi}^!(F_d)$ is a stable sheaf.*

Proof. Let us prove the statement by induction on $d \geq 6$. If $d = 6$, $\mathcal{F}_6 = \mathbf{\Phi}^!(F_6)$ is stable since it is a line bundle. Suppose now $d > 6$, assume that F_d and F_{d-1} fit into (3.17) for some line $L \subset X$, and that $\mathcal{F}_{d-1} = \mathbf{\Phi}^!(F_{d-1})$ is a stable bundle. Recall that $\mathcal{L} = \mathbf{\Phi}^!(\mathscr{O}_L)[-1]$ is a line bundle of degree 5 by Proposition 4.1. Applying the functor $\mathbf{\Phi}^!$ to the sequence (3.17), we get

$$0 \to \mathcal{L} \to \mathcal{F}_d \to \mathcal{F}_{d-1} \to 0. \tag{5.11}$$

Notice that the extension (5.11) is non-trivial because $\mathcal{F}_d$ is indecomposable since it is simple (see Proposition 5.4).

Since, by formulas (5.8), we know that $\mu(\mathcal{F}_d) = \frac{5d-24}{d-5} = 5 + \frac{1}{d-5}$, it is enough to prove that $\mathcal{F}_d$ is semistable. Assume by contradiction that there exists a subsheaf $\mathcal{K}$ destabilizing $\mathcal{F}_d$ of rank $r < d-5$ and degree c. Since $\mathcal{F}_{d-1}$ is stable, we must have:

$$5 + \frac{1}{d-5} < \frac{c}{r} \leq 5 + \frac{1}{d-6},$$

from which we get:

$$0 < \frac{c}{r} - \frac{5d-24}{d-5} \leq \frac{1}{(d-5)(d-6)}.$$

It is easy to check that the only possibility is $r = d-6$ and $c = 5d-29$, and so we would have $\mathcal{K} \cong \mathcal{F}_{d-1}$ and (5.11) would split, a contradiction. Hence $\mathcal{F}_d$ is stable. Therefore the same holds for a general point of $\mathsf{M}(d)$ by Maruyama's result [50]. □

Remark 5.8. We do not know whether the bundle $\mathcal{F} = \mathbf{\Phi}^!(F)$ is stable for *all* sheaves F in $\mathsf{M}_X(2,1,d)$ with $\mathrm{H}^1(X, F(-1)) = 0$, not even assuming that F lies in the component $\mathsf{M}(d)$.

In the next section we will study the space $\mathsf{M}_X(2,1,d)$, focusing first on the case $d \geq 7$, where we give a birational description. The case $d = 6$ will be treated in greater detail further on.

5.3. **The moduli spaces $\mathsf{M}_X(2,1,d)$, with $d \geq 7$.** Here we show that the component $\mathsf{M}(d)$ of the variety $\mathsf{M}_X(2,1,d)$ containing the sheaves arising from the construction of Theorem 3.7 is birational to a component $\mathsf{W}(d)$ of $W^{2\,d-11}_{d-5,5\,d-24}$. Recall that in Lemma 5.7 we have introduced the open set $\Omega(d) \subset \mathsf{M}(d)$. Every sheaf $F \in \Omega(d)$ satisfies the following two conditions:

i) the group $\mathrm{H}^1(X, F(-1))$ vanishes,
ii) the vector bundle $\mathcal{F} = \mathbf{\Phi}^!(F)$ is stable.

Then we have a morphism:

$$\varphi : \Omega(d) \to W^{2\,d-11}_{d-5,5\,d-24}, \qquad\qquad F \mapsto \mathbf{\Phi}^!(F).$$

which is well-defined by Proposition 5.4 and Lemmas 5.5, 5.7. We denote by $\mathsf{W}(d)$ the irreducible component of $W^{2\,d-11}_{d-5,5\,d-24}$ containing the image of φ. We can thus state the following result.

Theorem 5.9. *Let X be a smooth ordinary prime Fano threefold of genus 7, and let F be a sheaf in $\Omega(d)$ for $d \geq 7$. Then:*

i) the tangent space, respectively the space containing obstructions, to $\mathsf{W}(d)$ at the point $[\mathbf{\Phi}^!(F)]$ is naturally identified with $\mathrm{Ext}^1_X(F,F)$, respectively with $\mathrm{Ext}^2_X(F,F)$;
ii) the varieties $\mathsf{M}(d)$ and $\mathsf{W}(d)$ are birational, both generically smooth of dimension $2d-9$.

Proof. The goal of our proof will be to construct an inverse map ϑ to the morphism φ, defined on a suitable open subset of our component $\mathsf{W}(d)$. We let $B(d)$ be the subset of $\mathsf{W}(d)$ consisting of those sheaves $\mathcal{F}$ such that:

a) the following natural evaluation map is surjective:

$$e_{\mathcal{F}} = e_{\mathscr{O},\mathcal{F}} : \mathrm{H}^0(\Gamma, \mathcal{F}) \otimes \mathscr{O}_\Gamma \to \mathcal{F}$$

b) the kernel of $e_{\mathcal{F}}$ is isomorphic to $\mathcal{F}^*$;
c) the complex $\mathbf{\Phi}(\mathcal{F})$ is concentrated in degree zero.

Then we would like to define ϑ over $B(d)$ in the following way:

$$\vartheta : B(d) \to \mathsf{M}_X(2,1,d), \qquad\qquad \mathcal{F} \mapsto \mathrm{cok}(\mathcal{H}^0(\mathbf{\Phi}(e_{\mathcal{F}}))).$$

Let us prove that the sheaf $F = \mathrm{cok}(\mathcal{H}^0(\mathbf{\Phi}(e_{\mathcal{F}})))$ lies in $\mathsf{M}_X(2,1,d)$. First note that the duality (2.25) gives the isomorphism:

$$\mathbf{\Phi}(\mathcal{F}^*)[1] \cong \mathbf{\Phi}(\mathcal{F})^*(1),$$

where these are both locally free sheaves by (c) and by the same argument as in Proposition 5.4. By (a) and (b) we can apply the functor $\mathbf{\Phi}$ to the exact sequence:

$$0 \to \mathcal{F}^* \xrightarrow{e_{\mathcal{F}}^\top} \mathrm{H}^0(\Gamma, \mathcal{F}) \otimes \mathscr{O}_\Gamma \xrightarrow{e_{\mathcal{F}}} \mathcal{F} \to 0,$$

and get (by (c)) a long exact sequence of the form:

$$0 \to \mathrm{H}^0(\Gamma, \mathcal{F}) \otimes \mathcal{U}_+^* \xrightarrow{\mathcal{H}^0(\mathbf{\Phi}(e_{\mathcal{F}}))} \mathbf{\Phi}(\mathcal{F}) \to \mathbf{\Phi}(\mathcal{F})^* \otimes \mathscr{O}_X(1) \to \mathrm{H}^0(\Gamma, \mathcal{F}) \otimes \mathcal{U}_+(1) \to 0.$$

The sheaf F is then the image of the middle map in the above sequence. By [29, Proposition 1.1] the sheaf F is reflexive, and sits into the following exact sequence:

$$0 \to \mathrm{H}^0(\Gamma, \mathcal{F}) \otimes \mathcal{U}_+^* \xrightarrow{\mathcal{H}^0(\mathbf{\Phi}(e_{\mathcal{F}}))} \mathbf{\Phi}(\mathcal{F}) \to F \to 0. \tag{5.12}$$

By Grothendieck-Riemann-Roch one can calculate the rank and the Chern classes of $\mathbf{\Phi}(\mathcal{F})$, and deduce from the above exact sequence that the sheaf F has rank 2 and $c_1(F) = 1$, $c_2(F) = d$, $c_3(F) = 0$. Therefore F is locally free (because it is reflexive with vanishing c_3), so it is also stable, once we prove $\mathrm{Hom}_X(\mathscr{O}_X(1), F) = 0$. Now applying the functor $\mathrm{Hom}_X(\mathscr{O}_X(1), -)$ to (5.12), we have $\mathrm{Hom}_X(\mathscr{O}_X(1), F) \cong \mathrm{Hom}_X(\mathscr{O}_X(1), \mathbf{\Phi}(\mathcal{F}))$, and the latter group vanishes by the semiorthogonal decomposition (2.34). We have thus proved that F lies in $\mathsf{M}_X(2,1,d)$, so the map ϑ is defined on $B(d)$.

Applying the functor $\mathbf{\Phi}^!$ to (5.12) and using (2.31), we have $\mathbf{\Phi}^!(F) \cong \mathcal{F}$, so $\varphi(\vartheta(\mathcal{F})) = \mathcal{F}$. Let us now show that $\vartheta \circ \varphi$ is the identity over the open subset of $\Omega(d)$ consisting of locally free sheaves. For any sheaf E in $\Omega(d)$, the sheaf $\mathcal{F} = \mathbf{\Phi}^!(E)$ lies in $\mathsf{W}(d)$ and satisfies (c). Assume now in addition that E is locally free. Set again $A_E = \mathrm{Ext}^2_X(E, \mathcal{U}_+)^*$, and keep in mind the natural isomorphism of Lemma 5.5:

$$A_E \cong \mathrm{H}^0(\Gamma, E) \cong \mathrm{Hom}_X(\mathcal{U}_+^*, \mathbf{\Phi}(\mathcal{F})).$$

Further, by Lemma 5.6, since E is a locally free sheaf in $\mathsf{M}(d)$, we have that $\mathcal{F}$ satisfies also (a) and (b). We look at the resolution:

$$0 \to A_E^* \otimes \mathcal{U}_+^* \xrightarrow{\zeta_E} \mathbf{\Phi}(\mathcal{F}) \to E \to 0$$

given by Proposition 5.4. Notice that the map ζ_E agrees, up to a non-zero scalar, with the map $\mathcal{H}^0(\mathbf{\Phi}(e_{\mathcal{F}}))$. Indeed, both such maps are non-zero elements of $A_E \otimes A_E^*$, invariant under the natural $\mathsf{GL}(A_E)$-action, and such invariant is either zero, either unique up to non-zero scalar. Therefore:

$$E \cong \operatorname{cok}(\mathcal{H}^0(\mathbf{\Phi}(e_{\mathcal{F}}))),$$

which proves that $\vartheta \circ \varphi$ gives back E when applied to E.

We have thus proved that $B(d)$ is isomorphic to the open subset of $\Omega(d)$ consisting of locally free sheaves. This will prove a refinement of (ii) once we show that $B(d)$ is Zariski dense in $\mathsf{W}(d)$. Note that in view of (b) $B(d)$ is only open in a closed subset of $\mathsf{W}(d)$. But it will follow from (i) that $\mathsf{W}(d)$ is of dimension $2d-9$, then $B(d)$ and $\mathsf{W}(d)$ have the same dimension, so $B(d)$ is Zariski dense.

Therefore, it only remains to setup the natural identifications of tangent spaces required for (i). Let us accomplish this task. By using our adjoint functors, we have the natural isomorphisms:

$$\begin{aligned} &\operatorname{Ext}^1_X(\mathbf{\Phi}(\mathcal{F}), F) \cong \operatorname{Ext}^1_\Gamma(\mathcal{F}, \mathcal{F}), \\ &\operatorname{Ext}^1_X(\mathcal{U}_+^*, F) \cong \operatorname{Ext}^1_X(\mathbf{\Phi}(\mathscr{O}_\Gamma), F) \cong \operatorname{Ext}^1_\Gamma(\mathscr{O}_\Gamma, \mathcal{F}). \end{aligned} \tag{5.13}$$

Here, to prove (5.13), by (2.5) and (2.32) it suffices to show $\operatorname{Ext}^2_X(\mathcal{U}_+(1), F) = 0$ and $\operatorname{Ext}^3_X(\mathcal{U}_+(1), F) = 0$. By Serre duality we have $\operatorname{Ext}^k_X(\mathcal{U}_+(1), F)^* \cong \operatorname{Ext}^{3-k}_X(F, \mathcal{U}_+)$, which vanishes, for $k = 2, 3$, by Lemma 5.1. Now, applying the functor $\operatorname{Hom}_X(-, F)$ to (5.9) we obtain a long exact sequence:

$$\cdots \to \operatorname{Hom}_X(\mathcal{U}_+^*, F) \otimes A_F^* \to \operatorname{Ext}^1_X(F, F) \to \operatorname{Ext}^1_X(\mathbf{\Phi}(\mathcal{F}), F) \xrightarrow{\eta_F}$$
$$\xrightarrow{\eta_F} \operatorname{Ext}^1_X(\mathcal{U}_+^*, F) \otimes A_F^* \to \operatorname{Ext}^2_X(F, F) \to \operatorname{Ext}^2_X(\mathbf{\Phi}(\mathcal{F}), F) \to \cdots$$

where the map η_F is defined as $\operatorname{Ext}^1_X(\zeta_F, F)$. Note that the first and the last terms of the above sequence vanish, by Lemma 5.3 and since Γ is a curve. Hence we can identify $\operatorname{Ext}^1_X(F, F)$ with the kernel of the map η_F and $\operatorname{Ext}^2_X(F, F)$ with the cokernel of η_F. But since $\zeta_F = \mathcal{H}^0(\mathbf{\Phi}(e_{\mathcal{F}}))$, we have:

$$\eta_F = \operatorname{Ext}^1_X(\mathbf{\Phi}(e_{\mathcal{F}}), F) = \operatorname{Ext}^1_\Gamma(e_{\mathcal{F}}, \mathbf{\Phi}^!(F)) = \operatorname{Ext}^1_\Gamma(e_{\mathcal{F}}, \mathcal{F}) = \pi_{\mathcal{F}}^\top.$$

In view of the interpretation of the kernel and cokernel of $\pi_{\mathcal{F}}^\top$ (see Section 2.1.4), we have thus constructed the required identification of the tangent space to $W^{2\,d-11}_{d-5,5\,d-24}$ at the point $\mathcal{F}$ (i.e. of $\ker(\pi_{\mathcal{F}}^\top)$) with $\operatorname{Ext}^1_X(F, F)$. The same argument identifies the space containing obstructions with $\operatorname{Ext}^2_X(F, F)$. ☐

A relative version of the construction of the previous theorem gives the following result.

Corollary 5.10. *The moduli space $\Omega(d) \subset \mathsf{M}(d)$ is fine.*

Proof. For any $d \geq 6$, we let $\mathsf{P}(d)$ be the moduli space of stable vector bundles on Γ of rank $d-5$ and degree $5\,d-24$. Thus $\mathsf{W}(d)$ is a subvariety of $\mathsf{P}(d)$. Since the rank and the degree are coprime, it is well known that this moduli space is fine. So we denote by $\mathscr{P}$ the universal bundle over $\Gamma \times \mathsf{P}(d)$, and by abuse of notation its restriction to the product $\Gamma \times \mathsf{W}(d)$.

We would like to exhibit a universal sheaf $\mathscr{F}$ over $X \times \Omega(d)$ such that, for a given closed point z of $\Omega(d)$ representing a stable sheaf F, the restriction of $\mathscr{F}$ to $X \times \{z\}$ is isomorphic to F. Recall by Theorem 5.9 that φ maps $\Omega(d) \subset \mathsf{M}(d)$ to an open subset of $\mathsf{W}(d)$. Consider the projections:

$$\begin{array}{ccc} & X \times \Gamma \times \Omega(d) & \\ {}^{p\times 1}\swarrow & & \searrow^{q\times\varphi} \\ X \times \Omega(d) & & \Gamma \times \varphi(\Omega(d)) \end{array}$$

We consider the pull-back to $X \times \Gamma \times \Omega(d)$ of the map $\alpha : \mathcal{U}_+^* \to \mathscr{E}$ of (2.15). We tensor this map with $(q \times \varphi)^*(\mathscr{P})$. We have thus a morphism:

$$\mathcal{U}_+^* \boxtimes (q \times \varphi)^*(\mathscr{P}) \xrightarrow{\alpha \boxtimes 1} \mathscr{E} \boxtimes (q \times \varphi)^*(\mathscr{P}).$$

We define the universal sheaf $\mathscr{F}$ as the cokernel of the map $(p \times 1)_*(\alpha \boxtimes 1)$. Let us verify that $\mathscr{F}$ has the desired properties. So choose a closed point $z \in \Omega(d) \subset \mathsf{M}(d)$, and consider the corresponding sheaf F_z on X and the vector bundle $\mathscr{P}_{\varphi(z)} \cong \mathbf{\Phi}^!(F_z)$ on Γ. Notice that the sheaf $(q \times \varphi)^*(\mathscr{P}_{\varphi(z)})$ is just $q^*(\mathbf{\Phi}^!(F_z))$. Then, evaluating at the point z the map $(p \times 1)_*(\alpha \boxtimes 1)$ we obtain the map:

$$\mathrm{H}^0(\Gamma, \mathbf{\Phi}^!(F_z)) \otimes \mathcal{U}_+^* \to \mathbf{\Phi}(\mathbf{\Phi}^!(F_z)).$$

Recall the natural isomorphism $\mathrm{H}^0(\Gamma, \mathbf{\Phi}^!(F_z)) \cong \mathrm{Ext}^2_X(F_z, \mathcal{U}_+)^*$, and note that, by functoriality, this map agrees with the map ζ_{F_z} given by the resolution (5.9). Thus its cokernel is F_z. $\square$

5.4. **The moduli space $\mathsf{M}_X(2,1,6)$.** Here we focus on the moduli space $\mathsf{M}_X(2,1,6)$, and we prove that it is isomorphic to the Brill-Noether locus $W^1_{1,6}$ on the homologically projectively dual curve Γ. This makes more precise a result of Iliev-Markushevich, [37]. Then we investigate the subvariety of $\mathsf{M}_X(2,1,6)$ consisting of vector bundles which are not globally generated. We will see that these bundles are in one-to-one correspondence with non-reflexive sheaves in $\mathsf{M}_X(2,1,6)$. Finally we will see that these two subsets are interchanged by a natural involution of $\mathsf{M}_X(2,1,6)$.

5.4.1. *The moduli space $\mathsf{M}_X(2,1,6)$ as a Brill-Noether locus.* Here is the main result of this section.

Theorem 5.11. *Let X be a smooth prime Fano threefold of genus 7.*

A) The map $\varphi : F \mapsto \mathbf{\Phi}^!(F)$ gives an isomorphism of the moduli space $\mathsf{M}_X(2,1,6)$ onto the Brill-Noether variety $W^1_{1,6}$. In particular, $\mathsf{M}_X(2,1,6)$ is a connected threefold. Moreover it is a fine moduli space.

B) If X is not exotic, then $\mathsf{M}_X(2,1,6)$ has at most finitely many singular points. If X is general, then $\mathsf{M}_X(2,1,6)$ is smooth and irreducible.

We prove now part A. We postpone the proof of part B to the end of the subsection.

Proof of Theorem 5.11, part A. First of all the map $\varphi : F \mapsto \mathbf{\Phi}^!(F)$ is well-defined. Indeed, let F be any sheaf in $\mathsf{M}_X(2,1,6)$. By Proposition 3.4, part (i), we know that F satisfies the hypothesis (5.1). Then by Proposition 5.4, $\mathbf{\Phi}^!(F)$ is a line bundle of degree 6 on Γ. Set $\mathcal{L} = \mathbf{\Phi}^!(F)$. We have to prove that $\mathcal{L}$ admits at least two independent global sections. If F is locally free, by Lemma 5.6 we have that $\mathcal{L}$ is globally generated. Hence $\mathrm{h}^0(\Gamma, \mathcal{L}) = 1$ would imply $\mathcal{L} \cong \mathscr{O}_\Gamma$, which is impossible, i.e. $\mathrm{h}^0(\Gamma, \mathcal{L}) \geq 2$. This means that $\mathbf{\Phi}^!(F)$ lies in $W^1_{1,6}$. If F is not locally free, then it fits in the exact sequence (3.3). Recall that by Proposition 4.1 we know that $\mathbf{\Phi}^!(\mathscr{O}_L)[-1]$ is a line bundle $\mathcal{M}$ contained in $W^1_{1,5}$. Hence, applying $\mathbf{\Phi}^!$ to the exact sequence (3.3), we obtain:

$$0 \to \mathcal{M} \to \mathcal{L} \to \mathscr{O}_y \to 0, \tag{5.14}$$

where y is a point of Γ. Therefore we have again $\mathrm{h}^0(\Gamma, \mathcal{L}) \geq \mathrm{h}^0(\Gamma, \mathcal{M}) \geq 2$, and $\mathbf{\Phi}^!(F)$ lies in $W^1_{1,6}$. Note that the equality $\mathrm{h}^0(\Gamma, \mathcal{L}) = 2$ must be attained for all $\mathcal{L}$, since $W^2_{1,6}$ is empty in view of Mukai's result (see [59, Table 1]). Note that in this case the open subset $\Omega(6)$ coincides in fact with all of $\mathsf{M}_X(2,1,6)$.

Now we want to provide an inverse map $\vartheta : W^1_{1,6} \to \mathsf{M}_X(2,1,6)$ of φ. Take a line bundle $\mathcal{L}$ in $W^1_{1,6}$, and denote again by $e_{\mathcal{L}} = e_{\mathscr{O}_\Gamma, \mathcal{L}} : \mathrm{H}^0(\Gamma, \mathcal{L}) \otimes \mathscr{O}_\Gamma \to \mathcal{L}$ the natural evaluation map. We distinguish two cases according to whether $\mathcal{L}$ is globally generated or not.

In the former case, $\ker(e_{\mathcal{L}})$ is a reflexive sheaf of rank 1, so $\ker(e_{\mathcal{L}})$ is invertible and $c_1(\ker(e_{\mathcal{L}})) = -c_1(\mathcal{L})$ so $\ker(e_{\mathcal{L}}) \cong \mathscr{O}_\Gamma(-c_1(\mathcal{L})) \cong \mathcal{L}^*$. So we have an exact sequence:

$$0 \to \mathcal{L}^* \to \mathrm{H}^0(\Gamma, \mathcal{L}) \otimes \mathscr{O}_\Gamma \xrightarrow{e_{\mathcal{L}}} \mathcal{L} \to 0.$$

Since, for any $x \in X$ the vector bundle $\mathscr{E}_x$ is stable, we have $\mathrm{H}^0(\Gamma, \mathscr{E}_x \otimes \mathcal{L}^*) = \mathrm{H}^1(\Gamma, \mathscr{E}_x \otimes \mathcal{L}) = 0$. Hence the line bundle $\mathcal{L}$ satisfies all the conditions (a), (b), (c) of the proof of Theorem 5.9. Then the same proof of such theorem allows us to define $\vartheta(\mathcal{L}) = \mathrm{cok}(\mathcal{H}^0(\mathbf{\Phi}(e_{\mathcal{L}})))$ and to prove that $\varphi(\vartheta(\mathcal{L})) = \mathcal{L}$.

It remains to find an inverse image via φ of a non-globally generated sheaf $\mathcal{L}$. In this case, the image $\mathcal{M} \subset \mathcal{L}$ of $e_{\mathcal{L}}$ must be a line bundle, with $\mathrm{h}^0(\Gamma, \mathcal{M}) = \mathrm{h}^0(\Gamma, \mathcal{L}) = 2$. Then $\mathcal{M}$ must lie in $W^1_{1,5}$, since Γ has no g^1_4 by [59]. We have an exact sequence of the form (5.14), for some $y \in \Gamma$. Applying the functor $\mathbf{\Phi}$ to this sequence, by Proposition 4.1 and formula (4.6), we obtain:

$$0 \to A_L \otimes \mathcal{U}_+^* \xrightarrow{\mathcal{H}^0(\mathbf{\Phi}(e_{\mathcal{L}}))} \mathbf{\Phi}(\mathcal{L}) \to \mathscr{E}_y \to \mathscr{O}_L \to 0,$$

where L is the line contained in X such that $\mathcal{M} \cong \mathbf{\Phi}^!(\mathscr{O}_L)[-1]$ and $A_L = \mathrm{H}^0(\Gamma, \mathcal{M}) \cong \mathrm{H}^0(\Gamma, \mathcal{L})$. By Step 2 of the proof of Theorem 3.7, the image of the middle map in the exact sequence above is a sheaf $F \in \mathsf{M}_X(2,1,6)$. We define again $\vartheta(\mathcal{L}) = \mathrm{cok}(\mathcal{H}^0(\mathbf{\Phi}(e_{\mathcal{L}})))$ and since $\mathbf{\Phi}^!(F) \cong \mathcal{L}$, it follows $\varphi(\vartheta(\mathcal{L})) = \mathcal{L}$. Note that the map ϑ is defined in the same way as for globally generated bundles.

To show that $\vartheta \circ \varphi$ is the identity on $\mathsf{M}_X(2,1,6)$, we need to prove that the map ζ_F appearing in the resolution (5.9) provided by Proposition 5.4 agrees up to a scalar with $\mathcal{H}^0(\mathbf{\Phi}(e_{\mathcal{L}}))$. Applying the functor $\mathrm{Hom}_X(\mathcal{U}_+^*,-)$ to (5.9), and using adjunction and (2.30), we obtain

$$\mathrm{Ext}_X^2(F,\mathcal{U}_+)^* \subseteq \mathrm{Hom}(\mathcal{U}_+^*,\mathbf{\Phi}(\mathbf{\Phi}^!(F))) \cong \mathrm{H}^0(\Gamma,\mathbf{\Phi}^!(F)).$$

Now recall that $\dim \mathrm{Ext}_X^2(F,\mathcal{U}_+) = 2$, by (5.4), and $\mathrm{h}^0(\Gamma,\mathbf{\Phi}^!(F)) \leq 2$, since $W^2_{1,6}$ is empty, hence we conclude that $\mathrm{Ext}_X^2(F,\mathcal{U}_+)^* \cong \mathrm{Hom}(\mathcal{U}_+^*,\mathbf{\Phi}(\mathbf{\Phi}^!(F)))$, and the map ζ_F in the resolution (5.9) is uniquely determined. This proves that $\vartheta(\varphi(F)) = F$.

Now, with the same proof of Corollary 5.10 one can show that the moduli space $\mathsf{M}_X(2,1,6)$ is fine. The fact that $W^1_{1,6}$ is a connected threefold is well-known, see for instance [3, IV, Theorem 5.1 and V, Theorem 1.4]. This completes the proof of part (A). □

5.4.2. *A characterization of non globally generated sheaves in* $\mathsf{M}_X(2,1,6)$. Here we study in detail the subvariety of the moduli space $\mathsf{M}_X(2,1,6)$ consisting of sheaves that fail to be globally generated.

Lemma 5.12. *Let $F \in \mathsf{M}_X(2,1,6)$. Then either F is globally generated, or there is an exact sequence:*

$$0 \to I \to F \to \mathscr{O}_L(-1) \to 0, \tag{5.15}$$

where L is a line contained in X and I is a sheaf that, for some $y \in \Gamma$, fits into:

$$0 \to \mathscr{E}_y^* \to \mathrm{H}^0(X,F) \otimes \mathscr{O}_X \to I \to 0, \tag{5.16}$$

Proof. If the sheaf F fits into the exact sequence (5.15), it cannot be globally generated, since $\mathscr{O}_L(-1)$ has no global sections. So let us prove the converse implication.

Assume thus that F is not globally generated, let I (respectively, T and K) be the image (respectively, the cokernel and the kernel) of the natural evaluation map $e_{\mathscr{O},F} : \mathrm{H}^0(X,F) \otimes \mathscr{O}_X \to F$. By Proposition 3.4, we have $\mathrm{H}^k(X,F) = 0$, for each $k \neq 0$, and $\mathrm{h}^0(X,F) = 4$. We have the exact sequences:

$$0 \to K \to \mathscr{O}_X^4 \to I \to 0, \qquad 0 \to I \to F \to T \to 0, \tag{5.17}$$

where the induced maps $\mathrm{H}^0(X,\mathscr{O}_X^4) \to \mathrm{H}^0(X,I)$ and $\mathrm{H}^0(X,I) \to \mathrm{H}^0(X,F)$ compose to $e_{\mathscr{O},F}$, in particular $\mathrm{h}^0(X,I) = 4$.

Let us check that the torsion-free sheaf I must have rank 2 and $c_1(I) = 1$. By stability of $\mathscr{O}_X$ and F, we must have $c_1(I) = 1$ (and in this case $\mathrm{rk}(I) = 2$) or $c_1(I) = 0$. But in the latter case, by the uniqueness of the graded object associated to the Jordan-Hölder filtration of $\mathscr{O}_X^4$ ([32, Proposition 1.5.2]), we must have either that $I \cong \mathscr{O}_X$, or that I is semistable with Jordan-Hölder filtration:

$$0 \to \mathscr{O}_X \to I \to \mathscr{O}_X \to 0,$$

and so $I \cong \mathscr{O}_X^2$. In both cases we have a contradiction with $\mathrm{h}^0(X,I) = 4$. We have proved that I has rank 2 and $c_1(I) = 1$. Since F is stable, by (5.17) we deduce that I is stable with $c_2(I) \geq 6$.

Now, one easily sees that K is a stable reflexive (by [29, Proposition 1.1]) sheaf of rank 2 with $c_1(K) = -1$, $c_2(K) = 12 - c_2(I)$ (by (5.17)). Then we have $c_3(K) \geq 0$ and by Lemma 3.1 it follows $c_2(K) \geq 5$. This leaves two cases, namely $c_2(I) = 6$ or 7.

Assume first that $c_2(I) = 7$. Then we can apply Proposition 3.4 to the sheaf $K(1)$ to prove that K is locally free. It follows that K is of the form $\mathscr{E}_y^*$ for some y by virtue of Theorem 2.3. It follows that $\mathrm{H}^k(X,K) = 0$ for all k by Proposition 3.4, which by (5.17) implies $\mathrm{H}^k(X,I) = 0$ for $k \geq 1$ and in turn $\mathrm{H}^k(X,T) = 0$ for all k. We obtain that T is isomorphic to $\mathscr{O}_L(-1)$ by a Hilbert polynomial computation. This concludes the proof in case $c_2(I) = 7$.

Let us assume now that $c_2(I) = 6$, which implies $c_2(K) = 6$ and $c_3(K) \geq 0$ (recall that K is reflexive). In this case Proposition 3.4 implies that K is locally free so $c_3(K) = 0$. So by (5.17), using that $c_1(I) = 1$, $c_1(K) = -1$, $c_2(I) = c_2(K) = 6$, we compute $c_3(I) = -c_3(K)$. Hence $c_3(I)$ also vanishes. So $c_k(F) = c_k(I)$ for all k hence the sheaf T has $c_k(T) = 0$ for all k. Therefore $T = 0$ and F is globally generated. □

5.4.3. *An involution on the Brill-Noether locus* $W^1_{1,6}$. Here we will exhibit an involution τ on the Brill-Noether locus $W^1_{1,6}$. This is defined by:

$$\tau : W^1_{1,6} \to W^1_{1,6}, \qquad \mathcal{L} \mapsto \mathcal{L}^* \otimes \omega_\Gamma.$$

By Riemann-Roch, it is clear that τ sends $W^1_{1,6}$ to itself, and it obviously τ is an involution. If will turn out that τ interchanges the following closed subvarieties of $W^1_{1,6}$:

$$\mathsf{G} = \{\mathcal{L} \in W^1_{1,6} \,|\, \mathcal{L} \text{ is not globally generated}\}$$
$$\mathsf{C} = \{\mathcal{L} \in W^1_{1,6} \,|\, \mathcal{L} \text{ is contained in a line bundle lying in } W^2_{1,7}\}.$$

Note that any $\mathcal{N} \in W^2_{1,7}$ gives a map $\Gamma \to \mathbb{P}^2$ onto a septic and for general $(\Gamma, \mathcal{N})$ the image has 8 nodes, see e.g. [46]. When this happens for some $\mathcal{N}$ on a given Γ, we say that Γ can be represented as a plane septic with 8 nodes.

Proposition 5.13. *The sets* C *and* G *are interchanged by the involution* τ*, and are both isomorphic to the product* $\Gamma \times W^1_{1,5}$*.*

The intersection $\mathsf{C} \cap \mathsf{G} \subset W^1_{1,6}$ *is a finite cover of the curve* $W^1_{1,5}$*. If* Γ *can be represented as a plane septic with 8 nodes, the degree of this cover is* 16*.*

Proof. Given a line bundle $\mathcal{L}$ in G, we consider, as in the proof of Theorem 5.11, part (A), the image $\mathcal{M} \subset \mathcal{L}$ of the natural evaluation map $e_{\mathscr{O}_\Gamma,\mathcal{L}}$. We have $\mathcal{M} \in W^1_{1,5}$ and an exact sequence of the form (5.14), for some $y \in \Gamma$. This defines a map $\mathsf{G} \to \Gamma \times W^1_{1,5}$.

Let us define an inverse map. We first note that, given $\mathcal{M}$ in $W^1_{1,5}$ and $y \in \Gamma$, we have $\mathrm{Ext}^1_\Gamma(\mathscr{O}_y, \mathcal{M}) \cong \mathbb{C}$. The unique extension $\mathcal{F}$ from $\mathscr{O}_y$ to $\mathcal{M}$ must lie in $W^1_{1,6}$, since $W^2_{1,6}$ is empty by [59, Table 1].

To put this in family, we denote by $\mathscr{P}$ a Poincaré line bundle on $\Gamma \times W^1_{1,5}$. We consider the projection $\pi : \Gamma \times \Gamma \times W^1_{1,5} \to \Gamma \times W^1_{1,5}$ onto the last two components, the diagonal embedding $\Delta : \Gamma \times W^1_{1,5} \to \Gamma \times \Gamma \times W^1_{1,5}$ and the projection $p_\Gamma : \Gamma \times W^1_{1,5} \to \Gamma$. Set $\mathscr{L} = \mathscr{P} \otimes p^*_\Gamma(\omega^*_\Gamma)$. Since $\Delta^*(\pi^*(\mathscr{P})) \cong \mathscr{P}$, and $\Delta^*(\omega^*_{\Gamma\times\Gamma\times W^1_{1,5}}) \otimes \omega_{\Gamma\times W^1_{1,5}} \cong p^*_\Gamma(\omega^*_\Gamma)$, we get:

$$\mathrm{Ext}^1_{\Gamma\times\Gamma\times W^1_{1,5}}(\Delta_*(\mathscr{L}), \pi^*(\mathscr{P})) \cong \mathrm{Hom}_{\Gamma\times W^1_{1,5}}(\mathscr{L}, \Delta^!(\pi^*(\mathscr{P})[1]) \cong \mathrm{Hom}_{\Gamma\times W^1_{1,5}}(\mathscr{L}, \mathscr{P} \otimes p^*_\Gamma(\omega^*_\Gamma)).$$

The identity of this group corresponds thus to a line bundle $\mathscr{F}$ on $\Gamma \times \Gamma \times W^1_{1,5}$ fitting into:

$$0 \to \pi^*(\mathscr{P}) \to \mathscr{F} \to \Delta_*(\mathscr{L}) \to 0,$$

so that the one-dimensional space $\mathcal{F}_z$ is the fibre over $(z, y, \mathcal{M})$ of $\mathscr{F}$. Therefore $\mathscr{F}$ is a family of bundles in $W^1_{1,6}$ parametrized by $\Gamma \times W^1_{1,5}$, giving thus a classifying map $\Gamma \times W^1_{1,5} \to W^1_{1,6}$, which is clearly the desired inverse map.

To see the relation with $W^2_{1,7}$, we set $\mathcal{N} = \mathcal{M}^* \otimes \omega_\Gamma$, we have:

$$\mathrm{h}^0(\Gamma, \mathcal{M}) = \mathrm{h}^1(\Gamma, \mathcal{N}) = 2, \qquad \mathrm{h}^1(\Gamma, \mathcal{M}) = \mathrm{h}^0(\Gamma, \mathcal{N}) = 3.$$

It follows that $\mathcal{N}$ lies in $W^2_{1,7}$. Dualizing the sequence (5.14) and tensoring by ω_Γ, we obtain the exact sequence:

$$0 \to \tau(\mathcal{L}) \to \mathcal{N} \to \mathscr{O}_y \to 0.$$

So the line bundle $\tau(\mathcal{L})$ lies in C. Since this procedure is reversible, we have proved that the involution τ interchanges the subsets G and C. Note that the map $\tau : \mathcal{M} \mapsto \mathcal{M}^* \otimes \omega_\Gamma$ gives an isomorphism from $W^1_{1,5}$ to $W^2_{1,7}$.

Let us now describe the intersection $\mathsf{C} \cap \mathsf{G} \subset W^1_{1,6}$. Recall that the map $\varphi_{|\mathcal{N}|}$ associated to a given $\mathcal{N} \in W^2_{1,7}$ maps Γ to $\mathbb{P}^2$. This map is generically injective and the image is a curve of degree 7, smooth away from a subscheme of length 8. If $\mathcal{N}$ is general enough, then we get 8 distinct double points $y_1, \ldots, y_8$, [37, Lemma 2.6]. For each y_i we have a unique $\mathcal{M}_i \in W^1_{1,5}$ given by the projection from the double point y_i. On the other hand any subbundle $\mathcal{M} \in W^1_{1,5}$ of $\mathcal{N}$ must correspond to the projection from a double point y_i. Namely we will have:

$$0 \to \mathcal{M} \to \mathcal{N} \to \mathscr{O}_{Z_i} \to 0,$$

where Z_i is the subscheme of Γ over the double point y_i.

Now fix a line bundle $\mathcal{N}$ in $W^2_{1,7}$. A subbundle $\mathcal{L} \in \mathsf{C}$ of $\mathcal{N}$ corresponds to the projection from a smooth point y as soon as $\mathcal{L}$ is globally generated. Therefore, $\mathcal{L}$ lies in $\mathsf{C} \cap \mathsf{G}$ if and only if we have:

$$\mathcal{M}_i \subset \mathcal{L} \subset \mathcal{N}, \tag{5.18}$$

for some $i = 1, \ldots, 8$. Then $\mathcal{L}$ must be of the form $\mathcal{M}_i(z)$ for some point z in Γ lying over the double point y_i. The number of such z is finite for each y_i. Thus we have realized $\mathsf{C} \cap \mathsf{G}$ as a finite cover of $W^1_{1,5}$.

If there is one line bundle $\mathcal{N}$ such that the image of Γ under $\varphi_{|\mathcal{N}|}$ has 8 nodes, then for any i there are two points $z_{i,1}$, $z_{i,2}$ of Γ which are mapped to y_i by $\varphi_{|\mathcal{N}|}$. Then we set $\mathcal{L}_{i,j} = \mathcal{M}_i(z_{i,j})$, and the line bundles $\mathcal{L}_{i,j}$ fit into the inclusions (5.18). Hence the degree of the cover is 16. $\square$

5.4.4. *An involution on* $\mathsf{M}_X(2,1,6)$. We consider now the pull-back θ of τ to $\mathsf{M}_X(2,1,6)$, i.e. we set:

$$\theta : \mathsf{M}_X(2,1,6) \to \mathsf{M}_X(2,1,6), \qquad \theta = \varphi^{-1} \circ \tau \circ \varphi.$$

We will next show that θ can be seen on $\mathsf{M}_X(2,1,6)$ in terms of the functor T of Corollary 2.10.

Proposition 5.14. *Let F be an element of $\mathsf{M}_X(2,1,6)$. Then we have:*

i) the sheaf F is not locally free if and only if $\mathbf{\Phi}^!(F)$ lies in G.
ii) the sheaf F is not globally generated if and only if $\mathbf{\Phi}^!(F)$ lies in C.

Moreover the function θ is an involution which interchanges the two subsets of sheaves which are not locally free, and not globally generated. For each F in $\mathsf{M}_X(2,1,6)$ we have:

$$\theta(F) = T(F) = \varphi^{-1}\mathbf{\Phi}^!(\mathbf{R}\mathcal{H}om_X(F, \mathscr{O}_X))[1]. \tag{5.19}$$

Finally, $\theta(F)$ is isomorphic to both the following objects:

$$\mathbf{R}\mathcal{H}om_X\left((\mathrm{H}^0(X,F)\otimes\mathscr{O}_X \to F), \mathscr{O}_X\right)[-1] \qquad \textit{and:} \tag{5.20}$$

$$\mathbf{R}\mathcal{H}om_X(F, \mathscr{O}_X) \to \mathrm{H}^0(X,F)^* \otimes \mathscr{O}_X. \tag{5.21}$$

Proof. We have already proved the implication "$\Leftarrow$" of (i) in Lemma 5.6. To prove the converse, we consider a sheaf F which is not locally free. Then F fits into an exact sequence of the form (3.3). Applying the functor $\mathbf{\Phi}^!$ to this sequence and setting $\mathcal{L} = \mathbf{\Phi}^!(F)$ we obtain an exact sequence of the form (5.14) for some $\mathcal{M}$ in $W^1_{1,5}$ (see Proposition 4.1). Since $\mathrm{H}^0(\Gamma,\mathcal{M}) \cong \mathrm{H}^0(\Gamma,\mathcal{L})$, the evaluation map $\mathrm{H}^0(\Gamma,\mathcal{L})\otimes\mathscr{O}_\Gamma \to \mathcal{L}$ cannot be surjective, so $\mathcal{L}$ lies in G.

To prove (ii), in view of Lemma 5.12, we have to show that the sheaf F fits into (5.15), for some I fitting in (5.16) if and only if the line bundle $\mathbf{\Phi}^!(F)$ lies in C . To show "$\Rightarrow$", we consider a sheaf F fitting into an exact sequence of the form (5.15). Recall by Proposition 4.1 that $\mathcal{N} = \mathbf{\Phi}^!(\mathscr{O}_L(-1))$ lies in $W^2_{1,7}$. Since $\mathbf{\Phi}^!(\mathscr{O}_X) = 0$, by the exact sequence (5.16) we have $\mathbf{\Phi}^!(I)[1] \cong \mathbf{\Phi}^!(\mathscr{E}^*_y)[2]$, and using (2.25) we conclude that:

$$\mathbf{\Phi}^!(I)[1] \cong \mathbf{\Phi}^!(\mathscr{E}^*_y)[2] \cong \mathscr{O}_y.$$

Thus applying the functor $\mathbf{\Phi}^!$ to (5.15) we obtain an exact sequence:

$$0 \to \mathbf{\Phi}^!(F) \to \mathcal{N} \to \mathscr{O}_y \to 0,$$

and $\mathbf{\Phi}^!(F)$ lies in C.

To prove the converse implication, we consider a globally generated sheaf F and the exact sequence:

$$0 \to K \to \mathrm{H}^0(X,F)\otimes\mathscr{O}_X \to F \to 0. \tag{5.22}$$

Remark that K is a locally free sheaf and K^* lies in $\mathsf{M}_X(2,1,6)$ as well. We note that, applying (2.25), we get the natural isomorphism:

$$\mathbf{\Phi}^!(K) \cong \mathbf{\Phi}^!(K^*)^* \otimes \omega_\Gamma[-1].$$

On the other hand, by (5.22) we get $\mathbf{\Phi}^!(K) \cong \mathbf{\Phi}^!(F)[-1]$. Then we have:

$$\mathbf{\Phi}^!(F) \cong \tau(\mathbf{\Phi}^!(K^*)).$$

But $\mathbf{\Phi}^!(K^*)$ is globally generated by Lemma 5.6, hence we are done since τ interchanges C and G. We have thus established (i) and (ii).

It follows that θ interchanges the sheaves which are not locally free, and the sheaves which are not globally generated, and clearly θ is an involution.

To show the expression (5.19) of θ, recall that $\mathbf{\Phi}^! \circ T = \tau \circ \mathbf{\Phi}^!$ by Corollary 2.10. Therefore, for any F in $\mathsf{M}_X(2,1,6)$ we have $\theta(F) = \varphi^{-1}(\mathbf{\Phi}^!(T(F)))$. Since for any object a in $\mathbf{D^b}(X)$ we have $\mathbf{\Phi}^!(\mathbf{\Phi}(\mathbf{\Phi}^!(a))) \cong \mathbf{\Phi}^!(a)$, it follows that $\theta(F) = \varphi^{-1}\mathbf{\Phi}^!(\mathbf{R}\mathcal{H}om_X(F,\mathscr{O}_X))[1]$.

By (5.19), Corollary 2.10 and Remark 2.5, it follows that $\theta(F)$ is isomorphic to (5.20). Finally, by (2.35), we have that (5.20) is isomorphic to (5.21). $\square$

5.4.5. *Sheaves that are not locally free neither globally generated.* Here we will study the locus in $\mathsf{M}_X(2,1,6)$ given by sheaves where both the properties of being globally generated and locally free fail. In terms of $W^1_{1,6}$, this is the intersection of the two loci C and G in $W^1_{1,6}$.

Proposition 5.15. *Let F be a sheaf in $\mathsf{M}_X(2,1,6)$, and let $M,N \subset X$ be two lines in X. Set $\mathcal{L} = \mathbf{\Phi}^!(F)$, $\mathcal{M} = \mathbf{\Phi}^!(\mathscr{O}_M)[-1]$ and $\mathcal{P} = \mathbf{\Phi}^!(\mathscr{O}_N(-1))$. Then F is not globally generated over N and not locally free over M if and only if we have $\mathcal{M} \subset \mathcal{L} \subset \mathcal{P}$.*

Proof. If F is not locally free over M and not globally generated over N, then by Proposition 3.4 and Lemma 5.12 we have two surjective maps $F \to \mathscr{O}_M$ and $F \to \mathscr{O}_N(-1)$. By applying $\mathbf{\Phi}^!$, the first one gives $\mathcal{M} \subset \mathcal{L}$ and the second one $\mathcal{L} \subset \mathcal{P}$, so one implication is clear.

Conversely, if $\mathcal{M} \subset \mathcal{L}$, we have $\mathcal{L}/\mathcal{M} = \mathscr{O}_y$, with $y \in \Gamma$ and applying $\mathbf{\Phi}$, by Proposition 4.1 we get:

$$0 \to (\mathcal{U}_+^*)^2 \to \mathbf{\Phi}(\mathcal{L}) \to \mathscr{E}_y \to \mathscr{O}_M \to 0,$$

so using (5.9) we see that F is the image of the middle map in the sequence above, hence it is not locally free over M. Further if $\mathcal{L} \subset \mathcal{P}$, then we have a commutative exact diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{O}_\Gamma^2 & \longrightarrow & \mathscr{O}_\Gamma^3 & \longrightarrow & \mathscr{O}_\Gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{L} & \longrightarrow & \mathcal{P} & \longrightarrow & \mathscr{O}_z & \longrightarrow & 0,
\end{array}$$

with $z \in \Gamma$, and the vertical maps are natural evaluations. Applying $\mathbf{\Phi}$ to this diagram and taking cohomology, we get:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathcal{U}_+^*)^2 & \longrightarrow & (\mathcal{U}_+^*)^3 & \longrightarrow & \mathcal{U}_+^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{\Phi}(\mathcal{L}) & \longrightarrow & \mathbf{\Phi}(\mathcal{P}) & \longrightarrow & \mathscr{E}_z & \longrightarrow & 0
\end{array}$$

Taking cokernels, by using snake lemma, (5.9), (4.7), and (2.22) we obtain an exact sequence:

$$0 \to \mathscr{O}_X \to \mathscr{G}_z \to F \to \mathscr{O}_N(-1) \to 0.$$

The rightmost part of this sequence gives an exact sequence of the form (5.15), so F is not globally generated over N. ☐

In the above situation, if $N \neq M$ and $\mathcal{M} \subset \mathcal{P}$ then $N \cap M$ is a point by Lemma 4.3. In this case the conic $C = M \cup N$ satisfies $\mathbf{\Phi}^!(\mathscr{O}_C) \cong \mathscr{O}_{\{y,z\}}$. Further, note that the leftmost part of the diagram above gives rise to an exact sequence of the form (5.16). Indeed, if I is the image of middle map above, then I is also the image of the evaluation map $e_{\mathscr{O}_X,F}$ and, recalling (2.21), and the fact that for all $z \in \Gamma$, $(\mathcal{U}_-)_z \cong \mathscr{O}_X^5$, we get a commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & \mathscr{O}_X & = & \mathscr{O}_X & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{E}_z^* & \longrightarrow & \mathscr{O}_X^5 & \longrightarrow & \mathscr{G}_z & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{E}_z^* & \longrightarrow & \mathscr{O}_X^4 & \longrightarrow & I & \longrightarrow & 0
\end{array}$$

The bottom row thus gives (5.16).

Lemma 5.16 (Markushevich)**.** *The set of singular points $\mathcal{L}$ of $W^1_{1,6}$, such that $\mathcal{L}$ is globally generated, is in bijection with the set of even effective theta-characteristics on Γ. In particular, this set is finite. Moreover, it is empty if Γ is outside a divisor in the moduli space of curves of genus 7, and of cardinality 1 if Γ is general in that divisor.*

Proof. According to Mukai's classification in [59]), the smooth curve section Γ of the spinor 10-fold satisfies $W^2_{1,6} = \emptyset$, and a general curve of genus 7 is of this form. Now recall that a line bundle $\mathcal{L}$ lies in the singular locus of $W^1_{1,6}$ if and only if the Petri map:

$$\pi_{\mathcal{L}} : \mathrm{H}^0(\Gamma, \mathcal{L}) \otimes \mathrm{H}^0(\Gamma, \mathcal{L}^* \otimes \omega_\Gamma) \to \mathrm{H}^0(\Gamma, \omega_\Gamma)$$

is not injective. Since $\mathcal{L}$ is globally generated by two sections we know (see the proof of Theorem 5.11, part A) that $\ker(e_{\mathscr{O}_\Gamma,\mathcal{L}})$ is isomorphic to $\mathcal{L}^*$. Hence, the kernel of $\pi_{\mathcal{L}}$ is isomorphic to $\mathrm{H}^0(\Gamma, \mathcal{L}^* \otimes \mathcal{L}^* \otimes \omega_\Gamma)$. Therefore, the above map is injective unless $\mathcal{L} \otimes \mathcal{L} \cong \omega_\Gamma$, which means that $\mathcal{L}$

is an even effective theta-characteristic (even here means that $\mathrm{h}^0(\Gamma, \mathcal{L})$ is an even number, 2 in this case).

By [70, Theorem 2.16] the set of curves of genus 7 admitting an even effective theta-characteristic form a divisor in the moduli space of curves of genus 7, and the general curve in this divisor has precisely one even effective theta-characteristic. This concludes the proof. □

Proof of Theorem 5.11, part B. Let us assume X to be non-exotic, and prove that $\mathsf{M}_X(2,1,6)$ has at most finitely many singular points. In view of Theorem 5.11, part A, the space $\mathsf{M}_X(2,1,6)$ is isomorphic to $W^1_{1,6}$. The number of singular points of $W^1_{1,6}$ which correspond to globally generated line bundles is finite by Lemma 5.16.

Let us now study the non-globally-generated case. We consider thus a line bundle $\mathcal{L} \in \mathsf{G}$ which is a singular point of $W^1_{1,6}$, and fails to be globally generated at a point $y \in \Gamma$· Since τ is an isomorphism, we can also assume that $\tau(\mathcal{L})$ is not globally generated, i.e. $\mathcal{L} \in \mathsf{C} \cap \mathsf{G}$. In particular $\mathcal{L}$ contains a line bundle $\mathcal{M}$ in $W^1_{1,5}$, and there exists $\mathcal{N} \in W^2_{1,7}$ such that $\mathcal{M} \subset \mathcal{L} \subset \mathcal{N}$. Applying τ, we also get $\tau(\mathcal{N}) \subset \tau(\mathcal{L}) \subset \tau(\mathcal{M})$ with $\tau(\mathcal{N})$ in $W^1_{1,5}$. Set $e = e_{\mathscr{O}_\Gamma, \mathcal{L}}$. We have $\ker(e) \cong \mathcal{M}^*$ (see again the proof of Theorem 5.11, part A), hence an exact sequence:

$$0 \to \mathcal{M}^* \to \mathrm{H}^0(\Gamma, \mathcal{L}) \otimes \mathscr{O}_\Gamma \xrightarrow{e} \mathcal{L} \to \mathscr{O}_y \to 0.$$

We are assuming that the Petri map $\pi_{\mathcal{L}}$ is not injective, and $\pi_{\mathcal{L}} = \mathrm{H}^0(\Gamma, e \otimes \tau(\mathcal{L}))$, so tensoring the above sequence with $\tau(\mathcal{L})$ we get that $0 \neq \ker(\pi_{\mathcal{L}}) \cong \mathrm{H}^0(\Gamma, \mathcal{M}^* \otimes \tau(\mathcal{L}))$. We get then an inclusion $\mathcal{M} \hookrightarrow \tau(\mathcal{L})$. It follows that $\mathcal{M} \cong \tau(\mathcal{N})$, since $\tau(\mathcal{L})$ contains a unique line bundle lying in $W^1_{1,5}$, so also $\mathcal{N} \cong \tau(\mathcal{M})$.

We have thus an inclusion $\mathcal{M} \subset \tau(\mathcal{M})$, which means $\mathrm{H}^0(\Gamma, \mathcal{M}^* \otimes \mathcal{M}^* \otimes \omega_\Gamma) \neq 0$, so that $\mathcal{M}$ is a singular point of $W^1_{1,5}$ (see Remark 4.2). Since $W^1_{1,5} \cong \mathscr{H}^0_1(X)$ by Proposition 4.1, and since X is not exotic, the number of singular points of this form in $W^1_{1,6}$ is finite, and we are done.

Finally, note that if X is general, then the curve Γ is general. Then it is well-known that $W^1_{1,6}$ is smooth and irreducible, see for instance [3, V, Theorem 1.6]. It follows that $\mathsf{M}_X(2,1,6)$ is a smooth irreducible threefold. □

5.5. **The space $\mathsf{M}_X(2,1,6)$ as a subspace of $\mathsf{M}_S(2,1,6)$.** In this section we let X be a *general* prime Fano threefold of genus 7. Let S be a general hyperplane section of X. Assume in particular that S is a K3 surface of Picard number 1 and sectional genus 7. In this paragraph we will show that $\mathsf{M}_X(2,1,6)$ is isomorphic to a Lagrangian submanifold of $\mathsf{M}_S(2,1,6)$. This provides an instance of a general remark of Tyurin, [73].

Lemma 5.17. *Let S be a general hyperplane section of X. Then the restriction $F \mapsto F_S$ gives an everywhere defined immersion $\rho : \mathsf{M}_X(2,1,6) \to \mathsf{M}_S(2,1,6)$.*

Proof. First of all we would like to show that, given a sheaf $F \in \mathsf{M}_X(2,1,6)$, its restriction F_S is stable. Note that, by Proposition 3.4, part (iii), F_S is torsion-free as soon as S contains no lines, which is guaranteed by the assumption $\mathrm{Pic}(S) \cong \mathbb{Z}$. So, assuming F_S unstable, we can take a destabilizing rank-1 subsheaf K of F_S. We have then $c_1(K) \geq 1$, so $K^{**} \cong \mathscr{O}_S(t)$ with $t \geq 1$, hence $\mathrm{H}^0(S, F_S^{**}(-1)) \neq 0$. But, by Propositions 3.4 and 3.6, either F is locally free so $F \cong F^{**}$ and $\mathrm{H}^0(X, F(-1)) = \mathrm{H}^1(X, F(-2)) = 0$, or F^{**} lies in $\mathsf{M}_X(2,1,5)$ and satisfies the same vanishing. In any case, this implies $\mathrm{H}^0(S, F_S^{**}(-1)) = 0$, a contradiction.

It remains to prove that the differential $d(\rho_S)_F$ is injective at any point F of $\mathsf{M}_X(2,1,6)$. Applying the functor $\mathrm{Hom}_X(F, -)$ to (3.13) (with $t = 0$) we get:

$$\mathrm{Ext}^1_X(F, F(-1)) \to \mathrm{Ext}^1_X(F, F) \xrightarrow{\delta} \mathrm{Ext}^1_X(F, F_S).$$

Note that the leftmost term in the above sequence vanishes since by Serre duality $\mathrm{Ext}^1_X(F, F(-1)) \cong \mathrm{Ext}^2_X(F, F)^*$, and this group vanishes for $\mathsf{M}_X(2,1,6)$ is a non-singular threefold by Theorem 5.11, part B. So δ is injective. Recall that $\mathrm{Ext}^1_X(F, F)$ is naturally isomorphic to $T_F \mathsf{M}_X(2,1,6)$, hence we are done if we prove that $\mathrm{Ext}^1_X(F, F_S)$ is naturally isomorphic to $T_{F_S} \mathsf{M}_S(2,1,6)$. So we want $\mathrm{Ext}^1_X(F, F_S) \cong \mathrm{Ext}^1_S(F_S, F_S)$. To show it, denoting by ι the inclusion of the surface S in X, we have:

$$\mathrm{Ext}^1_X(F, F_S) \cong \mathrm{Ext}^1_X(F, \iota_* \iota^* F) \cong \mathrm{Ext}^1_S(\iota^* F, \iota^* F) \cong \mathrm{Ext}^1_S(F_S, F_S),$$

where the second isomorphism holds if $\mathbf{L}_k \iota^*(F) = 0$ for any $k > 0$, which follows from F torsion-free. □

Proposition 5.18. *Fix a general hyperplane section S of X, let $F \not\cong F' \in \mathsf{M}_X(2,1,6)$, and assume:*

$$F_S \cong F'_S.$$

i) Then, assuming F globally generated, we have $F' = \theta(F)$.
ii) Otherwise there are lines $L, L' \subset X$ with $L \cap L' \in S$ such that F (resp. F') is not globally generated over L' (resp. L) and not locally free over L (resp. L').

Moreover, the set of sheaves F in $\mathsf{M}_X(2,1,6)$ admitting a sheaf F' such that $F_S \cong F'_S$ is finite.

Proof. Let us prove (i). Considering the composition of the projection $F \to F_S$ and of an isomorphism $F_S \cong F'_S$ we obtain a non-zero map $F \to F'_S$. In view of the exact sequence:

$$0 \to F'(-1) \to F' \to F'_S \to 0,$$

we see that the map $F \to F'_S$ lifts to an isomorphisms $F \cong F'$ if $\mathrm{Ext}^1(F, F'(-1)) = 0$. We can assume thus that this group is non-trivial. Since F is globally generated by assumption, we have:

$$0 \to K \to \mathrm{H}^0(X, F) \otimes \mathscr{O}_X \to F \to 0, \tag{5.23}$$

where K is reflexive (see [29, Proposition 1.1]) hence $K(1)$ is a locally free sheaf in $\mathsf{M}_X(2,1,6)$, which is precisely $\theta(F)$ by Proposition 5.14. Applying $\mathrm{Hom}_X(-, F'(-1))$ to (5.23), one obtains $\mathrm{Hom}_X(K, F'(-1)) \neq 0$, so $F' \cong K(1) \cong \theta(F)$.

Let us now prove (ii). So we assume that F is not globally generated, say over a line $L' \subset X$, see Lemma 5.12. Let $e = e_{\mathscr{O}_X,F}$ and $e' = e_{\mathscr{O}_X,F'}$, so $\mathrm{cok}(e) \cong \mathscr{O}_{L'}(-1)$ and $\ker(e) \cong \mathscr{E}_y(-1)$ for some $y \in \Gamma$. We are assuming $F_S \cong F'_S$. Note that F_S fails to be globally generated over the point $x = L' \cap S$. Therefore F' is not globally generated either, say over a line $L \subset X$, and we must have either $L = L'$, or $L \cap L' = x$. Moreover $\ker(e') \cong \mathscr{E}_{y'}$ for some $y' \in \Gamma$.

Note that $\ker(e_{\mathscr{O}_S,F_S}) \cong \ker(e_{|S}) \cong (\mathscr{E}_y(-1))_S$, so also $\ker(e_{\mathscr{O}_S,F_S}) \cong \ker(e'_{|S}) \cong (\mathscr{E}_{y'}(-1))_S$, so $(\mathscr{E}_y(-1))_S \cong \mathscr{E}_{y'}(-1))_S$. This implies $\mathscr{E}_y \cong \mathscr{E}_{y'}$ (i.e. $y = y'$), because the restriction map $\mathsf{M}_X(2,1,5) \to \mathsf{M}_S(2,1,5)$ corresponds to the embedding of Γ as linear section of $\mathsf{M}_S(2,1,5)$.

So, if $L = L'$, we have $F \cong F'$, and we can thus look at the case $L \cap L' = x$. We have $\mathrm{Ext}^1_X(F, F'(-1)) \neq 0$ since $F_S \cong F'_S$. Applying the functor $\mathrm{Hom}_X(-, F'(-1))$ to (5.15) we get $\mathrm{Ext}^1_X(F, F'(-1)) = \mathrm{Ext}^1_X(\mathscr{O}_L, F')$. Indeed we compute $\mathrm{Hom}_X(I, F'(-1)) = 0$ by stability and we can easily check that $\mathrm{Ext}^1_X(I, F'(-1)) = 0$ by applying $\mathrm{Hom}_X(-, F'(-1))$ to (5.16) (where L should be replaced by L'). So $\mathrm{Ext}^1_X(\mathscr{O}_{L'}, F') \neq 0$, and by Proposition 3.4, part (iii) we have that F' is not locally free over L'. Likewise, F is not locally free over L. Moreover, the double duals of F and F' are both isomorphic to $\mathscr{E}_z$ for one $z \in \Gamma$ (the restrictions to S are isomorphic, and again the restriction $\mathsf{M}_X(2,1,5) \to \mathsf{M}_S(2,1,5)$ is injective). We have thus proved (ii). Note also that, by Proposition 5.15, the conic $C = L \cup L'$ is such that $\mathbf{\Phi}^!(\mathscr{O}_C) = \mathscr{O}_{y,z}$.

It remains to prove the last statement. Let us first do this in case F is locally free and globally generated. Set $F' = \theta(F) = K(1)$, and recall that F' is locally free. We consider the symmetric square of (5.23):

$$0 \to \mathrm{Sym}^2(F')^* \to \mathrm{H}^0(X, F) \otimes (F')^* \to \wedge^2 \mathrm{H}^0(X, F) \otimes \mathscr{O}_X \to \mathscr{O}_X(1) \to 0,$$

and we take global sections. Since $\mathrm{H}^0(X, (F')^*) = 0$, $\wedge^2(F')^* \cong \mathscr{O}_X(-1)$ and $\mathrm{H}^1(X, \mathrm{Sym}^2(F')^*) \cong \mathrm{H}^1(X, (F')^* \otimes (F')^*) \cong \mathrm{Ext}^2_X(F', F')^* = 0$, we obtain an injection $\iota_F : \wedge^2 \mathrm{H}^0(X, F) \hookrightarrow \mathrm{H}^0(X, \mathscr{O}_X(1))$. Note that $\dim(\mathrm{cok}(\iota_F)) = 3$, hence setting $\Lambda_F = \mathbb{P}(\mathrm{cok}(\iota_F)) \subset \mathbb{P}^8 = \mathbb{P}(\mathrm{H}^0(X, \mathscr{O}_X(1)))$, we define a correspondence:

$$\Lambda : \mathsf{M}_X(2,1,6) \to \mathbb{G}(\mathbb{P}^2, \mathbb{P}^8), \quad \Lambda : F \mapsto \Lambda_F,$$

Clearly we have $\dim(\mathrm{Im}(\Lambda)) \leq 3$.

Now we fix a general hyperplane section S. Taking global sections of the restriction to S of the symmetric square of (5.23), we obtain an exact commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \wedge^2 \mathrm{H}^0(X,F) \arrow[r, "\iota_F"] \arrow[d, equal] & \mathrm{H}^0(X, \mathscr{O}_X(1)) \arrow[d] \\
0 \arrow[r] & \mathrm{H}^1(S, \mathrm{Sym}^2 F_S^*) \arrow[r] & \wedge^2 \mathrm{H}^0(S, F_S) \arrow[r] & \mathrm{H}^0(S, \mathscr{O}_S(1)).
\end{tikzcd} \tag{5.24}$$

Note that $\mathrm{H}^1(S, \mathrm{Sym}^2 F_S^*) \neq 0$. Indeed since $K_S \cong F_S^*$, then the exact sequence (5.23) (restricted to S) provides a non-trivial element in $\mathrm{Ext}^1_S(F_S, F_S^*) \cong \mathrm{H}^1(S, F_S^* \otimes F_S^*) \cong \mathrm{H}^1(S, \mathrm{Sym}^2 F_S^*)$, where we use $\wedge^2 F_S^* \cong \mathscr{O}_S(-1)$.

Then the diagram (5.24) induces a projection $\mathrm{H}^0(S, \mathscr{O}_S(1)) \to \mathrm{cok}(\iota_F)$ and so the hyperplane defining the surface S must contain Λ_F. We denote by $\mathbb{G}_S \cong \mathbb{G}(2,7)$ the set of planes of $\mathbb{G}(2,8)$ contained in $\mathbb{P}(\mathrm{H}^0(S, \mathscr{O}_S(1))) = \mathbb{P}^7$, (recall our convention on projective space as space of quotients). We have proved that if ρ_S is not injective at F, then $\Lambda_F \in \mathbb{G}_S$. Clearly $\mathbb{G}_S \subset \mathbb{G}(2,8)$ is a subvariety of codimension 3 and corresponds to the choice of a general global section of the rank 3 universal bundle on $\mathbb{G}(2,8)$, which is globally generated. Hence, since the section corresponding to S is general, using Bertini theorem for globally generated vector bundles (see e.g. [42], or [64]), we conclude that the set of planes contained in $\mathrm{Im}(\Lambda) \cap \mathbb{G}_S$ must be finite.

It remains to prove the last statement in case one F fails to be locally free or globally generated. In the second case, we have already seen that F is not locally free either. We note that, since S is general, the curve spanned by the intersection points of lines in $\mathscr{H}_1^0(X)$ meets S at a finite number of points. Any pair of sheaves (F, F') such that we $F_S \cong F'_S$ with F not globally generated determines one such point x. And conversely given such x we may choose in a finite number of ways two lines L, L' through x (for there are finitely many lines through x), and a point z of $\mathbf{\Phi}^!(\mathscr{O}_{L\cup L'})$. This choice determines F and F' as kernels of $\mathscr{E}_z \to \mathscr{O}_L$ and $\mathscr{E}_z \to \mathscr{O}_{L'}$, so the set of pairs of non-globally generated sheaves (F, F') having isomorphic restrictions to S is finite.

Finally, if F is globally generated but not locally free, then by (i) we have $F' = \theta(F)$ so F' is not globally generated, so the previous argument shows again that the set of pairs of sheaves (F, F') with the same restrictions to S is finite. ☐

Recall that the moduli space $\mathsf{M}_S(2,1,6)$ is a holomorphic symplectic manifold with respect to the Mukai form, see [55].

Theorem 5.19. *Let X be a general prime Fano threefold of genus 7, S be a general hyperplane section of X, and let ρ_S be the restriction map from $\mathsf{M}_X(2,1,6)$ to $\mathsf{M}_S(2,1,6)$. The image $\rho_S(\mathsf{M}_X(2,1,6))$ is a Lagrangian subvariety of $\mathsf{M}_S(2,1,6)$ with finitely many double points.*

Proof. Recall that since X is general, then $\mathsf{M}_X(2,1,6)$ is smooth, by Theorem 5.11, part B. We have seen in Lemma 5.17 and Proposition 5.18 that ρ_S is a closed embedding outside a finite subset R of $\mathsf{M}_X(2,1,6)$.

By the proof of Proposition 5.18, we have that the preimage of a singular point of $\rho_S(\mathsf{M}_X(2,1,6))$ consists of precisely two points of $\mathsf{M}_X(2,1,6)$, hence the singular locus consists of double points. Finally, $\rho_S(\mathsf{M}_X(2,1,6) \setminus R)$ is a Lagrangian submanifold by a remark of Tyurin, see [73, Proposition 2.2]. ☐

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