Unsaturated incompressible flows in adsorbing porous media

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Abstract

We study a free boundary problem modelling the penetration of a liquid through a porous material in the presence of absorbing granules. The geometry is one-dimensional. The early stage of penetration is considered, when the flow is unsaturated. Since the hydraulic conductivity depends both on saturation and on porosity and the latter change due to the absorption, the main coefficient in the flow equation depends on the free boundary and on the history of the process. Some results have been obtained in [5] for a simplified version of the model. Here existence and uniqueness are proved in a class of weighted Hölder spaces in a more general situation. A basic tool are the estimates on a non-standard linear boundary value problem for the heat equation in an initially degenerate domain [10].

1 Introduction

The penetration of a wetting front in a porous medium is a classical free boundary problem. The oldest and best known example is the Green-Ampt model for water flow in soils [3]. There is an enormous variety of situations (chemically reacting media, deformable media, capillarity effects, mass exchange, flows of mixtures, media with complex structure, pollution, remediation, ground freezing, composite material manufacturing, coffee brewing, etc.).

Consequently the mathematical literature on that subject is immense and involves many different techniques. We quote the recent Lecture Notes [2] for a review of some problems of industrial relevance and a general introduction on the subject.

The problem discussed in this paper concerns the first stage of penetration of a liquid through a porous material in which hydrophile granules absorb and immobilize the liquid. The physical device we have in mind is a diaper. The effect of the absorbing granules on the flow is quite relevant, since the granules can swell up to about sixty times their original volume. During the first stage of penetration the medium is unsatured. Due to the presence of capillarity, saturation (i.e. the pore volume fraction occupied by the liquid) is a function of pressure. Even in one-dimensional setting the problem is complicated because of many reasons:

(i) The presence of a free boundary (the wetting front)

(ii) porosity is altered by the volume increase of the granules

(iii) the absorption process at a given point starts when such a point is reached by the wetting front.

Since the flow is governed by Darcy's law and the hydraulic conductivity of the system depends on porosity, the problem is going to be formulated as a free boundary problem for a parabolic equation in which the principal term depends on the free boundary and on the local history of the process.

The model was first formulated in [4], and some first results for a simplified version were given in [5].

A more detailed description involving various aspects (geometry, boundary conditions, etc.) can be found in [6]. An existence and uniqueness theorem for the one-dimensional case with the presence of an initially penetrated layer has been proved in [11]. The problem of the subsequent appearance of a saturation front has been studied in [1], proving existence and uniqueness of a classical solution.

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The one-dimensional case in which capillarity is neglected and the flow is saturated has been described in [6]

Here we show existence and uniqueness of a classical solution for the case in which a saturation zone has not appeared yet and starting (as in [5]) from an initial situation in which the medium is completely dry.

A basic tool is the use of estimates of time-weighted Hölder norms for the solutions of a non-standard linear problem for the heat equation, which has been derived in [10] and which provide more information on the behaviour of the solution for small times. A preliminary sketch of our results was presented in [9].

We remark that the physical structure we are referring to is such that porosity can be defined as a macroscopic quantity, depending on the size locally attained by the granules (i.e. there are sufficiently many granules in a representative element of the porous medium to take an average). A completely different approach has been followed in [7], where granules are considered large on the length scale of the pores (but still small on the macro-scale) with the following consequences: the domain occupied by the porous material is variable in time and is the complement of the set of the granules whose centers form a regular cubic lattice. The porosity at each point of the flow domain must be defined in terms of the size of the neighboring granules, the absorption rate is not uniform on the surface of a granule and its swelling rate is the result of an average. Such a problem is clearly three dimensional and existence and uniqueness of a weak solution has been proved.

On the contrary, in the present approach the sum of porosity and of the volume fraction of the granules is constant.

2 Statement of the Problem

The quantities defining the structure of the porous medium are the volume fraction occupied by the granules, $V(x,t)$, and the porous volume fraction (porosity) $\varepsilon(x,t)$. In this paper we assume that granules are dispersed in the pores and that granules and pores are of comparable size. The volume increase of the granules produces a corresponding reduction of porosity

$$\varepsilon(x,t) + V(x,t) = \varepsilon_0 + V_0, \tag{2.1}$$

where ε_0, V_0 are the initial values (i.e. those of the dry medium), taken constant for simplicity.

We suppose that absorption takes place in the wet medium when saturation exceeds a positive threshold S_0, until V has reached a maximum value $V_{\max} \in (V_0, \varepsilon_0 + V_0)$, so that fully swollen granules leave some porosity available for the flow.

Absorption occurs according to the law

$$\frac{\partial V}{\partial t} = f(V_{max} - V)(S - S_0)_+ \tag{2.2}$$

where $f(\xi)$ is a smooth function such that $f(0) = 0$, $f'(\xi) \geq 0$, $f(\xi) > 0$ for $\xi > 0$.

The quantities related to the flow are pressure, $p(x,t)$, and saturation $S(x,t) \in [0,1]$. We suppose that the medium is initially dry ($S = 0$) and the penetration occurs when $S > S_0$ (thus S_0 represent the moisture content that adheres to the porous skeleton, for simplicity we take the penetration threshold equal to the absorption threshold). We are interested in the unsaturated regime, i.e. $S < 1$, in which by capillarity saturation is related to pressure: $S = S(p)$, with $S(p)$ smooth, positive, increasing function, up to $p = p_S$ (saturation pressure), beyond which $S = 1$. Hence we choose the pressure scale such that $S_0 = S(0)$ and for simplicity we take $S(p)$ linear in $(0, p_S)$, namely

$$S = S_0 + \frac{1 - S_0}{p_S} p \equiv S_0 + c_0 p \quad , \quad 0 < p < p_S, \tag{2.3}$$

We remark that (2.3) is not a critical choice, since $S(p)$ can be any C^2 function with $S'(p)$ strictly positive.

The flow is governed by Darcy's law for the volumetric velocity q

$$q = -k(S, \varepsilon) \frac{\partial p}{\partial x} \tag{2.4}$$

(gravity is neglected), where $k(S,\varepsilon)$ is the hydraulic conductivity, supposed to be strictly positive and three times continuously differentiable.

Mass balance is expressed by

$$\frac{\partial S\varepsilon}{\partial t} + \frac{\partial V}{\partial t} = -\frac{\partial q}{\partial x}, \tag{2.5}$$

which together with (2.4) leads to a parabolic p.d.e. for pressure

$$\varepsilon c_0 \frac{\partial p}{\partial t} - (1-S)\frac{\partial \varepsilon}{\partial t} - \frac{\partial}{\partial x}\left[k(S,\varepsilon)\frac{\partial p}{\partial x}\right] = 0, \qquad 0 < x < s(t), \tag{2.6}$$

with S expressed by (2.3), and ε still to be determined.

We can obtain an expression for ε by formal integration of (2.2) and using (2.1). Introducing the monotone function

$$\Phi(V) = \int_{V_0}^{V} \frac{dy}{f(V_{max} - y)} \tag{2.7}$$

and its inverse Ψ, from (2.2) we obtain

$$V(x,t) = \Psi\left(\int_{\theta(x)}^{t} (S(x,\tau) - S_0)d\tau\right) = \Psi\left(\int_{\theta(x)}^{t} c_0 p(x,\tau)d\tau\right), \qquad t \geq \theta(x), \tag{2.8}$$

where $\theta(x)$ is the time at which the wetting front reaches the location x. Finally

$$\varepsilon(x,t) = \varepsilon_0 + V_0 - \Psi\left(\int_{\theta(x)}^{t} c_0 p(x,\tau)d\tau\right). \tag{2.9}$$

Thus in the governing p.d.e. (2.6), the main coefficient $k(S,\varepsilon)$ contains a causal functional of p and keeps track of the wetting front motion through the function $\theta(x)$.

We complete the statement of the problem with the boundary conditions. At the inflow surface the natural condition is to prescribe the volumetric velocity $q_0(t)$

$$-k(S,\varepsilon)\frac{\partial p}{\partial x}\Big|_{x=0} = q_0(t) \tag{2.10}$$

(note that S, ε are not known). On the wetting front $x = s(t)$, $s = \theta^{-1}$, S coincides with the threshold S_0, implying

$$p(s(t),t)) = 0, \tag{2.11}$$

while the penetration speed $\dot{s}$ is nothing but the molecular velocity, i.e.

$$\dot{s}(t) = -\mu \frac{\partial p}{\partial x}\Big|_{x=s(t)} \tag{2.12}$$

with $\mu = \dfrac{k_0}{S_0\varepsilon_0}$, $k_0 = k(S_0,\varepsilon_0)$.

The most realistic physical assumption on $s(0)$ is that, as we said, the medium is not penetrated for $t = 0$ so that

$$s(0) = 0, \tag{2.13}$$

which is source of considerable mathematical difficulties.

We take the compatibility condition

$$p(0,0) = 0, \tag{2.14}$$

and we look for classical solutions with enough regularity, so that the differential equation (2.6) is satisfied in the closure of the domain. In such a case we have some additional information: differentiating (2.11) we find $\dot{s}\dfrac{\partial p}{\partial x} + \dfrac{\partial p}{\partial t} = 0$ on $x = s(t)$ and in particular

$$\frac{\partial p}{\partial t} = -\dot{s}(0)p_0' = \mu p_0'^2, \quad \text{for } x = 0,\ t = 0, \tag{2.15}$$

with $p_0' = \dfrac{\partial p}{\partial x}$ at the origin, namely

$$p_0' = -\frac{1}{k_0} q_0(0). \tag{2.16}$$

From (2.15) and (2.6) we derive

$$\frac{\partial^2 p}{\partial x^2}(0,0) = \frac{c_0}{k_0}\Big(\varepsilon_0 \mu - \frac{\partial k}{\partial S}(S_0, \varepsilon_0)\Big) p_0'^2. \tag{2.17}$$

3 Statement of the Result

We first transform the problem into one with a fixed domain. We define l_0 the initial slope of the boundary, $l_0 = \dfrac{q_0(0)}{S_0 \varepsilon_0}$, and we take two positive numbers l_1, l_2 such that $0 < l_1 < l_2 < l_0$. We introduce a smooth monotone function $\eta(\xi)$ such that $\eta(\xi) = 0$ for $\xi < l_1$, $\eta(\xi) = 1$ for $\xi > l_2$.

The variable transformation

$$x = z + \eta(\frac{z}{t})\sigma(t) \tag{3.1}$$

$$\sigma(t) = s(t) - s_0(t), \quad s_0(t) = l_0 t \tag{3.2}$$

maps the interval $0 < x < s(t)$ into the interval $0 < z < s_0(t)$ for each $t > 0$.

Setting

$$\psi(z,t) = \eta(\frac{z}{t})\sigma(t), \quad Y(z,t) = z + \psi(z,t) \tag{3.3}$$

and calculating $\dfrac{\partial Y}{\partial z} = 1 + \eta'(\dfrac{z}{t})\dfrac{\sigma(t)}{t}$, we note that $\dfrac{\sigma(t)}{t} \to 0$ as $t \to 0$ by construction (provided $\dot{s}(t)$ is continuous for $t = 0$, as we are supposing). Therefore $\dfrac{\partial Y}{\partial z} > 0$ for t in a sufficiently small interval. Hence (3.1) is a 1-1 mapping in that interval.

We remark that we are interested in establishing existence locally in time, since for already penetrated media the results of [11] apply.

We denote by $\hat{p}, \hat{S}, \hat{\varepsilon}, \hat{k}$ the transformed functions ($\hat{p}(z,t) = p(Y(z,t),t)$, etc.). Equations (2.6), (2.10), (2.11), (2.12) take the form

$$c_0 \hat{\varepsilon}\left(\frac{\partial \hat{p}}{\partial t} - \frac{\partial \hat{p}}{\partial z}\frac{\frac{\partial \psi}{\partial t}}{1 + \frac{\partial \psi}{\partial z}}\right) - (1 - \hat{S})(\widehat{\frac{\partial \varepsilon}{\partial t}}) - \frac{1}{1 + \frac{\partial \psi}{\partial z}}\frac{\partial}{\partial z}\left(\frac{\hat{k}}{1 + \frac{\partial \psi}{\partial z}}\frac{\partial \hat{p}}{\partial z}\right) = 0, \tag{3.4}$$

$$-\hat{k}\frac{\partial \hat{p}}{\partial z}\bigg|_{z=0} = q_0(t) \tag{3.5}$$

$$\hat{p}(s_0(t), t) = 0 \tag{3.6}$$

$$\frac{d\sigma}{dt} = -\mu\left(\frac{\partial \hat{p}}{\partial z} - p_0'\right)\bigg|_{z=s_0(t)} \tag{3.7}$$

where from (2.9) we easily see that $\widehat{\varepsilon_t} = -\Psi'(I(z,t))c_0\widehat{p}(z,t)$.

Next, we note that if we introduce

$$r(z,t) = \hat{p}(z,t) - p_0'\psi(z,t) \tag{3.8}$$

and

$$\tilde{r}(t) = r(s_0(t), t) = -p_0'\sigma(t), \tag{3.9}$$

owing to (3.6), we may rewrite (3.4), (3.5), (3.7) as

$$c_0\hat{\varepsilon}\frac{\partial r}{\partial t} - \frac{\hat{k}}{(1 + \frac{\partial \psi}{\partial z})^2}\frac{\partial^2 r}{\partial z^2} - \left(\frac{\partial \hat{p}}{\partial z}\frac{1}{1 + \frac{\partial \psi}{\partial z}} - p_0'\right)\left(c_0\varepsilon\frac{\partial \psi}{\partial t} - \frac{\hat{k}\frac{\partial^2 \psi}{\partial z^2}}{(1 + \frac{\partial \psi}{\partial z})^2}\right) - \frac{1}{(1 + \dfrac{\partial \psi}{\partial z})^2}\frac{\partial \hat{k}}{\partial z}\frac{\partial \hat{p}}{\partial z} - (1 - \hat{S})\widehat{\varepsilon_t} = 0. \tag{3.10}$$

$$-\hat{k}\frac{\partial r}{\partial z}\Big|_{z=0} = q_0(t), \tag{3.11}$$

$$-\frac{1}{p_0'}\frac{d\tilde{r}}{dt} + \mu\frac{\partial r}{\partial z}\Big|_{z=s_0(t)} = \mu p_0'. \tag{3.12}$$

In writing (3.12), we have used the fact that $\eta'(\xi) = 0$ near $\xi = l_0$.

The next step in our transformation is to introduce the new unknown

$$u(x,t) = r(z,t) - r_0(z,t) \tag{3.13}$$

with $r_0(z,t)$ chosen in such a way that $r_0, \frac{\partial r_0}{\partial z}, \frac{\partial r_0}{\partial t}, \frac{\partial^2 r_0}{\partial z^2}$ take the corresponding values of $p, \frac{\partial p}{\partial z}, \frac{\partial p}{\partial t}, \frac{\partial^2 p}{\partial z^2}$ for $z = 0$, $t = 0$ (see (2.15)-(2.17)).

To be specific we take

$$r_0(z,t) = p_0' z + \mu p_0'^2 t + b_0 z^2 + b_1 z t + b_2 t^2 \tag{3.14}$$

with

$$b_0 = \frac{c_0}{2k_0}(\varepsilon_0\mu - \frac{\partial k}{\partial S}(S_0,\varepsilon_0))p_0'^2. \tag{3.15}$$

$$b_1 = -k_0^{-1}\Big(c_0\frac{\partial k}{\partial S}(S_0,\varepsilon_0)\mu p_0'^3 + q'(0)\Big), \tag{3.16}$$

$$b_2 = p_0'\Big(\frac{3}{2}b_1\mu - 2\mu^2 p_0' b_0\Big), \tag{3.17}$$

(with $q_0(t)$ continuously differentiable).

In such a way (3.10)-(3.12) become

$$c_0\varepsilon_0\frac{\partial u}{\partial t} - k_0\frac{\partial^2 u}{\partial z^2} = F[u] + f_0(z,t), \tag{3.18}$$

$$-k_0\frac{\partial u}{\partial z}\Big|_{z=0} = \Big((\hat{k}-k_0)\frac{\partial u}{\partial z} + \hat{k}\frac{\partial r_0}{\partial z}\Big)\Big|_{z=0} + q_0(t) \tag{3.19}$$

$$-\frac{1}{p_0'}\frac{d\tilde{u}}{dt} + \mu\frac{\partial u}{\partial z}\Big|_{z=s_0(t)} = \frac{1}{p_0'}\frac{d\tilde{r}_0}{dt} - \mu\frac{\partial r_0}{\partial z}\Big|_{z=s_0(t)} + \mu p_0' \tag{3.20}$$

where

$$\tilde{u}(t) = u(s_0(t),t) \quad , \quad \tilde{r}_0(t) = r_0(s_0(t),t) \tag{3.21}$$

and

$$F[u] = c_0(\varepsilon_0 - \hat{\varepsilon})\Big(\frac{\partial u}{\partial t} + \frac{\partial r_0}{\partial t}\Big) - \Big(k_0 - \frac{\hat{k}}{(1+\frac{\partial \psi}{\partial z})^2}\Big)\Big(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 r_0}{\partial z^2}\Big) +$$
$$\Big(\frac{\partial \hat{p}}{\partial z}\frac{1}{1+\frac{\partial\psi}{\partial z}} - p_0'\Big)\Bigg(c_0\hat{\varepsilon}\psi_t - \frac{\hat{k}\psi_{zz}}{\Big(1+\frac{\partial\psi}{\partial z}\Big)^2}\Bigg)$$
$$+ (1-\hat{S})\widehat{\varepsilon_t} + \Bigg(\frac{1}{\Big(1+\frac{\partial\psi}{\partial z}\Big)^2}\frac{\partial\hat{k}}{\partial z}\frac{\partial\hat{p}}{\partial z} - \frac{\partial k}{\partial S}(S_0,\varepsilon_0)c_0(\frac{\partial r_0}{\partial z})^2\Bigg) \equiv \sum_{i=1}^{5} F_i[u], \tag{3.22}$$

$$f_0(z,t) = -\Big(c_0\varepsilon_0\frac{\partial r_0}{\partial t} - k_0\frac{\partial^2 r_0}{\partial z^2} - \frac{\partial k}{\partial S}(S_0,\varepsilon_0)c_0(\frac{\partial r_0}{\partial z})^2\Big), \tag{3.23}$$

$$\hat{p} = u + r_0 + p_0'\psi, \tag{3.24}$$

$$\sigma(t) = -\frac{1}{p_0'}\Big(\tilde{u}(t) + \tilde{r}_0(t)\Big) = -\frac{1}{p_0'}\Big(\tilde{u}(t) + (b_0 l_0^2 + b_1 l_0 + b_2)t^2\Big). \tag{3.25}$$

It follows from the definition of r_0 that $f_0(0,0)=0$.

Taking into account (3.16), (3.17), we rewrite (3.19), (3.20) in the final form

$$-k_0\frac{\partial u}{\partial z}\Big|_{z=0}=g_0(t)+G[u], \tag{3.26}$$

$$-\frac{1}{p_0'}\frac{d\tilde{u}}{dt}+\mu\frac{\partial u}{\partial z}\Big|_{z=s_0(t)}=0, \tag{3.27}$$

with

$$\begin{aligned} g_0(t) &= \Big(k_0\frac{\partial r_0}{\partial z}+k_S'(S_0,\varepsilon_0)c_0r_0\frac{\partial r_0}{\partial z}\Big)\Big|_{z=0}+q(t) \\ &= c_0k_S'(S_0,\varepsilon_0)\Big(\mu p_0'^2b_1t^2+b_2p_0't^2+b_1b_2t^3\Big)+q(t)-q(0)-q'(0)t, \qquad (3.28) \\ G[u] &= c_0k_S'(S_0,\varepsilon_0)u\frac{\partial r_0}{\partial z}\Big|_{z=0}+\Big(\hat{k}-k_0-k_S'(S_0,\varepsilon_0)(\hat{S}-S_0)\Big)\frac{\partial r_0}{\partial z}\Big|_{z=0}+(\hat{k}-k_0)\frac{\partial u}{\partial z}\Big|_{z=0} \qquad (3.29) \end{aligned}$$

We also note that for $\tilde{r}_0(t)$ in (3.21) we have the expressions

$$\tilde{r}_0(t)=(b_0l_0^2+b_1l_0+b_2)t^2=(\mu^2p_0'^2b_0-\mu p_0'b_1+b_2)t^2, \tag{3.30}$$

$$\frac{1}{p_0'}\frac{d\tilde{r}_0}{dt}-\mu\frac{\partial r_0}{\partial z}\Big|_{z=s_0(t)}+\mu p_0'=0. \tag{3.31}$$

Thus, the final formulation of the problem consists of equations (3.18), (3.26), (3.27) and (3.25).

At this point we introduce the functional spaces to be used in the rest of the paper.

By J_t we denote the interval $(0,s(t))$.

- $C^l(J_t)$, $l\in([l],[l]+1)$, $l>0$, is the standard Hölder space with norm

$$\|u\|_{C^l(J_t)}=\sum_{j=0}^{[l]}\sup_{x\in J_t}\Big|\frac{d^ju(x)}{dx^j}\Big|+[u]_{J_t}^{(l)}, \tag{3.32}$$

- $C^l(0,T)$, similar space with norm

$$\|u\|_{C^l(0,T)}=\sum_{j=0}^{[l]}\sup_{t\in(0,T)}\Big|\frac{d^ju(t)}{dt^j}\Big|+[u]_{(0,T)}^{(l)}, \tag{3.33}$$

We remind that in (3.32), (3.33)

$$[u]_{J_t}^{(l)}=\sup_{x_1,x_2\in J_t}|x_1-x_2|^{[l]-l}\Big|\frac{d^{[l]}u(x_1)}{dx}-\frac{d^{[l]}u(x_2)}{dx}\Big| \tag{3.34}$$

and

$$[u]_{(0,T)}^{(l)}=\sup_{t_1,t_2\in(0,T)}|t_1-t_2|^{[l]-l}\Big|\frac{d^{[l]}u(t_1)}{dt}-\frac{d^{[l]}u(t_2)}{dt}\Big| \tag{3.35}$$

- $C^{l,l/2}(\Omega_{0T})$, $\Omega_{0T}=\{x\in J_t,\ t\in(0,T)\}$, with norm

$$\|u\|_{C^{l,l/2}(\Omega_{0T})}=\sum_{j+2k<l}\sup_{\Omega_{0T}}\Big|\frac{\partial^{j+k}u(x,t)}{\partial x^j\partial t^k}\Big|+[u]_{\Omega_{0T}}^{(l,l/2)} \tag{3.36}$$

where

$$[u]_{\Omega_{0T}}^{(l,l/2)}=\sup_{\tau<t}[u]_{J_t}^{(l)}+\sup_{\Omega_{0T}}\sup_{h\in(0,T-t)}h^{-l/2+[l]/2}\Big|\frac{\partial^{[l/2]}u(x,t+h)}{\partial t^{[l/2]}}-\frac{\partial^{[l/2]}u(x,t)}{\partial t^{[l/2]}}\Big| \tag{3.37}$$

(this definition requires $\dot{s}(t)>0$, that we are assuming).

Finally we introduce the weighted Hölder spaces $\hat{C}^{l,l/2}(\Omega_{0T})$, $\hat{C}^{l}(0,T)$, $\hat{C}^{l,l/2}_{\beta}(\Omega_{0T})$, $\hat{C}^{l}_{\beta}(0,T)$

- $\hat{C}^{l,l/2}(\Omega_{0T})$ has the norm

$$\|u\|_{\hat{C}^{l,l/2}(\Omega_{0T})} = \sup_{\tau\in(0,T)} [u]^{(l,l/2)}_{\Omega_{\tau/2,\tau}} + \sup_{\Omega_{0T}} t^{-l}|u(x,t)|. \tag{3.38}$$

- $\hat{C}^{l}(0,T)$ has the norm

$$\|u\|_{\hat{C}^{l}(0,T)} = [u]^{(l)}_{(0,T)} + \sup_{t\in(0,T)} t^{-l}|u(t)|. \tag{3.39}$$

- $\hat{C}^{l,l/2}_{\beta}(\Omega_{0T})$, $\beta \geq 0$, has the norm

$$\|u\|_{\hat{C}^{l,l/2}_{\beta}(\Omega_{0T})} = \sup_{\tau\in(0,T)} \tau^{\beta}[u]^{(l,l/2)}_{\Omega_{\tau/2,\tau}} + \sup_{\Omega_{0T}} t^{-l+\beta}|u(x,t)|, \tag{3.40}$$

- $\hat{C}^{l}_{\beta}(0,T)$ has the norm

$$\|u\|_{\hat{C}^{l}_{\beta}(0,T)} = \sup_{\tau\in(0,T)} \tau^{\beta}[u]^{(l)}_{(\tau/2,\tau)} + \sup_{t\in(0,T)} t^{-2l+\beta}|u(t)|. \tag{3.41}$$

Clearly, for $\beta = 0$ the spaces $\hat{C}_{\beta}$ coincide with the corresponding spaces $\hat{C}$.

At this point, we may state the results we are going to prove in the rest of the paper.

Theorem 1 *Let $k(S,\varepsilon)$ be a C^3 function in $[S_0,1]\times[\varepsilon_{\min},\varepsilon_0]$ and strictly positive ($k \geq \overline{k} > 0$). Suppose that $q_0 \in C^{3/2+\alpha_1/2}(0,T)$ for some $\alpha_1 \in (0,1)$ and strictly positive ($q_0 \geq \overline{q} > 0$). Then problem (3.18), (3.25), (3.26), (3.27) has one unique solution $u \in \hat{C}^{2+\alpha,1+\alpha/2}(\Omega_{0,T})$ such that $\dfrac{\partial u}{\partial t} \in \hat{C}^{\alpha,\alpha/2}(\Omega_{0T})$, $\dfrac{\partial u}{\partial z} \in \hat{C}^{1+\alpha,(1+\alpha)/2}(\Omega_{0T})$, $\dfrac{\partial^2 u}{\partial z^2} \in \hat{C}^{\alpha,\alpha/2}(\Omega_{0T})$, $\dfrac{\partial \tilde{u}}{\partial t} \in \hat{C}^{(1+\alpha)/2}(0,T)$, for any $\alpha \in (0,\alpha_1)$ and T sufficiently small. Moreover, the solution satisfies the inequality*

$$\begin{aligned}&\|u\|_{\hat{C}^{2+\alpha,1+\alpha/2}(\Omega_{0T})} + \|\frac{\partial u}{\partial x}\|_{\hat{C}^{1+\alpha,1/2+\alpha/2}(\Omega_{0T})} + \|\frac{\partial^2 u}{\partial x^2}\|_{\hat{C}^{\alpha,\alpha/2}(\Omega_{0T})} + \|\frac{\partial u}{\partial t}\|_{\hat{C}^{\alpha,\alpha/2}(\Omega_{0T})}\\ &+ \|\frac{\partial \tilde{u}}{\partial t}\|_{\hat{C}^{(1+\alpha)/2}(0,T)} \leq c\Big(\|f_0\|_{\hat{C}^{\alpha,\alpha/2}(\Omega_{0T})} + \|g_0\|_{\hat{C}^{(1+\alpha)/2}(0,T)}\Big).\end{aligned} \tag{3.42}$$

As a consequence we deduce the following properties for the pressure:

$$\hat{p}(z,t) = r_0(z,t) + p_0'\eta\Big(\frac{z}{t}\Big)\sigma(t) + u(z,t), \tag{3.43}$$

(i) the derivatives $\dfrac{\partial \hat{p}}{\partial z}, \dfrac{\partial \hat{p}}{\partial t}$ belong to $C^{\alpha,\alpha/2}(\Omega_{0T})$,

(ii) $\dfrac{\partial^2 \hat{p}}{\partial z^2}$ is bounded (although it has no limit for $t \downarrow 0$)

In terms of the original variables we can say that the functions

$$p(x,t), \quad \frac{\partial p}{\partial x} = \left(1 + \frac{\partial \psi}{\partial z}\right)^{-1} \frac{\partial \hat{p}}{\partial z}\bigg|_{z=Y^{-1}(x,t)},$$

$$\frac{\partial p}{\partial t} = \frac{\partial \hat{p}}{\partial t} + \frac{\partial \psi}{\partial t}\left(1 + \frac{\partial \psi}{\partial z}\right)^{-1} \frac{\partial \hat{p}}{\partial z}\bigg|_{z=Y^{-1}(x,t)}$$

all belong to $C^{\alpha,\alpha/2}(\Omega_{0T})$, and that

$$\frac{\partial^2 p}{\partial x^2} = \left(1+\frac{\partial\psi}{\partial z}\right)^{-2}\left[\frac{\partial^2 \hat{p}}{\partial z^2} - \frac{\partial \hat{p}}{\partial z}\frac{\partial^2\psi}{\partial z^2}\left(1+\frac{\partial\psi}{\partial z}\right)^{-1}\right]\Big|_{z=Y^{-1}(x,t)}$$

also belongs to $C^{\alpha,\alpha/2}(\Omega_{0T})$ and has the limit $c_0 k_0^{-1}\Big[\varepsilon_0\mu - \frac{\partial k}{\partial S}(S_0,\varepsilon_0)\Big]p_0'^2$ (see (2.16)) as $t\downarrow 0$.

The final conclusion about the regularity of the solution in the original variables is that

$$p \in C^{2+\alpha,1+\alpha/2}(\Omega_{0T}) \quad , \quad s \in C^{3/2+\alpha/2}(0,T). \tag{3.44}$$

Such a solution is unique, as we shall see.

Theorem 2 *The original problem* (2.6), (2.10)-(2.14) *has one unique solution satisfying* (3.44).

4 The basic steps in the Proof of Theorem 1

Let us introduce the space $\hat{\mathcal{C}}_\beta^{2+\alpha,1+\alpha/2}(\Omega_{0T})$ of the functions w defined on $\Omega_{0,T}$ and such that

$$w \in \hat{C}_\beta^{2+\alpha,1+\alpha/2}(\Omega_{0T}) \quad , \quad \frac{\partial w}{\partial z} \in \hat{C}_\beta^{1+\alpha,(1+\alpha)/2}(\Omega_{0T}),$$
$$\frac{\partial^2 w}{\partial z^2} \in \hat{C}_\beta^{\alpha,\alpha/2}(\Omega_{0T}) \quad , \quad \frac{\partial w}{\partial t} \in \hat{C}_\beta^{\alpha,\alpha/2}(\Omega_{0T}),$$
$$\frac{d\widetilde{w}}{dt} \in \hat{C}_\beta^{(1+\alpha)/2}(0,T),$$

with the usual meaning of $\widetilde{w}$. The norm in $\hat{\mathcal{C}}_\beta^{2+\alpha,1+\alpha.2}(\Omega_{0T})$ is the sum of the norms of the functions listed above in their respective weighted Hölder spaces. It will be denoted by $\mathcal{N}_{2+\alpha,\beta}[w]$. When $\beta=0$ we will use the symbol $\mathcal{N}_{2+\alpha}[w]$.

The basic idea of the existence proof is to look for a solution as a fixed point of a mapping of a closed set in $\hat{\mathcal{C}}_\beta^{2+\alpha,1+\alpha/2}(\Omega_{0T})$ into itself, defined as follows.

Consider the linear problem

$$c_0\varepsilon_0\frac{\partial w}{\partial t} - k_0\frac{\partial^2 w}{\partial z^2} = f(z,t), \tag{4.1}$$
$$-k_0\frac{\partial w}{\partial z}\Big|_{z=0} = g(t), \tag{4.2}$$
$$-\frac{1}{p_0'}\frac{d\tilde{w}}{dt} + \mu\frac{\partial w}{\partial z}\Big|_{z=s_0(t)} = h(t). \tag{4.3}$$

which, as we shall see, has one and only one solution (up to a constant) for $f\in\hat{C}_\beta^{\alpha,\alpha/2}(\Omega_{0T})$, $g,h\in\hat{C}_\beta^{(1+\alpha)/2}(0,T)$ satisfying the inequality

$$\mathcal{N}_{2+\alpha,\beta}[w] \le c\Big(\|f\|_{\hat{C}_\beta^{\alpha,\alpha/2}(\Omega_{0T})} + \|g\|_{\hat{C}_\beta^{(1+\alpha)/2}(0,T)} + \|h\|_{\hat{C}_\beta^{(1+\alpha)/2}(0,T)}\Big). \tag{4.4}$$

Such a problem defines the linear mapping $w=\mathcal{L}[f,g,h]$ and problem (3.18), (3.25), (3.26), (3.27) can be reformulated in the form

$$u = \mathcal{L}\Big[F[u]+f_0, G[u]+g_0, 0\Big] \equiv \mathcal{M}[u] \tag{4.5}$$

for u in a suitable ball of $\hat{\mathcal{C}}^{2+\alpha,1+\alpha/2}(\Omega_{0T})$.

We will prove that for sufficiently small T the operator $\mathcal{M}$ is contractive in the topology of $\mathcal{N}_{2+\alpha}$.

First we deal briefly with problem (4.1)-(4.3)

5 Some results on the linear problem (4.1)-(4.3)

Theorem 3 *Problem* (4.1)-(4.3) *(with* $w(0,0)=0$*) with data* $f \in \hat{C}_\beta^{\alpha,\alpha/2}(\Omega_{0T})$*,* $g,h \in \hat{C}_\beta^{(1+\alpha)/2}(0,T)$ *with* $\beta \in [0,1+\alpha]$ *has one unique solution in the space* $\hat{\mathcal{C}}_\beta^{2+\alpha,1+\alpha/2}(\Omega_{0,T})$*, where we have the estimate* (4.4)*.*

Proof. First of all we recall that (4.4) is an a-priori estimate. For $\beta=0$ its derivation has been performed in [10]. The extension to $\beta \le 1+\alpha$ needs only slight changes (see [9]).

Since the problem is linear the estimate above implies uniqueness. Now we turn our attention to the question of existence. Suppose for the moment that f is a smooth function and let us differentiate (4.1) formally with respect to z, setting $W=\dfrac{\partial w}{\partial z}$. Since

$$\frac{d\widetilde{w}}{dt}=\left(l_0\frac{\partial w}{\partial z}+\frac{\partial w}{\partial t}\right)\bigg|_{z=l_0t}=l_0\widetilde{W}+\left(\frac{k_0}{c_0\varepsilon_0}\frac{\partial W}{\partial z}+\frac{1}{c_0\varepsilon_0}f\right)_{z=l_0t},$$

we obtain for W the following problem

$$c_0\varepsilon_0\frac{\partial W}{\partial t}-k_0\frac{\partial^2 W}{\partial z^2}=\frac{\partial f}{\partial z}, \tag{5.1}$$

$$-k_0W(0,t)=g(t), \tag{5.2}$$

$$-\frac{1}{p_0'}\left[(l_0-p_0'\mu)W+\frac{k_0}{c_0\varepsilon_0}\frac{\partial W}{\partial z}\right]_{z=l_0t}=h(t)+\frac{1}{p_0'c_0\varepsilon_0}f(l_0t,t). \tag{5.3}$$

Existence for (5.1)-(5.3) can be stabilished using e.g. the techniques of [8]. Hence we can define

$$w(z,t)=-\int_z^{l_0t}W(\xi,t)\,d\xi+\omega(t) \tag{5.4}$$

where the function $\omega(t)$ plays the role of $\widetilde{w}(t)$. Indeed if we choose

$$-\frac{1}{p_0'}\dot{\omega}(t)=h-\mu\widetilde{W} \quad , \quad \omega(0)=0, \tag{5.5}$$

we immediately realize that $w(z,t)$ satisfies (4.1)-(4.3). At this point, taken a sequence $\{f_n\}$ of smooth functions converging to f in $\hat{C}_\beta^{\alpha,\alpha/2}(\Omega_{0T})$, with the help of (4.4) we conclude that the corresponding sequence $\{w_n\}$ converges in $\hat{\mathcal{C}}_\beta^{2+\alpha,1+\alpha/2}(\Omega_{0T})$ to the solution w of (4.1)-(4.3).

□

In order to get the estimates we need on the operator $\mathcal{M}[u]$ in (4.5) we must establish some inequality concerning the quantities $F[u], G[u]$.

6 Estimating $F[u], G[u]$

Definition 1 *We denote by* $\mathcal{U}_{2+\alpha}(\delta,T)$ *the set of the functions* $u(z,t)$ *having the following properties:*

(i) $\mathcal{N}_{2+\alpha}[u] \le \delta$*,*

(ii) $\sigma(t)$ *defined by (3.5) is such that* $\sigma(t)/t^2$ *is bounded*

(iii) T *is small enough (as a function of* δ*) so that* $\dot{s}(t)\ge\gamma_0$ *for some* $\gamma_0>0$ *and* $t\in(0,T)$*.*

We recall that $F[u]=\sum_{i=1}^{5}F_i[u]$ is defined by (3.22) and $G[u]$ is defined by (3.29).

Lemma 1 *For any u in the set $\mathcal{U}_{2+\alpha}(\delta,T)$ there exists a positive constant c depending on δ and on the data, such that*

$$\|F[u]\|_{\hat{C}^{\alpha,\alpha/2}(\Omega_{0T})} \le cT^{1-\alpha}, \tag{6.1}$$

$$\|G[u]\|_{\hat{C}^{1/2+\alpha/2}(0,T)} \le cT^{\gamma} \quad , \quad \gamma = \min(1-\alpha, \frac{1+\alpha}{2}). \tag{6.2}$$

The proof requires the following inequality, where we introduce the simpler notation

$$\{f\}_{\alpha,\Omega_{0T}} = \sup_{\Omega_{0T}} |f(z,t)| + \sup_{\tau<T} \tau^{\alpha}[f]^{(\alpha,\alpha/2)}_{\Omega_{\tau/2,\tau}}$$

Proposition 1 *If $f_1 \in \hat{C}^{\alpha,\alpha/2}(\Omega_{0T})$, $f_2 \in C^{\alpha,\alpha/2}(\Omega_{0T})$, then $f_1 f_2 \in \hat{C}^{\alpha,\alpha/2}(\Omega_{0T})$, and*

$$\|f_1 f_2\|_{\hat{C}^{\alpha,\alpha/2}(\Omega_{0T})} \le c\|f_1\|_{\hat{C}^{\alpha,\alpha/2}(\Omega_{0T})}\{f_2\}_{\alpha,\Omega_{0T}}. \tag{6.3}$$

Proof. Indeed,

$$\|f_1 f_2\|_{\hat{C}^{\alpha,\alpha/2}(\Omega_{0T})} \le \sup_{\tau<T}\Big(\sup_{\Omega_{\tau/2,\tau}} |f_1(z,t)|[f_2]^{(\alpha,\alpha/2)}_{\Omega_{\tau/2,\tau}} + \sup_{\Omega_{\tau/2,\tau}} |f_2(z,t)|[f_1]^{(\alpha,\alpha/2)}_{\Omega_{\tau/2,\tau}}\Big)$$
$$+ \sup_{\Omega_{0T}} |f_1(z,t)||f_2(z,t)| \le c\|f_1\|_{\hat{C}^{\alpha,\alpha/2}(\Omega_{0T})}\{f_2\}_{\alpha,\Omega_{0T}}.$$

□

Proof of Lemma 1. On the basis of the same arguments explained after (3.3), we may choose the time interval $[0,T]$ and the function $\eta(\xi)$ so that

$$\left|\frac{\partial\psi}{\partial z}\right| < \frac{1}{2} \tag{6.4}$$

which we will assume henceforth. In what follows we will denote by c constants depending on the data. Also, we introduce the slightly simpler notation

$$\|f\|_{\alpha,\Omega_{0T}} = \|f\|_{\hat{C}^{\alpha,\alpha/2}(\Omega_{0T})}$$

The estimate

$$\|F_1[u]\|_{\alpha,\Omega_{0T}} \le c\|\hat{\varepsilon} - \varepsilon_0\|_{\alpha,\Omega_{0T}}\Big(\{\frac{\partial u}{\partial t}\}_{\alpha,\Omega_{0T}} + \{\frac{\partial r_0}{\partial t}\}_{\alpha,\Omega_{0T}}\Big), \tag{6.5}$$

is an elementary application of Proposition 1.

The inequality

$$\|F_2[u]\|_{\alpha,\Omega_{0T}} \le c\Big(\|\hat{k} - k_0\|_{\alpha,\Omega_{0T}} + \|\frac{\partial\psi}{\partial z}\|_{\alpha,\Omega_{0T}}\Big)\Big(\{\frac{\partial^2 u}{\partial z^2}\}_{\alpha,\Omega_{0T}} + \{\frac{\partial^2 r_0}{\partial z^2}\}_{\alpha,\Omega_{0T}}\Big), \tag{6.6}$$

follows again from Proposition 1 and applying (6.4) in the identity

$$k_0 - \frac{\hat{k}}{(1+\frac{\partial\psi}{\partial z})^2} = \frac{k_0 - \hat{k}}{(1+\frac{\partial\psi}{\partial z})^2} + k_0\frac{2\frac{\partial\psi}{\partial z} + (\frac{\partial\psi}{\partial z})^2}{(1+\frac{\partial\psi}{\partial z})^2},$$

Likewise we obtain

$$\|F_3[u]\|_{\alpha,\Omega_{0T}} \le c\Big(\|r_{oz} - p_0'\|_{\alpha,\Omega_{0T}} + \|\frac{\partial\psi}{\partial z}\|_{\alpha,\Omega_{0T}} + \|\frac{\partial u}{\partial z}\|_{\alpha,\Omega_{0T}}\Big)\Big(\{\frac{\partial\psi}{\partial t}\}_{\alpha,\Omega_{0T}} + \{\frac{\partial^2\psi}{\partial z^2}\}_{\alpha,\Omega_{0T}}\Big), \tag{6.7}$$

just recalling (3.24).

Now we observe that all the norms $\{\cdot\}_{\alpha,\Omega_{0T}}$ in (6.5), (6.6), (6.7) are bounded. Therefore we work on the first factors on the r.h.s. We start by estimating $\left\|\frac{\partial u}{\partial z}\right\|_{\alpha,\Omega_{0T}}$, using the obvious inequality

$$\left[\frac{\partial u}{\partial z}\right]^{(\alpha)}_{J_t} \leq ct^{1-\alpha}\sup_{J_t}|\frac{\partial^2 u}{\partial z^2}(z,t)| \leq ct\|\frac{\partial^2 u}{\partial z^2}\|_{\alpha,\Omega_{0,t}}, \tag{6.8}$$

and also

$$\begin{aligned}
&\sup_{\Omega_{\tau/2,\tau}}\sup_{h\in(0,\tau-t)} h^{-\alpha/2}\left|\frac{\partial u}{\partial z}(z,t+h)-\frac{\partial u}{\partial z}(z,t)\right| \leq \\
&\leq \left(\sup_{\Omega_{\tau/2,\tau}}\sup_{h\in(0,\tau-t)} h^{-(1+\alpha)/2}\left|\frac{\partial u}{\partial z}(z,t+h)-\frac{\partial u}{\partial z}(z,t)\right|\right)^{\frac{\alpha}{1+\alpha}} 2^{\frac{1}{1+\alpha}}\left(\sup_{\Omega_{\tau/2,\tau}}\left|\frac{\partial u}{\partial z}\right|\right)^{\frac{1}{1+\alpha}} \\
&\leq c\tau\left\|\frac{\partial u}{\partial z}\right\|_{\hat{C}^{1+\alpha,(1+\alpha)/2}(\Omega_{0\tau})}
\end{aligned} \tag{6.9}$$

where last inequality is obtained thanks to Young's inequality. As a consequence of (6.8), (6.9) we get

$$\left\|\frac{\partial u}{\partial z}\right\|_{\alpha,\Omega_{0T}} \leq cT\left(\left\|\frac{\partial^2 u}{\partial z^2}\right\|_{\alpha,\Omega_{0T}} + \left\|\frac{\partial u}{\partial z}\right\|_{1+\alpha,\Omega_{0T}}\right).$$

As to the functions $\frac{\partial \psi}{\partial z}$, $\frac{\partial r_0}{\partial z} - p_0'$, $\hat{k}-k_0$, $\hat{\epsilon}-\varepsilon_0$ appearing in (6.5)-(6.7), we use the inequality

$$\|\phi\|_{\alpha,\Omega_{0T}} \leq cT^{1-\alpha}\left(\sup_{\Omega_{0T}}\left|\frac{\partial \phi}{\partial z}(z,t)\right| + \sup_{\Omega_{0T}}\left|\frac{\partial \phi}{\partial t}(z,t)\right|\right) \tag{6.10}$$

For instance, the derivatives of $\hat{\varepsilon}$ are computed as follows. Recalling (2.9) and (3.1)-(3.3), we have

$$\hat{\varepsilon}(z,t) = \varepsilon_0 + V_0 - \Psi(I(z,t)) \tag{6.11}$$

where

$$I(z,t) = \int_{\theta(Y(z,t))}^{t} c_0\hat{p}(X(z,t,\tau),\tau)d\tau, \qquad X(z,t,\tau) = Y^{-1}(Y(z,t),\tau) \tag{6.12}$$

so that

$$\frac{\partial \hat{\varepsilon}}{\partial z} = -\Psi'(I(z,t))\frac{\partial I}{\partial z}, \qquad \frac{\partial \hat{\varepsilon}}{\partial t} = -\Psi'(I(z,t))\frac{\partial I}{\partial t}, \tag{6.13}$$

with

$$\frac{\partial I}{\partial z} = c_0\int_{\theta(Y(z,t))}^{t}\frac{\partial \hat{p}}{\partial z}(X(z,t,\tau),\tau)\frac{\partial X}{\partial t}(z,t,\tau)d\tau, \tag{6.14}$$

$$\frac{\partial I}{\partial t} = c_0\int_{\theta(Y(z,t))}^{t}\frac{\partial \hat{p}}{\partial z}(X(z,t,\tau),\tau)\frac{\partial X}{\partial t}(z,t,\tau)d\tau + c_0\hat{p}(z,t). \tag{6.15}$$

Since

$$\frac{\partial}{\partial x}Y^{-1}(x,t) = \left(1+\frac{\partial \psi}{\partial \xi}\right)^{-1}\Bigg|_{\xi=Y^{-1}(x,t)} \tag{6.16}$$

we have

$$\frac{\partial X}{\partial z} = \frac{1+\frac{\partial \psi}{\partial z}}{1+A}, \qquad \frac{\partial X}{\partial t} = \frac{\frac{\partial \psi}{\partial t}}{1+A}, \qquad A = \frac{\partial \psi}{\partial z}\Bigg|_{z=X(z,t,\tau),\ t=\tau} \tag{6.17}$$

Again we recall (3.24) and that u is taken in $\mathcal{U}_{2+\alpha}(\delta,T)$ so that we can conclude that

$$\sum_{i=1}^{3}\|F_i[u]\|_{\alpha,\Omega_{0T}} \leq cT^{1-\alpha}. \tag{6.18}$$

which is a first step in the proof of (6.1). We continue our analysis considering $F_4[u], F_5[u]$, which we can rewrite in the form

$$F_4[u] = (1-S_0)(1-\hat{p}/p_S)(\widehat{\frac{\partial \varepsilon}{\partial t}}) = -(1-S_0)(1-\widehat{p}/p_S)\Psi'(I(z,t))c_0\widehat{p}(z,t), \tag{6.19}$$

$$F_5[u] = \frac{1}{(1+\dfrac{\partial \psi}{\partial z})^2}\frac{\partial k}{\partial \varepsilon}(\hat{S},\hat{\varepsilon})\frac{\partial \hat{\varepsilon}}{\partial z}\frac{\partial \hat{p}}{\partial z} + c_0\left(\frac{\partial k}{\partial S}(\hat{S},\hat{\varepsilon}) - \frac{\partial k}{\partial S}(S_0,\varepsilon_0)\right)\left(\frac{\partial r_0}{\partial z}\right)^2$$

$$+ c_0\frac{\partial k}{\partial S}(\hat{S},\hat{\varepsilon})(\frac{\partial \hat{p}}{\partial r} + \frac{\partial r_0}{\partial z})(\frac{\partial u}{\partial z} + p_0'\frac{\partial \psi}{\partial z}) - \frac{2\frac{\partial \psi}{\partial z} + (\frac{\partial \psi}{\partial z})^2}{(1+(\frac{\partial \psi}{\partial t})^2)}c_0\frac{\partial k}{\partial S}(\hat{S},\hat{\varepsilon})(\frac{\partial \hat{p}}{\partial r})^2 \tag{6.20}$$

From (6.19) we obtain immediately

$$\|F_4[u]\|_{\alpha,\Omega_{0T}} \le c\|\hat{p}\|_{\alpha,\Omega_{0T}}\left(1 + \left\{\frac{\partial p}{\partial z}\right\}_{\alpha,\Omega_{0T}}\right) \tag{6.21}$$

We remark that the presence of $\dfrac{\partial \hat{\varepsilon}}{\partial z}$ in $\|F_5[u]\|$ and the necessity of using (6.10) requires the boundedness of the derivatives of $\dfrac{\partial^2 \hat{\varepsilon}}{\partial z^2}, \dfrac{\partial^2 \hat{\varepsilon}}{\partial z \partial t}$. With the same symbols as in (6.12) we write

$$\frac{\partial^2 \hat{\varepsilon}}{\partial z^2} = -\Psi''(I)\left(\frac{\partial I}{\partial z}\right)^2 - \Psi'(I)\frac{\partial^2 I}{\partial z^2} \tag{6.22}$$

$$\frac{\partial^2 \hat{\varepsilon}}{\partial z \partial t} = -\Psi''(I)\frac{\partial I}{\partial z}\frac{\partial I}{\partial t} - \Psi'(I)\frac{\partial^2 I}{\partial z \partial t}, \tag{6.23}$$

where

$$\frac{\partial^2 I}{\partial z^2} = -c_0\frac{\partial \hat{p}}{\partial z}(X,\theta(Y))\frac{\partial X}{\partial z}(z,t,\theta(Y))\theta'(Y)(1+\frac{\partial \psi}{\partial z}(z,t))$$

$$+ c_0\int_{\theta(Y(z,t))}^{t}\Big(\hat{p}_{zz}(X,\tau)X_z^2 + \hat{p}_z(X,\tau)X_{zz}(z,t,\tau)\Big)d\tau, \tag{6.24}$$

$$\frac{\partial^2 I}{\partial z \partial t} = c_0\frac{\partial \hat{p}}{\partial z}(z,t) - c_0\frac{\partial \hat{p}}{\partial z}(X,\theta(Y))\frac{\partial X}{\partial z}(z,t,\theta(Y))\theta'(Y)\frac{\partial \psi}{\partial t}(z,t)$$

$$+ c_0\int_{\theta(Y(z,t))}^{t}\Big(\hat{p}_{zz}(X,\tau)X_zX_t + \hat{p}_z(X,\tau)X_{zt}\Big)d\tau, \tag{6.25}$$

$$\frac{\partial^2 X}{\partial z^2}(z,t,\tau) = \frac{\dfrac{\partial^2 \psi}{\partial z^2}(z,t)}{1+\dfrac{\partial \psi}{\partial z}(X,\tau)} - \frac{\dfrac{\partial^2 \psi}{\partial z^2}(X,\tau)(1+\dfrac{\partial \psi}{\partial z}(z,t))^2}{(1+\dfrac{\partial \psi}{\partial z}(X,\tau))^3}, \tag{6.26}$$

$$\frac{\partial^2 X}{\partial z \partial t}(z,t,\tau) = \frac{\dfrac{\partial^2 \psi}{\partial z \partial t}(z,t)}{1+\dfrac{\partial \psi}{\partial z}(X,\tau)} - \frac{\dfrac{\partial^2 \psi}{\partial z^2}(X,\tau)\psi_t^2(z,t)}{(1+\dfrac{\partial \psi}{\partial z}(X,\tau))^3}. \tag{6.27}$$

In the expression above

$$\frac{\partial^2 \psi}{\partial z^2} = \eta''\left(\frac{z}{t}\right)\frac{\sigma(t)}{t^2}, \quad \frac{\partial^2 \psi}{\partial z \partial t} = \eta'\left(\frac{z}{t}\right)\frac{\dot{\sigma}}{t} - \eta''\left(\frac{z}{t}\right)z\frac{\dot{\sigma}}{t^3} \tag{6.28}$$

are bounded functions (see (3.25)).

Therefore, proceeding as in the final step for the estimates of F_1, F_2, F_3, we obtain for $\sum_{i=4}^{5}\|F_i[u]\|_{\alpha,\Omega_{0T}}$ precisely the same estimate as (6.18), thus completing the proof of (6.1).

Inequality (6.2) is proved by means of a similar technique. Remembering the definition (3.39) of $G[u]$, we see that the main term can be conveniently rewritten in the form

$$\left[\hat{k} - k_0 - \frac{\partial k}{\partial S}(S_0, \varepsilon_0)(\hat{S} - S_0)\right] \frac{\partial r_0}{\partial z}\Big|_{z=0} =$$
$$\left\{ \int_0^1 \left[\frac{\partial k}{\partial S}\Big|_{\substack{S=S_0+\lambda(\hat{S}-S_0)\\ \varepsilon=\varepsilon_0+\lambda(\hat{\varepsilon}-\varepsilon_0)}} - \frac{\partial k}{\partial S}(S_0, \varepsilon_0)\right](\hat{S} - S_0) d\lambda \right.$$
$$\left. + \int_0^1 \frac{\partial k}{\partial \varepsilon}\Big|_{\substack{S=S_0+\lambda(\hat{S}-S_0)\\ \varepsilon=\varepsilon_0+\lambda(\hat{\varepsilon}-\varepsilon_0)}} (\hat{\varepsilon} - \varepsilon_0) d\lambda \right\} \frac{\partial r}{\partial z}\Big|_{z=0}$$

Owing to the assumptions made on $k(S, \varepsilon)$, we see that, for u taken in the usual set,

$$\sup_{(0,T)} t^{-1-\alpha} |G[u]| \le c \sup_{(0,T)} t^{-1-\alpha} \Big(|u(0,t)| + |p(0,t)|^2 + |\hat{\varepsilon}(0,t) - \varepsilon_0|$$
$$+ |\hat{\varepsilon}(0,t) - \varepsilon_0| \Big| \frac{\partial u}{\partial z}\Big|_{z=0} \Big) \le c T^{1-\alpha},$$

and

$$\Big[G[u]\Big]^{(1+\alpha)/2}_{(0,T)} \le c T^{(1-\alpha)/2} \sup_{(0,T)} \left(|\frac{\partial u}{\partial t}(0,t)| + |\hat{p}(0,t)| + |\frac{\partial \hat{\varepsilon}}{\partial t}(0,t)| + \sup_{(0,T)} |\frac{\partial \hat{k}}{\partial t}(0,t)| \right)$$
$$+ \sup_{(0,T)} |\hat{k}(0,t) - k_0| \left[\frac{\partial u}{\partial z}\right]^{(1+\alpha)/2}_{(0,T)} \le c T^{(1+\alpha)/2} \tag{6.29}$$

Therefore (6.2) is proved.

□

Our next aim is to prove the following estimates.

Lemma 2 *For u_1, u_2 in the set $\mathcal{U}_{2+\alpha}(\delta, T)$ the inequalities*

$$\|F_k[u_1] - F_k[u_2]\|_{\alpha,\Omega_{0T}} \le c T \mathcal{N}_{2+\alpha}[u_1 - u_2], \;\; k = 1, 2, 4 \tag{6.30}$$
$$\|F_k[u_1] - F_k[u_2]\|_{\alpha,\Omega_{0T}} \le c T^{1/2} \mathcal{N}_{l+\alpha}[u_1 - u_2], \;\; k = 3, 5 \tag{6.31}$$
$$\|G[u_1] - G[u_2]\|_{\hat{C}^{(1+\alpha)/2}(0,T)} \le c T^{(1+\alpha)/2} \mathcal{N}_{2+\alpha}[u_1, u_2] \tag{6.32}$$

hold true for a suitable constant c depending on the data.

The proof of this lemma is quite long and it requires comparisons between the pairs (σ_1, σ_2), (θ_1, θ_2), (Y_1, Y_2), (X_1, X_2), (I_1, I_2), with obvious definition of the symbols. We prefer to split it in several propositions, In all of them u_1, u_2 are assumed in the set $\mathcal{U}_{2+\alpha}(S, T)$.

Proposition 2

$$|Y_1(z,t) - Y_2(z,t)| \le |\sigma_1(t) - \sigma_2(t)| \le t \sup_{(0,t)} |\dot{\sigma}_1(\tau) - \dot{\sigma}_2(\tau)| \tag{6.33}$$

$$\left|\frac{\partial Y_1}{\partial z} - \frac{\partial Y_2}{\partial z}\right| + \left|\frac{\partial Y_1}{\partial t} - \frac{\partial Y_2}{\partial t}\right| \le c \sup_{(0,t)} |\dot{\sigma}_1(\tau) - \dot{\sigma}_2(\tau)|, \tag{6.34}$$

$$\left\|\frac{\partial \psi_1}{\partial z} - \frac{\partial \psi_2}{\partial z}\right\|_{\alpha,\Omega_{0T}} \le c[\dot{\sigma}_1 - \dot{\sigma}_2]^{(\alpha)}_{(0,T)}. \tag{6.35}$$

Proof. Since

$$Y_1 - Y_2 = \psi_1 - \psi_2 = \eta\left(\frac{z}{t}\right)(\sigma_1 - \sigma_2) \tag{6.36}$$

(6.33) is elementary. Differentiations with respect to z and t yield

$$\frac{\partial Y_1}{\partial z} - \frac{\partial Y_2}{\partial z} = \frac{\partial \psi_1}{\partial z} - \frac{\partial \psi_2}{\partial z} = \eta'\left(\frac{z}{t}\right)\frac{\sigma_1 - \sigma_2}{t} \tag{6.37}$$

$$\frac{\partial Y_1}{\partial t} - \frac{\partial Y_2}{\partial t} = \frac{\partial \psi_1}{\partial t} - \frac{\partial \psi_2}{\partial t} = -\eta'\left(\frac{z}{t}\right)\frac{z}{t}\frac{\sigma_1 - \sigma_2}{t} + r(\frac{z}{t})(\dot\sigma_1 - \dot\sigma_2),\ 0 < z < l_0 t \tag{6.38}$$

from which (6.34) follows immediately. Also (6.35) can be easily deduced from (6.37).

□

Proposition 3 *Setting* $s_{\min}(t) = \min(s_1(t), s_2(t))$*, and* $J'_t = (0, s_{\min}(t))$*, we have*

$$\sup_{J'_t} |\theta_1(x) - \theta_2(x)| \le ct \sup_{(0,t)} |\dot\sigma_1(\tau) - \dot\sigma_2(\tau)|, \tag{6.39}$$

$$[\theta_1 - \theta_2]^{(\alpha)}_{J'_t} \le ct^{1-\alpha} \sup_{(0,t)} |\dot\sigma_1(\tau) - \dot\sigma_2(\tau)|, \tag{6.40}$$

$$\|\theta_1(Y_1) - \theta_2(Y_2)\|_{\alpha,\Omega_{0T}} \le cT^{1-\alpha} \sup_{(0,T)} |\dot\sigma_1(t) - \dot\sigma_2(t)| \tag{6.41}$$

Proof. Let us define

$$\theta_{1,2}(x) = \theta_1(x) - \theta_2(x) \tag{6.42}$$

and use the identity $s(\theta(x)) = x$ to write

$$s_2(\theta_2) - s_1(\theta_2) = s_1(\theta_1) - s_1(\theta_2) = \theta_{1,2}\mathcal{I}(x) \tag{6.43}$$

with

$$\mathcal{I}(x) = \int_0^1 \dot s_1(\theta_2(x) + \lambda\theta_{1,2}(x))\, d\lambda \tag{6.44}$$

having the property

$$|\mathcal{I}(x) - \mathcal{I}(y)| \le c|x-y|^\alpha [\dot s_1]^{(\alpha)}_{(0,t)}, \qquad \forall\ x, y \in J'_t \tag{6.45}$$

thanks to the fact that $\dot s$ is separated from zero uniformly for u in the set $\mathcal{U}_{2+\alpha}(\delta, T)$. At this point (6.39) and (6.40) are easy consequences of (6.43), (6.44) and of (6.45).

Coming to the proof of (6.41), we define $\sigma_{\min}(t) = \min(\sigma_1(t), \sigma_2(t))$ and $Y_{\min}(z,t) = z + \eta\left(\frac{z}{t}\right)\sigma_{\min}(t)$, having L^∞ first time derivative, and we note that

$$\theta_i(Y_i(z,t)) - \theta_i(Y_{\min}(z,t)) =$$
$$(Y_i - Y_{\min}) \int_0^1 \theta'_i(Y_i - \lambda(Y_i - Y_{\min}))\, d\lambda,\ i = 1, 2.$$

Therefore, we can write

$$\theta_1(Y_1) - \theta_2(Y_2) = \theta_1(Y_{\min}) - \theta_2(Y_{\min}) +$$
$$+\eta\left(\frac{z}{t}\right)(\sigma_1 - \sigma_{\min}) \int_0^1 \theta'_1(Y_1 - \lambda(Y_1 - Y_{\min}))\, d\lambda$$
$$-\eta\left(\frac{z}{t}\right)(\sigma_2 - \sigma_{\min}) \int_0^1 \theta'_2(Y_2 - \lambda(Y_2 - Y_{\min}))\, d\lambda \tag{6.46}$$

From (6.46) and (6.39) we can easily deduce

$$|\theta_1(Y_1(z,t)) - \theta_2(Y_2(z,t))| \le ct \sup_{\tau \le t} |\dot\sigma_1(\tau) - \dot\sigma_2(\tau)|. \tag{6.47}$$

Similarly, with the help of (6.40) we find

$$[\theta_1(Y_1)-\theta_2(Y_2)]^{(\alpha,\alpha/2)}_{\Omega_{\tau,\tau/2}}\le c\tau^{1-\alpha}\sup_{(0,\tau)}|\dot\sigma_1-\dot\sigma_2|, \tag{6.48}$$

which, together with (6.47), leads to (6.41)

□

Remark 1 Defining $\theta_{\max}=\max(\theta_1(Y_1),\theta_2(Y_2))$ we see that estimate (6.41) applies also to $\|\theta_{\max}-\theta_i(Y_i)\|_{\alpha,\Omega_{0T}}$, $i=1,2$.

Proposition 4 *For arbitrary z,t,τ satisfying the conditions $\theta_{\max}(z,t)\le\tau\le t\le T$ there hold the inequalities*

$$|X_1(z,t,\tau)-X_2(z,t,\tau)|\le ct\sup_{(0,t)}|\dot\sigma_1(\xi)-\dot\sigma_2(\xi)|, \tag{6.49}$$

$$|\frac{\partial X_1}{\partial z}(z,t,\tau)-\frac{\partial X_2}{\partial z}(z,t,\tau)|+|\frac{\partial X_1}{\partial t}(z,t,\tau)-\frac{\partial X_2}{\partial t}(z,t,\tau)|\le c\sup_{\xi\le t}|\dot\sigma_1(\xi)-\dot\sigma_2(\xi)|. \tag{6.50}$$

Proof. Since $Y_i(X_i(z,t,\tau)\tau)=Y_i(z,t)$, we have

$$Y_1(X_1(z,t,\tau)\tau)-Y_2(X_2(z,t,\tau)\tau)=Y_1(z,t)-Y_2(z,t).$$

To be definite, assume that $s_2(\tau)\le s_1(\tau)$. Then $Y_1(X_2(z,t,\tau)\tau)$ is well defined, and

$$\begin{aligned}&Y_1(X_1(z,t,\tau)\tau)-Y_1(X_2(z,t,\tau)\tau)=Y_2(X_2(z,t,\tau)\tau)-Y_1(X_2(z,t,\tau)\tau)+Y_1(z,t)-Y_2(z,t)\\&=\eta(X_2/\tau)(\sigma_2(\tau)-\sigma_1(\tau))+\eta(z/t)(\sigma_1(t)-\sigma_2(t)).\end{aligned} \tag{6.51}$$

The left hand side can be written in the form

$$\begin{aligned}&(X_1(z,t,\tau)-X_2(z,t,\tau)\int_0^1\frac{d}{d\xi}Y_1(\xi,t)|_{\xi=X_2+\lambda(X_1-X_2)}d\lambda\\&=(X_1(z,t,\tau)-X_2(z,t,\tau)\int_0^1(1+\frac{\partial\psi_1}{\partial\xi}(\xi,\tau))|_{\xi=X_2+\lambda(X_1-X_2)}d\lambda\end{aligned} \tag{6.52}$$

where the integral is bounded from above and from below by some positive constants. Hence,

$$|X_1(z,t,\tau)-X_2(z,t,\tau)|\le ct\sup_{(0,t)}|\dot\sigma_1(\xi)-\dot\sigma_2(\xi)|. \tag{6.53}$$

For the difference $\frac{\partial X_1}{\partial z}-\frac{\partial X_2}{\partial z}$ we have the representation formula

$$\begin{aligned}&\frac{\partial X_1}{\partial z}(z,t,\tau)-\frac{\partial X_2}{\partial z}(z,t,\tau)=\frac{1}{1+\frac{\partial\psi_1}{\partial z}(X_1,\tau)}\eta'\Big(\frac{z}{t}\Big)\frac{\sigma_1(t)-\sigma_2(t)}{t}\\&+\frac{1+\frac{\partial\psi_2}{\partial z}(z,t)}{(1+\frac{\partial\psi 1}{\partial z}(X_1,\tau))(1+\frac{\partial\psi_2}{\partial z}(X_2,\tau))}\Big[\Big(\eta'\Big(\frac{X_2}{\tau}\Big)-\eta'\Big(\frac{X_1}{\tau}\Big)\Big)\frac{\sigma_2(\tau)}{\tau}-\eta'\Big(\frac{X_1}{\tau}\Big)\frac{\sigma_1(\tau)-\sigma_2(\tau)}{\tau}\Big],\end{aligned} \tag{6.54}$$

and, as a consequence, the estimate

$$\Big|\frac{\partial X_1}{\partial z}(z,t,\tau)-\frac{\partial X_2}{\partial z}(z,t,\tau)\Big|\le c\Big(\frac{|\sigma_1(t)-\sigma_2(t)|}{t}+\frac{|\sigma_1(\tau)-\sigma_2(\tau)|}{\tau}+\frac{|X_1-X_2|}{\tau}\frac{|\sigma_2(\tau)|}{\tau}\Big). \tag{6.55}$$

The same kind of estimate holds for $X_{1t}-X_{2t}$, so we arrive at (6.50). The proposition is proved.

□

Proposition 5 *Set* $X_{1,2} = X_1 - X_2$. *If* $\tau \in (\theta_{\max}(z,t),t) \cap (\theta_{\max}(z',t),t)$, *then*

$$|X_{1,2}(z,t,\tau) - X_{1,2}(z',t,\tau)| \le c|z-z'|^\alpha \Big(t^{-\alpha}|\sigma_1(t)-\sigma_2(t)| + \tau^{-\alpha}|\sigma_1(\tau)-\sigma_2(\tau)|\Big), \tag{6.56}$$

$$\left|\frac{\partial X_{1,2}}{\partial z}(z,t,\tau) - \frac{\partial X_{1,2}}{\partial z'}(z',t,\tau)\right| \le c|z-z'|^\alpha \tau^{-\alpha} \sup_{(0,t)} |\dot\sigma_1(\xi) - \dot\sigma_2(\xi)|, \tag{6.57}$$

If $\tau \in (\theta_{\max}(z,t),t) \cap (\theta_{\max}(z,t+h),t+h)$, $h \in (0,t)$, *then*

$$\begin{aligned}&|X_{1,2}(z,t+h,\tau) - X_{1,2}(z,t,\tau)| \le ch^{\alpha/2}\Big(t^{-\alpha/2}|\sigma_1(t)-\sigma_2(t)| + \tau^{-\alpha/2}|\sigma_1(\tau)-\sigma_2(\tau)|\Big)\\ &+c|\sigma_1(t+h) - \sigma_2(t+h) - \sigma_1(t) + \sigma_2(t)|,\end{aligned} \tag{6.58}$$

$$\left|\frac{\partial X_{1,2}}{\partial z}(z,t+h,\tau) - \frac{\partial X_{1,2}}{\partial z}(z,t,\tau)\right| \le ch^{\alpha/2}\tau^{-\alpha/2} \sup_{(\tau,t+h)} |\dot\sigma_1(\xi) - \dot\sigma_2(\xi)|, \tag{6.59}$$

Proof. The first inequality can be deduced directly from (6.55), using the following facts: $\dfrac{|z-z'|}{t} \le c|z-z'|^\alpha\, t^{-\alpha}$ on J'_t, $|X_1 - X_2| \le c \sup\limits_{\tau\in(0,t)} |\sigma_1(\tau) - \sigma_2(\tau)|$, (which is a more precise version of (6.53)), $\dfrac{\sigma_2(\tau)}{\tau}$ bounded.

The estimate (6.58) follows from (6.51), (6.52), because the integral factor in (6.52) is not only bounded but also satisfies the Hölder condition with respect to z and t (with the exponent α). To prove (6.57), we estimate the difference of the function (6.54) at the points z and z' and note that $f_j(z,t) = [1 + \psi'(X_j(z,t,\tau),\tau)]^{-1}$, $j = 1,2$, satisfy the inequality

$$|f_j(z,t) - f'_j(z',t)| \le c\tau^{-\alpha}|z-z'|^\alpha. \tag{6.60}$$

We also use the formula

$$\eta'\Big(\frac{X_1}{\tau}\Big) - \eta'\Big(\frac{X_2}{\tau}\Big) = \frac{X_1 - X_2}{\tau}\int_0^1 \eta''\Big(\frac{X_2 + \lambda(X_1 - X_2)}{\tau}\Big) d\lambda, \tag{6.61}$$

from which we obtain

$$\begin{aligned}&\left|\Big[\eta'\Big(\frac{X_1(y,t,\tau)}{\tau}\Big) - \eta'\Big(\frac{X_2(y,t,\tau}{\tau}\Big)\Big]\Big|_{y=z'}^{y=z}\right|\\ &\le c\Big(\frac{|X_{1,2}(z,t,\tau - X_{1,2}(z',t,\tau)|}{\tau} + \frac{|X_{12}(z,t,\tau)|}{\tau^{1+\alpha}}\Big).\end{aligned} \tag{6.62}$$

As a result, we arrive at

$$\begin{aligned}&|\frac{\partial X_{1,2}}{\partial z}(z,t,\tau - \frac{\partial X_{1,2}}{\partial z'}(z',t'\tau)| \le c|z-z'|^\alpha \Big[\Big(\frac{|\sigma_1(t)-\sigma_2(t)|}{t} + \frac{|\sigma_1(\tau)-\sigma_2(\tau)|}{\tau}\Big)\Big]\frac{1}{\tau^\alpha}\\ &+ \frac{|\sigma_1(t)-\sigma_2(t)|}{t^{1+\alpha}} + \frac{|\sigma_1(\tau)-\sigma_2(\tau)|}{\tau^{1+\alpha}} + |X_{1,2}(z,t,\tau)|\frac{|\sigma_2(\tau)|}{\tau^{2+\alpha}}\Big)\Big] + \frac{|\sigma_2(\tau)|}{\tau^2}|X_{12}(z,t,\tau X_{1,2}(z',t,\tau)|\\ &\le c|z-z'|^\alpha\tau^{-\alpha}\sup_{(0,t)}|\dot\sigma_1(\xi) - \dot\sigma_2(\xi)|\end{aligned} \tag{6.63}$$

which implies (6.57). Inequality (6.59) is obtained in the same way.

□

Proposition 6 *The differences* $I_1 - I_2$ *and* $I_{1z} - I_{2z}$ *satisfy the inequalities*

$$\|I_1 - I_2\|_{\alpha,\Omega_{0T}} \le cT^{1-\alpha}\Big(T\sup_{(0,T)}|\sigma'_1(t) - \sigma'_2(t)| + \sup_{(0,T)}\|u_1 - u_2\|_{C^\alpha(J_t)}\Big), \tag{6.64}$$

$$\|I_{1z} - I_{2z}\|_{\alpha,\Omega_{0T}} \le T^{1-\alpha}\Big(\sup_{(0,T)}|\dot\sigma_1 - \dot\sigma_2| + \sup_{(0,T)}\|\frac{\partial u_1}{\partial z} - \frac{\partial u_2}{\partial z}\|_{C^\alpha(J_t)}\Big). \tag{6.65}$$

Proof. Let us prove (6.65). By virtue of Remark 1,

$$\|I_{1z}-I_{2z}\|_{\alpha,\Omega_{0T}} \le cT^{1-\alpha}\sup_{(0,T)}|\dot\sigma_1(t)-\dot\sigma_2(t)|+\|K\|_{\alpha,\Omega_{0T}} \tag{6.66}$$

where $K(z,t)=\int_{\theta_{max}}^t P(z,t,\tau)d\tau$ and the function

$$P(z,t,\tau)=\frac{\partial\hat p_1}{\partial z}(X_1,\tau)\frac{\partial X_1}{\partial z}-\frac{\partial\hat p_2}{\partial z}(X_2,\tau)\frac{\partial X_2}{\partial z}=(\frac{\partial X_1}{\partial z}-\frac{\partial X_2}{\partial z})\frac{\partial\hat p_1}{\partial z}(X_1,\tau)$$
$$+\frac{\partial X_2}{\partial z}(\frac{\partial\hat p_1}{\partial z}(X_1,\tau)-\frac{\partial\hat p_2}{\partial z}(X_1,\tau))+\frac{\partial X_2}{\partial z}(X_1-X_2)\int_0^1\frac{\partial^2\hat p_2}{\partial z^2}(X_2+\lambda(X_1-X_2),\tau)d\tau \tag{6.67}$$

satisfies the inequalities

$$|P(z,t,\tau)|\le c\Big(|X_1(z,t,\tau)-X_2(z,t,\tau)|+|\frac{\partial X_1}{\partial z}(z,t,\tau)-\frac{\partial X_2}{\partial z}(z,t,\tau)|$$
$$+\sup_{J_\tau}|\frac{\partial\hat p_1}{\partial z}(z,\tau)-\frac{\partial\hat p_2}{\partial z}(z,\tau)|\Big)\le c\Big(\sup_{(0,t)}|\dot\sigma_1(\xi)-\dot\sigma_2(\xi)|+\sup_{J_\tau}|\frac{\partial u_1}{\partial z}(z,\tau)-\frac{\partial u_2}{\partial z}(z,\tau)|\Big), \tag{6.68}$$

$$|P(z,t,\tau)-P(z',t,\tau)|\le c|z-z'|^\alpha\Big(\tau^{-\alpha}\sup_{(0,t)}|\dot\sigma_1(\xi)-\dot\sigma_2(\xi)|+$$
$$[\frac{\partial u_1}{\partial z}(\cdot,\tau)-\frac{\partial u_2}{\partial z}(\cdot,\tau)]^{(\alpha)}_{J_\tau}\Big), \tag{6.69}$$

and, if $\tau\in(\theta_{max}(z,t+h),t+h)\cap(\theta_{max}(z,t),t)$, $h\in(0,t)$, then

$$|P(z,t+h,\tau)-P(z,t,\tau)|\le ch^{\alpha/2}\Big(\tau^{-\alpha/2}\sup_{(0,t)}|\dot\sigma_1(\xi)-\dot\sigma_2(\xi)|+[\frac{\partial u_1}{\partial z}(\cdot,\tau)-\frac{\partial u_2}{\partial z}(\cdot,\tau)]^{(\alpha)}_{J_\tau}\Big). \tag{6.70}$$

Hence,

$$t^{-\alpha}|K(z,t)|+[K(\cdot,t)]^{(\alpha)}_{J_t}\le ct^{1-\alpha}\Big(\sup_{(0,t)}|\dot\sigma_1(\xi)-\dot\sigma_2(\xi)|+\|\frac{\partial u_1}{\partial z}(\cdot,\tau)-\frac{\partial u_2}{\partial z}(\cdot,\tau)\|_{C^\alpha(J_\tau)}\Big), \tag{6.71}$$

$$h^{-\alpha/2}|K(z,t+h)-K(z,t)|\le h^{-\alpha/2}\int_t^{t+h}|P(z,t,\tau)d\tau|+h^{-\alpha/2}\Big|\int_{\theta_{max}(z,t)}^{\theta_{max}(z,t+h)}|P(z,t+h,\tau)|d\tau\Big|$$
$$+\int_{\Theta(z,t,h)}^t|P(z,t+h,\tau)-P(z,t,\tau)|d\tau\le ct^{1-\alpha}\Big(\sup_{(0,t)}|\dot\sigma_1(\xi)-\dot\sigma_2(\xi)|+\|\frac{\partial u_1}{\partial z}(\cdot,\tau)-\frac{\partial u_2}{\partial z}(\cdot,\tau)\|_{C^\alpha(J_\tau)}\Big). \tag{6.72}$$

where $\Theta(z,t,h)=\max(\theta_{max}(z,t),\theta_{max}(z,t+h)$. These inequalities yield (6.65), and (6.64) is obtained in the same way. The proposition is proved.

□

Proposition 7 *The following inequalities are valid:*

$$\|\hat\varepsilon_1-\hat\varepsilon_2\|_{\alpha,\Omega_{0T}}\le cT^{1-\alpha}\Big(\sup_{(0,T)}\|u_1(\cdot,t)-u_2(\cdot,t)\|_{C^\alpha(J_T)}+T\sup_{t<T}|\dot\sigma_1(t)-\dot\sigma_2(t)|\Big), \tag{6.73}$$

$$\Big\|\frac{\partial\hat\varepsilon_1}{\partial z}-\frac{\partial\hat\varepsilon_2}{\partial z}\Big\|_{\alpha,\Omega_{0T}}\le cT^{1-\alpha}\Big(\sup_{t<T}\Big\|\frac{\partial u_1}{\partial z}-\frac{\partial u_2}{\partial z}\Big\|_{C^\alpha(J_T)}+\sup_{t<T}|\dot\sigma_1(t)-\dot\sigma_2(t)|\Big), \tag{6.74}$$

$$\|\hat k_1-\hat k_2\|_{\alpha,\Omega_{0T}}+\Big\|\frac{\partial\hat k_1}{\partial S}-\frac{\partial\hat k_2}{\partial S}\Big\|_{\alpha,\Omega_{0T}}+\Big\|\frac{\partial\hat k_1}{\partial\varepsilon}-\frac{\partial\hat k_2}{\partial\varepsilon}\Big\|_{\alpha,\Omega_{0T}}$$
$$\le c\Big(\|\hat p_1-\hat p_2\|_{\alpha,\Omega_{0T}}+\|\hat\varepsilon_1-\hat\varepsilon_2\|_{\alpha,\Omega_{0T}}\Big)$$
$$\le c\Big(T^{1-\alpha}\sup_{t<T}|\dot\sigma_1(t)-\dot\sigma_2(t)|+\|u_1-u_2\|_{\alpha,\Omega_{0T}}\Big). \tag{6.75}$$

Proof. The first two inequalities follow from Prop. 6, using Prop. 1. The last inequality is a consequence of (6.73), (6.74).

□

We now return to the proof of Lemma 2

Proof of Lemma 2. A first series of estimates for the differences $F_i[u_1] - F_i[u_2]$ is obtained by simply applying Prop. 1:

$$\|F_1[u_1]-F_1[u_2]\|_{\alpha,\Omega_{0T}} \leq c\Big(\|\hat{\varepsilon}_1-\hat{\varepsilon}_2\|_{\alpha,\Omega_{0T}}\{\frac{\partial u_1}{\partial t}+\frac{\partial r_0}{\partial t}\}_{\alpha,\Omega_{0T}}+\|\frac{\partial u_1}{\partial t}-\frac{\partial u_2}{\partial t}\|_{\alpha,\Omega_{0T}}\{\hat{\varepsilon}_2-\varepsilon_0\}_{\alpha,\Omega_{0T}}\Big), \quad (6.76)$$

$$\|F_2[u_1]-F_2[u_2]\|_{\alpha,\Omega_{0T}} \leq c\big(\|\hat{k}_1-\hat{k}_2\|_{\alpha,\Omega_{0T}}+\|\frac{\partial \psi_1}{\partial z}-\frac{\partial \psi_2}{\partial z}\|_{\alpha,\Omega_{0T}}\big)\{\frac{\partial^2 u_1}{\partial z^2}+\frac{\partial^2 r_0}{\partial z^2}\}_{\alpha,\Omega_{0T}}$$
$$+c\|\frac{\partial^2 u_1}{\partial z^2}-\frac{\partial^2 u_2}{\partial z^2}\|_{\alpha,\Omega_{0T}}\Big(\{\hat{k}_2-k_0\}_{\alpha,\Omega_{0T}}+\{\frac{\partial \psi_2}{\partial z}\}_{\alpha,\Omega_{0T}}\Big), \quad (6.77)$$

$$\|F_3[u_1]-F_3[u_2]\|_{\alpha,\Omega_{0T}} \leq c\Big(\|\frac{\partial \psi_1}{\partial z}-\frac{\partial \psi_2}{\partial z}\|_{\alpha,\Omega_{0T}}+\|\frac{\partial u_1}{\partial z}-\frac{\partial u_2}{\partial z}\|_{\alpha,\Omega_{0T}}\Big)\Big(\{\frac{\partial \psi_1}{\partial t}\}_{\alpha,\Omega_{0T}}+\{\frac{\partial^2 \psi_1}{\partial z^2}\}_{\alpha,\Omega_{0T}}\Big)$$
$$+c\Big(\{\frac{\partial \hat{p}_2}{\partial z}-p_0'\}_{\alpha,\Omega_{0T}}+\{\frac{\partial \psi_2}{\partial z}\}_{\alpha,\Omega_{0T}}\Big)\Big(\|\varepsilon_1-\varepsilon_2\|_{\alpha,\Omega_{0T}}$$
$$+\|\frac{\partial \psi_1}{\partial t}-\frac{\partial \psi_2}{\partial t}\|_{\alpha,\Omega_{0T}}+\|\hat{k}_1-\hat{k}_2\|_{\alpha,\Omega_{0T}}+\|\frac{\partial^2 \psi_1}{\partial z^2}-\frac{\partial^2 \psi_2}{\partial z^2}\|_{\alpha,\Omega_{0T}}+\|\frac{\partial \psi_1}{\partial z}-\frac{\partial \psi_2}{\partial z}\|_{\alpha,\Omega_{0T}}\Big), \quad (6.78)$$

$$\|F_4[u_1]-F_4[u_2]\|_{\alpha,\Omega_{0T}} \leq c\Big(\|\hat{p}_1-\hat{p}_2\|_{\alpha,\Omega_{0T}}+\|I_1-I_2\|_{\alpha,\Omega_{0T}}\{\hat{p}_2\}_{\alpha,\Omega_{0T}}\Big), \quad (6.79)$$

$$\|F_5[u_1]-F_5[u_2]\|_{\alpha,\Omega_{0T}} \leq c\Big(\|\frac{\partial \hat{k}_1}{\partial S}-\frac{\partial \hat{k}_2}{\partial S}\|_{\alpha,\Omega_{0T}}+\|\frac{\partial \hat{k}_1}{\partial \varepsilon}-\frac{\partial \hat{k}_2}{\partial \varepsilon}\|_{\alpha,\Omega_{0T}}$$
$$+\|\frac{\partial \psi_1}{\partial z}-\frac{\partial \psi_2}{\partial z}\|_{\alpha,\Omega_{0T}}+\|\frac{\partial \hat{p}_1}{\partial z}-\frac{\partial \hat{p}_2}{\partial z}\|_{\alpha,\Omega_{0T}}+\|\frac{\partial \hat{\varepsilon}_1}{\partial z}-\frac{\partial \hat{\varepsilon}_2}{\partial z}\|_{\alpha,\Omega_{0T}}\Big). \quad (6.80)$$

Since

$$\{\hat{\varepsilon}_j-\varepsilon_0\}_{\alpha,\Omega_{0T}}+\{\hat{k}_j-k_0\}_{\alpha,\Omega_{0T}}+\{\frac{\partial \psi_j}{\partial z}\}_{\alpha,\Omega_{0T}}+\{\hat{p}_j\}_{\alpha,\Omega_{0T}} \leq cT, \quad (6.81)$$

$$\{\frac{\partial \hat{p}_j}{\partial z}-p_0'\}_{\alpha,\Omega_{0T}} \leq \{r_0-p_0'\}_{\alpha,\Omega_{0T}}+|p_0'|\{\frac{\partial \psi_j}{\partial z}\}_{\alpha,\Omega_{0T}}+\{\frac{\partial u_j}{\partial z}\}_{\alpha,\Omega_{0T}} \leq cT, \qquad j=1,2, \quad (6.82)$$

$$\|\frac{\partial \psi_1}{\partial z}-\frac{\partial \psi_2}{\partial z}\|_{\alpha,\Omega_{0T}} \leq cT^{-\alpha}\sup_{t<T}|\dot{\sigma}_1(t)-\dot{\sigma}_2(t)|, \quad (6.83)$$

$$\|\frac{\partial \psi_1}{\partial t}-\frac{\partial \psi_2}{\partial t}\|_{\alpha,\Omega_{0T}} \leq c\|\dot{\sigma}_1-\dot{\sigma}_2\|_{\hat{C}^{\alpha}(0,T)}, \quad (6.84)$$

$$\|\frac{\partial^2 \psi_1}{\partial z^2}-\frac{\partial^2 \psi_2}{\partial z^2}\|_{\alpha,\Omega_{0T}} \leq cT^{-\alpha}\|\dot{\sigma}_1-\dot{\sigma}_2\|_{\hat{C}^{(1+\alpha)/2}(0,T)}, \quad (6.85)$$

and

$$\|\frac{\partial u_1}{\partial z}-\frac{\partial u_2}{\partial z}\|_{\alpha,\Omega_{0T}} \leq cT^{1/2}\Big(\|\frac{\partial u_1}{\partial z}-\frac{\partial u_2}{\partial z}\|_{\hat{C}^{1+\alpha,(1+\alpha)/2}(\Omega_{0T})}$$
$$+\|\frac{\partial^2 u_1}{\partial z^2}-\frac{\partial^2 u_2}{\partial z^2}\|_{\hat{C}^{\alpha,\alpha/2}(\Omega_{0T})}\Big) \leq cT^{1/2}\mathcal{N}_{2+\alpha}[u], \quad (6.86)$$

we conclude that the norms $\|F_k[u_1]-F_k[u_2]\|_{\alpha,\Omega_{0T}}$ can be estimated by $cT\mathcal{N}_{2+\alpha}[u_1-u_2]$, if $k=1,2,4$, and by $cT^{1/2}\mathcal{N}_{2+\alpha}[u_1-u_2]$, if $k=3,5$.

□

Let us finally estimate

$$G[u_1] - G[u_2] = (\hat{k}_1 - \hat{k}_2)(\frac{\partial r_0}{\partial z} + \frac{\partial u_1}{\partial z}) + (\hat{k}_2 - k_0)(\frac{\partial u_1}{\partial z} - \frac{\partial u_2}{\partial z}), \qquad z = 0$$

(we took account of the fact that $\hat{p}_1 - \hat{p}_2 = u_1 - u_2$ for $z = 0$). We have

$$t^{-1-\alpha}|G[u_1] - G[u_2]| \leq ct^{-1-\alpha}(|u_1(0,t) - u_2(0,t)| + |\hat{\varepsilon}_1(0,t) - \hat{\varepsilon}_2(0,t)|)$$
$$+ct^{-\alpha}|\frac{\partial u_1}{\partial z}(0,t) - \frac{\partial u_2}{\partial z}(0,t)| \leq ct\mathcal{N}_{2+\alpha}[u_1 - u_2],$$

$$\Big[G[u_1] - G[u_2]\Big]^{(1+\alpha)/2}_{(0,T)} \leq cT^{(1-\alpha)/2}(\sup_{t<T}|\frac{\partial \hat{k}_1}{\partial t}(0,t) - \frac{\partial \hat{k}_2}{\partial t}(0,t)|$$
$$+\sup_{t<T}|\hat{k}_1(0,t) - \hat{k}_2(0,t)| + \sup_{t<T}|\frac{\partial \hat{k}_2}{\partial t}|\sup_{t<T}|\frac{\partial u_1}{\partial z}(0,t) - \frac{\partial u_2}{\partial z}(0,t)|)$$
$$+cT[\frac{\partial u_1}{\partial z}(0,\cdot) - \frac{\partial u_2}{\partial z}(0,\cdot)]^{(1+\alpha)/2}_{(0,T)} \leq cT^{(1+\alpha)/2}\mathcal{N}_{2+\alpha}[u_1 - u_2],$$

which proves (6.32), thus concluding the proof of Lemma 2.

□

7 Proof of theorems 1, 2

In Sect. 4 we have sketched the main arguments of the proof of Theorem 3.1. As we said, the existence proof is equivalent to showing that the operator $\mathcal{M}[u]$ acting on the set $\mathcal{U}_{2+\alpha}(\delta, T)$ is contractive in the topology defined by the norm $\mathcal{N}_{2+\alpha}$ for T small enough and δ compatible with the data.

Lemma 3 *There exists a positive constant c depending on δ and on the data such that for $u \in \mathcal{U}_{2+\alpha}(\delta, T)$*

$$\mathcal{N}_{2+\alpha}[\mathcal{M}[u]] \leq cT^{\gamma} \tag{7.1}$$

with γ as in (6.2) and for $u_1, u_2 \in \mathcal{U}_{2+\alpha}(\delta, T)$

$$\mathcal{N}_{2+\alpha}[\mathcal{M}[u_1] - \mathcal{M}[u_2]] \leq cT^{1/2}\mathcal{N}_{2+\alpha}[u_1 - u_2] \tag{7.2}$$

Proof. It is a straightforward consequence of the inequalities in Lemma 1, producing (7.1), and in Lemma 2, producing (7.2), along with the basic estimate (4.4) for the linear problem (4.1)-(4.3) (see [10]).

□

Obviously (7.1) allows to select T in such a way that $\mathcal{M}[u] \in \mathcal{U}_{2+\alpha}(\delta, T)$, once δ is given compatibly with the data, while (7.2) shows how to possibly reduce T in order to make $\mathcal{M}$ a contractive mapping. Thus Theorem 1 is proved.

Proof of Theorem 2. Theorem 1 provides existence and uniqueness of solutions to the system (2.6), (2.10)-(2.14) which can be derived from solutions of problem (3.18), (3.25)-(3.27).

Let us now suppose that $p_1(x,t)$ is the solution in the class above and that $p_2(x,t)$ is another solution belonging to $C^{2+\alpha,1+\alpha.2}(\Omega_{2T})$.

In Section 3 we remarked that $p_1 \in C^{2+\alpha,1+\alpha/2}(\Omega_{1T})$, Ω_{jT} denoting the domain of definition of p_j.

We construct the functions

$$u_j = \hat{p}_j - p_0'\psi_j - r_0 \in \hat{C}^{2+\alpha,1+\alpha/2}(\Omega_{0T})$$

and we consider the difference $u_1 - u_2$, satisfying

$$c_0\varepsilon_0\frac{\partial}{\partial t}(u_1 - u_2) - k_0\frac{\partial^2}{\partial z^2}(u_1 - u_2) = F[u_1] - F[u_2], \tag{7.3}$$

$$-k_0 \frac{\partial}{\partial z}(u_1 - u_2)\Big|_{z=0} = G[u_1] - G[u_2], \tag{7.4}$$

$$-\frac{1}{p_0'}\frac{d}{dt}(\tilde{u}_1 - \tilde{u}_2) + \mu\frac{\partial}{\partial z}(u_1 - u_2)\Big|_{z=l_0 t} = 0. \tag{7.5}$$

Exploiting the fact that $u_1 - u_2 \in \hat{C}_\alpha^{2+\alpha,1+\alpha/2}(\Omega_{0T})$ we will show that

- the norm $\|F[u_1] - F[u_2]\|_{\alpha,\Omega_{0T}}$ is bounded (7.6)
- the norm $\|G[u_1] - G[u_2]\|_{\hat{C}_{\alpha/2}^{(1+\alpha)/2}(0,T)}$ is bounded (7.7)

Then, by Theorem 3 we improve our knowledge of $u_1 - u_2$, concluding that $u_1 - u_2$ actually belongs to $\hat{C}_{\alpha/2}^{2+\alpha,1+\alpha/2}(\Omega_{0T})$.

On the basis of this information we will show that (7.7) can be improved to

- the norm $\|G[u_1] - G[u_2]\|_{\hat{C}^{(1+\alpha)/2}(0,T)}$ is bounded (7.8)

Back to Theorem 3, at this point we deduce that $u_1 - u_2 \in \mathcal{C}^{2+\alpha,1+\alpha/2}(\Omega_{0T})$, implying that u_2 belongs to the same space as u_1, implying uniqueness.

Now we proceed to the proof of (7.6).

The differences $F_i[u_1] - F_i[u_2]$ with $i = 1, 2$ are easily estimated

$$\|F_1[u_1]-F_1[u_2]\|_{\alpha,\Omega_{0T}} \le c\Big(\|\hat{\varepsilon}_1-\hat{\varepsilon}_2\|_{\alpha,\Omega_{0T}}\left\{\frac{\partial u_1}{\partial t}+\frac{\partial r_0}{\partial t}\right\}_{\alpha,\Omega_{0T}}+\left\{\frac{\partial u_1}{\partial t}-\frac{\partial u_2}{\partial t}\right\}_{\alpha,\Omega_{0T}}\|\hat{\varepsilon}_2-\varepsilon_0\|_{\alpha,\Omega_{0T}}\Big), \tag{7.9}$$

$$\begin{aligned}\|F_2[u_1] - F_2[u_2]\|_{\alpha,\Omega_{0T}} &\le c\Big(\|\hat{k}_1 - \hat{k}_2\|_{\alpha,\Omega_{0T}} + \left\|\frac{\partial\psi_1}{\partial z} - \frac{\partial\psi_2}{\partial z}\right\|_{\alpha,\Omega_{0T}}\Big)\left\{\frac{\partial^2 u_1}{\partial z^2} + \frac{\partial^2 r_0}{\partial z^2}\right\}_{\alpha,\Omega_{0T}} \\ &+c\Big\{\frac{\partial^2 u_1}{\partial z^2} - \frac{\partial^2 u_2}{\partial z^2}\Big\}_{\alpha,\Omega_{0T}}\Big(\|\hat{k}_2 - k_0\|_{\alpha,\Omega_{0T}} + \|\frac{\partial\psi_2}{\partial z}\|_{\alpha,\Omega_{0T}}\Big).\end{aligned} \tag{7.10}$$

Since $\left\|\frac{\partial\psi_j}{\partial z}\right\|_{\alpha,\Omega_{0T}} \le cT^{-\alpha}\sup_{(0,T)}|\dot{\sigma}_j(t)| \le cT^{(1-\alpha)/2}$, these norms are bounded. The boundedness of $\|F_4[u_1] - F_4[u_2]\|_{\alpha,\Omega_{0T}}$ and of $\|F_5[u_1] - F_5[u_2]\|_{\alpha,\Omega_{0T}}$ is clear from the above estimates. The difference $F_3[u_1] - F_3[u_2]$ contains the second derivatives $\frac{\partial^2\psi_j}{\partial z^2}$ which are singular in the case $\dot{\sigma}_j \in C^{1/2+\alpha/2}(0,T)$: $\left|\frac{\partial^2\psi_j}{\partial z^2}(z,t)\right| \le ct^{-(1+\alpha)/2}$, but this singularity is compensated by the factor $q(z,t) = \frac{\partial\hat{p}_2}{\partial z}/(1+\frac{\partial\psi_2}{\partial z}) - p_0'$, so the norm of the product $q\frac{\partial^2\psi_j}{\partial z^2}$ is bounded:

$$\begin{aligned}\left\|q\frac{\partial^2\psi_i}{\partial z^2}\right\|_{\alpha,\Omega_{0T}} &\le \sup_{(0,T)} t^{-\alpha}|q(z,t)|\left|\frac{\partial^2\psi_j}{\partial z^2}(z,t)\right| + \sup_{(0,T)}\sup_{\Omega_{\tau/2,\tau}}|q(z,t)|\left[\frac{\partial^2\psi_j}{\partial z^2}\right]^{(\alpha,\alpha/2)}_{\Omega_{\tau/2,\tau}} \\ &+ \sup_{(0,T)}\sup_{\Omega_{\tau/2,\tau}}\left|\frac{\partial^2\psi_j}{\partial z^2}(z,t)\right|[q]^{(\alpha,\alpha/2)}_{\Omega_{\tau/2,\tau}} < \infty.\end{aligned}$$

Other terms in $F_3[u_1] - F_3[u_2]$ are estimated in a similar way.

Clearly, not only $\|F[u_1] - F_2[u_2]\|_{\alpha,\Omega_{0T}}$ but also the norms $\|F[u_1] - F_2[u_2]\|_{\hat{C}_\beta^{\alpha,\alpha/2}(\Omega_{0T})}$ with arbitrary non-negative β are bounded.

Let us turn to the proof of (7.7). We have

$$|u_1(0,t) - u_2(0,t)| \le \int_0^t |\frac{\partial u_1}{\partial t}(0,\tau) - \frac{\partial u_2}{\partial t}(0,\tau)|d\tau| \le ct^{1+\alpha/2}, \tag{7.11}$$

so, repeating the above simple calculation we obtain:

$$\|G[u_1]-G[u_2]\|_{\hat C^{(1+\alpha)/2}_{\alpha/2}(0,T)} \le \sup_{(0,T)} t^{-1-\alpha/2} \sup_{(0,T)} |G[u_1](0,t)-G[u_2](0,t)| +$$

$$+ \sup_{(0,T)} t^{\alpha/2}[G[u_1]-G[u_2]]^{((1+\alpha)/2)}_{(\tau/2,\tau)}$$

$$\le \Big(\|u_1(0,\cdot)-u_2(0,\cdot)\|_{C^{1+\alpha/2}(0,T)} + \|\frac{\partial u_1}{\partial z}(0,\cdot)-\frac{\partial u_2}{\partial z}(0,\cdot)\|_{C^{(1+\alpha)/2}(0,T)}\Big) < \infty$$

At the second step, we have, instead of (5.1),

$$|u_1(0,t)-u_2(0,t)| \le \int_0^t |\frac{\partial u_1}{\partial t}(0,\tau)-\frac{\partial u_2}{\partial t}(0,\tau)|d\tau \le ct^{1+\alpha}\|\frac{\partial u_1}{\partial t}-\frac{\partial u_2}{\partial t}\|_{\hat C^{\alpha,\alpha/2}_{\alpha/2}(0,T)},$$

and, as a consequence,

$$\|G[u_1]-G[u_2]\|_{\hat C^{(1+\alpha)/2}(0,T)} \le c\|u_1-u_2\|_{\hat C^{2+\alpha,1+\alpha/2}_{\alpha/2}(\Omega_{0T})} < \infty,$$

which completes the proof.

□

Acknowledgments

This work was partially supported by the Italian project “Mathematical Analysis of Phase Transitions” co-financed by MIUR and by the Portuguese grant PRAXIS XXI/2/2.1/MAT/125/94.

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