RANK 2 ARITHMETICALLY COHEN-MACAULAY BUNDLES ON A GENERAL QUINTIC SURFACE

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Abstract. Rank 2 arithmetically Cohen-Macaulay bundles on a general quintic hypersurface of the three-dimensional projective space are classified.

1. Introduction

Let Y be an n-dimensional projective variety, embedded by $\mathscr{O}_Y(1)$, and take a vector bundle $\mathscr{E}$ on Y. Then $\mathscr{E}$ is called *arithmetically Cohen-Macaulay* (aCM) if:

$$\mathrm{H}^p(Y, \mathscr{E}(t)) = 0, \qquad \text{for } p \neq 0, n \text{ and for all } t \in \mathbb{Z}.$$

It is natural to ask whether it is possible to classify aCM bundles on a fixed variety Y. By the results of Horrocks [Hor64] and Knörrer in [Knö87] the answer is affirmative when Y is a projective space or a smooth quadric hypersurface. On the other hand, very few varieties are of *finite Cohen-Macaulay type*, namely almost never there exists a finite set of isomorphism classes of aCM indecomposable bundles up to twist by $\mathscr{O}_Y(t)$. In fact these varieties are completely classified, see [BGS87] and [EH88].

Considerable efforts have thus been driven toward a classification of aCM bundles, at least of low rank (say of rank 2), on some special classes of varieties. Several techniques are available to approach the problem. To mention a few: derived categories (see for instance the paper [AO91]); quivers and matrix factorization (widely investigated in the book by Yoshino [Yos90]); liaison theory (see. the work of Casanellas, Drozd and Hartshorne in [CH04], [CDH05]); computer-aided algebra (e.g. Schreyer's appendix to [Bea00]). One relevant instance of available classifications is the case of *prime* Fano threefolds, taken up in [Mad02], [AC00], [AF06], [Fae05a].

Even more attention has been paid to the case of hypersurfaces in $\mathbb{P}^m$ of degree d, which we denote by Y_d. Particularly pertinent to our study are the papers [Mad00], [CM00], [CM04], [CM05], [KRR05]. If we summarize the results obtained in the literature, we get that no aCM rank 2 indecomposable bundle exists on the general Y_d for $m \geq 5$, $d \geq 3$, plus evidence that the same should happen for $m = 4$, $d \geq 6$ (and this indeed holds true for $d = 6$). Moreover, the geometry of these bundles has been studied in great detail in case $m = 4$, $d \leq 4$, see for instance the papers [Dru00], [IM00a], [IM00b].

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Let us now consider the question for *surfaces in* $\mathbb{P}^3$ of degree d. For $d = 1, 2$, we have seen that the question is settled by Horrocks and Knörrer. For $d = 3, 4$ the classification follows from [Mad00], [Fae05b]. However, the problems remains wide open for $d \geq 5$, except the specific case of bundles admitting a resolution by linear forms (see Schreyer's appendix to [Bea00]).

This paper is devoted to the classification of aCM 2-bundles on a *general quintic surface*. After choosing an *initial* twist for the bundle $\mathscr{E}$, we completely classify the pairs of integers which are the Chern classes of an ACM indecomposable rank 2 bundle over a general quintic surface. More precisely, fix a twist the bundle $\mathscr{E}$ so that $\mathrm{H}^0(\mathscr{E}(-1)) = 0$, $\mathrm{H}^0(\mathscr{E}) \neq 0$ (then $\mathscr{E}$ is called *initialized*). Our main result is the following:

Theorem. *On the general quintic surface $X \subset \mathbb{P}^3$ there exist initialized indecomposable aCM bundles $\mathscr{E}$ of rank 2 with the following invariants:*

$c_1(\mathscr{E})$	-2	-1	0	0	0	1	1	1	2	2	2	2	3	4
$c_2(\mathscr{E})$	1	2	3	4	5	6	8	10	11	12	13	14	20	30

Moreover, these are the only possible Chern classes for such $\mathscr{E}$. The bundle $\mathscr{E}$ is stable for $c_1(\mathscr{E}) > 0$ and the moduli space $\mathsf{M}_X(2; c_1(\mathscr{E}), c_2(\mathscr{E}))$ is smooth of the expected dimension at a general point.

Observe that these are also the possible Chern classes of initialized indecomposable aCM rank 2 bundles on a general quintic threefold in $\mathbb{P}^4$. We do not know however, for general threefold hypersurfaces of degree 5, if bundles with all the listed Chern classes actually exist. See [CM05] for a discussion.

In the next section we fix some notation and review some basic results. In Section 3 we introduce an inductive method which takes care of the cases with low c_1, namely $c_1(\mathscr{E}) \leq 3$. In Section 4 we write down some general bounds for $c_i(\mathscr{E})$. Section 5 summarizes the classification of aCM 2-bundle on surfaces of degree ≤ 4, while Section 6 contains the proof of our result in the case $c_1(\mathscr{E}) \geq 3$.

2. Preliminaries

We will work on a projective variety Y defined over the field $\mathbb{C}$ of complex numbers, embedded by the line bundle $\mathscr{O}_Y(1)$.

As a matter of terminology, any claim about the *general element* of a given family will mean that there exists a Zariski closed subset of the relevant parameter space such that the claim holds in the complement of this set. The parameter space will often be implicit, for example we will choose the general hypersurface of degree d in $\mathbb{P}^3$ in an open subset of $|\mathscr{O}_{\mathbb{P}^3}(d)|$. The letter R will denote the coordinate ring of $\mathbb{P}^3$, i.e. the polynomial ring in 4 variables, and $R(t)$ will be its graded piece of degree t. Given a subscheme $Z \subset \mathbb{P}^3$, I_Z will denote the ideal of Z in R; we will write R_Z for the coordinate $\mathbb{C}$-algebra R/I_Z, and $R_Z(t)$ for its homogeneous degree t piece.

A subscheme $Z \subset \mathbb{P}^3$ is called *arithmetically Gorenstein* (aG) if R_Z is a Gorenstein ring. If the subscheme Z is a *complete intersection (ci)*, of hypersurfaces of degree d_1, d_2, d_3, then its *type* will be the triple of integers (d_1, d_2, d_3). We will consider the *difference Hilbert function* attached to Z:

$$\Delta\,\mathrm{h}(R_Z, t) = \dim_{\mathbb{C}} R_Z(t) - \dim_{\mathbb{C}} R_Z(t-1).$$

Let Y_d be a *general* surface of degree d in $\mathbb{P}^3$, given by a homogeneous polynomial F_d in $\mathrm{H}^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(d))$. We will assume that Y_d is smooth and, when $d \geq 4$, that the Picard group of Y_d is generated by H, namely the restriction to Y_d of the hyperplane class $c_1(\mathscr{O}_{\mathbb{P}^3}(1))$. We will identify $\mathrm{Pic}(Y_d)$ with $\mathbb{Z}$. The canonical class of Y_d is thus $\omega_{Y_d} \simeq \mathscr{O}_{Y_d}(d-4)$. Given a subscheme $Z \subset Y_d \subset \mathbb{P}^3$, the symbol $\mathscr{I}_{Z,Y_d}$, sometimes simplified to $\mathscr{I}_Z$, will denote the ideal sheaf of Z in Y_d. We have the exact sequence:

$$0 \to \mathscr{I}_Z \to \mathscr{O}_{Y_d} \to \mathscr{O}_Z \to 0. \tag{2.1}$$

Notice that, under our assumptions, for any sheaf $\mathscr{F}$ on Y_d, $c_1(\mathscr{F})$ is identifyed with an integer, while $c_2(\mathscr{F})$ will be a multiple of the class of a point in Y_d. So we identify the Chern classes of $\mathscr{F}$ with a pair of integers. We write $\mathscr{F}(t)$ for $\mathscr{F} \otimes \mathscr{O}_{Y_d}(1)^{\otimes t}$.

Definition 2.1. For a sheaf $\mathscr{F}$ on Y_d, define the *initial twist* as the integer n such that $\mathrm{h}^0(\mathscr{F}(n)) \neq 0$ and $\mathrm{h}^0(\mathscr{F}(n-1)) = 0$. $\mathscr{F}$ is called *initialized* if its initial twist is zero, i.e. if $\mathrm{h}^0(\mathscr{F}) > \mathrm{h}^0(\mathscr{F}(-1)) = 0$.

In some paper on the same subject (e.g. [Mad98]), bundles satisfying the previous property are called *normalized*. We prefer to switch to a new terminology, for in some classical paper on vector bundles (e.g. [Har77]), the word *normalized* has been used with a different meaning.

We will use the notation C_g^d for a reduced connected curve of arithmetic genus g and degree d contained in $\mathbb{P}^3$.

We will denote the moduli space of stable vector bundles on Y or rank r, with Chern classes c_1, c_2 by $\mathsf{M}_Y(r; c_1, c_2)$, see [HL97].

Remark 2.2. If $\mathscr{F}$ is a vector bundle on Y_d, of rank r, then we have:

$$c_1(\mathscr{F}(t)) = c_1(\mathscr{F}) + r\,t, \tag{2.2}$$

$$c_2(\mathscr{F}(t)) = c_2(\mathscr{F}) + c_1(\mathscr{F})\,(\mathrm{rk}(\mathscr{F}) - 1)\,d\,t + \binom{r}{2} d\,t^2, \tag{2.3}$$

$$\chi(\mathscr{F}) = -c_2(\mathscr{F}) + \frac{d}{6}\,(3\,c_1(\mathscr{F})\,(c_1(\mathscr{F}) + 4 - d) + r\,(11 - 6\,d + d^2)). \tag{2.4}$$

The following theorem summarizes well-known results about the Serre correspondence between (aG) zero-dimensional *locally complete intersection* (*lci* for short) subschemes $Z \subset Y_d$ and (aCM) rank 2 vector bundles on Y_d. We refer to [HL97, Theorem 5.1.1] for a proof.

Theorem 2.3. *Let $Z \subset Y_d$ be a zero-dimensional* lci *subscheme of Y_d, and consider $\mathscr{O}_{Y_d}(c)$. Then the following are equivalent:*

i) There exist a rank 2 vector bundle $\mathscr{F}$ with $c_1(\mathscr{F}) = c$ and an extension:

$$0 \to \mathscr{O}_{Y_d}(-c) \to \mathscr{F}^* \to \mathscr{I}_Z \to 0. \tag{2.5}$$

ii) The pair $(\mathscr{O}_{Y_d}(d + c - 4), Z)$ has the Cayley-Bacharach property i.e. for any section $s \in \mathrm{H}^0(\mathscr{O}_{Y_d}(d + c - 4))$, and for any $Z' \subset Z$ with $\mathrm{len}(Z') = \mathrm{len}(Z) - 1$, we have $s_{|Z} = 0 \Leftrightarrow s_{|Z'} = 0$.

Remark 2.4. The exact sequence (2.5) amounts locally to the Koszul complex associated to the section $s_Z \in \mathrm{H}^0(\mathscr{F})$ vanishing along Z. We say that

$\mathscr{F}$ is *associated* to Z. So, a general global section of $\mathscr{F}$ vanishes on a subscheme of length $\mathrm{len}(Z)$. Notice that dualizing (2.5) we obtain the exact sequence:

$$0 \to \mathscr{O}_{Y_d} \xrightarrow{s} \mathscr{F} \to \mathscr{I}_Z(c) \to 0. \tag{2.6}$$

In particular, when $\mathscr{F}$ is initialized, we have:

$$\mathrm{h}^0(\mathscr{I}_Z(c-t)) = 0 \quad \forall t > 0. \tag{2.7}$$

Equivalently, $\mathscr{F}$ is initialized if and only if Z lies in no surfaces of degree e for $e \leq c$.

The following theorem is essentially well-known, so we only sketch a bit of the proof. For general reference on aG subschemes and difference Hilbert functions we refer to the book [IK99].

Theorem 2.5. *In the previous setting, the following statements are equivalent:*

a) The scheme Z is aG;
b) For all t, we have $\dim_{\mathbb{C}} R_Z(t) + \dim_{\mathbb{C}} R_Z(c_1(\mathscr{F}) + d - 4 - t) = \mathrm{len}(Z)$;
c) For all t, we have $\Delta\,\mathrm{h}(R_Z, t) = \Delta\,\mathrm{h}(R_Z, c_1(\mathscr{F}) + d - 3 - t)$;
d) The bundle $\mathscr{F}$ is aCM.

Proof. The equivalence of (a) and (b) is proved in [DGO85]. To see (b) $\Leftrightarrow$ (c), just notice that:

$$\begin{aligned}\Delta\,\mathrm{h}(R_Z,t) - \Delta\,\mathrm{h}(R_Z, s-t) = &\dim_{\mathbb{C}} R_Z(t) + \dim_{\mathbb{C}} R_Z(s-1-t) - \\ &-(\dim_{\mathbb{C}} R_Z(t-1) + \dim_{\mathbb{C}} R_Z(s-t)),\end{aligned}$$

so, setting $s = c_1(\mathscr{F}) + d - 4$), we get that (c) holds if and only if $\dim_{\mathbb{C}} R_Z(t) + \dim_{\mathbb{C}} R_Z(c_1(\mathscr{F}) + d - 4 - t)$ is constant in t. But this constant equals $\mathrm{len}(Z)$ for $\dim_{\mathbb{C}} R_Z(t) = \mathrm{len}(Z)$, for big enough t. To see (b) $\Leftrightarrow$ (d), we twist by $\mathscr{O}_{Y_d}(t)$ the sequence (2.5) and take global sections. The cokernel of the transpose of the induced map $\mathrm{H}^2(\mathscr{I}_Z(t)) \simeq \mathrm{H}^2(\mathscr{O}_{Y_d}(t)) \to \mathrm{H}^2(\mathscr{F}^*(t))$ agrees, by Serre duality, with $\mathrm{H}^0(\mathscr{I}_Z(d + c_1(\mathscr{F}) - 4 - t))$. Then we compute $\mathrm{h}^1(\mathscr{F}^*(t)) = \mathrm{h}^1(\mathscr{I}_Z(t)) - \dim_{\mathbb{C}} R_Z(d + c_1(\mathscr{F}) - 4 - t)$. But since $\mathrm{h}^1(\mathscr{I}_Z(t)) = \mathrm{len}(Z) - \dim_{\mathbb{C}} R_Z(t)$, we see that $\mathrm{h}^1(\mathscr{F}^*(t))$ is zero for all t if and only if condition (b) is satisfied. ☐

Remark 2.6. Of course we have $\mathrm{H}^0(\mathscr{O}_Z(t)) = R_Z(t) = (0)$ for all negative t, so by Theorem 2.5, part (c) we get:

$$\begin{aligned} &\Delta\,\mathrm{h}(R_Z, t) = 0, && \text{for } t < 0 \text{ and for } t > c_1(\mathscr{F}) + d - 3; \\ &\dim_{\mathbb{C}} R_Z(t) = \mathrm{len}(Z), && \text{for } t \geq c_1(\mathscr{F}) + d - 3. \end{aligned} \tag{2.8}$$

Similarly one obtains $\Delta\,\mathrm{h}(R_Z, c_1(\mathscr{F}) + d - 3) = \Delta\,\mathrm{h}(R_Z, 0) = 1$. In turn, this implies $\dim_{\mathbb{C}} R_Z(c_1(\mathscr{F}) + d - 4) = d - 1$. We get:

$$\mathrm{h}^1(\mathscr{I}_Z(c_1(\mathscr{F}) + d - 4)) = 1. \tag{2.9}$$

Furthermore, we have:

$$\mathrm{len}(Z) = \sum_{t=0}^{c_1+d-3} \Delta\,\mathrm{h}(R_Z, t). \tag{2.10}$$

Following [Bea00, page 18], given an aG subscheme Z of $\mathbb{P}^3$, we will call *index* of Z the integer i_Z such that $\Delta\,\mathrm{h}(R_Z,t)=\Delta\,\mathrm{h}(R_Z,i_Z+1-t)$, for all t. It is the largest integer c such that $\dim_{\mathbb{C}} R_Z(c)<\mathrm{len}(Z)$.

Remark 2.7. The vector bundle $\mathscr{F}^*$ provides an element of the extension group $\mathrm{Ext}^1(\mathscr{I}_Z,\mathscr{O}_{Y_d}(-c_1))$. By Serre duality we have:

$$\mathrm{Ext}^1(\mathscr{I}_Z,\mathscr{O}_{Y_d}(-c_1))^*\simeq \mathrm{H}^1(\mathscr{I}_Z(d+c_1(\mathscr{F})-4)). \tag{2.11}$$

Hence by 2.9 the group $\mathrm{Ext}^1(\mathscr{I}_Z,\mathscr{O}_{Y_d}(-c_1))$ has dimension 1. Then, to the aG subscheme $Z\subset X$ we *associate* a pair $[\mathscr{F}_Z,s_Z]$, where $\mathscr{F}_Z$ fits in the extension (2.5) and is uniquely determined, and s_Z is a global section of $\mathscr{F}_Z$ (determined up to a scalar), and $Z=\{s_Z=0\}$.

Remark 2.8. A rank 2 bundle $\mathscr{F}$ on Y_d is decomposable into a direct sum of line bundles if and only if, for some integer a, there is a global section $s\in \mathrm{H}^0(\mathscr{F}(a))$ such that the cokernel of the induced map $s:\mathscr{O}_{Y_d}\to\mathscr{F}(a)$ is isomorphic to $\mathscr{O}_{Y_d}(c_1(\mathscr{F}(a)))$, i.e. the 0-locus is empty. From the point of view of subschemes, $Z\subset Y_d$ is a complete intersection *of Y_d and two more surfaces* if and only if its associated rank 2 bundle decomposes.

If an aG subscheme Z is contained in a plane H, it determines an aCM 2-bundle $\mathscr{E}$ defined on H, so $\mathscr{E}$ splits by Horrocks' criterion and Z is complete intersection in $\mathbb{P}^3$.

The following result, proved by Beauville in [Bea00], allows to relate rank 2 aCM bundles over the hypersurface Y_d to skew-symmetric matrices whose determinant has degree $2\,d$ defined on $\mathbb{P}^3$.

Theorem 2.9. *Let $Y_d\subset\mathbb{P}$ be a hypersurface defined by the homogeneous form F_d, $\mathscr{E}$ be a rank 2 aCM bundle on Y_d with $c_1(\mathscr{E})=c_1\,H$. Let $\iota:Y_d\hookrightarrow\mathbb{P}^3$ be the inclusion. Then the minimal graded free resolution of the sheaf $\iota_*(\mathscr{E})$ takes the form:*

$$0\to\mathsf{P}(\mathscr{E})^*(c_1-d)\xrightarrow{\mathsf{f}(\mathscr{E})}\mathsf{P}(\mathscr{E})\xrightarrow{\mathsf{p}(\mathscr{E})}\iota_*(\mathscr{E})\to 0,$$

where $\mathsf{P}(\mathscr{E})=\bigoplus_{i=1}^r\mathscr{O}_{\mathbb{P}^3}(a_i)$, $\mathsf{f}(\mathscr{E})$ is skew-symmetric, and $\mathrm{Pfaff}(\mathsf{f}(\mathscr{E}))=F_d$.

Conversely, given $\mathsf{P}=\bigoplus_{i=1}^r\mathscr{O}_{\mathbb{P}^3}(a_i)$ and a general skew-symmetric matrix $\mathsf{f}:\mathsf{P}^*(c-d)\to\mathsf{P}$, with $\mathrm{Pfaff}(\mathsf{f})=F_d$, the sheaf $\mathscr{E}=\mathrm{coker}(\mathsf{f})$ is a rank 2 aCM vector bundle, defined on the surface Y_d given by F_d, with $c_1(\mathscr{E})=c_1\,H$.

3. A degeneration result

For any variety Y, we denote with $\mathrm{Hilb}_m(Y)$ the Hilbert scheme parametrizing closed subschemes of Y of finite length m. Recall that the Hilbert scheme parametrizing closed subschemes of length m of any smooth surface, is a smooth irreducible projective variety of dimension $2\,m$ (see [HL97, pag. 104]).

Denote by $\mathscr{G}(m,i)$ the subset of the Hilbert schemes $\mathrm{Hilb}_m(\mathbb{P}^3)$ parametrizing length m subschemes of $\mathbb{P}^3$ consisting of aG schemes of index i. The number and the dimension of the irreducible components of this locally closed subset of $\mathrm{Hilb}_m(\mathbb{P}^3)$, are well-known, see [Die96] and [IK99]. In particular a component of $\mathscr{G}(m,i)$ is given once we fix the Hilbert function h of R_Z. We denote such a component by $\mathscr{G}_{\mathrm{h}}(m,i)$. Write $\mathscr{G}(m,i,d)$ for the incidence variety consisting of pairs (Z, Y), with $Y \in |\mathscr{O}_{\mathbb{P}^3}(d)|$, $Z \in \mathscr{G}(m, i)$ and $Z \subset Y$.

Remark 3.1. For any triple of integers (m, i, d), we have the incidence diagram:

$$
\begin{array}{ccc}
 & \mathscr{G}(m,i,d) & \\
 \swarrow^{q_d} & & \searrow^{p_{m,i,d}} \\
\mathscr{G}(m,i) & & |\mathscr{O}_{\mathbb{P}^3}(d)|.
\end{array}
\tag{3.1}
$$

The fibre $p_{m,i,d}^{-1}(Y)$ consists of the family of aG subschemes $Z \subset Y$, of length m and index i, while $q_d^{-1}(Z)$ is isomorphic to $\mathbb{P}(\mathrm{H}^0(\mathscr{I}_{Z,\mathbb{P}^3}(d)))$.

In view of Theorem 2.5, we conclude that the aCM vector bundle $\mathscr{E}$ having Chern classes $c_1 = c_1(\mathscr{E})$ and $c_2 = c_2(\mathscr{E})$ is defined on the *general* hypersurface of degree d if, setting $m = c_2$, $i = c_1 + d - 4$, the map $p_{m,i,d}$ in the diagram (3.1) is dominant. Computing dimensions in the same diagram, we get that $p = p_{m,i,d}$ is dominant if and only if, for some Hilbert function h, $p^{-1}(Y_d)$ has a component F where the following holds:

$$\binom{d+3}{3} + \dim(F) = \dim(\mathscr{G}_{\mathrm{h}}(m,i)) + \mathrm{h}^0(\mathscr{I}_{Z,Y_d}(d)).$$

Summing up we have the criterion:

$$\boxed{\begin{array}{c}\text{the map } p = p_{m,i,d}\\ \text{is dominant}\end{array}} \iff \boxed{\begin{array}{c}\text{there exists } Y_d \in |\mathscr{O}_{\mathbb{P}^3}(d)| \text{ such that } p^{-1}(Y_d) \text{ has a}\\ \text{component of dimension } \dim(\mathscr{G}_{\mathrm{h}}(m,i)) - \dim_{\mathbb{C}} R_Z(d)\end{array}}$$

The following lemma introduces an inductive method on the degree d, to prove that the map $p_{m,i,d}$ is dominant. Unfortunately the numerical assumption $c_1 \le 3$ limits its range of applicability.

Lemma 3.2. *Fix integers d, m and fix $c_1 \le 3$. Set $i = c_1 + d - 4$. Suppose that the map $p = p_{m,i,d}$ of Diagram (3.1) is dominant and $\mathrm{h}^0(\mathscr{I}_Z(1)) = 0$ for general subschemes $Z \subset \mathbb{P}^3$ contained in general fibres of p. Then $p' = p_{m,i,d+1}$ is also dominant.*

Proof. We would like to provide a surface of degree $d + 1$ satisfying the criterion introduced above. We start by taking a *general* surface Y_d of degree d, which we may assume to lie in the image of p by the hypothesis.

In view of the criterion of Remark (3.1), we are allowed to choose a component G of the quasi-projective scheme $\mathscr{G}(m, i)$, and a component F of $p^{-1}(Y_d)$ such that:

$$\dim(F) = \dim(G) - \dim_{\mathbb{C}} R_Z(d).$$

Now, take the reducible hypersurface $\overline{Y}_{d+1}$ defined as the union of the general hypersurface Y_d given above and a general plane in $\mathbb{P}^3$. The condition $c_1 \le 3$ entails $\dim_{\mathbb{C}} R_Z(d) = \dim_{\mathbb{C}} R_Z(d+1)$, indeed $\dim_{\mathbb{C}} R_Z(t) = \mathrm{len}(Z) = m$ for any $t \ge c_1 + d - 3$.

Notice that the map q_{d+1} is birational to a $\mathbb{P}^k$-bundle onto the given component G, where $k = \mathrm{h}^0(\mathscr{I}_Z(d+1)) - 1$, for a general element $Z \in G$. Indeed, the Hilbert function is constant on the components of $\mathscr{G}(m, i)$. Therefore the total space $q_{d+1}^{-1}(G)$ is irreducible, and we consider the restriction of p' to this space. Observe also that the component F can be seen as a component of p' as well, since a general hyperplane H will not meet the subscheme Z, so that a subscheme in the neighborhood of Z lies in Y_d if and only if it lies in $\overline{Y}_{d+1}$.

Then we are in position to apply [Har77, 3.22.b page 95]. This yields, for a general fibre of p' around $\overline{Y}_{d+1}$:

$$\begin{aligned} \dim(p')^{-1}(Y_{d+1}) \leq \dim p^{-1}(Y_d) = \\ = \dim(\mathscr{G}(m,i)) - \dim_{\mathbb{C}} R_Z(d) = \\ = \dim(\mathscr{G}(m,i)) - \dim_{\mathbb{C}} R_Z(d+1). \end{aligned} \tag{3.2}$$

We conclude that the dimension of the image of p' equals at least the dimension of $|\mathscr{O}_{\mathbb{P}^3}(d+1)|$. Then equality must hold and p' is dominant. □

4. Some general results

In the following proposition, we determine the range of $c_1(\mathscr{E})$, for an indecomposable rank 2 aCM bundle $\mathscr{E}$ on Y_d, and establish the value for $c_2(\mathscr{E})$ in the two maximal and minimal alternatives for $c_1(\mathscr{E})$.

Proposition 4.1. *Let Y_d as above, $d \geq 4$ and let $\mathscr{E}$ be a rank 2 initialized indecomposable aCM vector bundle over Y_d. Then any nonzero section of $\mathscr{E}$ vanishes over a zero-dimensional, locally complete intersection subscheme $Z \subset Y_d$ with $\operatorname{len}(Z) = c_2(\mathscr{E})$ and we have:*

$$3 - d \leq c_1(\mathscr{E}) \leq d - 1. \tag{4.1}$$

Moreover, we have the implications:

$$c_1(\mathscr{E}) = 3 - d \Longleftrightarrow c_2(\mathscr{E}) = 1, \tag{4.2}$$

$$c_1(\mathscr{E}) = 4 - d \Longleftrightarrow c_2(\mathscr{E}) = 2, \tag{4.3}$$

$$c_1(\mathscr{E}) = d - 2 \Longrightarrow c_2(\mathscr{E}) = \frac{d\,(d-1)\,(d-2)}{3}, \tag{4.4}$$

$$c_1(\mathscr{E}) = d - 1 \Longrightarrow c_2(\mathscr{E}) = \frac{d\,(d-1)\,(2\,d-1)}{6}. \tag{4.5}$$

Proof. The inequalities 4.1 are similar of the main theorem of [Mad98], where the same bounds are proved for hypersurfaces with Picard group $\mathbb{Z}$.

Fix $s \in \mathrm{H}^0(\mathscr{E})$. We have $\operatorname{Im}(s) \neq \mathscr{O}_{Y_d}$ by Remark 2.8. i.e. Z is not empty. If Z contains a divisor $A \in |\mathscr{O}_{Y_d}(a)|$, with $a \geq 1$, then $\mathscr{I}_Z \subset \mathscr{O}_{Y_d}(-a)$, so that $\mathrm{h}^0(\mathscr{E}(-a)) \neq 0$, contradicting the hypothesis that $\mathscr{E}$ is initialized. Thus s vanishes in codimension 2, so $\operatorname{len}(Z) = c_2(\mathscr{E}) \geq 1$.

Since $\dim_{\mathbb{C}} R_Z(t) = 0$ for $t \leq -1$, applying property (b) of Theorem 2.5 to $\mathscr{E}$, with $t = -1$, one gets $\dim_{\mathbb{C}} R_Z(c_1(\mathscr{E}) + d - 3) = \operatorname{len}(Z) > 0$, hence $c_1(\mathscr{E}) \geq 3-d$. Furthermore, when $c_1(\mathscr{E}) = 3-d$ (resp. $c_1(\mathscr{E}) = 4-d$), again by property (b) of Theorem 2.5 for $t = 0$, we get $\operatorname{len}(Z) = \dim_{\mathbb{C}} R_Z(0) = 1$ (resp. $\operatorname{len}(Z) = 2\dim_{\mathbb{C}} R_Z(0) = 2$). So (4.2) and (4.3) follow. On the other hand, if $c_1(\mathscr{E}) \geq d-1$, we have:

$$\mathrm{h}^2(\mathscr{E}(-2)) = \mathrm{h}^0(\mathscr{E}^* \otimes \omega_{Y_d}) = \mathrm{h}^0(\mathscr{E}(d - c_1(\mathscr{E}) - 2)) = 0,$$

for $\mathscr{E}$ is initialized. Since $\mathscr{E}$ is aCM, and by duality $\mathrm{h}^2(\mathscr{E}(-1)) \leq \mathrm{h}^2(\mathscr{E}(-2))$, this implies that:

$$\chi(\mathscr{E}(-2)) = \chi(\mathscr{E}(-1)) = 0.$$

Solving the above equations in $c_1(\mathscr{E})$ and $c_2(\mathscr{E})$, using Formula (2.4) one easily obtains (4.5) and (4.4). □

In Proposition 4.1, (4.5) and (4.4) are called, respectively, the *maximal* and *submaximal* cases.

Proposition 4.2. *Let Y_d and $\mathscr{E}$ be as above, $d \geq 4$, set $c_1 = c_1(\mathscr{E})$ and $c_2 = c_2(\mathscr{E})$. Then we have the following bounds on c_2.*

Lower bound: *We distinguish the cases:*

$$c_1 \leq 0 \Longrightarrow c_2 \geq c_1 + d - 2, \tag{4.6}$$

$$c_1 \geq 1 \Longrightarrow c_2 \geq \frac{c_1}{6}\,(c_1+1)\,(3\,d - c_1 - 2). \tag{4.7}$$

Upper bound: *According to the parity of $c_1 + d - 3$ we have:*

$$c_1 + d - 3 = 2\,\ell \Longrightarrow c_2 \leq \frac{(\ell+1)\,(\ell+2)}{6}\,(2\,\ell+3), \tag{4.8}$$

$$c_1 + d - 3 = 2\,\ell - 1 \Longrightarrow \begin{cases} c_2 \leq \frac{(\ell+1)\,(\ell+2)}{3}\,\ell, \\ c_2 \text{ is even.} \end{cases} \tag{4.9}$$

Proof. Take a nonzero section of $\mathscr{E}$ and consider its vanishing locus Z. We have $\mathrm{len}(Z) = c_2$. Recall that the difference Hilbert function $\Delta\,\mathrm{h}(R_Z, t)$ is concave and symmetric by Theorem 2.5, part (c).

In case $c_1 \leq 0$, the lower bound (4.6) amounts to $\Delta\,\mathrm{h}(R_Z, t)$ being greater or equal than 1 for all t between 0 and $i_Z+1 = c_1+d-3$. Equality is achieved for $\Delta\,\mathrm{h}(R_Z, t)$ constantly equal to 1, for $0 \leq t \leq c_1+d-3$. Such a subscheme has length $c_1 + d - 2$ and is contained in a line, for $\dim_{\mathbb{C}} R_Z(1) = 2$, hence $\mathrm{H}^0(\mathscr{I}_Z(1)) = 2$ by sequence 2.1.

Now, if $c_1 \geq 1$, by [IK99, Theorem 5.25], we can divide the summation interval $0 \leq t \leq c_1 + d - 3$ into three subintervals:

i) increasing: $t \in I_1 := [0, c_1 - 1]$;
ii) concave: $t \in I_2 := [c_1, d - 3]$;
iii) decreasing: $t \in I_3 := [d - 2, c_1 + d - 3]$;

Notice that:

$$t \in I_1 \Longleftrightarrow c_1 + d - 3 - t \in I_3.$$

So, using property (c) of Theorem 2.5, we obtain:

$$\sum_{t \in I_1} \Delta\,\mathrm{h}(R_Z, t) = \sum_{t \in I_3} \Delta\,\mathrm{h}(R_Z, t).$$

Putting this into (2.10), we get $\mathrm{len}(Z) = 2\sum_{t\in I_1} \Delta\,\mathrm{h}(R_Z, t) + \sum_{t\in I_2} \Delta\,\mathrm{h}(R_Z, t)$. Formula 2.7 shows that $\mathrm{h}^0(\mathscr{I}_Z(c_1 - 1 - t)) = 0$, for all $t \geq 0$, so $\Delta\,\mathrm{h}(R_Z, t)$ agrees with $\Delta\,\mathrm{h}(R, t) = \binom{t+2}{2}$ for all t in I_1 and $\sum_{t\in I_1} = \binom{c_1+2}{3}$.

In the interval I_2, see (ii), the function $\Delta\,\mathrm{h}(R_Z, t)$ takes value at least $\binom{c_1+1}{2}$ since $\Delta\,\mathrm{h}(R_Z)$ is concave (see [IK99]). So $\sum_{t\in I_2} \Delta\,\mathrm{h}(R_Z, t)$ is bounded below by this value, multiplied by the length of the interval (i.e. c_1+d-2). Summing this to the above value for the intervals I_1 and I_3 we obtain:

$$(4.10)\quad c_2(\mathscr{E}) = \mathrm{len}(Z) = 2\sum_{t\in I_1}\Delta\,\mathrm{h}(R_Z,t) + \sum_{t\in I_2}\Delta\,\mathrm{h}(R_Z,t) \geq$$
$$\geq \frac{c_1}{3}(c_1+1)(c_1+2) + \frac{c_1}{2}(c_1+1)(d-c_1-2) = \frac{c_1}{6}(c_1+1)(3\,d-c_1-2).$$

Now let us prove the upper bounds. For (4.8), observe that $\Delta\,\mathrm{h}(R_Z,t)$ is bounded above by $\binom{t+2}{2}$. Using again the symmetry of the difference Hilbert function and formula (2.10) we get:

$$(4.11)\quad c_2(\mathscr{E}) = \mathrm{len}(Z) = \Delta\,\mathrm{h}(R_Z,\ell) + \Big(\sum_{t=0}^{\ell-1}\Delta\,\mathrm{h}(R_Z,t) + \sum_{t=\ell+1}^{2\,\ell}\Delta\,\mathrm{h}(R_Z,t)\Big) =$$
$$= \Delta\,\mathrm{h}(R_Z,\ell) + 2\sum_{t=0}^{\ell-1}\Delta\,\mathrm{h}(R_Z,t) \leq \binom{\ell+2}{2} + 2\binom{\ell+2}{3} = \frac{(\ell+1)(\ell+2)}{6}(2\,\ell+3).$$

The case (4.9) is proved analogously; one just needs to observe that the summation interval for $\Delta\,\mathrm{h}$ is symmetric, so $\mathrm{len}(Z)$ is even. □

Remark 4.3. On any Y_d, $d \geq 2$ there exist initialized indecomposable aCM rank 2 bundles with minimal first Chern class $c_1(\mathscr{E}) = 3-d$, $c_2(\mathscr{E}) = 1$. Namely, just take a point $y \in Y_d$ and observe that $Z = \{y\}$ is an aG subscheme of Y_d, and the pair $(\mathscr{O}_{Y_d}(-1), Z)$ has the Cayley-Bacharach property. We obtain a bundle $\mathscr{E}_y$ (indecomposable for y is not *ci*), and the exact sequences:

$$(4.12)\qquad 0 \to \mathscr{O}_{Y_d}(d-3) \to \mathscr{E}_y^* \to \mathscr{I}_y \to 0,$$

$$(4.13)\qquad 0 \to \begin{matrix}\mathscr{O}(3-2\,d)\\ \oplus\\ \mathscr{O}(1-d)^3\end{matrix} \to \begin{matrix}\mathscr{O}\\ \oplus\\ \mathscr{O}(2-d)^3\end{matrix} \to \iota_*(\mathscr{E}_y) \to 0.$$

Similarly, a length-2 subscheme $Z_2 \subset Y_d$, Z_2 is aG and the pair $(\mathscr{O}_{Y_d}, Z_2)$ has the Cayley-Bacharach property. So, on Y_d it is defined an aCM rank-2 bundle $\mathscr{E}_{Z_2}$ with $c_1(\mathscr{E}_{Z_2}) = 4-d$, $c_2(\mathscr{E}_{Z_2}) = 2$. The bundle $\mathscr{E}_{Z_2}$ is indecomposable for $d \geq 3$ and we have the exact sequences:

$$(4.14)\qquad 0 \to \mathscr{O}_{Y_d}(d-4) \to \mathscr{E}_{Z_2}^* \to \mathscr{I}_{Z_2} \to 0.$$

$$(4.15)\qquad 0 \to \begin{matrix}\mathscr{O}(4-2\,d)\\ \oplus\\ \mathscr{O}(1-d)^2\\ \oplus\\ \mathscr{O}(2-d)\end{matrix} \to \begin{matrix}\mathscr{O}\\ \oplus\\ \mathscr{O}(3-d)^2\\ \oplus\\ \mathscr{O}(2-d)\end{matrix} \to \iota_*(\mathscr{E}_{Z_2}) \to 0.$$

These resolutions are obtained via a standard mapping cone construction from the obvious resolutions of the ideal sheaves $\mathscr{I}_y$, $\mathscr{I}_{Z_2}$. The fact that the previous bundles are initialized follows immediately from their resolutions.

Proposition 4.4. *Let $\mathscr{E}$ be an initialized indecomposable aCM rank 2 bundle on Y_d, $d \geq 4$, and suppose $c_1(\mathscr{E}) = 5-d$. Then we have: $c_2(\mathscr{E}) \in \{3,4,5\}$. A general section of $\mathscr{E}$ vanishes, respectively, along a length 3* ci *subscheme of type $(1,1,3)$, a length 4* ci *subscheme of type $(1,2,2)$, and a length 5 subscheme in Y_d contained in 5 independent quadrics.*

All the three possibilities take place on any reduced surface Y_d.

Proof. Let Z be a nonzero section of $\mathscr{E}$. Since $\mathscr{E}$ is aCM, property (c) of Theorem 2.5 gives $\Delta\,\mathrm{h}(R_Z,t) = \Delta\,\mathrm{h}(R_Z, 2-t)$, so $\Delta\,\mathrm{h}(R_Z, 2) = 1$. Furthermore, we have $1 \leq \Delta\,\mathrm{h}(R_Z, 1) \leq 3$, hence $c_2(\mathscr{E}) = \mathrm{len}(Z)$ runs between 3 and 5.

If $c_2(\mathscr{E}) = 3$ (i.e. if $\leq \Delta\,\mathrm{h}(R_Z, 1) = 1$), then $\mathrm{h}^0(\mathscr{I}_Z(1)) = 2$. This means that Z is contained in a line and therefore it is *ci*. Similarly, $c_2(\mathscr{E}) = 4$ implies $\mathrm{h}^0(\mathscr{I}_Z(1)) = 1$, so Z is contained in a plane and thus it is complete intersection. In case $c_2(\mathscr{E}) = 5$, we have $\mathrm{h}^0(\mathscr{I}_Z(1)) = 0$ and $\mathrm{h}^0(\mathscr{I}_Z(2)) = 5$, which means 5 independent quadrics.

Clearly, any hypersurface Y_d contains 3 collinear points. Analogously, 4 general coplanar points in Y_d are aG. Finally, 5 general points in Y_d are also aG, indeed it suffices that they are non coplanar, for in this case $\Delta\,\mathrm{h}(R_Z, t)$ takes the form $1, 3, 1$. □

Proposition 4.5. *Let $\mathscr{E}$ be an initialized indecomposable aCM rank 2 bundle on Y_d, $d \geq 5$, and suppose $c_1(\mathscr{E}) = 6 - d$. Let Z be the vanishing locus of a nonzero section of $\mathscr{E}$. Then we have: $c_2(\mathscr{E}) \in \{4, 6, 8\}$. If $c_2(\mathscr{E}) \in \{4, 6\}$, then Z is a* ci *subscheme of type $(1, 1, 4)$, $(1, 2, 3)$. If $c_2(\mathscr{E}) = 8$, then Z is contained in three independent quadrics (and no hyperplane).*

All the above possibilities take place over a general hypersurface Y_d.

Proof. By Theorem 2.5, in our hypothesis the function $\Delta\,\mathrm{h}(R_Z)$ takes one of the three possible forms: $1, 1, 1, 1$ or $1, 2, 2, 1$ or $1, 3, 3, 1$. In the first two cases the aG subscheme Z is contained in a plane and thus it is complete intersection. In the third case, i.e. if $c_2(\mathscr{E}) = 8$, then looking at the function $\Delta\,\mathrm{h}(R_Z)$, one sees that Z is contained in three independent quadrics and no hyperplane.

For the second statement, of course any Y_d contains 4 collinear points. Moreover, intersecting Y_d with a conic C contained in a general plane, one gets $2\,d$ points on C. Choosing Z to be the union of 6 of them, one can find a cubic intersecting C at Z, so we find the subscheme $Z \subset Y_d$ of type $(1, 2, 3)$. To deal with the case $c_2(\mathscr{E}) = 8$, we can use Lemma 3.2, indeed Z is contained in no hyperplane and $c_1 \leq 3$. It suffices to prove that a general cubic surface Y_3 contains the subscheme Z_8 in question. This is clear: just take an elliptic quartic contained in Y_3 and intersect with a general quadric to obtain Z_8. □

Remark 4.6. We can say something more precise about the geometry of the subschemes $Z \subset Y_d$ which are 0-loci of sections of an initialized aCM rank 2 bundle $\mathscr{E}$ on Y_d, $d \geq 4$, with $c_1(\mathscr{E}) = 6 - d$ and $c_2(\mathscr{E}) = 8$.

Namely, assume that Z is a set of 8 points in very uniform position, on an irreducible quadric Q. Then, up to generalization, either Z is a *ci* subscheme of type $(2, 2, 2)$, or it lies on a rational cubic curve $C \subset Q$.

The first case holds when three independent quadrics containing Z intersect in a finite set of points. The second one takes place when all the quadrics through Z contain a common curve.

Observe that both cases occur a general Y_d. Indeed, we proved above that the second one occurs. But since the second case is a degeneration of the first one (see [IK99, Theorem 5.25]), then also the first case takes place by diagram (3.1).

5. aCM bundles on surfaces of degree up to 4

We summarize here the classification of aCM rank 2 bundles over a general surface $Y_d \subset \mathbb{P}^3$ of degree $d \leq 4$. For $d = 2$, Y_2 is a smooth quadric $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, and any aCM bundle splits as a direct sum of twists of $\mathscr{O}_Q(0,0)$, $\mathscr{O}_Q(1,0)$ and $\mathscr{O}_Q(0,1)$. So there is no such $\mathscr{E}$ on Q.

For $d = 3$ the result is well-known after [Fae05b]. The case $d = 4$ follows immediately from the classification of aCM rank 2 bundles on a smooth quartic threefold in $\mathbb{P}^4$, achieved by Madonna in [Mad00]. We reproduce here these results and sketch an easy proof for the case $d = 4$. In the following theorem, in the column *Difference Hilbert Function* column, the t^{th} place represents the integer $\Delta\,\mathrm{h}(R_Z, t)$ (only non-zero values are displayed). The column *Resolution* illustrates (an instance of) the generators $\mathsf{P}(\mathscr{E})$ in the minimal graded free resolution of $\iota_*(\mathscr{E})$, in the framework of Theorem 2.9. The syzygies each resolution are given by $\mathsf{P}^*(c_1 - d)$.

Theorem 5.1. *Let $\mathscr{E}$ be an initialized, indecomposable, aCM rank 2 bundle over a general surface Y_d of degree $d \in \{3, 4\}$ in $\mathbb{P}^3$, with $c_1(\mathscr{E}) = c_1 H$, let s be a general section of $\mathscr{E}$ and set $Z = \{s = 0\}$. Then $\mathscr{E}$ and Z are among the types summarized by the following table.*

(5.1)

degree	*Chern* $c_1(\mathscr{E})$	*Chern* $c_2(\mathscr{E})$	*Diff. Hilbert Function* 0	1	2	3	4	*Resolution* $\mathsf{P}(\mathscr{E})$
3	0	1	1					$\mathscr{O} \oplus \mathscr{O}(-1)^3$
	1	2	1	1				$\mathscr{O}^3 \oplus \mathscr{O}(-1)$
	2	5	1	3	1			$\mathscr{O}^6$
4	-1	1	1					$\mathscr{O} \oplus \mathscr{O}(-2)^3$
	0	2	1	1				$\mathscr{O} \oplus \mathscr{O}(-1)^2 \oplus \mathscr{O}(-2)$
	1	3	1	1	1			$\mathscr{O}^3 \oplus \mathscr{O}(-2)$
	1	4	1	2	1			$\mathscr{O}^2 \oplus \mathscr{O}(-1)^2$
	1	5	1	3	1			$\mathscr{O} \oplus \mathscr{O}(-1)^5$
	2	8	1	3	3	1		$\mathscr{O}^4$
	2	8	1	3	3	1		$\mathscr{O}^4 \oplus \mathscr{O}(-1)^2$
	3	14	1	3	6	3	1	$\mathscr{O}^8$

Moreover, any class in the table is non-empty.

Proof. All the possibilities for the c_1 are listed in the table, by the inequalities 4.1 of Proposition 4.1. The list of possible c_2 follows then by 4.2, 4.2, 4.3, 4.4, 4.5 of the same proposition, and by Proposition 4.4. The statement about the difference Hilbert Function follows directly from Theorem 2.5.

Remark 4.3 and Proposition 4.4 prove the existence for $c_1 < 2$. The remaining cases are a consequence of the main theorem of [Mad00]. Indeed, in the paper, it is proved that indecomposable aCM bundles with $c_1 = 2$, $c_2 = 8$ and $c_1 = 3$, $c_2 = 14$ exist on any smooth quartic hypersurface $\tilde{Y}_4 \subset \mathbb{P}^4$. Then just take as Y_4 a hyperplane section of $\tilde{Y}_4$.

Alternatively, in order to prove existence for $c_1 = 2$, $c_2 = 8$ one just needs to prove that a general quartic can be obtained as the Pfaffian of a skew-symmetric matrix $\mathsf{f} : \mathscr{O}^4(-2) \to \mathscr{O}^4$. Given a general matrix f, set $\mathscr{E}_0 = \operatorname{coker}(\mathsf{f})$, $Y_4^0 = \operatorname{Pfaff}(\mathsf{f})$, and observe that $\mathscr{E}_0$ is a stable bundle. By a parameter count, our claim is equivalent to the moduli space $\mathsf{M}_{Y_4^0}(2; 2, 8)$

being smooth of the expected dimension 10 at the point $[\mathscr{E}_0]$. But since Y_4^0 is a smooth K3 surface, this is indeed the case. The same approach has been used by Beauville to prove existence in the case $c_1 = 3$, $c_2 = 14$, see [Bea00, Lemma 7.7].

To compute the resolutions, whenever $\Delta\,\mathrm{h}(R_Z, t)$ starts with $1, s$, with $s \leq 2$, the scheme Z is contained in a plane, hence it is *ci*. The resolution of the ideal sheaf $\mathscr{I}_Z$ follows immediately. By a standard mapping cone construction, it is easy to obtain the resolution of $\mathscr{E}$ as well. One settles similarly the case $c_2(\mathscr{E}) = 5$. For the case $c_2(\mathscr{E}) = 14$, we see that the matrix $\mathsf{f}(\mathscr{E})$ of Theorem 2.9 can only have linear entries, working as in [Fae05b, Theorem 4.1]. Finally, in case $c_2(\mathscr{E}) = 8$, we have two cases, according to whether Z is contained or not in a twisted cubic curve: in both cases the resolution of $\mathscr{E}$ follows easily from that of $\mathscr{I}_Z$. On the other hand, the matrix $\mathsf{f}(\mathscr{E})$ cannot have bigger size (say a square matrix of order 8), for in this case Y_4 would be a determinantal quartic, contradicting generality. □

6. The list of aCM bundles on the quintic surface

From now on, we will denote by X a general surface in $\mathbb{P}^3$ of degree $d = 5$, given by the homogeneous polynomial F, and we will write $\mathscr{E}$ for an *initialized indecomposable rank* 2 *aCM bundle* on X. We have the following result, whose proof amounts to writing the conditions of Proposition (4.2).

Proposition 6.1. *Let $\mathscr{E}$ and X be as above. Then the Chern classes of $\mathscr{E}$, the minimal graded free resolution of $\iota_*(\mathscr{E})$ and the difference Hilbert function of the zero locus of a nonzero section Z of $\mathscr{E}$ fall into one of the types summarized by the following table.*

(6.1)

Chern		*Difference Hilbert Function*						
$c_1(\mathscr{E})$	$c_2(\mathscr{E})$	0	1	2	3	4	5	6
−2	1	1						
−1	2	1	1					
0	3	1	1	1				
0	4	1	2	1				
0	5	1	3	1				
1	4	1	1	1	1			
1	6	1	2	2	1			
1	8	1	3	3	1			
2	11	1	3	3	3	1		
2	12	1	3	4	3	1		
2	13	1	3	5	3	1		
2	14	1	3	6	3	1		
3	20	1	3	6	6	3	1	
4	30	1	3	6	10	6	3	1

In the Difference Hilbert Function *column, the t^{th} place represents the integer $\Delta\,\mathrm{h}(R_Z, t)$ (only non-zero values are displayed).*

The rest of the paper is devoted to a detailed analysis of the above cases. We will prove that all of these alternatives take place over the general hypersurface X.

Remark 6.2. In fact, we only need to prove the existence of bundles for the cases where $c_1 \geq 2$. In fact, the cases $c_1 = -2$ and $c_1 = -1$ follow by Remark 4.3. The cases $c_1 = 0, 1$ are covered by propositions 4.4 and 4.5.

6.1. **The case $c_1 = 2$.** Here we assume $c_1(\mathscr{E}) = 2$ and prove the existence of aCM 2-bundles whose second Chern class appears in Table (6.1).

Proposition 6.3. *On a general quintic surface X there exists an indecomposable aCM rank 2 bundle $\mathscr{E}$ with Chern classes $c_1 = 2$ and c_2, for any $c_2 \in \{11, \dots, 14\}$.*

Proof. Going back to Diagram 3.1 of Remark 3.1, we need to prove that the map $p_{m,i,d}$ is dominant, for any triple $(m, i, d) = (m, 3, 5)$, with $m = 11, \dots, 14$. In other words, we need to prove the existence, on a general quintic surface, of an aG set of points with index 3 and degree $m = 11, \dots, 14$.

We will use Lemma 3.2, and start working on a surface of degree 4. Indeed, notice that all the involved subschemes are contained in no hyperplane, and we are assuming $c_1 = 2$. Take a general hypersurface Y_4 in $\mathbb{P}^3$ and let Z_m be an aG subscheme $Z_m \subset Y_4$ with $i_{Z_m} = 3$ and $\operatorname{len}(Z_m) = m$. Then, by Theorem 2.5, we have an exact sequence:

$$0 \to \mathscr{O}_{Y_4}(-3) \to \mathscr{F}_m^* \to \mathscr{I}_{Z_m} \to 0, \tag{6.2}$$

where $\mathscr{F}_m$ is an aCM rank 2 stable bundle, with $c_1(\mathscr{F}_m) = 3$ and $c_2(\mathscr{F}_m) = m$. One sees easily that $\mathrm{h}^0(\mathscr{F}_m^*(2)) = 14 - m$. Thus, $\mathscr{F}_m$ is initialized if and only if $m = 14$. On the other hand, looking at Table (6.1), we assume $m \geq 11$. So we distinguish the two cases $c_2(\mathscr{F}_m) = m = 14$, or $c_2(\mathscr{F}_m) = m \in \{11, 12, 13\}$.

Case 1. *On a general surface X of degree 5 there exists an aCM stable initialized indecomposable rank 2 bundle $\mathscr{E}_m$ with $c_1(\mathscr{E}_m) = 2$ and $c_2(\mathscr{E}_m) = m$, for $m \in \{11, 12, 13\}$.*

Proof. In view of the previous discussion, it suffices to check that a general hypersurface Y_4 contains an aG subscheme Z_m with the required Hilbert function.

By Theorem 5.1, over Y_4 one has aCM vector bundles $\overline{\mathscr{F}}_\ell$ with $c_1(\overline{\mathscr{F}}_\ell) = 1$ and $c_1(\overline{\mathscr{F}}_\ell) = \ell$, for $\ell \in \{3, 4, 5\}$. By Lemma 2.2, we have:

$$c_1(\overline{\mathscr{F}}_\ell(1)) = 3, \qquad c_2(\overline{\mathscr{F}}_\ell(1)) = \ell + 8 \in \{11, 12, 13\}.$$

Set $\mathscr{F}_{\ell+8} := \overline{\mathscr{F}}_\ell(1)$. Of course, the bundle $\mathscr{F}_m$ is also aCM, so the zero locus of a general section of $\mathscr{F}_m$ vanishes along the required aG subscheme. □

Case 2. *On a general surface X of degree 5 there exists an aCM stable initialized rank 2 bundle $\mathscr{E}$ with $c_1(\mathscr{E}) = 2$ and $c_2(\mathscr{E}) = 14$.*

Proof. We proceed similarly, considering a general hypersurface Y_4. This time we have to provide 14 aG points of index 4 in Y_4, with difference Hilbert function $1, 3, 6, 3, 1$. Equivalently, we have to provide a rank-2 aCM initialized bundle with $c_1 = 3$, $c_2 = 14$. As we have already pointed out, this question has been addressed by Beauville. The existence follows from Theorem 5.1. □

The proof of Proposition 6.3 is thus established. □

6.2. **The submaximal case:** $c_1 = 3$**, 20 points.** Unfortunately, we cannot use the degeneration technique of the previous section, because on a quartic surface, an aG set of 20 points with index 4 yields a vector bundle with $c_1 = 4$. Instead, we construct a set of 20 points with the *wrong* Hilbert function on a general quintic surface and deform it to the required set of points.

Remark 6.4. We would like to point out that the existence of such $\mathscr{E}$ over the general quintic surface can be addressed by `Macaulay2`, see Schreyer's appendix in [Bea00]. Indeed, it suffices to take a random skew-symmetric matrix $\mathsf{f} : \mathscr{O}^5(-2) \oplus \mathscr{O}(-1) \to \mathscr{O}(-1) \oplus \mathscr{O}^5$ and check that the differential of the rational map Pfaff is surjective at $[\mathsf{f}]$. This happens if the space of quintic forms in the four variables $x_0, \ldots, x_3$ is generated by the polynomials $P_{i\,j\,k} = x_k \operatorname{Pfaff}(\mathsf{f}_{i,j})$, where $\mathsf{f}_{i,j}$ is obtained by f removing the i-th column and the j-th row.

This computation is carried out easily by `Macaulay 2` and in fact it proves that on the general surface of degree $d \leq 13$ it is defined an initialized aCM rank 2 bundle $\mathscr{E}$ with $c_1(\mathscr{E}) = d - 2$ and $c_2(\mathscr{E}) = d\,(d-1)\,(d-2)/3$.

In view of the previous Remark, the question about existence is thus settled. However, we give here an abstract proof, in the hope of clarifying why the phenomenon occurs.

Lemma 6.5. *A general quintic surfaces in* $\mathbb{P}^3$ *contain a smooth set of* 10 *points* A_0 *in uniform position, with difference Hilbert function of the form* $1, 3, 4, 2$. *Moreover we may assume that* A_0 *lie in a smooth irreducible complete intersection curve* $\mathscr{C} = C_6^4$.

Proof. Take the intersection of a general quartic or quintic surface with a general quartic elliptic curve $\mathscr{D} = C_4^1$ and consider a length 10 subscheme A_0 of the intersection. Since $\mathscr{D}$ is complete intersection of two general quadrics and A_0 lies in no hyperplanes, one computes $\Delta\,\mathrm{h}(R_{A_0}, 2) = 4$. The claim on the Hilbert function follows. Notice that A_0 is contained in a smooth quadric surface $Q \supset \mathscr{D}$. Consider the linear system $|\mathscr{L}|$ cut on Q by cubic surfaces through A_0. Since cubics separate the points of A_0, $|\mathscr{L}|$ has no fixed components. Furthermore, $|\mathscr{L}|$ contains properly all the unions of C_4^1 with hyperplane sections of Q, so $|\mathscr{L}|$ is not composed with a pencil. Thus a general element $\mathscr{C}$ in the linear system $|\mathscr{L}|$ is irreducible, by Bertini's theorem. The curve $\mathscr{C}$ is also smooth, since A_0 is smooth and separated (sheaf-theoretically) by cubics. □

Corollary 6.6. *A general quintic surface in* $\mathbb{P}^3$ *contains a set of* 20 *distinct points* A*, in uniform position, with difference Hilbert function of the form* $1, 3, 5, 6, 4, 1$.

Proof. Fix a general quintic X and consider the set $A_0 \subset X$ and the complete intersection curve $\mathscr{C}$, of degree 6, containing A_0, given by the previous lemma. Then the residual intersection $A = (C \cap X) \setminus A_0$ is formed by 20 points. It is easy to compute the Hilbert function of A and check that it has the desired form.

The linear series cut on $\mathscr{C}$ by quintics through A_0 has no base points, for A_0 is cut sheaf-theoretically by quintics. Therefore its general element is smooth. Then, replacing X with another quintic surface X' in a neighbourhood of X, we may assume A consists of distinct points in uniform position.

Finally, since the curve $\mathscr{C}$ is contained in a smooth quartic surface, the same happens to A. □

Recall the notation introduced at the beginning of section 3: $\mathscr{G}(m,i)$ denotes the subset $\mathrm{Hilb}_m(\mathbb{P}^3)$ parametrizing aG sets of points with index i. Denote by $\mathcal{U}$ the locally closed subvariety of $\mathrm{Hilb}_{20}(\mathbb{P}^3)$ parametrizing smooth sets of 20 points given by the previous lemma.

We want to prove that $\mathcal{U}$ sits in the closure of $\mathscr{G}(20,4)$. To perform the task, we take a general $A \in \mathcal{U}$ and we examine carefully the behavior of sections of a bundle on Y_4 associated to A. We need a series of lemmas.

Lemma 6.7. *Take a general $A \in \mathcal{U}$, contained in a smooth general quartic Y_4. Then:*

i) the subscheme A induces a unique semistable vector bundle $\mathscr{F}_A$ of rank 2 on Y_4;
ii) the bundle $\mathscr{F}_A(-2)$ has a global section vanishing on a smooth set B of 4 non-aligned points lying in some plane π;
iii) the bundle $\mathscr{F}_A(-1)$ has a 5-dimensional space of sections, the general one vanishing on a smooth set W of 8 points in π, with difference Hilbert function $1,2,3,2$ (i.e. with general Hilbert function on π);
iv) we may assume that the curve $\Gamma = \pi \cap Y_4$ is smooth.

Proof. From the difference Hilbert function, one sees immediately that A is not separated by quartics, and $\mathrm{H}^1(\mathscr{I}_A(4)) = 1$. Then by Serre duality, $\mathrm{Ext}^1(\mathscr{I}_A(4), \mathscr{O}_{Y_4})$ is 1-dimensional and provides an extension (as in sequence (2.6):

$$0 \to \mathscr{O}_{Y_4} \to \mathscr{F}_A \to \mathscr{I}_A(4) \to 0, \tag{6.3}$$

where $\mathscr{F}_A$ is a rank 2 bundle with Chern classes $c_1(\mathscr{F}_A) = 4$, $c_2(\mathscr{F}_A) = 20$.

Looking at the Hilbert function of A, we get $\mathrm{H}^0(\mathscr{F}_A(-3)) = 0$, $\mathrm{h}^0(\mathscr{F}_A(-2)) = 1$, $\mathrm{h}^0(\mathscr{F}_A(-1)) = 5$. It turns out then that $\mathscr{F}_A$ is semistable, and $\mathscr{F}_A(-2)$ has a global section s vanishing on a finite set B. We have $c_1(\mathscr{F}_A(-2)) = 0$, $c_2(\mathscr{F}_A(-2)) = 4$, so B has length 4. We have an exact sequence:

$$0 \to \mathscr{O}_{Y_4} \xrightarrow{s} \mathscr{F}_A(-2) \to \mathscr{I}_B \to 0. \tag{6.4}$$

We obtain that $\mathrm{h}^0(\mathscr{I}_B(1)) = \mathrm{h}^0(\mathscr{F}_A(-1)) - \mathrm{h}^0(\mathscr{O}_{Y_4}(1)) = 5 - 4 = 1$, so B is not aligned. Since $\mathrm{h}^0(\mathscr{F}_A(-1)) > \mathrm{h}^0(\mathscr{O}_{Y_4}(1))$, then $\mathscr{F}_A(-1)$ has global sections which are not multiple of s. Consequently, a general global section $t \in \mathrm{H}^0(\mathscr{F}_A(-1))$ vanishes on a set W of finite length $8 = c_2(\mathscr{F}_A(-1))$, and we have an exact sequence:

$$0 \to \mathscr{O}_{Y_4} \xrightarrow{t} \mathscr{F}_A(-1) \to \mathscr{I}_W(2) \to 0. \tag{6.5}$$

One easily computes the value of $\mathrm{h}^0(\mathscr{F}_A(t))$ for all t, hence the Hilbert functions of B and W are known. Let us show that W belongs to π. By construction the plane π is induced in $\mathrm{H}^0(\mathscr{I}_B(1))$ by a general section $t \in \mathrm{H}^0(\mathscr{F}_A(-1))$, thus it corresponds to the vanishing set of the wedge product

$s \wedge t$. More precisely, W corresponds to the intersection of the vanishing loci of $s\,f$ and t, for any choice of $f \in \mathrm{H}^(O_{Y_4}(1)))$. Since the plane containing W is induced by any global section $s \in \mathrm{H}^0(\mathscr{F}_A(-2))$, the two planes coincides.

Finally, we need to show that we may assume B, W and $\Gamma = \pi \cap Y_4$ to be smooth. Indeed on π the Hilbert scheme of subschemes of finite length is smooth. Then we may move generically B in π to a smooth set of 4 points. Since the Hilbert function is constant in the family, the space $\mathrm{H}^0(\mathscr{I}_{B,\mathbb{P}^2}(4))$ defines a vector bundle over the deformation. Thus Y_4 moves in a family of quartic surfaces containing the 4 points. Similarly, $\mathrm{Ext}^1(\mathscr{I}_B, \mathscr{O}_{Y_4})$, which is dual to $\mathrm{H}^1(\mathscr{I}_B)$, has constant dimension through the deformation. Thus also $\mathscr{F}_A$ moves in a family of vector bundles. Possibly replacing A with some neighbouring general element of $\mathcal{U}$, we can now assume that the schemes B and Γ are smooth. A similar procedure proves that we may presume W to be smooth. □

Lemma 6.8. *Fix the previous notation. The sections* $\mathrm{H}^0(\mathscr{F}_A(-1))$ *determine a non-complete linear series of degree* 8 *and dimension at most* 4 *on the curve* $\Gamma = \pi \cap Y_4$*. The dimension of the series is* 4*, unless* $\mathscr{F}_A$ *has infinitely many sections (modulo the* $\mathbb{C}^*$ *action) vanishing on* W*.*

Proof. From the following diagram, it turns out that the datum of $W + B$ corresponds to a linear section of Γ:

(6.6)
$$\begin{array}{ccccccccc}
 & & & & \mathscr{O}_{Y_4}(-1) & = & \mathscr{O}_{Y_4}(-1) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{O}_{Y_4} & \to & \mathscr{F}_A(-1) & \to & \mathscr{I}_W(2) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathscr{O}_{Y_4} & \to & \mathscr{I}_B(1) & \to & \mathscr{I}_{B,\Gamma}(1) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

The claim follows, since $\mathrm{h}^0(\mathscr{F}_A(-1)) = 5$. □

Next, we need a technical result, for deformations of 8 points. We are grateful to Ciro Ciliberto, who pointed us an elegant a quick proof for it.

Proposition 6.9. *Let* W *be a general set of* 8 *points in a plane* $\pi \subset \mathbb{P}^3$*. Then in* $\mathrm{Hilb}_8(\mathbb{P}^3)$*,* W *sits in the closure of the subvariety defined by complete intersections of three quadrics.*

Proof. In fact, we are going to prove that W is the projection to π of a complete intersection W' of three quadrics $Q_1, Q_2, Q_3 \subset \mathbb{P}^3$. Then the procedure of projection determines a deformation of the quadrics which, in turn, defines a deformation of W' to W.

To show the claim, fix a cubic plane curve $\mathscr{B} \subset \pi$ which contains W. As W is general, we may assume $\mathscr{B}$ smooth. Call $\mathscr{N}$ the linear series cut on $\mathscr{B}$ by the lines of π and consider the complete linear series $\mathscr{M} = \mathscr{O}_{\mathscr{B}}(W - 2\,\mathscr{N})$ on $\mathscr{B}$. This series is a pencil for it has degree 2. Fix a Weierstrass point y for $\mathscr{M}$ and consider the complete linear series $\mathscr{N}' = \mathscr{O}_{\mathscr{B}}(\mathscr{N} + y)$ on $\mathscr{B}$.

Since $\mathscr{N}'$ has degree 4, it defines a map $\phi : \mathscr{B} \to \mathbb{P}^3$, whose image is a quartic curve $\mathscr{B}' \subset \mathbb{P}^3$. The curve $\mathscr{B}$ corresponds to the projection of $\mathscr{B}'$ from $z = \phi(y)$. The subscheme W sits in the series $2\,\mathscr{N}'$ by construction. Hence $W' = \phi(W)$ is the intersection of $\mathscr{B}'$ and a quadric. Since $\mathscr{B}'$ is itself a complete intersection of two quadrics and the projection from z maps W' to W, the claim follows. □

Remark 6.10. In general one may ask for which values of a, b, c, a general set of $a\,b\,c$ points in π is the limit of a complete intersection of type (a, b, c) in $\mathbb{P}^3$. A simple parameter count proves that the answer is negative as soon as $(a, b, c) > (2, 3, 3)$. The previous lemma provides a positive answer for $(a, b, c) = (2, 2, 2)$. A similar argument works for $(a, b, c) = (2, 2, n)$.

Now we are ready to prove:

Proposition 6.11. *The subscheme $\mathcal{U} \subset \mathrm{Hilb}_{20}(\mathbb{P}^3)$ sits in the closure of $\mathscr{G}(20, 4)$.*

Proof. Call A a general element in $\mathcal{U}$ and fix a quartic surface Y containing A. Let $\mathscr{F}$ be a rank 2 bundle on Y associated to A, with Chern classes $c_1 = 4$ and $c_2 = 20$. Consider also a smooth set W of 8 points in a plane π, given by a general section of $\mathscr{F}(-1)$ (see Lemma 6.7).

By Proposition 6.9, we have a flat family $\eta : \mathscr{W} \to \Delta$ of smooth length-8 subschemes of $\mathbb{P}^3$, parametrized by a neighbourhood Δ of 0, whose fibre η^{-1} is W and whose general fibre $\eta^{-1}(g)$ is a set W_g, complete intersection of three quadrics. Since $\mathrm{h}^0(\mathscr{I}_{W,\mathbb{P}^3}(4)) = \mathrm{h}^0(\mathscr{I}_{W_g,\mathbb{P}^3}(4))$, we may lift Y to a family of quartic surfaces whose general element Y_g contains W_g. Notice that the family W_g is produced via the choice of a cubic plane curve through W (which moves in a pencil) plus a Weierstrass point on the curve (they are interchanged in the pencil). Thus it depends continuously on one parameter.

The rank 2 bundle $\mathscr{F}$ corresponds, up the $\mathbb{C}^*$ action, to the choice of an element in the 2-dimensional space $\mathrm{Ext}^1(\mathscr{I}_W(2), \mathscr{O}_Y) \cong \mathrm{H}^1(\mathscr{I}_W(2))^*$. Our statement amounts to the following:

Claim. *The deformation η of the 8 points $W \subset \Gamma = Y \cap \pi$ can be chosen in such a way that $\mathscr{F}$ extends to a rank 2 bundle $\mathscr{F}_g$ on Y_g, where $\mathscr{F}_g(-1)$ is associated with a complete intersection set of 8 points.*

□

Notice that $\mathrm{Ext}^1(\mathscr{I}_{W_g}(2), \mathscr{O}_{Y_g}) \cong \mathrm{H}^1(\mathscr{I}_{W_g}(2))^*$, thus $\mathrm{Ext}^1(\mathscr{I}_{W_g}(2), \mathscr{O}_{Y_g})$ defines a 1-dimensional subset of the 2-dimensional space $\mathrm{Ext}^1(\mathscr{I}_W(2), \mathscr{O}_Y)$, as W_g goes to W. So the claim is non trivial.

Proof of the Claim. The statement is obvious when there are infinitely many sections of $\mathscr{F}(-1)$ vanishing on W (mod $\mathbb{C}^*$), for in this case all the elements of $\mathrm{Ext}^1(\mathscr{I}_W(2), \mathscr{O}_Y)$ correspond to the same rank 2 bundle $\mathscr{F}$ (only the section varies). Assume this is not the case. Thus $\mathscr{F}$ defines a 4-dimensional linear series in $|W|$.

Changing the element in $\mathrm{Ext}^1(\mathscr{I}_W(2), \mathscr{O}_Y)$ has the effect of choosing a vector bundle $\mathscr{F}'$ on Y, associated with a 4-dimensional linear series in $|W|$. This linear series cannot be fixed as we vary the extension, for $|W|$ has dimension 5, so it is not the union of disjoint 4-dimensional subspaces, while a general set of 8 points on Γ determines a rank 2 bundle on Y, hence a 4-dimensional linear subseries. Since a general set of 8 points can be lifted to a *ci* subscheme in $\mathbb{P}^3$, we have at least a 1-dimensional family of bundles on Y, associated with W, which lifts to a deformation of a set of 8 points on Γ. Possibly replacing now W with a neighbouring element in $|W|$, we may assume that W admits a deformation to a complete intersection. This completes the proof. □

We are now in position to prove:

Theorem 6.12. *The map $q : \mathscr{G}(20,4,5) \to |\mathscr{O}_{\mathbb{P}^3}(5)|$ of diagram (3.1) is dominant. In other words, on a general quintic surface X there exists an indecomposable aCM rank 2 bundle $\mathscr{E}$ with Chern classes $c_1 = 3$ and $c_2 = 20$.*

Proof. Consider the set $\mathcal{U}'$ of pairs (A, Y) where $A \in \mathcal{U}$ and Y is a quintic surface containing A. Since $\mathcal{U}$ sits in the closure of $\mathscr{G}(20,4)$ and elements of both $\mathcal{U}$ and $\mathscr{G}(20,4)$ are separated by quintics, then $\mathcal{U}'$ sits in the closure of $\mathscr{G}(20,4,5)$. So Corollary 6.6 proves that the map $q_{|\mathcal{U}'} : \mathcal{U}' \to |\mathscr{O}_{\mathbb{P}^3}(5)|$ is dominant. The claim follows. □

6.3. **The maximal case: $c_1 = 4$, 30 points.** Here we prove the existence of an initialized aCM vector bundle on the general surface X, achieving the maximal value of c_1 and c_2. This result is already known, and due to Beauville and Schreyer, see the appendix of [Bea00]. However their proof relies on a `Macaulay2` computation, while we outline a geometric approach which makes use of Beilinson's theorem and a deformation argument.

Proposition 6.13. *On a general quintic surface X there exists an initialized indecomposable aCM rank 2 bundle $\mathscr{E}$ with Chern classes $c_1 = 4$ and $c_2 = 30$.*

The rest of this section contains the proof of this proposition. Consider the sheaves of differentials $\Omega^p(p) = \wedge^p \Omega_{\mathbb{P}^3}(p)$, for $0 \le p \le 3$. Define the vector bundle $\mathsf{P} = \mathscr{O}^{10} \oplus \Omega^1(1) \oplus \Omega^2(2)$ and the vector space $V = \mathrm{H}^0(\mathbb{P}^3, \wedge^2(\mathsf{P})(1))$. We can view an element in V as a skew-symmetric matrix $\mathsf{f} : \mathsf{P}^*(-1) \to \mathsf{P}$. So we can define a rational map $\mathrm{Pfaff} : \mathbb{P}(V) \to \mathbb{P}(\mathrm{H}^0(\mathbb{P}^3, \mathscr{O}(5)))$, which associates to an element of V the square root of its determinant, defined up to a nonzero scalar.

Lemma 6.14. *Assume that on the general quintic surface X there exists an initialized rank 2 bundle $\mathscr{F}$, with $c_1(\mathscr{F}) = 2$, $c_2(\mathscr{F}) = 15$, and with $\mathrm{h}^1(\mathscr{F}(t)) = 0$, for each $t \in \mathbb{Z}$, except for $\mathrm{h}^1(\mathscr{F}) = \mathrm{h}^1(\mathscr{F}(-1)) = 1$. Then the map* Pfaff *defined above is dominant.*

Proof. Applying Beilinson's theorem to the sheaf $\mathscr{F}(1)$, extended by zero to $\mathbb{P}^3$, one can write down the following resolution:

$$0 \to \begin{matrix} \mathscr{O}(-1)^{10} \\ \oplus \\ \Omega^2(2) \\ \oplus \\ \Omega^1(1) \end{matrix} \xrightarrow{\mathsf{f}(\mathscr{F}(1))} \begin{matrix} \mathscr{O}^{10} \\ \oplus \\ \Omega^1(1) \\ \oplus \\ \Omega^2(2) \end{matrix} \to \mathscr{F}(1) \to 0. \tag{6.7}$$

We still denote by $\mathsf{P}(\mathscr{F}(1))$ the target of the map $\mathsf{f}(\mathscr{F}(1))$, while we write $\mathsf{Q}(\mathscr{F}(1))$ for the domain of $\mathsf{f}(\mathscr{F}(1))$. Notice that $\mathsf{P}(\mathscr{F}(1)) \simeq \mathsf{P}$, and $\mathsf{Q}(\mathscr{F}(1)) \simeq \mathsf{P}^*(-1)$, with P defined above.

Observe that there is no obstruction to the lifting of any map $\mathsf{P}(\mathscr{F}(1)) \to \mathscr{F}(1)$ to an endomorphism of $\mathsf{P}(\mathscr{F}(1))$, thanks to the vanishing:

$$\mathrm{Ext}^1(\mathsf{P}(\mathscr{F}(1)), \mathsf{Q}(\mathscr{F}(1))) = 0.$$

Using this fact one can easily prove, following step by step the proof of 2.9 contained in [Bea00], that there is an isomorphism $\phi : \mathsf{P}(\mathscr{F}(1)) \to \mathsf{Q}(\mathscr{F}(1))^*(-1)$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathsf{Q}(\mathscr{F}(1)) & \xrightarrow{\mathsf{f}(\mathscr{F}(1))} & \mathsf{P}(\mathscr{F}(1)) & \longrightarrow & \mathscr{F}(1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi^\top} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \kappa} & & \\
0 & \longrightarrow & \mathsf{P}(\mathscr{F}(1))^*(-1) & \xrightarrow{-\mathsf{f}(\mathscr{F}(1))^\top} & \mathsf{Q}(\mathscr{F}(1))^*(-1) & \longrightarrow & \mathscr{F}^*(3) & \longrightarrow & 0.
\end{array}$$

where κ is a skew-symmetric duality of $\mathscr{F}$. Equivalently, the matrix $\mathsf{f}(\mathscr{F}(1))$ is skew-symmetric, i.e. $\mathsf{f}(\mathscr{F}(1))$ sits in the vector space V defined above. Now $\mathrm{Pfaff}(\mathsf{f}(\mathscr{F}(1)))$ is the equation of the support of $\mathscr{F}$, which is defined over the general quintic surface. So our hypothesis implies that the map Pfaff is dominant. □

Now set $\mathsf{P}' = \mathscr{O}^{10}$, $V' = \mathrm{H}^0(\mathbb{P}^3, \wedge^2(\mathsf{P}')(1))$ and consider the restriction Pfaff' of the map Pfaff to $\mathbb{P}(V')$, where we think of an element $[\mathsf{f}']$ of $\mathbb{P}(V')$ in $\mathbb{P}(V)$ as a block matrix made up of f', the identity map on $\Omega^1(1)$ and $\Omega^2(2)$, and zero everywhere else.

Roughly speaking, the next lemma allows to deform the matrix $\mathsf{f}(\mathscr{F}(1))$ in such a way that we can *erase* the undesired summands from the resolution (namely the copies of $\Omega^1(1)$ and $\Omega^2(2)$), whereby obtaining an initialized aCM bundle as a generalization of $\mathscr{F}(1)$.

Lemma 6.15. *Set hypothesis as in the previous lemma. Then* Pfaff' *is dominant. In particular, Proposition 6.13 holds.*

Proof. Recall by Lemma 6.14 that Pfaff is dominant, so its differential at a general element $[\mathsf{f}]$ of $\mathbb{P}(V)$ is surjective. Representing $[\mathsf{f}]$ as a skew-symmetric matrix $\mathsf{f} : \mathsf{P}^*(-1) \to \mathsf{P}$, we can write f in the form:

$$\begin{pmatrix} \mathsf{f}_0 & a & b \\ -a^\top & c & d \\ -b^\top & -d^\top & 0 \end{pmatrix},$$

with:

$$\begin{aligned}
a &: \Omega^2(2) \to \mathsf{P}', & a^\top &: (\mathsf{P}')^*(-1) \to \Omega^1(1), \\
b &: \Omega^1(1) \to \mathsf{P}', & b^\top &: (\mathsf{P}')^*(-1) \to \Omega^2(2), \\
c &: \Omega^2(2) \to \Omega^1(1), & c^\top &: \Omega^2(2) \to \Omega^1(1), \\
d &: \Omega^1(1) \to \Omega^1(1), & d^\top &: \Omega^2(2) \to \Omega^2(2).
\end{aligned}$$

and with $\mathsf{f}_0 = -\mathsf{f}_0^\top$ and $c = -c^\top$. Consider a matrix f of the form (6.7) provided by Lemma 6.14. It satisfies $d = 0$, and the differential of Pfaff is surjective at the point represented by such f. In general d must be a scalar multiple of the identity, so we may set $\mathsf{f}_\varepsilon = \mathsf{f} + \varepsilon\,\mathrm{id}_{\Omega^1(1)} - \varepsilon\,\mathrm{id}_{\Omega^2(2)}$. The differential of Pfaff will be surjective at f_ε.

Since c is skew-symmetric, we can choose $\zeta \in \mathrm{Hom}(\Omega^2(2), \Omega^1(1))$ with $\zeta - \zeta^\top = c$. Consider now the automorphism g of P, written in the form of a block matrix like:

$$\begin{pmatrix} 1 & -b/\varepsilon & 1/\varepsilon^2(\varepsilon\, a - b\, c) \\ 0 & \varepsilon & \zeta \\ 0 & 0 & 1 \end{pmatrix}.$$

We obtain the matrix $h_\varepsilon = g^\top \cdot \mathsf{f}_\varepsilon \cdot g$. The only nonvanishing blocks of the matrix h_ε are a block of linear forms f' and identity maps of $\Omega^1(1)$ and $\Omega^2(2)$, which we can now factor out. This means that $[h_\varepsilon]$ sits in $\mathbb{P}(V')$, and the differential of Pfaff is surjective at $[h_\varepsilon]$, hence we are done. □

Lemma 6.16. *On the general quintic surface X it is defined a vector bundle $\mathscr{F}$ satisfying the hypothesis of Lemma 6.14.*

Proof. Our claim is equivalent to the fact that the general quintic surface contains a length 15 subscheme Z having difference Hilbert function of the form $1, 3, 6, 4, 1$.

We need a slight refinement of Lemma 3.2, where we allow the subscheme Z to be of the above form, (in particular Z is not aG). Namely we claim that if Z is given as the vanishing locus of a section of a rank 2 bundle $\mathscr{G}$ with $c_1 \leq 3$ defined on the general quartic surface Y_4, and $\mathrm{H}^0(\mathscr{I}_Z(1)) = 0$, then the general quintic surface X contains a subscheme with the same Hilbert function as Z. The proof of this claim is similar to that of Lemma 3.2, so we omit it here.

Counting dimensions, we see that our claim amounts to the fact that on a given smooth quartic surface Y_4 containing Z, with $\mathrm{Pic}(Y_4) = \mathbb{Z}$, the family of subschemes of Y_4 with difference Hilbert function $1, 3, 6, 4, 1$ has dimension 24. Notice that $\mathscr{G}$ is stable and has 7 independent sections, hence it suffices to show that on such Y_4 the moduli space $\mathsf{M}_{Y_4}(2; 3, 15)$ has dimension 18. But since Y_4 is a smooth K3 surface and $\mathscr{G}$ is a stable bundle, this follows immediately by Riemann-Roch. □

Corollary 6.17. *For general X, and for $\mathscr{E}$ as is Proposition 6.1, with $c_1(\mathscr{E}) \geq 1$, the component of the moduli space $\mathsf{M}_X(2; c_1(\mathscr{E}), c_2(\mathscr{E}))$ containing $[\mathscr{E}]$ is smooth of the expected dimension at a general point.*

Proof. The map $p_{m,i,5}$ is dominant, so its differential has maximal rank at a general point (Z, X). The fibre $p_{m,i,5}^{-1}(X)$ is thus generically smooth of the expected dimension. Now Theorems 2.3 and 2.5 provide a map $\zeta : p_{m,i,5}^{-1}(X) \to \mathsf{FM^s}_X(2; c_1(\mathscr{E}), c_2(\mathscr{E}))$, where the target space is the so-called moduli space of *framed sheaves*, i.e. pairs $[\mathscr{F}, s]$ with $\mathscr{F} \in \mathsf{M}_X(2; c_1(\mathscr{E}), c_2(\mathscr{E}))$ and $s \in \mathbb{P}(\mathrm{H}^0(\mathscr{F}))$. The map ζ is a locally closed immersion. Since being aCM is an open condition, the differential at $[\mathscr{E}]$ of the natural projection $\mathsf{FM^s}_X(2; c_1(\mathscr{E}), c_2(\mathscr{E})) \to \mathsf{M}_X(2; c_1(\mathscr{E}), c_2(\mathscr{E}))$ is surjective on the tangent space of $\mathsf{M}_X(2; c_1(\mathscr{E}), c_2(\mathscr{E}))$ at $[\mathscr{E}]$. Our claim follows. □

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