MODULI SPACE OF A FAMILY OF HOMOGENEOUS BUNDLES ON $\mathbb{P}^2$

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Abstract. We describe explicitly a family of moduli spaces of homogeneous bundles over $\mathbb{P}^2(\mathbb{C})$ via representations of the quiver $\mathcal{Q}_{\mathbb{P}^2}$, making use of toric geometry.

1. Introduction

Let $X = \mathsf{G}/\mathsf{P}$ be a Hermitian symmetric variety and let $\mathsf{P} = \mathsf{R} \cdot \mathsf{N}$ be the Levi decomposition, where R is reductive and N is nilpotent. Then we define the (infinite) quiver $\mathcal{Q}_X$ whose points are the dominant weights of R. To any vertex v_λ it is associated a unique irreducible homogeneous vector bundle E_λ. Arrows $v_\lambda \to v_\mu$ in $\mathcal{Q}_X$ are defined by G-equivariant extensions $\mathrm{Ext}^1_X(E_\lambda, E_\mu)^{\mathsf{G}}$.

After Bondal and Kapranov in [1] first introduced these quivers as a tool to investigate homogeneous bundles on X, the study was taken up again by Hille in [7] and Ottaviani and Rubei in [13] and [12]. After these papers, we know that the category of G-homogeneous bundles over X is equivalent to the category of representations of $\mathcal{Q}_X$ with certain relations. In [12] these relations are made precise, and a method to compute the cohomology of homogeneous bundles is also developed.

As a consequence, one can describe the moduli space of homogeneous bundles on X making use of the relevant (semi)invariant theory, see [10], [14], [15], [16], [8], [9]. In this situation, deep computational tools are available after [2], [3], [4], [5].

Here we describe explicitly the moduli space of homogeneous deformations of a special type of homogeneous bundles on $X = \mathbb{P}^2$, given by a representation of the quiver $\mathcal{Q}_{\mathbb{P}^2}$ having only a chain of boxes meeting at points sitting on the line representing completely reducible homogeneous bundles with slope zero (see Definition 2). This moduli space is isomorphic to an iterated $\mathbb{P}^1$-bundle over a single point. In particular it turns out to be an irreducible smooth projective toric Fano variety. This is the content of Theorem 1.

In Section (2) we provide the general framework about homogeneous bundles on $\mathbb{P}^2$ and state the main result. In Section (3) we describe the ring of semi–invariants in terms of toric coordinates and prove our statement. In section (4) we recover the torus action on the quiver representation and draw some side remarks.

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2. Moduli space of homogeneous bundles

Let $\mathcal{Q} = \mathcal{Q}_{\mathbb{P}^2}$ be the connected component containing $\mathcal{O} = \mathcal{O}_{\mathbb{P}^2}$ of the quiver with relations associated to $\mathbb{P}^2 = \mathsf{SL}(2)/\mathsf{P}(\alpha_1)$ as defined in [12, Definition 5.2]. Let $\mathcal{Q}_0$ (resp. $\mathcal{Q}_1$) be the set of vertices (resp. arrows) of $\mathcal{Q}$. The set $\mathcal{Q}_0$ is in one–to–one correspondence with the subset of the lattice $\mathbb{Z}^2$ given by points (i, j) with $i \leq j$. The point (i, j) represents a homogeneous vector bundle $E_{i,j}$ associated to a completely reducible representation of $\mathsf{P}(\alpha_1)$. Setting $Q = \Omega_{\mathbb{P}^2}(2)$, we have $E_{i,j} \simeq \mathrm{S}^{i-j} Q(i + 2j)$. Thus we have

$$c_1(E_{i,j}) = 3/2\,(i + j)\,(i - j + 1) \tag{1}$$

An arrow $E_{i,j} \to E_{h,k}$ is given by an element of $\mathrm{Ext}^1(E_{i,j}, E_{h,k})^{\mathsf{SL}(2)} \simeq \mathrm{Hom}(E_{i,j}, E_{h,k})^{\mathsf{SL}(2)}$.

Let $\alpha \in \mathbb{N}^{\mathcal{Q}_0}$ be nonzero for a finite number of elements (i, j) in $\mathcal{Q}_0$. We call α a weight vector. Define a homogeneous bundle $F(\alpha) = \oplus_{i,j} V_{i,j} \otimes E_{i,j}$, where $V_{i,j}$ is a complex vector space of dimension $\alpha_{i,j}$ and $E_{i,j}$ is defined above. Accordingly define $\mathsf{G}(\alpha) = \prod_{i,j} \mathsf{GL}(V_{i,j})$. A character $\theta : \mathsf{G}(\alpha) \to \mathbb{C}^*$ is defined for any $g = (g_{i,j}) \in \mathsf{G}$ by $\theta(g_{i,j}) = \prod_{i,j} \det(g_{i,j})^{\theta_{i,j}}$, with $\theta_{i,j} \in \mathbb{Z}$.

According to [10] there is a natural choice for the character θ namely

$$\theta_{i,j} = \mathrm{rk}(F(\alpha))\, c_1(E_{i,j}) - c_1(F(\alpha))\, \mathrm{rk}(E_{i,j}) \tag{2}$$

With this choice for θ, semistability of representations according to King is equivalent to slope semistability according to Mumford and Takemoto, check [12, Theorem 8.1 and 8.2]. We will always assume θ to be of this form.

Definition 1 (Moduli space of homogeneous bundles)**.** Define the space $K(\mathcal{Q}, V(\alpha)) = \bigoplus_{a \in \mathcal{Q}_1} \mathrm{Hom}(V_{\mathsf{t}\,a}, V_{\mathsf{h}\,a})$ where $\mathsf{t}\,a \in \mathcal{Q}_0$ (resp. $\mathsf{h}\,a \in \mathcal{Q}_0$) denotes the tail (resp. head) of $a \in \mathcal{Q}_1$, and consider the closed subset $V(\alpha) \subset K(\mathcal{Q}, V(\alpha))$ given by the relations in $\mathcal{Q}$ (see [12, Section 8]). The group $\mathsf{G}(\alpha)$ acts on $V(\alpha)$ and we define the space of semi–invariants of degree m as

$$\begin{aligned} A_m(\alpha) =& \mathbb{C}[V(\alpha)]^{\mathsf{G}(\alpha), m\theta} = \\ =& \{f \in \mathbb{C}[V(\alpha)] \mid f(g\,x) = \theta(g)^m\, f(x),\ \forall x \in V(\alpha)\} \end{aligned}$$

Given a weight vector α, define the associated moduli space

$$M(\alpha) := \mathrm{Proj}(\bigoplus_m A_m(\alpha))$$

The variety $M(\alpha)$ is projective and parametrizes homogeneous bundles F over $\mathbb{P}^2$ such that the completely reducible bundle associated to F is isomorphic to $F(\alpha)$.

Definition 2. For $n \geq 1$ and for any sequence of natural numbers $k_0, \ldots, k_{n+1}$ with $k_0 \geq 1$, $k_{n+1} \geq 1$, and $k_i \geq 2$ for $2 \leq i \leq n$, define the weight vector α_n as

$$\alpha_n = (k_0, 1, 1, k_1, 1, 1, k_2, 1, \ldots, 1, k_n, 1, 1, k_{n+1})$$

according to the diagram

$$
\begin{tikzcd}
\mathbb{C}^{k_0} \arrow[d, "d_0"'] & \mathbb{C} \arrow[l, "c_0"'] \arrow[d, "a_1"] & & & & \\
\mathbb{C} & \mathbb{C}^{k_1} \arrow[l, "b_1"'] \arrow[d, "d_1"'] & \mathbb{C} \arrow[l, "c_1"'] \arrow[d, "a_2"] & & & \\
& \mathbb{C} & \mathbb{C}^{k_2} \arrow[l, "b_2"'] \arrow[d, "d_2"'] & \cdots \arrow[l, "c_2"'] & & \\
& & \vdots & & \vdots \arrow[d, "a_n"] & \\
& & & \cdots & \mathbb{C}^{k_n} \arrow[l, "b_n"'] \arrow[d, "d_n"'] & \mathbb{C} \arrow[l, "c_n"'] \arrow[d, "a_{n+1}"] \\
& & & & \mathbb{C} & \mathbb{C}^{k_{n+1}} \arrow[l, "b_{n+1}"]
\end{tikzcd}
\tag{3}
$$

We call the vertex corresponding to k_0 (i.e. the upper left corner) the *origin* of α_n. We denote $M_n := M(\alpha_n)$.

Up to twisting by a line bundle $\mathcal{O}(3\,t)$ we can suppose that the main diagonal in Definition 2 sits either on the line $i = -j$, either on the line $i = -j + 1$. For simplicity, we will consider only the former case, the latter being completely analogous.

By formulas (1) and (2) one derives the following straightforward lemma.

Lemma 1. *Given a homogeneous vector bundle F with weight vector α, the character coefficients of θ corresponding to slope semistability are, up to scalar, given by*

$$\theta_{i,j} = (i - j + 1)(i + j). \tag{4}$$

Formula (4) says that the coefficient $\theta_{i,j}$ is $\mathrm{rk}(E_{i,j}) = i - j + 1$, times the distance from the diagonal $i + j = 0$.

Theorem 1. *The moduli space M_n is isomorphic to a $\mathbb{P}^1$-bundle over M_{n-1}, where M_0 is a single point. M_n is an irreducible smooth projective Fano variety of dimension n.*

The rest of the paper is devoted to the proof of the above result.

3. Semi–Invariants via the toric picture

Consider α_n and fix a complex vector space $W = V_{j,-j}$ for some j with and $n_j = \dim(W) \neq 0$. By Lemma 1, we can write the value of θ at the vector spaces $V_{i,j}$ associated to the dimension vector α_n

$$\theta = (0, 1, -1, 0, 2, -2, \ldots, 0, n + 1, -n - 1, 0) \tag{5}$$

Then θ takes value 0 at the point corresponding to W; and we look for semi–invariants of weight zero at W i.e. for invariants under the $\mathsf{GL}(W)$-action. For convenience rename a_j, b_j, c_j and d_j of diagram (3) as a, b, c, d

in order to draw the following picture

$$\begin{tikzcd} & V_{\mathsf{t}\,a} \arrow[d, "a"] & \\ V_{\mathsf{h}\,b} & W \arrow[l, "b"'] \arrow[d, "d"] & V_{\mathsf{t}\,c} \arrow[l, "c"'] \\ & V_{\mathsf{h}\,d} & \end{tikzcd}$$

Lemma 2. *Let f be a regular function on $V(\alpha_n)$ satisfying $f(g\,x) = f(x)$ for all $g \in \mathsf{GL}(W)$. Then f is a linear combination of forms of the type $f'\,h$ where h is a function independent of a, b, c, d and f' lies in the ring $\mathbb{C}[b\,a, d\,a, b\,c, d\,c]$.*

Proof. Write $K(\mathcal{Q}, V(\alpha_n)) = K^W(\alpha_n) \oplus K'(\alpha_n)$, where no arrow in $K'(\alpha_n)$ has tail or head in W and

$$K^W(\alpha_n) = \mathrm{Hom}(V_{\mathsf{t}\,a}, W) \oplus \mathrm{Hom}(V_{\mathsf{t}\,c}, W) \oplus \mathrm{Hom}(W, V_{\mathsf{h}\,b}) \oplus \mathrm{Hom}(W, V_{\mathsf{h}\,d})$$

A regular function f is represented by an element of the polynomial algebra $\mathrm{Sym}(K(\mathcal{Q}, V(\alpha_n))) \simeq \mathrm{Sym}(K^W(\alpha_n)) \otimes \mathrm{Sym}(K'(\alpha_n))$. Clearly any element in $\mathrm{Sym}(K'(\alpha_n))$ is independent of a, b, c, d. Put $S(W) := \mathrm{Sym}(K^W(\alpha_n))$. We need to identify the $\mathsf{GL}(W)$-invariant elements in $S(W)$.

Decomposing $S(W)$ with respect to $\mathsf{GL}(W)$-action we find

$$S(W) \simeq \bigoplus_{\eta,\xi} \mathrm{S}^\eta\, W \otimes \mathrm{S}^\xi\, W^\vee \otimes (\mathrm{S}^\eta(V_{\mathsf{t}\,a}^\vee \oplus V_{\mathsf{t}\,c}^\vee) \otimes \mathrm{S}^\xi(V_{\mathsf{h}\,b} \oplus V_{\mathsf{h}\,d}))$$

where η and ξ run through Young tableaux with at most $\dim(W)$ rows and S^η, S^ξ are the associated Schur functors.

Let $\mathsf{T}_{a,c} := V_{\mathsf{t}\,a}^\vee \oplus V_{\mathsf{t}\,c}^\vee$ and $\mathsf{H}_{b,d} := V_{\mathsf{h}\,b} \oplus V_{\mathsf{h}\,d}$. The spaces $\mathsf{T}_{a,c}$ and $\mathsf{H}_{b,d}$ are 2-dimensional, so η and ξ run through Young tableaux with at most 2 rows, respectively of length $p + q$ and p. Since we look for $\mathsf{GL}(W)$-invariant elements, Schur Lemma gives $\eta = \xi$ and we get

$$S(W)^{\mathsf{GL}(W)} \simeq \wedge^2(\mathsf{T}_{a,c})^{\otimes p} \otimes \wedge^2(\mathsf{H}_{b,d})^{\otimes p} \otimes \mathrm{S}^q(\mathsf{T}_{a,c}) \otimes \mathrm{S}^q(\mathsf{H}_{b,d})$$

Now the unique element in $\wedge^2(\mathsf{T}_{a,c})^{\otimes p} \otimes \wedge^2(\mathsf{H}_{b,d})^{\otimes p}$ represents the p-th power of the determinant D of the matrix

$$\mathsf{T}_{a,c} \xrightarrow{\begin{pmatrix} b\,a & b\,c \\ d\,a & d\,c \end{pmatrix}} \mathsf{H}_{b,d}$$

On other hand $\bigoplus_q \mathrm{S}^q(\mathsf{T}_{a,c}) \otimes \mathrm{S}^q(\mathsf{H}_{b,d})$ can be identified with the subspace of degree 0 elements in the subring $\mathrm{Sym}(\mathsf{T}_{a,c}) \otimes \mathrm{Sym}(\mathsf{H}_{b,d})$, where $\deg(a) = \deg(c) = -1$, $\deg(b) = \deg(d) = 1$. It is immediate to check that $b\,a, d\,a, b\,c, d\,c$ have degree zero and that they generate this subring. Finally, we observe that $D = b\,a \cdot d\,c - b\,c \cdot d\,a$, so the generator D is actually redundant. $\square$

For a quiver representation as in diagram (3) with dimension vector α_n, we set

$$x_i = b_i\, c_i \cdot d_i\, a_i \qquad \text{for } i = 1 \dots n \tag{6}$$

$$y_i = b_i\, a_i \qquad \text{for } i = 1 \dots n+1 \tag{7}$$

Lemma 3. *Let α_n be as in Definition 2 and suppose the origin of α_n sits on $(0,0)$ i.e. $F(\alpha_n)$ contains $\mathcal{O}^{k_0}$. Then a basis of the vector space $A_1(\alpha_n)$ is given by the elements*

$$x_1^{e_1} \cdot x_2^{e_2} \cdots x_n^{e_n} \cdot y_1^{1-e_1} \cdot y_2^{2-e_1-e_2} \cdots y_{n+1}^{n+1-e_n} \tag{8}$$

with the conditions

$$\begin{cases} 0 \le e_1 \le 1, \\ 0 \le e_1 + e_2 \le 2, \\ \vdots \\ 0 \le e_{n-1} + e_n \le n, \\ e_i \ge 0 \end{cases} \tag{9}$$

Furthermore, the ring $\bigoplus_m A_m(\alpha_n)$ is generated by $A_1(\alpha_n)$.

Proof. Let f be an element of $A_m(\alpha_n)$. By the expression (5) of θ, f is an invariant element for the $\mathsf{GL}(V_{j,-j})$-action for every j. Consider the subring of the ring of regular functions on $K(\mathcal{Q}, V(\alpha_n))$ consisting of the elements f' satisfying $f'(g\,x) = f'(x)$ for all $g \in \prod_j \mathsf{GL}(V_{j,-j})$. Lemma 2 provides generators for this subring of the form $b_i\, a_i$, $d_i\, a_i$, $b_i\, c_i$, $d_i\, c_i$.

Now, on the sequence of 1-dimensional vector spaces lying above (resp. below) the main diagonal, the character θ takes values $1, 2, \dots, n+1$ (resp. $-1, -2, \dots, -n-1$). On the other hand, notice that $b_i\, a_i$ and $d_i\, a_i$ are semi–invariants of weight 1 for the $\mathbb{C}^*$-action on $V_{\mathsf{t}\, a_i}$, while $b_i\, c_i$, $d_i\, c_i$ have weight 0 for this action. Analogously for $V_{\mathsf{t}\, c_i}$. Respectively, $b_i\, a_i$, $b_i\, c_i$ are semi–invariants of weight -1 for the $\mathbb{C}^*$-action on $V_{\mathsf{h}\, b_i}$, and analogously for $V_{\mathsf{h}\, d_i}$. Finally, notice that the only semi–invariants of weight 0 (i.e. the only invariants) for the $\mathsf{GL}(V_{\mathsf{h}\, c_0})$-action on $V_{\mathsf{h}\, c_0}$ involving c_0 and d_0 (resp. for the $\mathsf{GL}(V_{\mathsf{t}\, b_{n+1}})$-action on $V_{\mathsf{t}\, b_{n+1}}$ involving a_{n+1} and b_{n+1}) are of the form $(d_0\, c_0)^p$ (resp. are of the form $(b_{n+1}\, a_{n+1})^p$) for some natural number p.

Using the commutativity relations in $V(\alpha)$ we can replace any occurrence of $d_j\, c_j$ by $b_j\, a_j$.

For $n = 1$ it is easy to see that $A_1(\alpha_1)$ is generated by $x_1^{e_1} y_1^{1-e_1} y_2^{2-e_1}$ for $e_1 = 0, 1$. Suppose now $n \ge 2$. It follows that any f in $A_m(\alpha_n)$, is a linear combination of monomials of the form

$$\prod (b_j\, a_j)^{p_j} \cdot (b_j\, c_j)^{q_j} \cdot (d_j\, a_j)^{t_j} \tag{10}$$

for certain exponents p_j, q_j, t_j and $0 \le j \le n+1$ (we assume $d_0\, a_0 = b_0\, c_0 = d_{n+1}\, a_{n+1} = b_{n+1}\, c_{n+1} = 1$). In order for the above monomial to have weight j at $V_{\mathsf{t}\, a_j}$ and weight $-j$ at $V_{\mathsf{h}\, b_j}$ for all $j = 2, \dots, n$ (i.e. in order for f to lie in $A_1(\alpha_n)$), we have $p_j + q_{j-1} + t_j = p_{j-1} + q_j + t_j = j$. It follows that $p_j - q_j = p_{j-1} - q_{j-1}$. Since $p_0 + t_0 = q_0 + t_0 = 1$, we have that $q_j = t_j$ for all j.

Since no arrow in $V(\alpha_n)$ has head in $V_{\mathfrak{t}a_j}$, for f to lie in $A_1(V(\alpha_n))$ the term of (10) involving a_j occur at most j times. Hence we obtain the expression (8) and the conditions (9) for the generators of $A_1(V(\alpha_n))$. Finally, for f in $A_m(V(\alpha_n))$, it suffices to put $p_j + q_{j-1} + t_j = p_{j-1} + q_j + t_j = mj$ to deduce that a basis for $A_m(V(\alpha))$ is provided by the elements of the form

$$x_1^{e_1} \cdot x_2^{e_2} \cdots x_n^{e_n} \cdot y_1^{m-e_1} \cdot y_2^{2m-e_1-e_2} \cdots y_{n+1}^{m(n+1)-e_n} \tag{11}$$

with the conditions $0 \leq e_1 \leq m, \dots, 0 \leq e_{n-1} + e_n \leq mn$ and $e_i \geq 0$ for all i. Since this space is generated by $A_1(V(\alpha_n))$, the last statement is proved. $\square$

Lemma 4. *In the hypothesis of Lemma 3, the conditions (9) of Lemma 3 cut a rational convex polyhedron $\Delta(n)$. The fan $F(n)$ corresponding to $\Delta(n)$ is defined by the following rays in the lattice $\mathbb{Z}^n$*

$$\begin{cases} r_1 = (1, 0, \dots, 0), & s_1 = (-1, 0, \dots, 0), \\ r_2 = (0, 1, \dots, 0), & s_2 = (-1, -1, 0, \dots, 0), \\ \vdots & \vdots \\ r_n = (0, \dots, 0, 1), & s_n = (0, \dots, 0, -1, -1) \end{cases} \tag{12}$$

Proof. The rays of the fan $F(n)$ correspond to the facets of $\Delta(n)$. In fact, up to the sign, they are the coefficients of the hyperplane supporting a facet (see [6], pages 23–27). The inductive construction of $F(n)$ out of $F(n-1)$ goes as follows. The facets of $\Delta(n-1)$ yield the facets of $\Delta(n)$. The new facets appearing in $\Delta(n)$ are $\Delta(n-1)$ itself and a facet defined by the equation $e_{n-1} + e_n = n$. Hence the cones of $F(n)$ are exactly the cones generated by the cones in $F(n-1)$ and a ray $(0, \dots, 1)$ (resp. $(0, \dots, -1, -1)$). Hence if the primitive generators of the rays in the cones of $F(n-1)$ form the $\mathbb{Z}^{n-1}$-basis also $F(n)$ describes a smooth variety. The fan $F(1)$ describes $\mathbb{P}^1$. This finishes the proof. $\square$

Lemma 5. *The fan $F(n)$ corresponding to $\Delta(n)$ as defined in Lemma 4 is associated to the toric variety P_n recursively defined by*

$$\begin{aligned} P_n &= \mathbb{P}(\mathcal{O}_{P_{n-1}} \oplus \mathcal{O}_{P_{n-1}}(-S_1)) \to P_{n-1} \\ P_0 &= \{\mathrm{pt}\}, \end{aligned}$$

where S_1 is a divisor corresponding to the ray s_1. The variety P_n is naturally equipped with the very ample divisor L_n taking value $1, 2, 3, \dots, n$ at the rays $s_1, \dots, s_n$ and 0 at the rays $r_1, \dots, r_n$ defined in Lemma 4.

Proof. The coefficients of the very ample divisor follow easily from the inequalities describing the polyhedron $\Delta(n)$ (see [6], page 66). The $\mathbb{P}^1-$bundle structure follows from the description of projective line bundles given by Oda in [11], pages 58–59. $\square$

Proof of the main result. By Lemmas 3, 4, 5, the moduli space M_n is isomorphic to the variety P_n i.e. it is identified with a $\mathbb{P}^1$-bundle over M_{n-1}. Hence it is clearly an irreducible smooth n-dimensional Fano manifold. This concludes the proof of Theorem 1.

4. Further remarks

We relate first the weight vector α_n to representation–theoretic bundles. Given $n \in \mathbb{N}$, write $o(n)$ for the round down of $(n-1)/2$ and $p(n)$ for the round down of $n/2$, so that $o(n)+p(n)+1 = n$. The proof of the following remark is straightforward.

Remark 1. i) Given the weight vector α_0 with vertex in $(i,-i)$, the unique point of M_0 corresponds to the homogeneous bundle $\mathcal{B}_i$ defined as cokernel of the unique $\mathsf{SL}(3)$-equivariant map

$$\Gamma^{i,1}V \otimes \mathcal{O}(-i-2) \longrightarrow \Gamma^{i+2,1}V \otimes \mathcal{O}(-i)$$

ii) The weight vector α_n with vertex in $(0,0)$ corresponds the tensor product $\mathcal{B}_{o(n)} \otimes E_{p(n),-p(n)}$.

On the other hand, one can ask wether M_n parametrizes stable, semistable or even decomposable homogeneous vector bundles. This has been investigated by Kac in [8], [9] and we give account of this in the following remark.

Remark 2. Putting $k_0 = k_{n+1} = 1$ and $k_i = 2$ for $1 \le i \le n$, the general element of $M(\alpha_n)$ represents a stable homogeneous bundle. Indeed, for general maps (a_i, b_i, c_i, d_i) in (3), corresponding to a general representation V, one easily checks that there exists no subrepresentation V' with $\theta(V') \ge \theta(V)$.

On the other hand, for $k_i > 2$, $1 \le i \le n$, the general element V of $M(\alpha_n)$ decomposes as $V \simeq \overline{V} \oplus V'$, where $\overline{V}$ is a representation with dimension vector $\overline{\alpha_n}$ and $\overline{k_i} = 2$ for $1 \le i \le n$, while V' is a direct sum $\oplus_i \mathrm{S}^{2\,i}\, Q(-i)^{\oplus k_i - \overline{k_i}}$. Indeed in this case it is easy to find, for general maps (a_i, b_i, c_i, d_i) of V', a splitting injection $\overline{V} \hookrightarrow V$. However, the injection might not split at special points.

A geometric interpretation of the previous fact can be given as follows. Letting $\alpha_1 = (1,1,1,k_1,1,1,1)$ have origin at $E_{0,0} = \mathcal{O}$, one obtains $M_1 \simeq \mathbb{P}^1$. For $k_1 = 2$, M_1 is parametrized by the projective invariant of the 4 maps a_1, b_1, c_1, d_1, i.e. by the cross ratio of 4 points in $\mathbb{P}^1$, check [12]. For $k_1 > 2$, the maps a_1, b_1, c_1, d_1 yield 2 points P_1, P_2 and 2 hyperplanes Z_1, Z_2 in $\mathbb{P}^{k_1-1}$. Hence again M_1 is parametrized by the cross ratio of the 4 points P_1, P_2, $\overline{P_1\,P_2} \cap Z_1$, $\overline{P_1\,P_2} \cap Z_2$. The corresponding vector bundle is indecomposable if $\overline{P_1\,P_2}$ meets $Z_1 \cap Z_2$.

Remark 3. Let z_j be a map $z_j : V_{\mathsf{t}\,a_j} \to V_{\mathsf{h}\,a_j}$ such that $d_j\, z_j = 0$ and $b_j\, z_j \neq 0$. Choose such a map z_j for $1 \le j \le n$ and let $(\lambda_1, \dots, \lambda_n)$ be an element of $(\mathbb{C}^*)^n$. Define the $(\mathbb{C}^*)^n$-action

$$(\lambda_1, \dots, \lambda_n) : (a_i, b_i, c_i, d_i) \mapsto (a_i + (\lambda_i - 1)\frac{b_i\, a_i}{b_i\, z_i}\, z_i, b_i, c_i, d_i)$$

Then $(\mathbb{C}^*)^n$ acts on $V(\alpha)$. This action lifts to $A_m(\alpha)$ multiplying x_i in (6) by λ_i. So the induced $(\mathbb{C}^*)^n$-action on M_n is equivalent to the standard $(\mathbb{C}^*)^n$-action on M_n as an n-dimensional toric variety (cfr. [11], page 95).

Observe that different values of $\theta_{i,j}$ yield different linear systems on M_n.

Remark 4. Let α_n be as in Definition 2 and let the origin of α_n be at $E_{p,-p}$ i.e. let $F(\alpha_n)$ contain $E^{k_0}_{p,-p} = \mathrm{S}^{2p}\, Q(-p)^{k_0}$. Then, tracing back the proof of Lemma 3, one shows that M_n is isomorphic to P_n (confer Lemma 5), equipped with the very ample line bundle L_p taking value $(1+p, 2+p, \ldots, n+p)$ at the rays $s_1, \ldots, s_n$ and 0 elsewhere.

For example, letting $\alpha_1 = (1,1,1,2,1,1,1)$ have origin at $E_{p,-p}$, the coordinate ring of the space M_1 is $\mathbb{C}[y_p^p\, y_{p+1}^{p+1}, x_p\, y_p^{p-1}\, y_{p+1}^p, \ldots, x_p^p\, y_{p+1}]$, so M_1 is a rational normal curve of degree p.

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