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On derivations of lattices

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Abstract. The notion of lattice derivation, introduced by Szász in [S1, S2], has been recently resumed in the study of several different problems. Our theoretical investigations ideally pursue and complete the ones initiated by Szász, who only scratched the surface of this subject.

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1 Introduction

Given an algebraic structure $[A;+,\cdot]$, where $+$ and $\cdot$ denote two arbitrary binary operations, we call *derivation* of A any function $f : A \longrightarrow A$ such that:

$$\begin{aligned} f(a+b) &= f(a)+f(b), \\ f(a\cdot b) &= f(a)\cdot b + a\cdot f(b). \end{aligned}$$

This definition clearly coincides with the usual (algebraic) notion of derivation when $[A;+,\cdot]$ is a ring. However, it can be formally stated for every algebraic structure endowed with two binary operations. In this paper, we will consider the special case in which $[A;+,\cdot]$ is a lattice, so that $+$ and $\cdot$ are, respectively, the join and the meet operations.

These ideas have been introduced and developed by Szász in a series of papers (here we recall [S1, S2]), in which he established the main properties of derivations of lattices. Also Kolibiar [K] gave his contribution, for example in the study of the case of the chain of natural numbers. However, it seems that these investigations only scratched the surface of the subject.

Several years later, some works appeared in which these concepts are considered again, even if, in most of the cases, only implicitly.

In [C] the author introduces the notion of Γ-lattice, in order to study the lattice of the submodules of a module over a commutative ring with identity. Using the terminology of the above paper, it turns out that the maps φ_γ, though in general not lattice derivations, have many properties in common with them (for instance, $\varphi_\gamma(0) = 0$ and φ_γ preserves joins). Moreover, the definition of c-purity recalls the characterization of lattice derivations in a lattice with maximum (see theorem 3 in the present paper). In [NP] the authors study the concept of translation on a graph. They explicitly cite the works of Szász [S1, S2]. It is clear that, if one considers the lattice of convex subsets of a graph such that the meet operation is given by taking the convex hull and the join operation by taking the greatest included convex (so that the usual order by inclusion is reversed), then the defining condition of a translation on a graph is essentially the same of the axiom concerning the behaviour of a lattice derivation with respect to the meet operation. Finally, the study of nonexpansive multipliers is undertaken in [PS]. Also in this case it is evident how the concept of multiplier derives from and is strictly related to that of lattice derivation (or, better, of lattice translation).

We think that the above examples fully justify the investigation of the properties of derivations of lattices from a purely theoretical point of view. In the present paper we first review the main results, due essentially to Szász, also providing some interesting examples. Then we embed any lattice having some additional properties into the lattice of its derivations, thus introducing a new kind of completion of a lattice. Next we construct an interesting derivation which can be defined in a vast class of lattices. These two last constructions are both believed to be new. Finally, we examine a particular derivation of the divisor lattice, which we have called the *radical*, and we show its numerous appearances in combinatorics, algebra and geometry.

2 Definition and basic properties

Let $\mathbf{L} = [L; \vee, \wedge]$ be a lattice; a *derivation* of $\mathbf{L}$ [S2] is a function $f : L \longrightarrow L$ satisfying:

1) $f(x \vee y) = f(x) \vee f(y)$,

2) $f(x \wedge y) = (f(x) \wedge y) \vee (x \wedge f(y))$, for any $x, y \in L$.

Before giving the basic instances of derivations of particular classes of lattices, it could be useful to learn some simple properties of such functions; we remark that some of them have already been stated in [S2]. However, a very elementary example can be given.

Let $\mathbf{B}_2 = \{0,1\} \times \{0,1\}$ be the Boolean algebra having 4 elements. Apart from the trivial derivations $\mathbf{0}$ and id (which are derivations in every lattice with minimum), we find the two functions f and g defined as follows:

$$f(a,b) = f(a,0), \qquad g(a,b) = g(0,b).$$

Moreover, if we denote $\mathcal{D}(\mathbf{B}_2)$ the set of the derivations of $\mathbf{B}_2$, it can be easily seen that $\mathcal{D}(\mathbf{B}_2) = \{\mathbf{0}, id, f, g\}$.

Now we start with the very first elements of the theory of lattice derivations. First of all, let us have a look to the definition of derivation, and particularly to condition 2), i.e.

$$f(x \wedge y) = (f(x) \wedge y) \vee (x \wedge f(y)).$$

We can prove that this condition is redundant, and we find a suitable way to simplify it. Indeed, since $x \wedge y \leq x$, we have $f(x \wedge y) \leq f(x)$, and we also have $f(x \wedge y) \leq x \wedge y \leq y$. These two last facts are simple consequences of the definition of derivation, and can be immediately derived, for example, from the next theorem 1. Therefore we have obtained $f(x \wedge y) \leq f(x) \wedge y$. In the same way we also get $f(x \wedge y) \leq x \wedge f(y)$, so we can conclude that condition 2) is equivalent to the following:

$$2') \qquad f(x \wedge y) = f(x) \wedge y = x \wedge f(y).$$

A map satisfying condition 2') is called a *(meet-)translation* in [S1, S2]. The dual concept of join-translation has been introduced and studied for the first time in [S1].

The first concept we need to recall from general lattice theory is that of dual closure.

Given a lattice $\mathbf{L}$, a function $f : L \longrightarrow L$ is said to be a *dual closure* when the following conditions hold:

i) f is an order morphism;

ii) $f(x) \leq x$, for any $x \in L$;

iii) $f(f(x)) = f(x)$, for any $x \in L$.

The following theorem is a collection of some results scattered throughout [S1, S2].

THEOREM 1 *Every lattice derivation is a dual closure.*

Proof. Let $f : L \longrightarrow L$ be a lattice derivation.

i) Given $x, y \in L$, if $x \leq y$, then $x \wedge y = x$, and so $f(x) = f(x \wedge y) = x \wedge f(y) \leq f(y)$.

ii) From 2) of the definition of derivation we get $f(x) = f(x \wedge x) = f(x) \wedge x$, and so $f(x) \leq x$.

iii) We have:

$$\begin{aligned} f^2(x) &= f(f(x \wedge x)) = f(f(x) \wedge x) \\ &= f(x) \wedge f(x) = f(x). \end{aligned} \qquad \square$$

Remark. Notice that in the above proof we have never used the fact that f is a join-endomorphism. This means that every meet-translation is a dual closure.

It is very easy to see that the above theorem is not invertible. Indeed, consider the Boolean algebra $\mathbf{B}_2$ and define the function $f : B_2 \longrightarrow B_2$ as follows:

$$f(x, y) = \begin{cases} (0,0), \text{if } (x, y) \neq (1,1) \\ (1,1), \text{if } (x, y) = (1,1) \end{cases}$$

It is immediately seen that f is a dual closure but not a derivation.

An element $a \in L$ is said to be *fixed* for the derivation f when $a = f(a)$; we denote by Fix f the set of the fixed elements of f. Observe that Fix $f = \mathrm{Im}\, f$, as a simple consequence of theorem 1, *iii)*.

PROPOSITION 2 [S2] *A derivation f is a lattice homomorphism of* $\mathbf{L}$ *and also preserves the minimum* 0. *In particular,* $\ker f = \{a \in L | f(a) = 0\}$ *is an ideal of* $\mathbf{L}$. *Furthermore,* Fix f *is an ideal of* $\mathbf{L}$ *too.*

Proof. We show that f preserves meets. For x, y in L, we get:

$$\begin{aligned} f(x \wedge y) &\leq f(x) \wedge f(y), \quad \text{for } f \text{ is an order-morphism,} \\ f(x) \wedge f(y) &\leq f(x) \wedge y = f(x \wedge y), \quad \text{from 2').} \end{aligned}$$

Now consider $x, y \in$ Fix f; clearly $f(x \vee y) = f(x) \vee f(y) = x \vee y$. So it remains only to prove that, for any $x \in$ Fix f and $y \leq x$, we have $y \in$ Fix f. Indeed, we get $y = y \wedge x = y \wedge f(x) = f(x \wedge y) = f(y)$ which is enough to conclude. □

Remark. Notice that, if $\mathbf{L}$ is a modular lattice, condition 2) can be rewritten in the following way:

$$\begin{aligned} f(x \wedge y) &= (f(x) \wedge y) \vee (x \wedge f(y)) = y \wedge (f(x) \vee (x \wedge f(y))) \\ &= y \wedge (x \wedge (f(x) \vee f(y))) = (x \wedge y) \wedge f(x \vee y). \end{aligned}$$

Recall ([CD]) that an element a of a lattice $\mathbf{L}$ is said to be *distributive* whenever, for every $x, y \in L$, $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$. The set of all the distributive elements of a lattice is called the *center* of the lattice.

The next, fundamental result states that for a very large class of lattices all the derivations are of the same (very simple) form.

THEOREM 3 [S2] *Consider a lattice* $\mathbf{L}$ *with a maximum 1. Then* $f : L \longrightarrow L$ *is a derivation of* $\mathbf{L}$ *if and only if there exists a distributive element* $a \in L$ *such that* $f(x) = a \wedge x$, *for every* $x \in L$. *Obviously, in this case we have* $a = f(1)$.

Proof. Clearly, if $a \in L$ is a distributive element, every function of the form $f(x) = a \wedge x$ is a derivation (even if $\mathbf{L}$ does not have a maximum). Now suppose $f : L \longrightarrow L$ is a derivation; by definition we have:

$$f(x) = f(x \wedge 1) = x \wedge f(1).$$

Clearly $f(1) \in L$ must be a distributive element, since f is a derivation, and this completes our proof. □

Therefore, for a distributive lattice $\mathbf{L}$ with maximum 1, the class of the derivations of $\mathbf{L}$ coincides with the class of the maps $f_a : L \longrightarrow L$ defined by $f_a(x) = a \wedge x$. More precisely, f_a is the only derivation of $\mathbf{L}$ such that $f(1) = a$. We will call these derivations *simple derivations* (even if the lattice $\mathbf{L}$ is not distributive), and we will say that f_a is the simple derivation *associated with* a.

Thus, in the rest of the paper we will be mainly interested in derivations of lattices either unbounded or nondistributive.

Recall that in [S1, S2] distributivity in lattices is characterized by means of translations. More precisely, it is shown that a lattice is distributive if and only if the sets of meet-translations and of derivations coincide.

To close this section, we find interesting to state a few further properties of derivations, in order to investigate their behaviour with respect to some common operations. Part of the following proposition has been proved in [S2].

Proposition 4 *If f, g are derivations of the lattice $\mathbf{L}$, then $f \circ g$ is a derivation too. Moreover, if $\mathbf{L}$ is distributive, then $f \vee g$ and $f \wedge g$ are derivations, and we also have $f \wedge g = f \circ g$.*

Proof. A simple computation shows that

$$\begin{aligned} (f \circ g)(x \vee y) &= f(g(x) \vee g(y)) = (f \circ g)(x) \vee (f \circ g)(y), \\ (f \circ g)(x \wedge y) &= f(g(x) \wedge y) = (f \circ g)(x) \wedge y, \end{aligned}$$

which is the first thesis.

For the second part of the proposition, we have:

$$\begin{aligned} (f \vee g)(x \vee y) &= f(x \vee y) \vee g(x \vee y) = f(x) \vee f(y) \vee g(x) \vee g(y) \\ &= (f \vee g)(x) \vee (f \vee g)(y); \end{aligned}$$

$$\begin{aligned} (f \vee g)(x \wedge y) &= f(x \wedge y) \vee g(x \wedge y) = (f(x) \wedge y) \vee (g(x) \wedge y) \\ &= y \wedge (f(x) \vee g(x)) = y \wedge (f \vee g)(x); \end{aligned}$$

$$\begin{aligned} (f \wedge g)(x \vee y) &= f(x \vee y) \wedge g(x \vee y) = (f(x) \vee f(y)) \wedge (g(x) \vee g(y)) \\ &= (f(x) \wedge g(y)) \vee (f(y) \wedge g(x)) \vee (f \wedge g)(x) \vee (f \wedge g)(y) \\ &=^{(*)} (f \wedge g)(x) \vee (f \wedge g)(y), \end{aligned}$$

$$\begin{aligned} (f \wedge g)(x \wedge y) &= f(x \wedge y) \wedge g(x \wedge y) = f(x) \wedge y \wedge g(x) \wedge y \\ &= (f \wedge g)(x) \wedge y. \end{aligned}$$

Equality (*) can be explained as follows. We have $(f \wedge g)(x \wedge y) = f(x \wedge y) \wedge g(x \wedge y) = f(x) \wedge y \wedge x \wedge g(y) = f(x) \wedge g(y)$, and so both $f(x) \wedge g(y) \leq (f \wedge g)(x)$ and $f(x) \wedge g(y) \leq (f \wedge g)(y)$.

Finally, we immediately have:

$$(f \circ g)(x) = f(x \wedge g(x)) = f(x) \wedge g(x) = (f \wedge g)(x)$$

so our proof is complete. □

Remark. Note that the equality $f \circ g = f \wedge g$ does not depend on the distributivity of $\mathbf{L}$; this means that the map $f \wedge g$ is a derivation even if the lattice $\mathbf{L}$ is not distributive.

3 Some detailed examples

In this section we study in some details a few concrete examples of derivations in very special lattices.

As we have said in the above section, we will only consider lattices either unbounded or nondistributive, which are the only cases in which the derivations can be nontrivial.

3.1 The chain of natural numbers

Consider the lattice $[\mathbf{N}; \max, \min]$, whose associated poset is the chain $[\mathbf{N}; \leq]$ with the usual total order. This lattice does not have a maximum, nevertheless it possesses only trivial derivations. We remark that this example has also been considered in [K].

THEOREM 5 *A map* $f : \mathbf{N} \longrightarrow \mathbf{N}$ *is a derivation if and only if either* $f = id_{\mathbf{N}}$ *or* $f(x) = \min(a, x)$ *for some* $a \in \mathbf{N}$.

Proof. $\Leftarrow$) Trivial.

$\Rightarrow$) Consider a derivation $f : \mathbf{N} \longrightarrow \mathbf{N}$ and suppose that $f \neq id_{\mathbf{N}}$. Then there exists an element $n+1$ such that $f(n+1) \leq n$; in particular, suppose that $\overline{n}+1$ is the minimum integer with this property. Clearly $f(\overline{n}+1) = \overline{n}$ otherwise we would have $f(\overline{n}+1) < \overline{n} = f(\overline{n}) \leq f(\overline{n}+1)$. Furthermore, if $m \in \mathbf{N}$, we get $f(\overline{n}+m) = \overline{n}$; indeed, if we had $f(\overline{n}+m) \geq \overline{n}+1$, we would get $\overline{n}+1 \leq \min(f(\overline{n}+m), \overline{n}+1) = f(\min(\overline{n}+m, \overline{n}+1)) = f(\overline{n}+1) = \overline{n}$, a contradiction. Thus, if we set $a = \overline{n}$, we have $f(n) = \min(a, n)$, for every $n \in \mathbf{N}$, as desired. □

Thanks to the above theorem, we can assert that the case of the chain of natural numbers is completely analogous to that of a bounded distributive lattice, in the sense that all the derivations are simple. The only exception is the identity, which obviously cannot be simple. However, if one “completes" the chain $[\mathbf{N}; \leq]$ by adding a maximum ∞ (this is usually called the Dedekind-McNeille completion of $[\mathbf{N}; \leq]$ and we will denote it $\overline{\mathbf{N}}$, see [DP]), it turns out that every derivation of $\mathbf{N}$ is the restriction of a simple derivation of its

Dedekind-McNeille completion $\overline{\mathbf{N}}$ (the identity of $\mathbf{N}$ is simply the restriction of the identity of $\overline{\mathbf{N}}$, which is the simple derivation of $\overline{\mathbf{N}}$ associated with the maximum ∞).

3.2 Direct products

Another interesting case to investigate is that of a direct product of lattices. The first result we obtain is the following proposition, which provides a special class of derivations, namely the "projections" onto a finite subproduct.

PROPOSITION 6 *Consider a family $(L_i)_{i\in I}$ of lattices with minimum 0, and denote by $\prod_{i\in I} L_i$ the direct product of the family. Consider a finite subset $I_0 \subseteq I$ and take the function $f_{I_0} : \prod_{i\in I} L_i \longrightarrow \prod_{i\in I} L_i$ defined by $f((x_i)_{i\in I}) = (y_i)_{i\in I}$, where $y_i = x_i$ if $i \in I_0$ and $y_i = 0$ if $i \notin I_0$. Then f_{I_0} is a derivation of the lattice $\prod_{i\in I} L_i$.*

Proof. We examine only the case in which $I = \{1, 2\}$, since the general case can be treated exactly in the same way. So consider, for example, the map $f_1 : L_1 \times L_2 \longrightarrow L_1 \times L_2$ defined by $f_1(x_1, x_2) = (x_1, 0)$. The proof that f_1 is a derivation is a straightforward verification:

$$\begin{aligned} f_1((x_1, x_2) \vee (y_1, y_2)) &= f_1(x_1 \vee y_1, x_2 \vee y_2) \\ &= (x_1 \vee y_1, 0) = (x_1, 0) \vee (y_1, 0) \\ &= f_1(x_1, x_2) \vee f_1(y_1, y_2); \end{aligned}$$

$$\begin{aligned} f_1((x_1, x_2) \wedge (y_1, y_2)) &= f_1(x_1 \wedge y_1, x_2 \wedge y_2) \\ &= (x_1 \wedge y_1, 0) = (x_1, 0) \wedge (y_1, y_2) \\ &= f_1(x_1, x_2) \wedge (y_1, y_2) = (x_1, x_2) \wedge (y_1, 0) \\ &= (x_1, x_2) \wedge f_1(y_1, y_2). \square \end{aligned}$$

The derivations found in the above proposition are not simple derivations in general (it can be easily seen by observing that, if the L_i are unbounded, then $\mathrm{Im} f_{I_0}$ is unbounded as well, for $I_0 \neq \emptyset$). So the situation is slightly different from the case of a chain, since now we have instances of nontrivial derivations (i.e., different from the identity) which are not simple. Furthermore, in this case also the concept of Dedekind-McNeille completion cannot help: indeed, if we consider, e.g., the direct product $\mathbf{N}\times\mathbf{N}$, we observe that its Dedekind-McNeille completion is obtained simply by adding a maximum ∞, but it is clear that the two projections (which are derivations thanks to the last proposition) are not simple even in $(\mathbf{N} \times \mathbf{N}) \cup \{\infty\}$. However, if we denote by $\overline{\mathbf{N}} = \mathbf{N} \cup \{\infty\}$ the Dedekind-McNeille completion of $\mathbf{N}$ (as we did in the previous subsection), we see that the projections are simple derivations of the complete lattice $\overline{\mathbf{N}} \times \overline{\mathbf{N}}$ (associated with the elements $(\infty, 0)$ and $(0, \infty)$). More generally, taken a family $(L_i)_{i\in I}$ of lattices, if $\overline{L_i}$ is any completion of L_i, then the projections associated with any finite subset of I are simple derivations of the lattice $\prod_{i\in I} \overline{L_i}$.

For the rest of this subsection we will use the following notations:

$$\begin{aligned} \pi_j &: \prod_{i\in I} L_i \longrightarrow L_j \\ &: (a_i)_{i\in I} \longmapsto a_j \\ \iota_j &: L_j \longrightarrow \prod_{i\in I} L_i \\ &: a_j \longmapsto (\delta_{ij} a_j)_{i\in I} \end{aligned}$$

where δ_{ij} is the Kronecher delta. As usual, we will call the maps π_j *projections* and the maps ι_j *immersions*[1]. Given a family of functions $(f_i)_{i\in I}$, with $f_i : L_i \longrightarrow L_i$, for every $i \in I$, we define the *direct product* of the above family to be the map

$$\begin{aligned} \bigotimes_{i\in I} f_i &: \prod_{i\in I} L_i \longrightarrow \prod_{i\in I} L_i \\ &: (a_i)_{i\in I} \longmapsto (f_i(a_i))_{i\in I}. \end{aligned} \tag{1}$$

Proposition 7 *Let $f = \bigotimes_{i\in I} f_i$ as in (1). If f is a derivation of $\prod_{i\in I} L_i$, then f_j is a derivation of L_j, for every $j \in I$.*

Proof. Fix $j \in I$. Then it is immediate to see that

$$f_j = \pi_j \circ f \circ \iota_j.$$

The axioms of a derivation are now easy to prove. We only show that $f_j(a \wedge b) = a \wedge f_j(b)$, leaving the fact that f_j is a join-homomorphism to the reader:

$$\begin{aligned} f_j(a \wedge b) &= \pi_j(f(\iota_j(a \wedge b))) = \pi_j(\iota_j(a) \wedge f(\iota_j(b))) \\ &= a \wedge f_j(b). \end{aligned}$$

□

Since the converse of the above proposition is trivial, we have that a direct product of functions is a derivation if and only if each factor is a derivation.

More generally, we can characterize the derivations of a (finite) direct product of lattices as follows. Define a *π-derivation* as a function $f : \prod_{i\in I} L_i \longrightarrow L_j$ (for some $j \in I$) satisfying:

i) $f((a_i)_{i\in I} \vee (b_i)_{i\in I}) = f((a_i)_{i\in I}) \vee f((b_i)_{i\in I})$;

ii) $f((a_i)_{i\in I} \wedge (b_i)_{i\in I}) = f((a_i)_{i\in I}) \wedge b_j = a_j \wedge f((b_i)_{i\in I})$.

Then it is easy to prove the following:

Theorem 8 *The map $f : \prod_{i\in I} L_i \longrightarrow \prod_{i\in I} L_i$ is a derivation if and only if all the projections $\pi_j \circ f : \prod_{i\in I} L_i \longrightarrow L_j$ are π-derivations.*

[1] recall that they are lattice homomorphisms.

3.3 The divisor lattice $[\mathbf{N}^*; \mathrm{lcm}, \mathrm{gcd}]$

The set $\mathbf{N}^*$ of positive integers endowed with the well-known operations of lcm and gcd is a distributive lattice without maximum. It is clear that this lattice is isomorphic to a sublattice of the direct product of $|\mathbf{N}|$ copies of $\mathbf{N}$ considered as a chain with its natural order. More precisely, it is the sublattice of $\mathbf{N}^{\mathbf{N}}$ constituted by all the elements having finite support (i.e., of the form $(a_n)_{n\in\mathbf{N}}$ where $a_n \neq 0$ only for a finite number of $n \in \mathbf{N}$). The interest of this example lies in the possibility of defining and studying a particular function, whose appearences in algebra, geometry and combiatorics are plentiful. We call *radical* of a positive integer $n = p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}$ (uniquely factorized into product of primes) the positive integer $r(n) = r(p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}) = p_1 \cdot \ldots \cdot p_r$. Therefore the radical of a number is the product of the primes of its factorization.

Proposition 9 *The radical function $r : \mathbf{N} \longrightarrow \mathbf{N}$ is a derivation of the lattice* $[\mathbf{N}^*; \mathrm{lcm}, \mathrm{gcd}]$.

Proof. Let $n = p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r} \cdot p_{r+1}^{\alpha_{r+1}} \cdot \ldots \cdot p_s^{\alpha_s}$, $m = p_1^{\beta_1} \cdot \ldots \cdot p_r^{\beta_r} \cdot q_{r+1}^{\beta_{r+1}} \cdot \ldots \cdot q_t^{\beta_t}$ (in this way we distinguish the common primes belonging to both n and m from the other primes occurring in the two factorizations). It is not difficult to show that the following equalities hold:

$$\begin{aligned}
& r(\mathrm{lcm}(n,m)) = \\
= \; & r\left(p_1^{\max(\alpha_1,\beta_1)} \cdot \ldots \cdot p_r^{\max(\alpha_r,\beta_r)} \cdot p_{r+1}^{\alpha_{r+1}} \cdot \ldots \cdot p_s^{\alpha_s} \cdot q_{r+1}^{\beta_{r+1}} \cdot \ldots \cdot q_t^{\beta_t}\right) \\
= \; & p_1 \cdot \ldots \cdot p_r \cdot p_{r+1} \cdot \ldots \cdot p_s \cdot q_{r+1} \cdot \ldots \cdot q_t \\
= \; & \mathrm{lcm}(p_1 \cdot \ldots \cdot p_s, p_1 \cdot \ldots \cdot p_r \cdot q_{r+1} \cdot \ldots \cdot q_t) \\
= \; & \mathrm{lcm}(r(n), r(m));
\end{aligned}$$

$$\begin{aligned}
r(\gcd(n,m)) &= r\left(p_1^{\min(\alpha_1,\beta_1)} \cdot \ldots \cdot p_r^{\min(\alpha_r,\beta_r)}\right) \\
&= p_1 \cdot \ldots \cdot p_r \\
&= \gcd\left(p_1^{\alpha_1} \cdot \ldots \cdot p_s^{\alpha_s}, p_1 \cdot \ldots \cdot p_r \cdot q_{r+1} \cdot \ldots \cdot q_t\right) \\
&= \gcd(n, r(m)) \quad (= \gcd(r(n), m)).\square
\end{aligned}$$

Some applications concerning the radical function will be given at the end of the paper.

4 The Der-completion of a lattice

In many of the examples considered in the previous section it happens that, for a given lattice $\mathbf{L}$, it is possible to define a suitable completion $\overline{\mathbf{L}}$ of $\mathbf{L}$ such that the set $\mathcal{D}(\overline{\mathbf{L}})$ of the derivations of $\overline{\mathbf{L}}$ coincides with the set $\mathcal{D}(\mathbf{L})$ of the derivations of $\mathbf{L}$ (and, of course, all such derivations are simple in $\overline{\mathbf{L}}$). This suggests the idea that, if the lattice $\mathbf{L}$ possesses nice properties, then the set

$\mathcal{D}(\mathbf{L})$ of derivations of $\mathbf{L}$ is a completion of $\mathbf{L}$. The next theorem is a first result in this direction.

THEOREM 10 *Let* $\mathbf{D}$ *be a locally finite and distributive lattice with minimum 0. Consider on* $\mathcal{D}(\mathbf{D})$ *the usual meet and join operations as defined in section 2. Then* $\mathcal{D}(\mathbf{D})$ *is a complete lattice containing (a copy of)* $\mathbf{D}$*: it will be called the* Der-completion *of* $\mathbf{D}$.

Proof. Given $(f_i)_{i\in I} \subseteq \mathcal{D}(\mathbf{D})$, we can define:

$$\left(\bigvee_{i\in I} f_i\right)(x) = \bigvee_{i\in I} f_i(x), \tag{2}$$

$$\left(\bigwedge_{i\in I} f_i\right)(x) = \bigwedge_{i\in I} f_i(x). \tag{3}$$

The r.h. sides of (2) and (3) are both well-defined. Indeed, $f_i(x) \leq x, \forall i \in I$, hence $f_i(x) \in [0,x], \forall i \in I$. Since $\mathbf{D}$ is locally finite, there exists a finite subset $\overline{I}$ of I such that $\{f_i(x) \mid i \in I\} = \{f_i(x) \mid i \in \overline{I}\}$, and so $\bigvee_{i\in I} f_i(x) = \bigvee_{i\in \overline{I}} f_i(x)$ and $\bigwedge_{i\in I} f_i(x) = \bigwedge_{i\in\overline{I}} f_i(x)$. Therefore arbitrary sups and infs of elements of $\mathcal{D}(\mathbf{D})$ are defined, at least as functions from D to itself. Besides, it is clear that the set $\mathcal{S}(\mathbf{D}) \subseteq \mathcal{D}(\mathbf{D})$ of simple derivations is isomorphic to $\mathbf{D}$.

Thus it only remains to show that $\mathcal{D}(\mathbf{D})$ is a complete lattice, that is arbitrary joins and meets of derivations are derivations too. Let's start with the case of the join, and consider the function $\bigvee_{i\in I} f_i$, where all the f_i are derivations of $\mathbf{D}$. We have:

$$\begin{aligned}\left(\bigvee_{i\in I} f_i\right)(x\vee y) &= \bigvee_{i\in I} f_i(x\vee y) = \bigvee_{i\in I}\left(f_i(x)\vee f_i(y)\right)\\ &= \bigvee_{i\in I} f_i(x) \vee \bigvee_{i\in I} f_i(y) = \left(\bigvee_{i\in I} f_i\right)(x) \vee \left(\bigvee_{i\in I} f_i\right)(y);\end{aligned}$$

so the function $\bigvee_{i\in I} f_i$ is a join-homomorphism. Its behaviour with respect to the meet operation is a bit more difficult to investigate. We have:

$$\left(\bigvee_{i\in I} f_i\right)(x\wedge y) = \bigvee_{i\in I} f_i(x\wedge y);$$

obviously, since $f_i(x\wedge y) \leq x\wedge y \leq x\vee y$, we can consider a finite set $\overline{I} \subseteq I$ such that $\{f_i(z) \mid i \in I, z \leq x\vee y\} = \{f_i(z) \mid i \in \overline{I}, z \leq x\vee y\}$. It is then clear that, for any $z \leq x\vee y$, we have $\bigvee_{i\in I} f_i(z) = \bigvee_{i\in\overline{I}} f_i(z)$, where the r. h. s. is now a finite join. Therefore we get:

$$\begin{aligned}\left(\bigvee_{i\in I} f_i\right)(x\wedge y) &= \bigvee_{i\in \overline{I}} f_i(x\wedge y) = \bigvee_{i\in \overline{I}} (x\wedge f_i(y))\\ &= x\wedge \bigvee_{i\in \overline{I}} f_i(y) = x\wedge \left(\bigvee_{i\in I} f_i\right)(x),\end{aligned}$$

which is enough to conclude that $\bigvee_{i\in I} f_i$ is a derivation. The argument to be used for the function $\bigwedge_{i\in I} f_i$ is analogous. As far as the join operation is concerned, we have

$$\left(\bigwedge_{i\in I} f_i\right)(x\vee y) = \bigwedge_{i\in I} f_i(x\vee y) = \bigwedge_{i\in \overline{I}} f_i(x\vee y),$$

where, as before, $\overline{I}\subseteq I$ is a finite set such that $\{f_i(z) \mid i\in I, z\leq x\vee y\} = \{f_i(z) \mid i\in \overline{I}, z\leq x\vee y\}$. From proposition 4 we know that any finite meet of derivations is a derivation, and so

$$\begin{aligned}\bigwedge_{i\in \overline{I}} f_i(x\vee y) &= \bigwedge_{i\in \overline{I}} f_i(x)\vee \bigwedge_{i\in \overline{I}} f_i(y)\\ &= \left(\bigwedge_{i\in I} f_i\right)(x)\vee \left(\bigwedge_{i\in I} f_i\right)(y).\end{aligned}$$

Finally, for the meet we have immediately:

$$\begin{aligned}\left(\bigwedge_{i\in I} f_i\right)(x\wedge y) &= \bigwedge_{i\in I} f_i(x\wedge y) = \bigwedge_{i\in I} x\wedge f_i(y)\\ &= x\wedge \left(\bigwedge_{i\in I} f_i\right)(y),\end{aligned}$$

so the proof is complete. □

Remark. Clearly, if $\mathbf{D}$ is a distributive lattice with maximum, then $\mathcal{D}(\mathbf{D})\simeq \mathbf{D}$.

Examples. As we have seen in section 3.1, the Der-completion of the chain of natural numbers is the chain $\overline{\mathbf{N}} = \mathbf{N}\cup\{\infty\}$, where ∞ is greater than any natural number: this is nothing else than the Dedekind-Mc Neille completion of $\mathbf{N}$. If one considers a finite direct product of the form $\mathbf{N}^r$ (endowed with coordinatewise meet and join), then its Der-completion is the lattice $\overline{\mathbf{N}}^r$ (which is not the Dedekind-Mc Neill completion of $\mathbf{N}^r$). Finally, the Der-completion of the lattice $[\mathbf{N}; \mathrm{lcm}, \gcd]$ is (isomorphic to) the lattice of all the infinite sequences of elements of $\overline{\mathbf{N}}$ (with coordinatewise meet and join).

5 Infinitely distributive lattices

Let $\mathbf{D}$ be a distributive lattice with minimum 0. We say that $\mathbf{D}$ is *infinitely distributive* when, for every $a, x_i \in D$, the following equalities hold:

$$a \vee \bigwedge_i x_i = \bigwedge_i (a \vee x_i),$$
$$a \wedge \bigvee_i x_i = \bigvee_i (a \wedge x_i),$$

provided that the above infs and sups exist. There is a nice example of derivation, which can be defined in any infinitely distributive lattice.

THEOREM 11 *Let* $\mathbf{D}$ *be an infinitely distributive lattice with minimum 0. The function*

$$\begin{aligned} f \quad &: \quad D \longrightarrow D \\ &: \quad x \longmapsto \bigvee_{\substack{a \leq x \\ a \text{ atom}}} a \end{aligned}$$

is a derivation of $\mathbf{D}$*.*

Proof. As usual, we have to study the behaviour of f with respect to the meet and join operations. We have:

$$f(x \vee y) = \bigvee_{\substack{c \leq x \vee y \\ c \text{ atom}}} c;$$

now observe that in a distributive lattice, if c is an atom, then $c \leq x \vee y$ if and only if $c \leq x$ or $c \leq y$. Indeed, if $c \leq x \vee y$, then $c \wedge (x \vee y) = c$ and, using distributivity, $(c \wedge x) \vee (c \wedge y) = c$. Now use the fact that c is an atom to conclude that $c \wedge x = c$ or $c \wedge y = c$, that is $c \leq x$ or $c \leq y$. Thus we have

$$\bigvee_{\substack{c \leq (x \vee y) \\ c \text{ atom}}} c = \bigvee_{\substack{a \leq x \\ a \text{ atom}}} a \vee \bigvee_{\substack{b \leq y \\ b \text{ atom}}} b = f(x) \vee f(y),$$

which is enough to conclude that $f(x \vee y) = f(x) \vee f(y)$.

Next we have

$$f(x) \wedge y = \left(\bigvee_{\substack{a \leq x \\ a \text{ atom}}} a \right) \wedge y = \bigvee_{\substack{a \leq x \\ a \text{ atom}}} (a \wedge y). \tag{4}$$

In the last equality we have used the hypothesis that $\mathbf{D}$ is infinitely distributive. Now observe that, if a is an atom and $a \not\leq y$, then clearly $a \wedge y = 0$.

Therefore, in the r. h. s. of (4) we can replace the atoms less than x with the atoms less than $x \wedge y$, so obtaining:

$$\bigvee_{\substack{a \le x \\ a \text{ atom}}} (a \wedge y) = \bigvee_{\substack{c \le x \wedge y \\ c \text{ atom}}} (c \wedge y) = \bigvee_{\substack{c \le x \wedge y \\ c \text{ atom}}} c = f(x \wedge y).$$

Thus we can conclude that $f(x \wedge y) = f(x) \wedge y$, and this completes our proof. □

Example. The radical function of $[\mathbf{N}; \mathrm{lcm}, \gcd]$ is clearly a derivation of this form.

6 The radical function in combinatorics, algebra and geometry

6.1 Multisets

A *multiset* $\mathcal{M}$ is a set of pairs $(m, \alpha) \in M \times \mathbf{N}$, where M is any set. This definition is much less rigorous than many other ones, however it will be enough for our purpouses. If $(m, \alpha) \in \mathcal{M}$, then α is called the *multiplicity* of m in M. The set M is called the *support* of $\mathcal{M}$. It is clear that any set can be viewed as a special multiset whose elements all have multiplicity 1. Given a family of multisets, one can endow it with an obvious partial order, by saying that $\mathcal{M} \le \mathcal{N}$ whenever, for any $(m, \alpha) \in \mathcal{M}$, there exists $\beta \ge \alpha$ such that $(m, \beta) \in \mathcal{N}$. The operations $\vee$ and $\wedge$ of sup and inf between two multisets induced by the above partial order are lattice operations; so, if a family of multisets is closed under $\vee$ and $\wedge$, we will call it a *lattice of multisets*. Observe that any lattice of multisets is trivially distributive.

PROPOSITION 12 *Let $\mathfrak{M}$ be a lattice of multisets. Then the function*

$$\begin{array}{rcl} r & : & \mathfrak{M} \longrightarrow \mathfrak{M} \\ & : & \mathcal{M} \longmapsto r(\mathcal{M}) \end{array}$$

which maps any $\mathcal{M} \in \mathfrak{M}$ to its support $M = r(\mathcal{M})$ is a derivation of $\mathfrak{M}$. In particular, if $\mathfrak{M}$ is the family of all finite multisets, then $\mathfrak{M}$ is isomorphic to the divisor lattice $[\mathbf{N}^*; \mathrm{lcm}, \gcd]$ and r is precisely the radical function.

6.2 Arithmetical functions

Consider the set of all the functions $f : \mathbf{N} \longrightarrow \mathbf{C}$ (these are called *arithmetical functions*) endowed with the usual sum and scalar multiplication ($(f + g)(n) = f(n) + g(n)$, $\alpha f(n) = f(\alpha n)$) and with the *convolution* operation: $(f \star g)(n) = \sum_{d|n} f(d) g(\frac{n}{d})$. The algebra obtained this way is usually called *Dirichlet algebra*, and we will denote it by $\mathcal{D}$.

The radical function is clearly an element of $\mathcal{D}$. It is called the *core function* in [MC], where it is also stated the following identity (μ and ϕ are the usual Möbius and Euler functions of number theory):

$$r(n) = \sum_{d|n} |\mu(d)|\phi(d), \qquad \forall n.$$

One of the most remarkable properties of the radical function from the point of view of Dirichlet algebra is stated in the next proposition.

PROPOSITION 13 *r is a* multiplicative function, *i.e. for any n, m such that* $\gcd(n,m) = 1$, $r(n \cdot m) = r(n) \cdot r(m)$.

Proof. It is an immediate consequence of the fact that r is a join-homomorphism in the divisor lattice. □

As a byproduct of the above proposition, we have that r is invertible in $\mathcal{D}$. It is quite easy to determine r^{-1}.

PROPOSITION 14 *We have:*

$$r^{-1}(1) = 1, \qquad r^{-1}(p^n) = (-1)^n p \cdot (p-1)^{n-1}, \quad n > 0.$$

Proof. Clearly $r^{-1}(1) = 1$. By induction, suppose that

$$r^{-1}(p^k) = (-1)^k p \cdot (p-1)^{k-1}, \quad \text{if} \quad 0 < k < n\,.$$

Since $(r \star r^{-1})(p^n) = 0$ ($n \neq 0$), we must have:

$$\begin{aligned} 0 &= (r \star r^{-1})(p^n) = \sum_{k=0}^{n} r(p^k) \cdot r^{-1}(p^{n-k}) \\ &= r^{-1}(p^n) + p \cdot \sum_{k=1}^{n-1} (-1)^k p(p-1)^{k-1} + p, \end{aligned}$$

whence

$$\begin{aligned} r^{-1}(p^n) &= p \cdot \sum_{k=1}^{n-1} (-1)^{k-1} p(p-1)^{k-1} - p \\ &= p^2 \cdot \sum_{k=0}^{n-2} (1-p)^k - p = p^2 \cdot \frac{1-(1-p)^{n-1}}{p} - p \\ &= -p(1-p)^{n-1} = (-1)^n p(p-1)^{n-1}, \end{aligned}$$

which is the desired expression for $r^{-1}(p^n)$. □

Remark. The function r^{-1} is interesting from a combinatorial point of view. Indeed for any prime p, the polynomial $(-1)^n r^{-1}(p^n)$ (in p) is the chromatic polynomial of a tree having n vertices.

In some cases, the convolution of r with well-known arithmetical functions gives some interesting results.

PROPOSITION 15 *If μ is the usual Möbius function on* $\mathbf{N}$*, then:*

$$(r \star \mu)(n) = \begin{cases} 1, & \text{if } n = 1, \\ \phi(n), & \text{if } n \text{ is square-free}, \\ 0, & \text{otherwise}, \end{cases}$$

where ϕ is the usual Euler function.

Proof. Clearly $(r \star \mu)(1) = 1$, since the convolution of multiplicative functions is again multiplicative. Then we have:

$$\begin{aligned} (r \star \mu)(p) &= p - 1 = \phi(p), \\ (r \star \mu)(p^n) &= p - p = 0 \qquad (n > 1). \end{aligned}$$

This is enough to conclude thanks to the multiplicativity of $r \star \mu$. □

PROPOSITION 16 *If ζ is the usual zeta function on* $\mathbf{N}$*, then:*

$$(r \star \zeta)(p_1^{\alpha_1} \cdot \ldots \cdot p_n^{\alpha_n}) = (1 + \alpha_1 p_1) \cdot \ldots \cdot (1 + \alpha_n p_n). \tag{5}$$

Proof. Computing $r \star \zeta$ on a generic prime-power we get

$$(r \star \zeta)(p^n) = \sum_{d | p^n} r(d) \zeta \left(\frac{p^n}{d} \right) = 1 + np,$$

and the conclusion follows by multiplicativity. □

Remark. The function $r \star \zeta$ is usually called the *arithmetical integral* of r, since $(r \star \zeta)(n) = \sum_{d|n} r(d)$. From the last proposition it follows that $(r \star \zeta)(n)$ is the sum of the square-free divisors of n each considered with its own multiplicity (i.e., how many times it appears in n).

6.3 Commutative algebra (and algebraic geometry)

Let K be a field and $K[x_1, \ldots, x_n]$ the polynomial ring on n indeterminates over K. Using the preorder induced by divisibility and then considering the canonically associated partial order, $K[x_1, \ldots, x_n]$ is a poset and, in fact, a lattice. Take $f \in K[x_1, \ldots, x_n]$, and suppose that its unique decomposition into irreducible factors (up to invertible elements) is $f = f_1^{\alpha_1} \cdot \ldots \cdot f_r^{\alpha_r}$. We call *radical* of f the polynomial $r(f) = f_1 \cdot \ldots \cdot f_r$. From a geometric point of view, it is clear that $V(f) = V(r(f))$ (where, by definition, $V(f) = \{P \in K^n \mid f(P) = 0\}$), so that the polynomials f and $r(f)$ defines the same algebraic variety. A simple result which relates f and $r(f)$ is the following:

$$f = r(f) \cdot (f \wedge f_{x_1} \wedge \ldots \wedge f_{x_n}),$$

where f_y denotes the partial derivative with respect to y. So the radical function plays a important role in "cleaning" algebraic objects, retaining only the essential geometric informations. It is an easy exercise to verify that r is a derivation of the lattice of polynomials.

The above considerations can be brought to a more abstract level. Let A be a commutative ring with unity, and $\mathcal{I}(A)$ the set of its ideals. If $I \in \mathcal{I}(A)$, the *radical* of I is, by definition, the set $r(I) = \{x \in A \mid \exists n \in \mathbf{N} : x^n \in I\}$. A well-known characterization says that

$$r(I) = \bigcap_{\substack{\mathcal{P} \trianglelefteq A \\ \mathcal{P} \text{ prime} \\ \mathcal{P} \supseteq I}} \mathcal{P}.$$

It is clear that the above definition introduces a function r (the *radical function*) from $\mathcal{I}(A)$ to itself.

The theory of ideals study, among other things, various operations which can be introduced on $\mathcal{I}(A)$, such as sum, product, intersection, etc. . In general, $\mathcal{I}(A)$ is not a lattice with respect to any two operation one can define; moreover, even if it is, it seldom happens that the radical function introduced above is a derivation in the lattice so obtained. However, there is at least one special case in which everything works.

A *Dedekind domain* is an integral domain in which every ideal is a product of prime ideals. It is a standard exercise in commutative algebra to show that, in this case, every ideal has a *unique* decomposition as a product of prime ideals, except for the order of the factors. The next proposition, stated without proof, collects some known facts about Dedekind domains. They can be found, for example, in [LMC].

PROPOSITION 17 *Let A be a Dedekind domain. Then:*

i) *A is an* arithmetical ring*, i.e. the operations of sum and intersection of ideals are distributive one with respect to the other; this means that $[\mathcal{I}(A); \cap, +]$ is a distributive lattice;*

ii) *A is a* multiplication ring*, i.e. if $I, J \in \mathcal{I}(A)$ and $I \subseteq J$, then there exists $L \in \mathcal{I}(A)$ such that $I = JL$;*

iii) *A is an* almost multiplication ring*, i.e. each ideal of A which has prime radical is a power of its radical;*

iv) *A is Noetherian and every nonzero proper prime ideal of A is a maximal ideal.*

The next property of Dedekind domains is crucial for our last result, so we will give a proof of it.

LEMMA 18 *Let A be a Dedekind domain and $I, J \trianglelefteq A$ having no common factor in their decompositions as a product of prime ideals. Then I and J are* comaximal, *that is $I + J = A$.*

Proof. If I and J are distinct prime ideal, then the lemma is proved thanks to proposition 17, iv). Otherwise, without loss of generality, suppose that $I = \mathcal{P}_1 \cdot \mathcal{P}_2$ and $J = \mathcal{Q}_1 \cdot \mathcal{Q}_2$. If $\mathcal{P}_1 \cdot \mathcal{P}_2 + \mathcal{Q}_1 \cdot \mathcal{Q}_2 \subset A$, then there exists a maximal ideal $\mathcal{M}$ such that $\mathcal{P}_1 \cdot \mathcal{P}_2 + \mathcal{Q}_1 \cdot \mathcal{Q}_2 \subseteq \mathcal{M}$, hence $\mathcal{P}_1 \cdot \mathcal{P}_2 \subseteq \mathcal{M}$ and $\mathcal{Q}_1 \cdot \mathcal{Q}_2 \subseteq \mathcal{M}$. From proposition 17, ii), there exist $\mathcal{A}, \mathcal{B} \trianglelefteq A$ such that $\mathcal{P}_1 \cdot \mathcal{P}_2 = \mathcal{M} \cdot \mathcal{A}$ and $\mathcal{Q}_1 \cdot \mathcal{Q}_2 = \mathcal{M} \cdot \mathcal{B}$. Therefore I and J have the common factor $\mathcal{M}$ in their decomposition, which is a contradiction. $\square$

Thanks to proposition 17, i), if A is a Dedekind domain, then $[\mathcal{I}(A)^* = \mathcal{I}(A) \setminus \{\mathbf{0}\}; \cap, +]$ is a distributive lattice. If we interpret the operation $\cap$ as the *join* operation and the operation $+$ as the *meet* operation (so reversing the usual order given by inclusion), we have that $\mathcal{I}(A)^*$ has minimum A and does not have maximum.

THEOREM 19 *If A is a Dedekind domain, then the radical function $r : \mathcal{I}(A)^* \longrightarrow \mathcal{I}(A)^*$ defined on nonzero ideals is a derivation of $\mathcal{I}(A)^*$.*

Proof. We have to show the following equalities, for any $I, J \in \mathcal{I}(A)^*$:

1) $r(I \cap J) = r(I) \cap r(J)$;

2) $r(I + J) = r(I) + J = I + r(J)$.

Equality 1) is true in any commutative ring, as it is well known. As far as equality 2) is concerned, assume that, in the expressions of I and J as products of prime ideals, there are some common factors. Thanks to the distributivity of the product with respect to the sum (which is valid in any commutative ring), we can write:

$$I + J = \mathcal{P}_1^{\alpha_1} \cdot \ldots \cdot \mathcal{P}_r^{\alpha_r} \cdot (\overline{I} + \overline{J}),$$

where $\overline{I}, \overline{J} \trianglelefteq A$ having no common prime factor. Therefore, using the above lemma, $\overline{I} + \overline{J} = A$, and so

$$r(I + J) = \mathcal{P}_1 \cdot \ldots \cdot \mathcal{P}_r \cdot r(\overline{I} + \overline{J}) = \mathcal{P}_1 \cdot \ldots \cdot \mathcal{P}_r.$$

(Here we have used the fact that $r(\mathcal{A} \cdot \mathcal{B}) = r(\mathcal{A}) \cdot r(\mathcal{B})$). On the other hand, consider the ideal $r(I) + J$. It is clear that

$$r(I) + J = \mathcal{P}_1 \cdot \ldots \cdot \mathcal{P}_r \cdot \mathcal{A} + \mathcal{P}_1^{\beta_1} \cdot \ldots \cdot \mathcal{P}_r^{\beta_r} \cdot \mathcal{B},$$

where $\mathcal{P}_1, \ldots, \mathcal{P}_r$ are the prime ideals appearing in both the factorizations of I and J, $\mathcal{A}$ is a product of prime ideals different from $\mathcal{P}_1, \ldots, \mathcal{P}_r$ and $\mathcal{B}$ is an ideal whose factorization does not contain $\mathcal{P}_1, \ldots, \mathcal{P}_r$. Therefore we have immediately:

$$r(I) + J = \mathcal{P}_1 \cdot \ldots \cdot \mathcal{P}_r \cdot (\mathcal{A} + \mathcal{P}_1^{\beta_1 - 1} \cdot \ldots \cdot \mathcal{P}_r^{\beta_r - 1} \cdot \mathcal{B}).$$

(If any of the $\beta_i - 1$ is zero, then $\mathcal{P}_i^{\beta_i - 1} = A$ and can be removed). Now it is clear that $\mathcal{A}$ and $\mathcal{P}_1^{\beta_1 - 1} \cdot \ldots \cdot \mathcal{P}_r^{\beta_r - 1} \cdot \mathcal{B}$ do not have common prime factors, so (lemma 18) their sum is A, whence

$$r(I) + J = \mathcal{P}_1 \cdot \ldots \cdot \mathcal{P}_r,$$

which concludes the proof. □

Remark. It could be interesting to wonder whether the last theorem remains true by relaxing the hypotheses on the ring A. It could be possible to use some of the conditions stated in proposition 17, so avoiding the stronger hypothesis that A is a Dedekind domain.

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