BUNDLES OVER FANO THREEFOLDS OF TYPE V_{22}

DANIELE FAENZI

Abstract. An explicit resolution of the diagonal over V_{22} is given, making use of some observations about instanton bundles with $c_2 = 3$. Different descriptions of V_{22} are interpreted in terms of mutations of vector bundles.

1. Introduction

Among smooth complex Fano threefolds with $\rho = 1$, those with $h^{2,1} = 0$ play a special rôle. According to Iskosvskih's classification [Isk77], [Isk78], there exist four classes of such varieties, namely $\mathbb{P}^3$, Q_3, V_5 and V_{22}, respectively of index $4, 3, 2$ and 1, where the genus-12 variety V_{22} is the only one with non–trivial moduli. Their degree, respectively $1, 2, 5$ and 22, is maximal in each index class; they are all rational and deformation equivalent to a smooth orbit closure of the group $\mathsf{SL}(2, \mathbb{C})$. Moreover, their K-theory ring is isomorphic to $\mathbb{Z}^4$. The geometry of these varieties has been studied in an enormous number of papers, and we refer to [IP99] for an exhaustive treatment of them and to [Sch01], [Muk03], [AF93] for more important results.

Here we will study the variety V_{22} in terms of bundles over it and we will prove the following result.

Theorem. *The general variety X of type V_{22} admits the resolution of the diagonal*

$$0 \to G_3 \boxtimes G^3 \to G_2 \boxtimes G^2 \to G_1 \boxtimes G^1 \to G_0 \boxtimes G^0 \to \mathcal{O}_\Delta \to 0$$

where $(G_3, \dots, G_0)$ (respectively $(G^3, \dots, G^0)$) is an exceptional collection of stable aCM bundles of rank $2, 3, 4, 1$ (respectively of rank $2, 5, 3, 1$).

This gives an analogue of Beĭlinson's theorem over the projective space, see [Bei78]. Further instances of this fact were found e.g. by Kapranov in [Kap88], by Canonaco for weighted projective spaces in [Can00], by Orlov in [Orl91] for the threefold V_5, by the author in [Fae03] still for the threefold V_5.

The main tools are results obtained by Schreyer in [Sch01] and by Mukai in [Muk92] involving nets of quadrics, 3-instanton bundles on $\mathbb{P}^3$ and nets of alternating 2-forms. It will turn out that mutations of the bundles G_i of the above Theorem are closely related to the different descriptions of the threefold V_{22}.

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The paper is organized as follows. In Section 2 we give basic definitions and lemmas. In Section 3 we provide the first description of V_{22} by means of nets of quadrics, recall its relation with the moduli space of twisted cubics and with 3-instanton bundles on $\mathbb{P}^3$. Section 4 takes care of the definition of V_{22} via nets of alternating 2-forms and contains the technical core of the paper i.e. Theorem 4.5. This theorem is crucial in Proposition 6.4, which in turn is the key to prove Theorem 7.2, our main result. Section 5 is devoted to the description of V_{22} via polar hexagons to a plane quartic. In Section 6 we give several results concerning bundles on X and describe the homomorphism groups between them, while in Section 7 we state precisely and prove the main result (cfr. Theorem 7.2), together with some corollaries (cfr. Corollary 7.3 and 7.4). Finally in Section 8 we draw some remarks, including helices and the Mukai–Umemura case i.e. a threefold of type V_{22} with an $\mathsf{SL}(2)$ quasi–homogeneous structure.

Remark. After this paper was finished, the author learned of the existence of an interesting preprint by Alexander Kuznetsov, [Kuz97], where similar questions are investigated, although making use of different methods.

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2. Generalities

We will always assume that the ambient variety X is a compact complex algebraic smooth variety with $\mathrm{Pic}(X) \simeq \mathbb{Z} = \langle \mathcal{O}_X(1) \rangle$, $\mathcal{O}_X(1)$ being a very ample line bundle.

Definition 2.1. For a pair of vector bundles $\mathcal{F}$ and $\mathcal{G}$ on a variety X, define $p_{\mathcal{F},\mathcal{G}} : \mathrm{Hom}(\mathcal{F},\mathcal{G}) \otimes \mathcal{F} \to \mathcal{G}$ and $i_{\mathcal{F},\mathcal{G}} : \mathcal{F} \to \mathrm{Hom}(\mathcal{F},\mathcal{G})^* \otimes \mathcal{G}$ as the canonical evaluations. If $p_{\mathcal{F},\mathcal{G}}$ (resp. $i_{\mathcal{F},\mathcal{G}}$)) is surjective (resp. injective) define the left mutation $\mathrm{L}_{\mathcal{F}}\,\mathcal{G} = \ker(p_{\mathcal{F},\mathcal{G}})$ (resp. the right mutation $\mathrm{R}_{\mathcal{G}}\,\mathcal{F} = (\mathrm{coker}\, i_{\mathcal{F},\mathcal{G}})$).

We refer to the book [hel90] useful properties of mutations, to [Dre86] and [GR87] for their original use over projective spaces.

For any complex vector space V denote by 1_V (resp. by χ_V) the identity map of V (resp. the canonical map $V^* \otimes V \to \mathbb{C}$). We write S^i, where i is a finite sequence of nondecreasing integers, for the Schur functor associated to the Young tableau defined by the partition given by i. More precisely, the tableau defined by i has i_j boxes on the j-th row. For example $\mathrm{S}^j\, V$, where j is an integer and V is a vector space, is the j-th symmetric power of V.

Definition 2.2. Given a sheaf $\mathcal{F}$ over X, we say that $\mathcal{F}$ is aCM (for arithmetically Cohen–Macaulay) if $\mathrm{H}^p(X, \mathcal{F}(t)) = 0$, for all $t \in \mathbb{Z}$ and for $0 < p < \dim(X)$.

We will write $\mathrm{H}^p(-)$ or $\mathrm{Hom}(-,-)$ instead of $\mathrm{H}^p(X,-)$ or $\mathrm{Hom}_X(-,-)$ unless the ambient variety X is not clear from the context.

Given a subvariety $Z \subset X$ we denote its ideal sheaf by $J_{Z,X}$ and, by abuse of notation, the ideal of Z in the coordinate ring of X.

Definition 2.3. A variety X of type V_{22} is a smooth projective threefold with $\mathrm{Pic}(X) = \langle \mathcal{O}_X(1)\rangle = \langle \omega_X^*\rangle$ and $\deg(\mathcal{O}_X(1)) = 22$.

We refer to [HL97] for the definition of stability of bundles, in the sense of Mumford–Takemoto, with respect to the positive generator $c_1(\mathcal{O}_X(1))$ of $\mathrm{Pic}(X)$.

Recall from [IP99] that the Chow ring $\mathrm{CH}(X)$ is isomorphic to $\mathbb{Z}^4$, where $\mathrm{CH}^2(X)$ (resp. $\mathrm{CH}^3(X)$) is generated by the class of a line (resp. of a point) in X.

Given a vector bundle $\mathcal{F}$ on X we denote its Chern classes by $c_i(\mathcal{F})$, for $1 \leq i \leq 3$ by $c_i \in \mathbb{Z}$, meaning $c_i(\mathcal{F}) = c_i\xi_i$, where $\xi_1 = c_1(\mathcal{O}_X(1))$, ξ_2 is the cohomology class of a line in X and ξ_3 is the cohomology class of a point in X. Denote by $\mu(\mathcal{F})$ the rational number $c_1(\mathcal{F})/\operatorname{rk}(\mathcal{F})$, called the *slope* of $\mathcal{F}$. We say that a bundle $\mathcal{F}$ is normalized if $-1 < \mu(\mathcal{F}) \leq 0$. We write $\mathcal{F}_n$ for the unique normalized twist of $\mathcal{F}$. The proof of following lemma can be adapted from $\mathbb{P}^n$ since $\mathrm{Pic}(X) \simeq \mathbb{Z}$, see [OSS80].

Lemma 2.4 (Hoppe). *Let $\mathcal{F}$ be a rank r vector bundle on X. Then $\mathcal{F}$ is stable if $\mathrm{h}^0((\wedge^p\mathcal{F})_n) = 0$ for $1 \leq p < r$.*

3. Nets of dual quadrics and 3-instanton bundles on $\mathbb{P}^3$

Let $A \simeq \mathbb{C}^4$ and $B \simeq \mathbb{C}^3$ be complex vector spaces, and let $R = \mathbb{C}[A] = \mathbb{C}[x_0, \ldots, x_3]$ and $T = \mathbb{C}[B]$ be polynomial algebras over them. Considering the dual ring $R^* = \mathbb{C}[A^*]$ we have $R^* \simeq \mathbb{C}[\partial_0, \ldots, \partial_3]$. Then define the *apolarity action* of R^* on R by differentiation $\partial^i(x_j) = i!\ j/i\ x_{j-i}$, where i, j are multiindeces and $\partial^i(x_j) = 0$ if $j \not\geq i$. Then for $\partial \in \mathrm{S}^i A^*$ we have the apolarity map $\partial : \mathrm{S}^j A \to \mathrm{S}^{j-i} A$. In the same way T^* acts on T by apolarity and we have perfect pairings between degree d polynomials over R (resp. over T) and degree d differential operators over R (resp. over T).

We define the variety H to be the irreducible component of $\mathrm{Hilb}_{3t+1}(\mathbb{P}(A))$ containing rational normal cubics in $\mathbb{P}(A)$, as constructed in [EPS87]. The open subset H_c consisting of points which are Cohen–Macaulay embeds in $\mathbb{G}(\mathbb{C}^3, \mathrm{S}^2 A)$ by means of the vector bundle U_{H}^* whose fiber over $[\Gamma] \in \mathsf{H}_c$ is $\mathrm{Tor}_1^R(R/J_{\Gamma,\mathbb{P}^3}, \mathbb{C})_2 \simeq \mathbb{C}^3$. Equivalently, we associate to any $[\Gamma] \in \mathsf{H}_c$ the net of quadrics on $\mathbb{P}(A)$ vanishing on Γ.

Moreover, there exists a rank-2 bundle on H_c whose fiber over Γ is $\mathrm{Tor}_2^R(R/J_{\Gamma,\mathbb{P}^3}, \mathbb{C})_3 \simeq \mathbb{C}^2$. Namely we take the space of first order syzygies of $J_{\Gamma,\mathbb{P}^3}$. We denote this bundle by E_{H}.

Lemma 3.1. *Over the variety H_c, the bundle U_{H}^* (resp. the bundle E_{H}) is globally generated with $\mathrm{H}^0(U^*)^* \simeq \mathrm{S}^2 A$. (resp. with $\mathrm{H}^0(E_{\mathsf{H}}^*)^* \simeq \mathrm{S}^{2,1} A = \ker(\mathrm{S}^2 A \otimes A \to \mathrm{S}^3 A)$). The vector bundle E_{H}^* embeds H_c into $\mathbb{G}(\mathbb{C}^2, \mathrm{S}^{2,1} A) = \mathbb{G}(\mathbb{P}^1, \mathbb{P}^{19})$.*

We have the canonical isomorphisms $\mathrm{Hom}(E_{\mathsf{H}}, U_{\mathsf{H}}) \simeq A^*$, $\mathrm{Hom}(\wedge^2 U_{\mathsf{H}}, E_{\mathsf{H}}) \simeq A^*$ and $\mathrm{H}^0(\wedge^2 U^*)^* \simeq \wedge^2 \mathrm{S}^2 A \simeq \mathrm{S}^{3,1} A$. The morphism $i_{\wedge^2 U_{\mathsf{H}}, E_{\mathsf{H}}}$ is induced by the map $\wedge^2 \mathrm{S}^2 A \to A \otimes \mathrm{S}^{2,1} A$ in the diagram *below*

$$
\begin{array}{ccccc}
 & & \wedge^2 \mathrm{S}^2 A & & \\
 & \swarrow & \downarrow & & \\
A \otimes \mathrm{S}^{2,1} A & \longrightarrow & \mathrm{S}^2 A \otimes \mathrm{S}^2 A & \xrightarrow{m} & \mathrm{S}^4 A
\end{array} \tag{1}
$$

where the map m is the multiplication in $\mathbb{C}[A]$ and the maps $\wedge^2 \mathrm{S}^2 A \to \mathrm{S}^2 A \otimes \mathrm{S}^2 A$ and $A \otimes \mathrm{S}^{2,1} A \to \mathrm{S}^2 A \otimes \mathrm{S}^2 A$ are the canonical injections.

Proof. Over a point $[\Gamma]$ in H_c we take the minimal graded free resolution of $J_{\Gamma,\mathbb{P}^3}$ in degree 3. This yields the exact sequence

$$0 \leftarrow \mathrm{H}^0(J_{\Gamma,\mathbb{P}^3}(3)) \leftarrow A \otimes U_{\mathsf{H},\Gamma} \leftarrow E_{\mathsf{H},\Gamma} \leftarrow 0 \tag{2}$$

The above map $A \otimes U_{\mathsf{H},\Gamma} \leftarrow E_{\mathsf{H},\Gamma}$ is induced by $i_{E,U}$ and the isomorphism $\mathrm{Hom}(E_{\mathsf{H}}, U_{\mathsf{H}}) \simeq A^*$ is clear. Since $\det(U_{\mathsf{H}}) \simeq \mathcal{O}_{\mathsf{H}}(1)$ and $\det(E_{\mathsf{H}}) \simeq \mathcal{O}_{\mathsf{H}}(1)$ we have $\mathrm{Hom}(\wedge^2 U_{\mathsf{H}}, E_{\mathsf{H}}) \simeq \mathrm{Hom}(E_{\mathsf{H}}^*, \wedge^2 U_{\mathsf{H}}^*) \simeq \mathrm{Hom}(E_{\mathsf{H}}, U_{\mathsf{H}}) \simeq A^*$.

Since any quadratic form on A contains a twisted cubic in H_c, we have $\mathrm{H}^0(U_{\mathsf{H}}^*) \simeq \mathrm{S}^2 A^*$ and globalizing (2) we get $\mathrm{H}^0(E_{\mathsf{H}}^*)^* \simeq \mathrm{S}^{2,1} A = \ker(\mathrm{S}^2 A \otimes A \to \mathrm{S}^3 A)$. Since the 2×2 minors of the matrix $A \otimes U_{\mathsf{H}} \leftarrow E_{\mathsf{H}}$ in (2) define the twisted cubic Γ, E_{H} provides an embedding into $\mathbb{G}(\mathbb{C}^2, \mathrm{S}^{2,1} A)$ and thus it is globally generated.

Finally, there are $\mathsf{SL}(A)$-equivariant isomorphisms $\mathrm{S}^2 A \otimes \mathrm{S}^2 A \simeq \mathrm{S}^{2,2} A \oplus \mathrm{S}^{3,1} A \oplus \mathrm{S}^4 A$, $A \otimes \mathrm{S}^{2,1} A \simeq \mathrm{S}^{2,1,1} A \oplus \mathrm{S}^{2,2} A \oplus \mathrm{S}^{3,1} A$, $\wedge^2 \mathrm{S}^2 A \simeq \mathrm{S}^{2,1,1} A$. Then by Schur's Lemma the inclusion $\mathrm{S}^{2,1,1} A \hookrightarrow \mathrm{S}^2 A \otimes \mathrm{S}^2 A$ composes to zero with m and therefore factors injectively through $A \otimes \mathrm{S}^{2,1} A$, so the last statement is proved. □

Definition 3.2. A net of dual quadrics Ψ (parametrized by B) on $\mathbb{P}(A)$ is defined as a surjective map $\Psi : \mathrm{S}^2 A \to B$. We also denote by Ψ the composition $A \otimes A \to \mathrm{S}^2 A \xrightarrow{\Psi} B$. Let $\Psi^\top : B^* \to \mathrm{S}^2 A^*$ be the transpose of Ψ and let $V_\Psi = \ker(\Psi)$. For $[\Gamma] \in \mathsf{H}$ consider $J_{\Gamma,\mathbb{P}^3}$. Given a general net Ψ define

$$
\begin{aligned}
X_\Psi = & \{[\Gamma] \in \mathsf{H} \subset \mathrm{Hilb}_{3t+1}(\mathbb{P}^3) | \Psi(\mathrm{H}^0(J_{\Gamma,\mathbb{P}^3}(2))) = 0\} = \\
= & \{[\Gamma] \in \mathsf{H} \subset \mathrm{Hilb}_{3t+1}(\mathbb{P}^3) | \, \mathrm{H}^0(J_{\Gamma,\mathbb{P}^3}(2)) \subset V_\Psi\}
\end{aligned}
$$

Lemma 3.3. *Given a general net of dual quadrics $\Psi : \mathrm{S}^2 A \to B$, X_Ψ is a Fano threefold of type V_{22}, equipped with a rank-2 vector bundle E_{H} and a rank-3 vector bundle U_{H}.*

Consider a net of dual quadrics Ψ as defined in 3.2. We take the ideal J^Ψ of polynomials in R annihilated by Ψ i.e.

$$J^\Psi = \{p \in R \mid \Psi^\top(\beta)(p) = 0, \ \forall \, \beta \in B^*\} \tag{3}$$

where $\Psi^\top(\beta)$ sits in $\mathrm{S}^2 A^*$ and for $\partial \in \mathrm{S}^2 A^*$, $p \in \mathrm{S}^2 A$ we define $\partial(p)$ by apolarity action as at the beginning of this section.

Definition 3.4. For general Ψ define the Artinian ring $R^\Psi = R/J^\Psi$. Taking its minimal graded free resolution, put $V_\Psi^{i,j} = \mathrm{Tor}_i^R(R^\Psi, \mathbb{C})_j$. As shown in

[Sch01, Lemma 4.1], the minimal graded free resolution of R^Ψ reads

(4)

$$0 \leftarrow R/J^\Psi \leftarrow R \xleftarrow{p_\Psi} V_\Psi^{1,2} \otimes R(-2) \xleftarrow{q_\Psi} V_\Psi^{2,3} \otimes R(-3) \oplus V_\Psi^{2,4} \otimes R(-4) \xleftarrow{r_\Psi}$$
$$\xleftarrow{r_\Psi} V_\Psi^{3,5} \otimes R(-5) \leftarrow (V_\Psi^{4,6}) \otimes R(-6) \leftarrow 0$$

We have $R_1^\Psi \simeq A$, $R_2^\Psi \simeq B$ and $R_d^\Psi = 0$ for $d \geq 3$.

Recall by [Sch01, Corollary 4.3] that there is an isomorphism $(V_\Psi^{4,6})^* \simeq V_\Psi^{2,4}$. The dimensions of the spaces $V_\Psi^{i,j}$ are the following

$$\dim(V_\Psi^{1,2}) = 7 \quad \dim(V_\Psi^{2,3}) = 8 \quad \dim(V_\Psi^{2,4}) = 3 \quad \dim(V_\Psi^{3,5}) = 8$$

There is a canonical isomorphism $V_\Psi^{1,2} \simeq V_\Psi = \ker(\Psi)$, indeed we have $V_\Psi^{1,2} = \{p \in \mathrm{S}^2 A \mid \Psi^\top(\beta)(p) = 0, \forall \beta \in B^*\} \simeq \ker \Psi$. Thus we will identify these spaces from now on.

Definition 3.5. Given a general net of dual quadrics Ψ as in 3.2 and the ring R^Ψ defined in 3.4, consider the vector bundle $\ker(p_\Psi)$ over $\mathbb{P}(A)$ obtained sheafifying p_Ψ

$$0 \to \ker(p_\Psi) \to V \otimes \mathcal{O}_{\mathbb{P}(A)}(-2) \xrightarrow{p_\Psi} \mathcal{O}_{\mathbb{P}(A)} \to 0 \tag{5}$$

We get $\mathrm{H}^0(\ker(p_\Psi)(t)) = 0$ for $t < 3$.

Lemma 3.6 (Schreyer)**.** *Given a general net of dual quadrics Ψ as in 3.2, the sheafification of the map q_Ψ gives an instanton bundle $\mathcal{E}_\Psi$ defined by*

$$0 \to \mathcal{E}_\Psi(-5) \to V_\Psi^{2,3} \otimes \mathcal{O}_{\mathbb{P}(A)}(-3) \xrightarrow{q_\Psi} \ker(p_\Psi) \to 0 \tag{6}$$

We have $c_2(\mathcal{E}_\Psi) = 3$ and $\mathrm{h}^1(\mathcal{E}_\Psi(t)) = 0$ except for $t = 0, 1, 2$. Furthermore we have the canonical isomorphisms

$$\mathrm{H}^1(\ker(p_\Psi)) \simeq \mathbb{C} \qquad \mathrm{H}^1(\ker(p_\Psi)(1)) \simeq A \qquad \mathrm{H}^1(\ker(p_\Psi)(2)) \simeq B$$
$$\mathrm{H}^1(\mathcal{E}_\Psi(-1)) \simeq V_\Psi^{2,4} \qquad \mathrm{H}^1(\mathcal{E}_\Psi) \simeq A^* \qquad \mathrm{H}^1(\mathcal{E}_\Psi(1)) \simeq \mathbb{C}$$

There exists an isomorphism $\mathrm{H}^1(\Omega^1_{\mathbb{P}(A)} \otimes \mathcal{E}_\Psi) \simeq V_\Psi^{3,5}$ and the vector space $V_\Psi^{3,5}$ is endowed with a canonical alternating duality. Finally, the instanton bundle $\mathcal{E}_\Psi$ is isomorphic to the cohomology of the monad

$$(V_\Psi^{2,4})^* \otimes \mathcal{O}_{\mathbb{P}(A)}(-1) \xrightarrow{r_\Psi} V_\Psi^{3,5} \otimes \mathcal{O}_{\mathbb{P}(A)} \xrightarrow{r_\Psi^\top} V_\Psi^{2,4} \otimes \mathcal{O}_{\mathbb{P}(A)}(1)$$

where the map r_Ψ is defined by the minimal graded free resolution (4) and we recall $(V_\Psi^{4,6})^* \simeq V_\Psi^{2,4}$ and $(V_\Psi^{3,5})^* \simeq V_\Psi^{3,5}$.

The relation between nets of quadrics and 3-instanton bundles has been throughly investigated by Gruson and Skiti in [GS94]. We give account of this in the following remark.

Remark 3.7 (Gruson–Skiti)**.** For a general instanton bundle $\mathcal{E}$ on $\mathbb{P}(A)$ with $c_2(\mathcal{E}) = 3$, the homomorphism $\mathrm{H}^1(\mathcal{E}(-1)) \otimes A \to \mathrm{H}^1(\mathcal{E})$ gives a map $\Psi_\mathcal{E}^\top : \mathrm{H}^1(E(-1)) \simeq \mathbb{C}^3 \hookrightarrow A^* \otimes A^*$ since $\mathrm{H}^1(\mathcal{E}) \simeq A^*$. The map $\Psi_\mathcal{E}^\top$ factors through $\mathrm{S}^2 A^*$ and for $\mathcal{E} \simeq \mathcal{E}_\Psi$ it agrees with $\Psi^\top$. Then we may indifferently start with a general net Ψ or with a general 3-instanton $\mathcal{E}$.

4. Nets of dual quadrics and nets of alternating 2-forms

Remark 4.1. Given a general net of dual quadrics Ψ as in 3.2, the space $\mathrm{Tor}^R_*(R^\Psi, \mathbb{C})_*$ is endowed with a natural skew–commutative algebra structure, see [Sch01, Section 5]. In particular, we define the net of alternating 2-forms σ_Ψ as the tor-multiplication $\wedge^2 V_\Psi \to V_\Psi^{2,4}$.

By construction, see [Eis95, exercise A.3.20], the map σ_Ψ fits in the commutative diagram below.

$$\begin{array}{ccccc} \wedge^2 V_\Psi^{1,2} \otimes R(-4) & \xrightarrow{\wedge^2 p_\Psi} & (V_\Psi^{1,2} \otimes R(-2))^{\oplus 2} & \xrightarrow{(p_\Psi, p_\Psi)} & R \\ \Big\downarrow{\scriptstyle \sigma_\Psi \otimes 1_R} & & \Big\downarrow & & \Big\| \\ V_\Psi^{2,4} \otimes R(-4) & \xrightarrow[q_\Psi]{} & V_\Psi^{1,2} \otimes R(-2) & \xrightarrow[p_\Psi]{} & R \end{array} \tag{7}$$

It will turn out from Theorem 4.5 that $V_\Psi^{2,4} \simeq B^*$ so we will be able to write with no ambiguity σ_Ψ as a map $\wedge^2 V_\Psi \to B^*$.

Let V be a complex vector space of dimension 7 and consider the Grassmannian $\mathsf{G} = \mathbb{G}(\mathbb{C}^3, V)$ endowed with the rank-3 universal subbundle U_G. Given a 3-dimensional complex vector space B, let $\sigma : \wedge^2 V \to B^*$ be a net of alternating 2-forms.

Definition 4.2. Given a general net of alternating forms $\sigma : \wedge^2 V \to B^*$ define

$$X_\sigma = \{\mathbb{C}^3 \subset V \mid \sigma^\top(b)(u \wedge v) = 0 \text{ for any } u, v \in \mathbb{C}^3, \text{ for any } b \in B\}$$

The variety X_σ is a Fano threefold of type V_{22} given in G as the zero locus of the section σ of $B^* \otimes \wedge^2(U_\mathsf{G}^*)$.

Lemma 4.3 (Schreyer)**.** *Given a general net dual quadrics Ψ as in Definition 3.2, and the net of alternating 2-forms σ_Ψ of Remark 4.1, we have an isomorphism $X_\Psi \simeq X_{\sigma_\Psi}$. Under this isomorphism U_H is taken to U_G.*

Remark 4.4. For a general net σ define the map $\varsigma : V \otimes B \to V^*$ associated to $\sigma^\top : B \to \wedge^2 V^*$ by $\varsigma(u \otimes b)(v) = \sigma^\top(b)(u \wedge v)$ for $u \in B$ and $u, v \in V$. There is an isomorphism

$$\mathrm{Hom}_X(U_\mathsf{G}, Q_\mathsf{G}^*) \simeq B$$

The map ς is the transpose of the map on global sections of the dual of the surjective map $p_{U,Q^*} : B \otimes U \to Q^*$.

Proof. The definition of ς is clear. Considering the Koszul complex of X in G one computes easily $\mathrm{H}^0(X, \mathrm{S}^2 U_\mathsf{G}^*) \simeq \mathrm{S}^2 V^*$ and $\mathrm{H}^0(X, \wedge^2 U_\mathsf{G}^*) \simeq \mathrm{coker}\, \sigma^\top$, obtaining $\mathrm{Hom}(U_\mathsf{G}, Q_\mathsf{G}^*) \simeq \mathrm{H}^0(U_\mathsf{G}^* \otimes Q_\mathsf{G}^*) \simeq B$. Now for $b \in B$, $u \in U_\mathsf{G}$, $q \in Q_\mathsf{G}$, we have $p_{U_\mathsf{G}, Q_\mathsf{G}^*}(b \otimes u)(q) = \sigma^\top(b)(u \otimes q)$. Therefore $p_{U_\mathsf{G}, Q_\mathsf{G}^*}$ agrees with ς. The map ς is surjective for general σ so $p_{U_\mathsf{G}, Q_\mathsf{G}^*}$ is also surjective for U_G^* and Q_G are globally generated. □

Theorem 4.5. *For a general net of dual quadrics Ψ we have the following natural exact sequence*

$$0 \to Y_\Psi \xrightarrow{\ell_1} A \otimes V_\Psi^{2,3} \xrightarrow{\ell_2} B \otimes V_\Psi \xrightarrow{\ell_3} V_\Psi^* \to 0 \tag{8}$$

where the vector space Y_Ψ and the map ℓ_2 are given by

(9) $$Y_\Psi = \ker(\sigma_\Psi : \wedge^2 V_\Psi \to V_\Psi^{2,4})$$

(10) $$\ell_2 : A \otimes V_\Psi^{2,3} \xrightarrow{q_\Psi} \mathrm{S}^2 A \otimes V_\Psi \xrightarrow{\Psi \otimes 1_{V_\Psi}} B \otimes V_\Psi$$

The map ℓ_3 is defined as ς as in Remark 4.4 with $\sigma = \sigma_\Psi$. The map ℓ_1 is defined lifting the inclusion $Y_\Psi \hookrightarrow \mathrm{S}^2 A \otimes V_\Psi$ to $A \otimes V_\Psi^{2,3}$ via the map q_Ψ i.e. ℓ_1 makes the following diagram commutative

(11) $$\begin{array}{ccccccc} & & & & & & A \otimes V_\Psi^{2,3} \\ & & & \nearrow^{\ell_1} & & & \downarrow q_\Psi \\ Y_\Psi & \to & \wedge^2 V_\Psi & \hookrightarrow & V_\Psi \otimes V_\Psi & \to & \mathrm{S}^2 A \otimes V_\Psi \end{array}$$

There is a canonical isomorphism $V_\Psi^{2,4} \simeq B^*$.

Proof. To prove the exactness in $A \otimes V_\Psi^{2,3}$ we need to use the definition of σ_Ψ in 4.1. In fact, considering the resolution of the ideal J^Ψ, taken in degree 4, we write

(12) $$0 \to A \otimes V_\Psi^{2,3} \oplus V_\Psi^{2,4} \xrightarrow{q_\Psi} \mathrm{S}^2 A \otimes V_\Psi \to \mathrm{S}^4 A \to 0$$

and we need to consider the composition (10). Notice that the kernel of $\Psi \otimes 1_{V_\Psi}$ in $\mathrm{S}^2 A \otimes V_\Psi$ is $V_\Psi \otimes V_\Psi$. Moreover it is mapped to zero by $m : \mathrm{S}^2 A \otimes \mathrm{S}^2 A \to \mathrm{S}^4 A$, so it must lie in $\wedge^2 \mathrm{S}^2 A$. Therefore we have $\ker(\Psi \otimes 1_{V_\Psi}) = \wedge^2 \mathrm{S}^2 A \cap V_\Psi \otimes V_\Psi = \wedge^2 V_\Psi$. So we obtain the following exact sequence

(13) $$0 \to \wedge^2 V_\Psi \to A \otimes V_\Psi^{2,3} \oplus V_\Psi^{2,4} \xrightarrow{\Psi \otimes 1_{V_\Psi} \circ q_\Psi} B \otimes V_\Psi$$

where the map ℓ_1 of the statement is the restriction to Y_Ψ of the above map $\wedge^2 V_\Psi \to A \otimes V_\Psi^{2,3} \oplus V_\Psi^{2,4}$ and we still have to prove that ℓ_1 takes image in $A \otimes V_\Psi^{2,3}$. Now Tor multiplication identifies the map $\sigma_\Psi : \wedge^2 V_\Psi \to V_\Psi^{2,4}$ as the arrow making diagram (7) commutative. This diagram, taken in degree 4, boils down to the following commutative diagram.

$$\begin{array}{ccccc} \wedge^2 V_\Psi & \longrightarrow & A \otimes V_\Psi^{2,3} \oplus V_\Psi^{2,4} & \longrightarrow & \mathrm{S}^2 A \otimes V_\Psi \\ & \searrow_{\sigma_\Psi} & \downarrow \pi_{V_\Psi^{2,4}} & & \\ & & V_\Psi^{2,4} & & \end{array}$$

This, together with (13), proves at the same time that the map ℓ_1 is well defined in Y_Ψ and that the the required sequence is exact in $A \otimes V_\Psi^{2,3}$. Consequently it is exact also in Y_Ψ. In order to prove exactness in $B \otimes V_\Psi$ we will need to use the instanton bundle $\mathcal{E}_\Psi$.

Denote the kernel sheaf $\ker(p_\Psi)$ by $\mathcal{K}_\Psi^1$. Taking the symmetrized powers of the sequence (5) we get

(14) $$0 \to \mathcal{K}_\Psi^2 \to \mathrm{S}^2 V_\Psi \otimes \mathcal{O} \to \mathcal{O}(4) \to 0$$

(15) $$0 \to \wedge^2 \mathcal{K}_\Psi^1(4) \to V_\Psi \otimes \mathcal{K}_\Psi^1(2) \to \mathcal{K}_\Psi^2 \to 0$$

(16) $$0 \to \mathrm{S}^2 \mathcal{K}_\Psi^1(4) \to \mathrm{S}^2 V_\Psi \otimes \mathcal{O} \to V_\Psi \otimes \mathcal{O}(2) \to 0$$

(17) $$0 \to \wedge^2 \mathcal{K}_\Psi^1(4) \to \wedge^2 V_\Psi \otimes \mathcal{O} \to \mathcal{K}_\Psi^1(4) \to 0$$

for some vector bundle $\mathcal{K}^2_\Psi$. In turn the symmetrized square of the sequence (6) gives the following

$$0 \to \mathcal{O}(-6) \to \wedge^2 V^{2,3}_\Psi \otimes \mathcal{O}(-2) \to \mathcal{K}^3_\Psi \to 0 \tag{18}$$

$$0 \to \mathcal{K}^3_\Psi \to V^{2,3}_\Psi \otimes \mathcal{K}^1_\Psi(1) \to \mathrm{S}^2\,\mathcal{K}^1_\Psi(4) \to 0 \tag{19}$$

for some vector bundle $\mathcal{K}^3_\Psi$. So we get the the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{S}^2\,V_\Psi & \longrightarrow & \mathrm{S}^4\,A & \longrightarrow\!\!\!\!\rightarrow & \mathrm{H}^1(\mathcal{K}^2_\Psi) \\
 & \uparrow & & \uparrow & & \uparrow\!\!\uparrow \\
0 \longrightarrow & V_\Psi \otimes V_\Psi & \longrightarrow & \mathrm{S}^2\,A \otimes V_\Psi & \xrightarrow{\Psi \otimes 1_{V_\Psi}} & B \otimes V_\Psi \\
 & \uparrow & & \uparrow & & \uparrow \\
0 \longrightarrow & \wedge^2 V_\Psi & \longrightarrow & A \otimes V^{2,3}_\Psi \oplus V^{2,4}_\Psi & \longrightarrow\!\!\!\!\rightarrow & \mathrm{H}^1(\wedge^2 \mathcal{K}^1_\Psi(4))
\end{array}$$

Here the left vertical column is the canonical decomposition $V_\Psi^{\otimes 2}$ into symmetric and skew–symmetric tensors, the central vertical row is (12) and the bottom row is the cohomology sequence of (17). This yields two presentations of $\mathrm{H}^1(\mathcal{K}^2_\Psi)$ (the vertical one from (15) and the horizontal one from (14)).

On the other hand $\mathcal{K}^2_\Psi$ defined in (14) and $\mathrm{S}^2\,\mathcal{K}^1_\Psi(4)$ fit into the following short exact sequence

$$0 \to \mathrm{S}^2\,\mathcal{K}^1_\Psi(4) \to \mathcal{K}^2_\Psi \to \mathcal{K}^1_\Psi(4) \to 0 \tag{20}$$

Summing up we can then build the following commutative diagram (we omit surrounding zeroes for brevity)

$$\begin{array}{ccccc}
\mathcal{K}^3_\Psi & \longrightarrow & V^{2,3}_\Psi \otimes \mathcal{K}^1_\Psi(1) & \longrightarrow & \mathrm{S}^2\,\mathcal{K}^1_\Psi(4) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{K}^4_\Psi & \longrightarrow & V^{2,3}_\Psi \otimes V_\Psi \otimes \mathcal{O}(-1) & \longrightarrow & \mathcal{K}^2_\Psi \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{E}_\Psi(-1) & \longrightarrow & V^{2,3}_\Psi \otimes \mathcal{O}(1) & \longrightarrow & \mathcal{K}^1_\Psi(4)
\end{array} \tag{21}$$

where the top (resp. bottom) horizontal row is (19) (resp. (6)). The central row defines some bundle $\mathcal{K}^4_\Psi$ as the kernel of the composition of the two projections $V^{2,3}_\Psi \otimes V_\Psi \otimes \mathcal{O}(-1) \to V_\Psi \otimes \mathcal{K}^1_\Psi(2)$ and $V_\Psi \otimes \mathcal{K}^1_\Psi(2) \to \mathcal{K}^2_\Psi$ of (6) and (15). The right (resp. central) vertical column comes from (20) (resp. comes from (5)) . Use (18) to show $\mathrm{H}^2(\mathcal{K}^3_\Psi) \simeq \mathrm{S}^2\,A^*$ and Lemma 3.6 for $\mathrm{H}^1(\mathcal{E}_\Psi(-1)) \simeq B^*$. Now taking cohomology in the diagram (21) we get

$$\begin{array}{ccccccc}
0 \longrightarrow & A \otimes V^{2,3}_\Psi & \longrightarrow & A \otimes V^{2,3}_\Psi \oplus V^{2,4}_\Psi & \longrightarrow\!\!\!\!\rightarrow & B^* \simeq \mathrm{H}^1(\mathcal{E}_\Psi(-1)) \\
 & \| & & \downarrow & & \downarrow {\scriptstyle \Psi^\top} \\
0 \longrightarrow & A \otimes V^{2,3}_\Psi & \longrightarrow & \mathrm{H}^1(\mathrm{S}^2\,\mathcal{K}^1_\Psi(4)) & \longrightarrow\!\!\!\!\rightarrow & \mathrm{S}^2\,A^* \simeq \mathrm{H}^2(\mathcal{K}^3_\Psi) \\
 & & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\
 & & & \mathrm{H}^1(\mathcal{K}^2_\Psi) & \xrightarrow{\ \simeq\ } & V^*_\Psi
\end{array}$$

This provides the following isomorphisms

$$V_\Psi^{2,4} \simeq B^*$$
$$V_\Psi^* \simeq \operatorname{coker}(\mathrm{S}^2 V_\Psi \to \mathrm{S}^4 A) \simeq \mathrm{H}^1(\mathcal{K}_\Psi^2)$$
$$V_\Psi^* \simeq \operatorname{coker}(A \otimes V_\Psi^{2,3} \to B \otimes V_\Psi) \simeq \mathrm{H}^1(\mathcal{K}_\Psi^2)$$

Thus we proved the exactness of the sequence (8). □

5. Polar hexagons to a plane quartic

5.1. **The Variety of Sums of Powers.** Let B be a 3-dimensional $\mathbb{C}$-vector space and $f \in \mathrm{S}^4 B$ be a plane quartic. Put $\check{\mathbb{P}}^2 = \mathbb{P}(B^*)$. According to Mukai [Muk92], we define the subvariety of $\mathrm{Hilb}_6(\check{\mathbb{P}}^2)$ consisting of polar hexagons to f.

Definition 5.1. Given a general quartic form $f \in \mathrm{S}^4 B = \mathrm{H}^0(\mathbb{P}^2, \mathcal{O}(4))$ define the variety of sums of powers as

$$\mathrm{VSP}(6, f) = \overline{\{f_1, \ldots, f_6 \mid f_1^4 + \ldots + f_6^4 = f\}}$$

where the closure is taken in $\mathrm{Hilb}_6(\check{\mathbb{P}}^2)$.

Lemma 5.2 (Mukai, Schreyer)**.** *For general f the variety* $\mathrm{VSP}(6, f)$ *is a Fano threefold of type V_{22}. Given a net of dual quadrics Ψ as in Definition 3.2, there exists a quartic such that* $\mathrm{VSP}(6, f) \simeq X_\Psi$*.*

Remark 5.3. Considering the apolarity action of T^* on T (cfr. section 3) we may view f as the map $f : B^* \to \mathrm{S}^3 B$ taking ∂ to $\partial(f)$. This map is injective for general f so we can define $V_f = \mathrm{S}^3 B / f(B^*)$. Under the hypothesis of Lemma 5.2, there is a natural isomorphism $V_\Psi \simeq \mathrm{S}^3 B / f(B^*)$.

Definition 5.4. Let f be a general plane quartic and let $X = \mathrm{VSP}(6, f)$. Then there is a rank-3 vector bundle U_{VSP} (resp. a rank-5 vector bundle K_{VSP}) on $\mathrm{VSP}(6, f)$, whose fiber over an element $\Lambda = (f_1, \ldots, f_6) \in \mathrm{VSP}(6, f)$ is $\langle f_1^3, \ldots, f_6^3 \rangle / f(B^*)$ (resp. the fiber is $\langle f_1^4, \ldots, f_6^4 \rangle / f$). This bundle embeds X into $\mathbb{G}(\mathbb{C}^3, V_f)$ (resp. into $\mathbb{G}(\mathbb{C}^3, \mathrm{S}^4 B / f)$) (see Remark 5.3). Denote by Q_{VSP}^* the restriction to X of the universal rank-4 quotient bundle on $\mathbb{G}(\mathbb{C}^3, V_f)$.

5.2. **The Hilbert Scheme.** For any $\Lambda \in \mathrm{Hilb}_6(\check{\mathbb{P}}^2)$ we can consider the resolution of the ideal $J_{\Lambda, \check{\mathbb{P}}^2}$ over the ring $T^* = \mathbb{C}[B^*]$. For a general length-6 subscheme Λ the resolution reads

$$0 \leftarrow J_{\Lambda, \check{\mathbb{P}}^2} \leftarrow T^*(-3)^4 \leftarrow T^*(-4)^3 \leftarrow 0$$

The resolution has this shape whenever no conic of $\check{\mathbb{P}}^2$ passes through Λ and no line cuts a length-3 subscheme of Λ. This open set, which we denote by $\mathrm{Hilb}_6(\check{\mathbb{P}}^2)^\circ$, embeds into $\mathbb{G}(\mathbb{C}^4, \mathrm{S}^3 B^*)$ by means of a rank-4 vector bundle Q_{L}^*. The fiber of Q_{L}^* over Λ defined as $\mathrm{Tor}_1^{T^*}(T^*/J_{\Lambda, \check{\mathbb{P}}^2}, \mathbb{C})_3 \simeq \mathbb{C}^4$ i.e. we take the space of cubics vanishing on Λ. We have $\mathrm{H}^0(Q_{\mathsf{L}})^* \simeq \mathrm{S}^3 B^*$.

Moreover we have a rank-3 vector bundle U_{L} on $\mathrm{Hilb}_6(\check{\mathbb{P}}^2)^\circ$ whose fiber over Λ is the 3-space of first–order syzygies of Λ. Equivalently we take $\mathrm{Tor}_2^{T^*}(T^*/J_{\Lambda, \check{\mathbb{P}}^2}, \mathbb{C})_4 \simeq \mathbb{C}^3$. We have $\mathrm{H}^0(U_{\mathsf{L}}^*)^* \simeq \mathrm{S}^{3,1} B^* \simeq \ker(\mathrm{S}^3 B^* \otimes B^* \to \mathrm{S}^4 B^*)$. One computes $\dim(\mathrm{S}^{3,1} B^*) = 15$. The bundle U_{L}^* provides an embedding $\mathrm{Hilb}_6(\check{\mathbb{P}}^2)^\circ \subset \mathbb{G}(\mathbb{C}^3, \mathrm{S}^{3,1} B^*)$. The following Lemma is proved in [Sch01, Theorem 2.3] except for the last statement that follows from [Sch01, Theorem 2.6].

Lemma 5.5 (Schreyer)**.** *Let $f \in \mathrm{S}^4 B$ be a general quartic and, making use of the apolarity pairing (cfr Section 3), define the ideal*

$$J^f = \{s \in T^* \mid s(f) = 0\}$$

Then the ring $(T^*)^f = T^*/J^f$ is Artinian Gorenstein and its minimal graded free resolution over T^* takes the form

$$0 \leftarrow (T^*)^f \leftarrow T^* \leftarrow V_f^{1,3} \otimes T^*(-3) \leftarrow V_f^{2,4} \otimes T^*(-4) \leftarrow T^*(-7) \leftarrow 0 \qquad (22)$$

where $V_f^{i,j} = \mathrm{Tor}_i^{T^*}((T^*)^f, \mathbb{C})_j$. We have $\dim(\mathrm{V}_f^{1,3}) = \dim(\mathrm{V}_f^{2,4}) = 7$ and there is a canonical duality $(V_f^{1,3})^* \simeq V_f^{2,4}$.

The map $\varsigma_f^\top : V_f^{2,4} \to B^* \otimes V_f^{1,3}$ defined by (22) is skew–symmetric and induces $\sigma_f^\top : B \to \wedge^2(V_f^{2,4})^*$ (cfr. Remark 4.4). We have $X_{\sigma_f} \simeq \mathrm{VSP}(6, f)$ (see Definitions 4.2 and 5.1).

Thus, the rank-4 bundle Q_{L} provides an embedding $X \hookrightarrow \mathbb{G}(\mathbb{C}^4, V_f^{1,3})$ while the rank-3 bundle U_{L}^* gives $X \hookrightarrow \mathbb{G}(\mathbb{C}^4, V_f^{2,4})$ where $V_f^{1,3}$ and $V_f^{2,4}$ are dual 7-dimensional complex vector spaces.

Remark 5.6. After restriction to $X = \mathrm{VSP}(6, f)$ there are natural isomorphisms $Q_{\mathrm{VSP}}^* \simeq Q_{\mathsf{L}}^*$ and $U_{\mathrm{VSP}} \simeq U_{\mathsf{L}}$. On $X = \mathrm{VSP}(6, f)$ we have $\mathrm{H}^0(Q_{\mathsf{L}})^* \simeq V_f^{1,3} \simeq \ker(f^\top : \mathrm{S}^3 B^* \twoheadrightarrow B)$, $\mathrm{H}^0(U_{\mathsf{L}}^*)^* \simeq V_f^{2,4} \simeq V_f$.

Proof. The isomorphism $V_f^{1,3} \simeq \ker(f^\top : \mathrm{S}^3 B^* \twoheadrightarrow B)$ is clear, indeed, by the definition of J_f, the cubic forms that generate J^f (i.e. the space $V_f^{1,3}$ by Lemma 5.5) are those annihilated by $f^\top : \mathrm{S}^3 B^* \twoheadrightarrow B$ under the apolarity pairing (i.e. the space $\ker(f^\top)$). Also, we have $\mathrm{H}^0(Q_{\mathsf{L}})^* \simeq V_f^{1,3}$ since by Lemma 5.5 the cubic forms vanishing on a length-6 subscheme Λ with $[\Lambda] \in \mathrm{VSP}(6, f)$ lie in $V_f^{1,3}$.

Now, given $\Lambda = (f_1, \ldots, f_6) \in \mathrm{VSP}(6, f)$, the fiber of $Q_{\mathsf{L},\Lambda}^*$ consists of those elements in $\mathrm{S}^3 B^*$ (and actually in $V_f^{1,3} \subset \mathrm{S}^3 B^*$) that vanish identically on Λ i.e. that annihilate $f_1^3, \ldots, f_6^3$ under the apolarity pairing (cfr. Remark 5.3). Equivalently we take the degree-3 generators of the ideal $J_{\Lambda,\check{\mathbb{P}}^2}$ i.e. $Q_{\mathrm{VSP},\Lambda}^*$. Thus we have $Q_{\mathsf{L},\Lambda}^* \simeq Q_{\mathrm{VSP},\Lambda}^*$.

Then we also have $U_{\mathsf{L}} \simeq U_{\mathrm{VSP}}$ since since they are both isomorphic to $\ker p_{\mathcal{O},Q_{\mathrm{VSP}}}$. By virtue of the duality in Lemma 5.5, we also have $\mathrm{H}^0(U^*)^* \simeq V_f^{2,4} \simeq (V^{1,3})_f^* \simeq V_f$. □

5.3. **The variety of Kronecker modules.** Following Drezet we introduce the following variety of Kronecker modules. Consider the space of 3×4 matrices with entries in B^* and the G.I.T. quotient

$$\mathsf{K} = \mathrm{M}_{3\times 4}(B^*) // \mathsf{SL}(3) \times \mathsf{SL}(4)$$

An element $[\gamma] \in \mathsf{K}$ is represented by $\gamma : V_{s(\gamma)} \to V_{t(\gamma)} \otimes B^*$ where $V_{s(\gamma)} \simeq \mathbb{C}^3$ and $V_{t(\gamma)} \simeq \mathbb{C}^4$ denote the source and target vector spaces of the map γ.

The variety K has been studied in detail in [Dre88]. It is endowed with two natural bundles, Q_{K}^* (resp. U_{K}) of rank 4 (resp. 3), whose fiber over $[\gamma] \in \mathsf{K}$ is $V_{s(\gamma)}$ (resp. $V_{t(\gamma)}$). The bundles U_{K} and Q_{K}^* are related by

$$\mathrm{Hom}_{\mathsf{K}}(U_{\mathsf{K}}, Q_{\mathsf{K}}^*) \simeq B$$

Lemma 5.7 (Drezet). *There is a birational map* $\delta : \mathrm{Hilb}_6(\check{\mathbb{P}}^2) \dashrightarrow \mathsf{K}$ *defined over* $\mathrm{Hilb}_6(\check{\mathbb{P}}^2)^\circ$ *associating to* Λ *the map* $\gamma : \mathrm{Tor}_2^{T^*}(T^*/J_{\Lambda,\check{\mathbb{P}}^2}, \mathbb{C})_4 \to B^* \otimes \mathrm{Tor}_1^{T^*}(T^*/J_{\Lambda,\check{\mathbb{P}}^2}, \mathbb{C})_3$. *Denote by* K° *the open subset of* K *isomorphic via* δ *to* $\mathrm{Hilb}_6(\check{\mathbb{P}}^2)^\circ$.

Notice that under the isomorphism $\delta_{|\,\mathrm{Hilb}_6(\check{\mathbb{P}}^2)^\circ}$, the bundle U_{K} is pulled back to U_{L} and Q_{K}^* is pulled back to Q_{L}^*.

Lemma 5.8. *Define* P_{K} *as* $P_{\mathsf{K}} = \mathrm{coker}(i_{U_{\mathsf{K}},Q_{\mathsf{K}}^*} : U_{\mathsf{K}} \to B^* \otimes Q_{\mathsf{K}}^*)$. *Then* P_{K} *is locally free of rank 9 over* K°. *The fiber of* P_{K} *is identified with* $\mathrm{H}^0(J_{\Lambda,\check{\mathbb{P}}^2}(4))/f$.

The bundle P_{K}^* *is globally generated with* $\mathrm{H}^0(P_{\mathsf{K}}^*) \simeq \mathrm{S}^4 B$. *The zero locus in* $\mathbb{G}(3 \times 4, B^*)$ *of its general section* f *is a Fano threefold of type* V_{22} *of the form* $\mathrm{VSP}(6, f)$ *defined in 5.1.*

Proof. The bundle P_{K}^* is globally generated since Q_{K} and U_{K}^* are. Recall that $\mathrm{H}^0(Q_{\mathsf{L}})^* \simeq \mathrm{S}^3 B^*$ and $\mathrm{H}^0(U_{\mathsf{L}}^*)^* \simeq \mathrm{S}^{3,1} B^*$. Computing global sections of P_{K}^* via the map δ defined in 5.7 we get the $\mathrm{H}^0(P_{\mathsf{K}}^*) \simeq \mathrm{S}^4 B$. Now the condition for a point $\Lambda \in \mathrm{Hilb}_6(\check{\mathbb{P}}^2)$ to lie in X is that the generators of its ideal, as elements of $\mathrm{S}^3 B^*$, multiplied by any linear form $\partial \in B^*$, map to zero under the evaluation with $f \in (\mathrm{S}^4 B^*)^* \simeq \mathrm{S}^4 B$.

This means that Λ lies in the zero locus of the section f of the kernel bundle P_{K}^*, since the map induced on global sections by the evaluation $B \otimes Q_{\mathsf{K}} \to U_{\mathsf{K}}^*$ is just the multiplication m in T. So the zero locus of a section f of P_{K}^* is isomorphic to the variety $\mathrm{VSP}(6, f)$. □

Remark 5.9. In the framework of Lemma 5.8, there exists a rank-5 bundle $K_{\mathsf{K}} = \ker(p_{\mathcal{O},P_{\mathsf{K}}^*} : \mathrm{H}^0(P_{\mathsf{K}}^*) \otimes \mathcal{O}_{\mathsf{K}} \to P_{\mathsf{K}}^*)$. Under the identifications of Remark 5.6 we have an isomorphism on X, $K_{\mathrm{VSP}} \simeq K_{\mathsf{K}}$.

Proof. It is easy to show that there exists the following commutative diagram with exact rows and columns (we omit zeroes surrounding all the diagram for brevity).

$$\begin{array}{ccccc}
\mathfrak{sl}(B) & \longrightarrow & B^* \otimes B & \xrightarrow{\chi_B} & \mathbb{C} \\
\uparrow & & \uparrow{\scriptstyle 1_{B^*} \otimes f} & & \uparrow{\scriptstyle f} \\
\mathrm{S}^{3,1} B^* & \longrightarrow & B^* \otimes \mathrm{S}^3 B^* & \xrightarrow{m} & \mathrm{S}^4 B^* \\
\uparrow & & \uparrow & & \uparrow \\
V_f & \xrightarrow[\varsigma^\top]{} & B^* \otimes V_f^* & \longrightarrow & \ker(f)
\end{array} \tag{23}$$

Here f is considered alternatively as a map $\mathrm{S}^3 B^* \to B$ (in the central column) or as an element of $\mathrm{S}^4 B \simeq (\mathrm{S}^4 B^*)^*$ (in the right column) and the bottom row is defined by Remark 4.4. Since K_{VSP}^* is globally generated with $\mathrm{H}^0(K^*_{\mathrm{VSP}})^* \simeq \ker f \subset \mathrm{S}^4 B^*$ by Definition 5.4, the bottom row of (23) proves the following

$$K_{\mathrm{VSP}} \simeq \ker(p_{U_{\mathrm{VSP}},Q^*_{\mathrm{VSP}}} : B \otimes U_{\mathrm{VSP}} \to Q^*_{\mathrm{VSP}}) \tag{24}$$

On the other hand by definition of K_{K} we have

$$K_{\mathsf{K}} \simeq \ker(p_{U_{\mathsf{K}},Q^*_{\mathsf{K}}} : B \otimes U_{\mathsf{K}} \to Q^*_{\mathsf{K}}) \tag{25}$$

Thus we conclude keeping in mind Lemma 5.7. □

6. Bundles on X

Throughout the rest of the paper, X will be a Fano threefold of type V_{22} defined by a *general* net of dual quadrics Ψ as $X = X_\Psi$ according to Definition 3.2. In particular, we fix a 3-dimensional (respectively, 4-dimensional) $\mathbb{C}$-vector space B (respectively, A). We will keep in mind the isomorphisms of Lemmas 4.3, 5.2 and of Remarks 5.3, 5.6 and 5.9.

Then, we denote by U, (resp. Q^* and K) the rank-3 (resp. rank-4 and rank-5) bundles on X defined by any of the constructions of Sections (3), (4) and (5). We will often drop the subscript Ψ e.g. $V^{2,3}_\Psi = V^{2,3}$ and we will write E for the bundle E_{H} restricted to X.

Lemma 6.1. *There are the following natural isomorphisms*

$$\mathrm{Hom}(U,Q^*) \simeq B \qquad \mathrm{Hom}(E,U) \simeq A^* \qquad \mathrm{Hom}(K,U) \simeq B^* \tag{26}$$

Furthermore there are the following natural exact sequences

$$0 \to K \to B \otimes U \to Q^* \to 0 \tag{27}$$

$$0 \to U \to B^* \otimes Q^* \to P \to 0 \tag{28}$$

$$0 \to K \to \mathrm{H}^0(K^*)^* \otimes \mathcal{O} \to P^* \to 0 \tag{29}$$

$$0 \to U \to V \otimes \mathcal{O} \to Q^* \to 0 \tag{30}$$

Finally we have the following Chern classes

$$\begin{array}{lll}
c_1(U) = -1 & c_2(U) = 10 & c_3(U) = -2 \\
c_1(Q^*) = -1 & c_2(Q^*) = 12 & c_3(Q^*) = -4 \\
c_1(K) = -2 & c_2(K) = 40 & c_3(K) = -20 \\
c_1(P) = -2 & c_2(P) = 48 & c_3(P) = -36
\end{array}$$

Proof. It is straightforward to compute the Chern classes of the bundles involved in our statement. Further, the isomorphisms in (26) follow the from Remarks 5.9, 4.4 and Lemma 3.1 by restriction to X, as well as the exact sequences (27). The exact sequence (28) follows from Lemma 5.8. Combining (27) and (28) one obtains the following commutative diagram with exact rows and columns (we omit surrounding zeroes for brevity)

$$\begin{array}{ccccc}
U & \xrightarrow{i_{U,Q^*}} & B^* \otimes Q^* & \xrightarrow{p_{Q^*,P}} & P \\
\downarrow{\scriptstyle i_{U,\mathcal{O}}} & & \downarrow{\scriptstyle 1_{B^*} \otimes i_{Q^*,\mathcal{O}}} & & \downarrow{\scriptstyle i_{P,\mathcal{O}}} \\
V \otimes \mathcal{O} & \xrightarrow{\varsigma^\top \otimes 1_{\mathcal{O}}} & B^* \otimes V^* \otimes \mathcal{O} & \longrightarrow & \mathrm{coker}(\varsigma^\top) \otimes \mathcal{O} \simeq \mathrm{H}^0(K^*) \otimes \mathcal{O} \\
\downarrow{\scriptstyle p_{\mathcal{O},Q}} & & \downarrow{\scriptstyle 1_{B^*} \otimes p_{\mathcal{O},U^*}} & & \downarrow{\scriptstyle p_{\mathcal{O},K^*}} \\
Q & \xrightarrow[i_{Q,U^*}]{} & B^* \otimes U^* & \xrightarrow[p_{U^*,K^*}]{} & K^*
\end{array}$$

where ς is defined in Remark 4.4 and the vertical arrows in the first two columns are the obvious ones. Then the last column yields the exact sequence (29). □

Lemma 6.2. *The bundles U, Q, K are aCM stable sheaves on X.*

Proof. It follows from Definition 4.2 that X is the zero locus in $\mathbb{G}(\mathbb{C}^3, V)$ of a section of the globally generated bundle $U(1)^3$. Taking the Koszul complex associated to this section tensorized by $U(t)$ and using Bott theorem (see [Bot57]) over $\mathbb{G}(\mathbb{C}^3, V)$ one computes the required vanishing for U and Q in order to show that they are aCM modules. Using the exact sequence (27) it is immediate to show that K is also an aCM module.

Since $c_1(U) = c_1(Q^*) = -1$ and $\mathrm{h}^0(U) = \mathrm{h}^0(Q^*) = 0$, and since U^* and Q are globally generated, it follows easily from Lemma 2.4 that U and Q are stable. From the exact sequence (27) one sees that $\mathrm{h}^0(K) = \mathrm{h}^0(\wedge^2 K) = 0$. Finally from $\wedge^p K(1) \simeq \wedge^{5-p} K^*(-1)$ and again using (27) it follows that $\mathrm{h}^0((\wedge^3 K)_n) = \mathrm{h}^0((\wedge^4 K)_n) = 0$. Thus we conclude by Lemma 2.4. □

Lemma 6.3. *The bundle E^* is globally generated with $\mathrm{H}^0(E^*)^* \simeq V^{2,3} \simeq \mathbb{C}^8$. There is a rank-6 bundle L defined by the exact sequence*

$$0 \to E \to V^{2,3} \otimes \mathcal{O} \to L^* \to 0 \tag{31}$$

There exists a rank-10 vector bundle M with $\mathrm{H}^0(M^*)^* \simeq \mathrm{S}^3 A$, whose fiber over $[\Gamma] \in X = X_\Psi$ is $\mathrm{H}^0(J_{\Gamma,\mathbb{P}(A)}(3)) \simeq \mathbb{C}^{10}$ according to Definition 3.2. There are the exact sequences

$$0 \to E \xrightarrow{i_{E,U}} A \otimes U \xrightarrow{p_{U,M}} M \to 0 \tag{32}$$

$$0 \to V^{2,3} \to A \otimes V \xrightarrow{m} \mathrm{S}^3 A \to 0 \tag{33}$$

where (33) is obtained dualizing global sections of the dual of (32) and m is the composition of the obvious maps $A \otimes V \hookrightarrow A \otimes \mathrm{S}^2 A \to \mathrm{S}^3 A$.

Finally, M^* is globally generated and there exists a rank-10 vector bundle N defined by the exact sequence

$$0 \to M \to \mathrm{S}^3 A \otimes \mathcal{O} \to N^* \to 0 \tag{34}$$

Proof. By the discussion in Section 3 and Definition 3.4, the fiber of the bundle E over any point of X embeds in $V_\Psi^{2,3} \simeq \mathbb{C}^8$, so $\mathrm{H}^0(E^*)^* \simeq V_\Psi^{2,3}$ and E^* is globally generated and we have the exact sequence (31).

The map $E \to A \otimes U$ in (32) is obtained globalizing q_Ψ in the resolution (4). Equivalently over any $[\Gamma] \in \mathsf{H}_c$ we take the linear map $A \otimes \mathrm{Tor}_2(R/J_{\Gamma,\mathbb{P}^3}, \mathbb{C})_3 \to \mathrm{Tor}_1(R/J_{\Gamma,\mathbb{P}^3}, \mathbb{C})_2$ given by the 2×3 matrix of linear forms whose 2×2 minors define Γ. Therefore (33) follows at once from (4).

Finally, it is clear that M^* is globally generated and $\mathrm{rk}(N) = \dim(\mathrm{S}^3 A) - 10 = 10$. □

Proposition 6.4. *We have* $\mathrm{Hom}(\wedge^2 U, E) \simeq A^*$ *and we define the following maps (cfr. Lemma 6.1)*

$$e_1 = i_{\wedge^2 U, E} : \wedge^2 U \to A \otimes E \tag{35}$$

$$e_2 = \Psi \circ i_{E,U} : A \otimes E \to B \otimes U \tag{36}$$

$$e_3 = p_{U,Q^*} : B \otimes U \to Q^* \tag{37}$$

Then the following sequence is exact

$$0 \to \wedge^2 U \xrightarrow{e_1} A \otimes E \xrightarrow{e_2} B \otimes U \xrightarrow{e_3} Q^* \to 0 \tag{38}$$

Proof. The dual of all the bundles appearing in the sequence (38) are globally generated, hence the sequence is exact if we prove that transpose of the maps e_1, e_2 and e_3 induce an exact sequence on global sections of the dual bundles. Denote these maps by $\overline{e_j}$ for $j = 1, 2, 3$.

It is clear that $\overline{e_3}^\top = \ell_3$ of Theorem 4.5. Since $i_{E,U}$ maps the syzygy of a twisted cubic Γ with $[\Gamma] \in X$ to the 2×3 matrix of linear forms in the minimal graded free resolution of $J_{\Gamma,\mathbb{P}^3}$ (see Lemma 6.3), the map on global section $A \otimes \mathrm{H}^0(E^*)^* \to \mathrm{S}^2 A \otimes \mathrm{H}^0(U^*)^*$ induced by $i_{E,U}$ agrees with q_Ψ. Therefore we have $\overline{e_2}^\top = \ell_2$.

Now recall that $\mathrm{Hom}(\wedge^2 U, E) \simeq \mathrm{Hom}(E^*, \wedge^2 U^*) \simeq \mathrm{Hom}(E, U) \simeq A^*$, (cfr. Lemma 6.1 and Lemma 3.1), thus we have the map $i_{\wedge^2 U,E} : \wedge^2 U \to A \otimes E$. Since $\sigma = \sigma_\Psi$ by Definition 4.2 and Lemma 4.3 we have $\mathrm{H}^0(\wedge^2 U^*)^* \simeq \ker(\sigma : \wedge^2 V \to B^*) \simeq Y_\Psi$. Thus we have the map $\overline{e_1}^\top : Y_\Psi \to A \otimes V^{2,3}_\Psi$, and we need to prove that is coincides with ℓ_1. Observe that the map $i_{\wedge^2 U,E}$ is defined by restriction from H of the map $i_{\wedge^2 U_{\mathsf{H}},E_{\mathsf{H}}}$. Now by Lemma 3.1 $i_{\wedge^2 U,E}$ is induced by the diagram (1). On the other hand by Theorem 4.5 $i_{\wedge^2 U,E}$ is induced by the the diagram (11). Since the diagram (11) is obtained restricting to X the global sections spaces appearing in the diagram (1), we have $\overline{e_1}^\top = \ell_1$. Thus we conclude by Theorem 4.5. □

Remark 6.5. We can define a map $\overline{\psi} : A \to A^* \otimes B$ associated to the net $\Psi^\top : B^* \to \mathrm{S}^2 A^*$. Indeed we put $\overline{\psi}(a)(b \otimes \partial) = \Psi(\partial(a \otimes b))$ for $a, b \in A$ and $\partial \in B^*$. In turn we have a map

$$\psi : A \otimes U \to A^* \otimes Q^* \tag{39}$$

and the map $\psi^\top : A \otimes Q \to A^* \otimes U^*$ is defined by the formula $\psi^\top(a \otimes q)(b \otimes u) = \Psi(a \otimes b)(u)(q)$ under the identification $B \simeq \mathrm{Hom}(U, Q^*)$.

Lemma 6.6. *Given a general net of dual quadrics $\Psi : A \otimes A \to B$, using $\sigma = \sigma_\Psi$ of Remark 4.1 define a map $\kappa : A \otimes V \to A^* \otimes V^*$ by*

$$\kappa(a \otimes u)(b \otimes v) = \sigma^\top(\Psi(a \otimes b))(u \otimes v) \tag{40}$$

The map κ is induced on global sections by ψ. There is an exact sequence

$$0 \to V^{2,3} \xrightarrow{q_\Psi} A \otimes V \xrightarrow{\kappa} A^* \otimes V^* \xrightarrow{q_\Psi^\top} (V^{2,3})^* \to 0$$

Furthermore, there is a skew–symmmetric duality $\overline{\kappa} : \mathrm{S}^3 A \to \mathrm{S}^3 A^*$ such that the following diagram is commutative

$$\begin{array}{ccc} A \otimes V & \xrightarrow{m} & \mathrm{S}^3 A \\ \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \overline{\kappa}} \\ A^* \otimes V^* & \xleftarrow[m^\top]{} & \mathrm{S}^3 A^* \end{array} \tag{41}$$

Proof. The definition of κ is clear and implies $\kappa(a \otimes u)(b \otimes v) = \kappa(b \otimes u)(a \otimes v) = -\kappa(b \otimes v)(a \otimes u)$. Since $\kappa^\top(a \otimes u)(b \otimes v) = \kappa(b \otimes v)(a \otimes u)$, we have that κ is skew–symmetric.

Taking the minimal graded free resolution (4) of R^Ψ in degree 3 and 4 we get the exact sequences (33) and (12), and we denote, for the sake of this proof, by q^3_ψ (resp. by q^4_ψ) the map q_Ψ in degree 3 (resp. in degree 4).

Now, by the exact sequence (8) we have $\sum \alpha_{i,j}\sigma^\top(\Psi(a_i \otimes b))(u \otimes v_j) = 0$ if $\sum \alpha_{i,j} b \otimes a_i \otimes v_j \in \mathrm{Im}(q^4_\Psi)$, for some coefficients $\alpha_{i,j}$. Then $\sum \alpha_{i,j}\sigma^\top(\Psi(a_i \otimes b))(u \otimes v_j) = 0$ if $\sum \alpha_{i,j} a_i \otimes v_j \in \mathrm{Im}(q^3_\Psi)$. Thus there exists a map $\phi : \mathrm{S}^3 A \to A^* \otimes V^*$ with $\phi \circ m = \kappa$. On the other hand, again by (8), if $\sum \alpha_{i,j}\kappa(a_i \otimes u_j) = 0$ then $\sum \alpha_{i,j}\, a_i \otimes u_j \in \mathrm{Im}(q^3_\Psi)$. It follows that ϕ is injective.

Now, since $(q^3_\Psi)^\top \circ \kappa = 0$, there exists a map $\overline{\kappa}$ such that $m^\top \circ \overline{\kappa} = \phi$. This map is bijective since ϕ is injective and it is skew–symmetric since κ is. □

Lemma 6.7. *We have the following exact sequences.*

$$0 \to \wedge^2 U \to A \otimes E \to K \to 0 \tag{42}$$

$$0 \to M \to A^* \otimes Q^* \to L \to 0 \tag{43}$$

Furthermore, in the notation of Lemmas 6.3 and 6.6, there is a natural isomorphism $\eta : M \simeq N$ which makes the following diagram commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i_{M,\mathcal{O}}} & \mathrm{S}^3 A \otimes \mathcal{O} & \xrightarrow{p_{\mathcal{O},N^*}} & N^* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \overline{\kappa}} & & \downarrow{\scriptstyle \eta^\top} & & \\ 0 & \longrightarrow & N & \xrightarrow{i_{N,\mathcal{O}}} & \mathrm{S}^3 A^* \otimes \mathcal{O} & \xrightarrow{p_{\mathcal{O},M^*}} & M^* & \longrightarrow & 0 \end{array} \tag{44}$$

Proof. The exact sequence (42) follows immediately by (38) and (27).

It is easy to check that the exact sequences (30), (31), (32), (33) and (34) induce the following exact commutative diagram (ommitting surrounding zeroes)

$$\begin{array}{ccccc} E & \xrightarrow{i_{E,U}} & A \otimes U & \xrightarrow{p_{U,M}} & M \\ \downarrow{\scriptstyle i_{E,\mathcal{O}}} & & \downarrow{\scriptstyle 1_A \otimes i_{U,\mathcal{O}}} & & \downarrow{\scriptstyle i_{M,\mathcal{O}}} \\ V^{2,3} \otimes \mathcal{O} & \xrightarrow{q_\Psi} & A \otimes V \otimes \mathcal{O} & \xrightarrow{m} & \mathrm{S}^3 A \otimes \mathcal{O} \\ \downarrow{\scriptstyle p_{\mathcal{O},L^*}} & & \downarrow{\scriptstyle 1_A \otimes p_{\mathcal{O},Q}} & & \downarrow{\scriptstyle p_{\mathcal{O},N^*}} \\ L^* & \longrightarrow & A \otimes Q & \longrightarrow & N^* \end{array} \tag{45}$$

thus the dual of the bottom row provides (43).

Further, since the diagram (41) is commutative, and since the homomorphism κ is induced by the map ψ (cfr. Lemma 6.6), we get the following exact commutative diagram (again we omit surrounding zeroes)

$$\begin{array}{ccccc} M & \xrightarrow{\overline{\kappa} \circ i_{M,Q^*}} & A^* \otimes Q^* & \xrightarrow{p_{Q^*,L}} & L \\ \downarrow{\scriptstyle i_{M,\mathcal{O}}} & & \downarrow{\scriptstyle 1_{A^*} \otimes i_{Q^*,\mathcal{O}}} & & \downarrow{\scriptstyle i_{L,\mathcal{O}}} \\ \mathrm{S}^3 A^* \otimes \mathcal{O} & \xrightarrow{m^\top} & A^* \otimes V^* \otimes \mathcal{O} & \xrightarrow[q^\top_\Psi]{} & (V^{2,3})^* \otimes \mathcal{O} \\ \downarrow{\scriptstyle p_{\mathcal{O},N^*} \circ \overline{\kappa}} & & \downarrow{\scriptstyle 1_{A^*} \otimes p_{\mathcal{O},U^*}} & & \downarrow{\scriptstyle p_{\mathcal{O},E^*}} \\ N^* & \longrightarrow & A^* \otimes U^* & \longrightarrow & E^* \end{array} \tag{46}$$

where the central column is the dual of (30), tensorized with 1_A, and the last column is induced by the first two (and in turn it is the same as the dual of (31)).

It is easy to prove that the two maps in the bottom row of (45) (resp. of (46)) agree with $i_{L^*,Q}$ and p_{Q,N^*} (resp. agree with i_{N^*,U^*} and p_{U^*,E^*}). Since $p_{U^*,E^*} = i^\top_{E,U}$, the bottom row of (46) and the first row of (45) give the isomorphism η. It is clear also that η is induced by $\overline{\kappa}$, so that (44) is commutative. □

Tracing back the above proof, it is straightforward to prove the following corollary.

Corollary 6.8. *There are the following natural isomorphisms*

$$\operatorname{Hom}(E, K) \simeq A \tag{47}$$
$$\operatorname{Hom}(E, Q^*) \simeq \operatorname{coker}(\overline{\psi}) \tag{48}$$
$$\operatorname{Hom}(Q^*, L) \simeq A^* \tag{49}$$
$$\operatorname{Hom}(M, Q^*) \simeq A \tag{50}$$
$$\operatorname{Hom}(U, L) \simeq \operatorname{coker}(\overline{\psi}) \tag{51}$$
$$\operatorname{Hom}(U, M) \simeq A \tag{52}$$

Lemma 6.9. *The bundle E is stable aCM with $c_1(E) = -1$ and $c_2(E) = 7$. The bundle L is also stable and aCM.*

Proof. The invariants of E are clear by Lemma The Cohen–Macaulay condition for E follows (42) and Lemma 6.2. The bundle L is also aCM by the dual of (31) since the map $p_{\mathcal{O},E^*}$ is surjective on global sections for any twist.

Stability of E and L is obvious from Lemma 2.4 and (31) since $c_1(E) = c_1(L) = -1$. □

7. Resolution of the diagonal

Define the collection $(G_3, \dots, G_0) = (E, U, Q^*, \mathcal{O})$.

Lemma 7.1 (Kuznetsov)**.** *The collection $(G_3, \dots, G_0) = (E, U, Q^*, \mathcal{O})$ is strongly exceptional i.e. $\operatorname{Ext}^p(G_j, G_i) = 0$ if $p > 0$ or if $i > j$ and $\operatorname{Hom}(G_i, G_i) \simeq \mathbb{C}$.*

For the original proof we refer to [Kuz96]. However it is easy to reprove Lemma 7.1 using Corollary 6.8 and the exact sequences of Lemmas 6.1, 6.3 and 6.7. The *dual* collection is defined as $(G^3, \dots, G^0) = (E, K, U, \mathcal{O})$.

Theorem 7.2. *The general variety X admits the following resolution of the diagonal*

$$0 \longrightarrow \underset{\mathcal{P}_0}{E \boxtimes E} \xrightarrow{d_0} \underset{\mathcal{P}_1}{U \boxtimes K} \xrightarrow{d_1} \underset{\mathcal{P}_2}{Q^* \boxtimes U} \xrightarrow{d_2} \underset{\mathcal{P}_3}{\mathcal{O}} \longrightarrow \mathcal{O}_\Delta \tag{53}$$

where the arrows are given by the following natural elements

$$\begin{aligned}\mathrm{Hom}(\mathcal{P}_0,\mathcal{P}_1) &\simeq A^*\otimes A \ni 1_A\\ \mathrm{Hom}(\mathcal{P}_1,\mathcal{P}_2) &\simeq B\otimes B^* \ni 1_B\\ \mathrm{Hom}(\mathcal{P}_2,\mathcal{P}_3) &\simeq V^*\otimes V \ni 1_V\end{aligned}$$

Proof. Let us look at the maps in more detail

$$d_2: \quad \begin{array}{ccc} Q^*\boxtimes U & \xrightarrow{d_2} & \mathcal{O} \\ {\scriptstyle i_{Q^*,\mathcal{O}}\boxtimes i_{U,\mathcal{O}}}\downarrow & \nearrow{\scriptstyle \chi_V\otimes 1_{\mathcal{O}\boxtimes\mathcal{O}}} & \\ V\otimes V^*\otimes\mathcal{O}\boxtimes\mathcal{O} & & \end{array}$$

$$d_1: \quad \begin{array}{ccc} U\boxtimes K & \xrightarrow{d_1} & Q^*\boxtimes U \\ {\scriptstyle 1_U\boxtimes i_{K,U}}\downarrow & \nearrow{\scriptstyle p_{U,Q^*}\otimes 1_U} & \\ B\otimes U\boxtimes U & & \end{array}$$

$$d_0: \quad \begin{array}{ccc} E\boxtimes E & \xrightarrow{d_0} & U\boxtimes K \\ {\scriptstyle i_{E,U}\boxtimes i_{E,U}}\downarrow & & \downarrow{\scriptstyle 1_U\boxtimes i_{K,U}} \\ A\otimes A\otimes U\boxtimes U & \xrightarrow[\Psi\otimes 1_{U\boxtimes U}]{} & B\otimes U\boxtimes U \end{array}$$

The map d_2 is the restriction from $\mathsf{G}=\mathbb{G}(\mathbb{C}^3,V)$ of a map $\tilde{d}: U_{\mathsf{G}}\boxtimes Q^*_{\mathsf{G}}\to\mathcal{O}_{\mathsf{G}}$ and it is a classical fact that $\mathrm{coker}(\tilde{d})\simeq\mathcal{O}_{\Delta(\mathsf{G})}$. Thus we have $\mathrm{coker}(d_2)\simeq\mathcal{O}_\Delta$ and the sequence is exact in $\mathcal{P}_3$.

Let us now look at the composition in $\mathcal{P}_2$. It is convenient to prove exactness for the dualized maps which we then write

$$\begin{array}{ccccc} \mathcal{O} & \xrightarrow{d_2^\top} & Q\boxtimes U^* & \xrightarrow{d_1^\top} & U^*\boxtimes K^* \\ {\scriptstyle i_{\mathcal{O},U}\boxtimes i_{\mathcal{O},U}}\downarrow & & {\scriptstyle i_{Q,U^*}\boxtimes 1_U}\downarrow & \nearrow{\scriptstyle 1_U\otimes p_{U^*,K^*}} & \\ V\otimes V\otimes U^*\boxtimes U^* & \xrightarrow{\sigma\otimes 1_{U^*\boxtimes U^*}} & B^*\otimes U^*\boxtimes U^* & & \end{array}$$

This yields

$$\ker(d_1^\top)=Q\boxtimes U^*\cap U^*\boxtimes Q\subset B^*\otimes U^*\boxtimes U^*$$

Then the mixed tensor products can be separated by factoring out the identity over U^*. If τ is the involution interchanging factors in $X\times X$, then we have the symmetry $U^*\boxtimes Q=\tau^*(Q\boxtimes U^*)$. So exactness in $\mathcal{P}_2$ is proved if we prove surjectivity of the map $\overline{p}$ below

(54)
$$\begin{array}{ccc} V\otimes\mathcal{O} & \xrightarrow{\overline{p}} & Q \\ {\scriptstyle 1_V\otimes i_{\mathcal{O},U^*}}\downarrow & & \downarrow{\scriptstyle i_{Q,U^*}} \\ V\otimes V\otimes U^* & \xrightarrow[\sigma\otimes 1_{U^*}]{} & B^*\otimes U^* \end{array}$$

Hence we are done if we prove that the map $\overline{p}$ is the universal quotient (which is clearly surjective) i.e. if we prove that $p_{\mathcal{O},Q^*}$ makes the diagram (54) commutative when replacing $\overline{p}$. And indeed this holds since any morphism $b : U \to Q^*$ comes from a skew–symmetric homomorphism $\overline{b} : V \to V^*$ of the ambient space and we have $\sigma^\top(b)(u \wedge v) = \overline{b}(v)(u)$.

Let us turn to $\mathcal{P}_1$. Looking at the definition we have

$$\ker(d_2) = U \boxtimes K \cap K \boxtimes U \subset B \otimes U \boxtimes U$$

Just as in the case of $\mathcal{P}_2$ we are allowed to separate the mixed tensor products by factoring out the identity over U and so reduce to prove surjectivity below

$$\begin{array}{ccc} A \otimes E & \xrightarrow{\overline{q}} & K \\ \downarrow{\scriptstyle 1_A \otimes i_{E,U}} & & \downarrow{\scriptstyle i_{K,U}} \\ A \otimes A \otimes U & \xrightarrow[1_A \otimes 1_A \otimes \Psi]{} & B \otimes U \\ & & \downarrow{\scriptstyle p_{U,Q}} \\ & & Q^* \end{array} \tag{55}$$

Thus we are done if we prove that the following sequence is exact in $B \otimes U$

$$0 \to \wedge^2 U \to A \otimes E \xrightarrow{\Psi \circ i_{E,U}} B \otimes U \xrightarrow{p_{U,Q^*}} Q^* \to 0$$

But this is proved in Proposition 6.4. $\square$

By the classical argument in [Bei78], we get the following corollary.

Corollary 7.3. *Any coherent sheaf $\mathcal{F}$ on X is functorially isomorphic to the cohomology a complex $\mathcal{C}_{\mathcal{F}}$ whose terms are given by*

$$\mathcal{C}^k_{\mathcal{F}} = \bigoplus_{i-j=k} \mathrm{H}^i(\mathcal{F} \otimes G^j) \otimes G_j$$

Alternatively $\mathcal{F}$ is functorially isomorphic to the cohomology a complex $\mathcal{D}_{\mathcal{F}}$ whose terms are given by

$$\mathcal{D}^k_{\mathcal{F}} = \bigoplus_{i-j=k} \mathrm{H}^i(\mathcal{F} \otimes G_j) \otimes G^j$$

We have the following standard consequence of Theorem 7.2, namely Castelnuovo–Mumford regularity associated to the collection $(G_3, \dots, G_0)$.

Corollary 7.4. *Let $\mathcal{F}$ be a coherent sheaf on X and suppose $\mathrm{H}^p(G_p \otimes \mathcal{F}) = 0$ for $p > 0$. Then $\mathcal{F}$ is globally generated.*

Proof. Again by a standard argument, one looks at the term $\mathcal{D}^0_{\mathcal{F}}$ in the complex $\mathcal{D}_{\mathcal{F}}$. which is isomorphic to $\mathrm{H}^0(\mathcal{F}) \otimes \mathcal{O}$ in the hypothesis. On the other hand, in the complex $\mathcal{D}_{\mathcal{F}}$, any differential with source in $\mathrm{H}^0(\mathcal{F}) \otimes \mathcal{O}$ vanishes. Therefore the evaluation map $p_{\mathcal{O},\mathcal{F}}$ is surjective and the statement is proved. $\square$

8. Further remarks

Remark 8.1. The diagram (55) can be completed to the following exact diagram, where we omit surrounding zeroes for brevity.

$$
\begin{array}{ccccc}
\wedge^2 U & \xrightarrow{i_{\wedge^2 U,E}} & A \otimes E & \xrightarrow{p_{E,K}} & K \\
{\scriptstyle i_{\wedge^2 U,U}}\downarrow & & \downarrow{\scriptstyle 1_A \otimes i_{E,U}} & & \downarrow{\scriptstyle i_{K,U}} \\
(V \oplus \wedge^2 A) \otimes U & \longrightarrow & A \otimes A \otimes U & \xrightarrow{\Psi \otimes 1_U} & B \otimes U \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle 1_A \otimes \psi} & & \downarrow{\scriptstyle p_{U,Q}} \\
\mathfrak{sl}(A) \otimes Q^* & \longrightarrow & A \otimes A^* \otimes Q^* & \xrightarrow{\chi_A \otimes 1_{Q^*}} & Q^* \\
{\scriptstyle p_{Q^*,L}}\downarrow & & \downarrow{\scriptstyle 1_A \otimes p_{Q^*,L}} & & \\
A \otimes L & = & A \otimes L & &
\end{array}
$$

where j is defined by the inclusion $V \oplus \wedge^2 A \hookrightarrow A \otimes A$ followed by ψ, defined by (39) in Remark 6.5.

Proof. Exactness of the horizontal sequences is straightforward. The central column follows from the exact sequences (32) and (43). The right column is (27). The left column is induced by the central and right ones, where the isomorphism $\mathrm{Hom}(\wedge^2 U, U) \simeq \ker(A \otimes A \xrightarrow{\Psi} B) \simeq V \oplus \wedge^2 A$, is clear. Commutativity of all the squares is left to the reader. □

8.1. **Helices.** We refer to [hel90] and [Bon90] for general definitions and properties concerning helices and to [Nog94] for the study of helices on Fano threefolds.

Consider the collection $(G_3, \ldots, G_0)$ of Section (7) (strongly exceptional by Lemma 7.1) and extend it defining $G_{j+4k} = G_j \otimes \mathcal{O}(-1)$ for any $j = 0, 1, 2, 3$ and any k. We will show in the following remark that $G_{j+1} \simeq \mathrm{L}_{G_j} \mathrm{L}_{G_{j-1}} \mathrm{L}_{G_{j-2}} G_{j-3}$, for any j, according to Definition 2.1. All of the sequences from (56) to (59) below are obtained resolving $G_i \otimes \mathcal{O}(1)$ with respect to the basis $(G_3, \ldots, G_0)$ according to Corollary 7.3.

Remark 8.2. There are the following exact sequences.

$$\mathcal{O}(-1) \xrightarrow{i_{\mathcal{O}(-1),E}} V^{2,3} \otimes E \xrightarrow{h_2^0} \ker(\varsigma) \otimes U \xrightarrow{h_1^0} V \otimes Q^* \xrightarrow{p_{Q^*,\mathcal{O}}} \mathcal{O} \tag{56}$$

$$Q^*(-1) \xrightarrow{i_{Q^*,\mathcal{O}}} V^* \otimes \mathcal{O}(-1) \xrightarrow{h_2^1} A \otimes E \xrightarrow{h_1^1} B \otimes U \xrightarrow{p_{U,Q^*}} Q^* \tag{57}$$

$$U(-1) \xrightarrow{i_{U,Q^*}} B^* \otimes Q^*(-1) \xrightarrow{h_2^2} \ker(\varsigma)^* \otimes \mathcal{O}(-1) \xrightarrow{h_1^2} A^* \otimes E \xrightarrow{p_{E,U}} U \tag{58}$$

$$E \xrightarrow{i_{E,U}} A \otimes U \xrightarrow{h_2^3} A^* \otimes Q^* \xrightarrow{h_1^3} (V^{2,3})^* \otimes \mathcal{O} \xrightarrow{p_{\mathcal{O},E^*}} E^* \tag{59}$$

where the first map in each sequence is injective and the last one is surjective.

In (56) ς is defined in Remark 4.4, h_1^0 is the defined by the composition $\ker(\varsigma) \otimes U \hookrightarrow V \otimes B \otimes U$ followed by $1_V \otimes p_{U,Q^*}$ and h_2^0 is given by

$$V^{2,3} \otimes E \xrightarrow{q_\Psi \otimes i_{E,U}} V \otimes A \otimes A \otimes U \xrightarrow{1_V \otimes \Psi \otimes 1_U} V \otimes B \otimes U$$

In (57), h^1_1 is the composition

$$A \otimes E \xrightarrow{1_A \otimes \imath_{E,U}} A \otimes A \otimes U \xrightarrow{\Psi \otimes 1_U} B \otimes U$$

while h^1_2 is identified with the projection $V^* \otimes \mathcal{O}(-1) \to \wedge^2 U \simeq U^*(-1)$.

In (58) we have $\ker(p_{E,U}) \simeq K^*(-1)$ by (42). This gives back the isomorphisms $\ker(\varsigma)^* \simeq \mathrm{coker}(\varsigma^\top) \simeq \mathrm{H}^0(K^*)$, established in the proof of Lemma 6.1. and the map h^2_1. Then h^2_2 is given by (28).

In (58) we have to glue together the exact sequences (31), (32) and (43).

8.2. **Quasi–homogeneous case.** In this section we restrict our attention to U_{22}, the Mukai–Umemura 3-fold i.e. the $\mathsf{SL}(2)$-quasi homogeneous case. This is throughly studied in [MU83], [AF93], [Muk92]. Let us denote Y_1 the standard representation space of $\mathsf{SL}(2)$ and Y_n the weight$-n$ representation, so that and $Y_n = \mathrm{S}^n Y_1$.

In terms of plane quartics U_{22} corresponds to a double conic. The action of $\mathsf{SO}(3)$ preserves this conic so we may view B as Y_2 and the stabilizer in $\mathsf{SO}(3)$ of a polar hexagon is the order 60 icosahedral group, isomorphic to A_5.

In terms of the net Ψ of dual quadrics, U_{22} corresponds to a net containing a twisted cubic in the dual space, on which $\mathsf{SL}(2)$ naturally acts. In this case there are isomorphisms of $\mathsf{SL}(2)$-modules

$$B \simeq Y_2 \qquad A \simeq Y_3 \tag{60}$$

The net of dual quadrics Ψ is itself equivariant. Therefore by the isomorphism $\mathrm{S}^2 A \simeq Y_6 \oplus Y_2$ we deduce $V \simeq Y_6$. Further, the resolution of R^Ψ takes the form (3), so one computes $V^{2,3} \simeq Y_7$. The instanton $\mathcal{E}_\Psi$ of Section (3) is endowed with an $\mathsf{SL}(2)$-action in this case and $\mathrm{H}^1(\Omega_{\mathbb{P}(A)} \otimes \mathcal{E}_\Psi)$ is isomorphic to $Y_1 \oplus Y_5$.

The threefold U_{22} also corresponds to the (smooth) closure of the $\mathsf{SL}(2)$-orbit of the polynomial $x^{11}\, y + 11\, x^6\, y^6 - x\, y^{11}$ in Y_{12}. This appeared first in [MU83]. The roots of this polynomial can be drawn in the Riemann sphere when to form the vertices of a regular icosahedron. For a quick sketch of how this relates to the other Fano threefolds with $b_3 = 0$ see also [Fae03].

Proposition 8.3. *The variety U_{22} admits the resolution of the diagonal (53), where all maps are $\mathsf{SL}(2)$-equivariant.*

Proof. The maps we have defined over the product $X \times X$ in Theorem 7.2 are equivariant under the $\mathsf{SL}(2)$-action. Since d_i represents the identity in $\mathrm{Hom}_{X\times X}(\mathcal{P}_i, \mathcal{P}_{i+1}) \simeq Y_{w(i)} \otimes Y_{w(i)}$, where $w(0) = 3$, $w(1) = 2$, $w(2) = 6$, it lies in the unique 1-dimensional $\mathsf{SL}(2)$-invariant subspace of $\mathrm{Hom}_{X\times X}(\mathcal{P}_i, \mathcal{P}_{i+1})$.

Computing the weights of $\mathrm{Hom}_{X\times X}(\mathcal{P}_i, \mathcal{P}_{i+2})$, one sees that there are no $\mathsf{SL}(2)$-invariant subspaces. Thus the composition $d_i \circ d_{i-1}$ is zero for all i.

The sequence (8) in this case can be read in terms of $\mathsf{SL}(2)$-modules and it boils down to

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Y_{10} & \longrightarrow & Y_{10} & & & & \\
 & & & & \oplus & & & & \\
 & & \oplus & & Y_8 & \longrightarrow & Y_8 & & \\
 & & & & \oplus & & \oplus & & \\
 & & Y_6 & \longrightarrow & Y_6 & & Y_6 & \longrightarrow & Y_6 \longrightarrow 0 \\
 & & & & \oplus & & \oplus & & \\
 & & & & Y_4 & \longrightarrow & Y_4 & &
\end{array}
$$

This sequence is clearly exact. However the proof of exactness in (55) is forced since the induced map $A \otimes E \to K$ is $\mathsf{SL}(2)$-invariant, hence it coincides (up to a scalar) with the projection from $A \otimes E$ onto the cokernel of $\wedge^2 U \to A \otimes E$ and as such it it surjective. □

Since all the maps defined in Theorem 7.2 are functorial, they lift to the moduli space of V_{22} threefolds. So, once we prove that the sequence of morphisms (53) is a complex, by semicontinuity we can deduce general exactness from exactness over a point of the moduli space. By the above proposition this point can be taken to be $[U_{22}]$.

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E-mail address: faenzi@math.unifi.it

Dipartimento di Matematica "U. Dini", Università di Firenze, Viale Morgagni 67/a, I-50134, Florence, Italy

URL: http://www.math.unifi.it/~faenzi/