Signal description: Process or Gibbs?
I. General introduction

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The issue

A signal with a stochastic component is detected

$$\cdots\ \omega_{-n-1}\ \omega_{-n} \cdots \omega_{-1}\ \omega_0\ \omega_1\ \cdots\ \omega_n\ \omega_{n+1} \cdots$$

ω_i belongs to some finite "alphabet" $\mathcal{A}$

E.g. biological signals:

- Spike sequence of a neuron, $\mathcal{A} = \{0, 1\}$
- DNA string, $\mathcal{A} = \{A, C, G, T\}$

Basic tenets

Stochastic description due to signal variability

Full description = probability measure μ on $\mathcal{A}^{\mathbb{Z}}$

Key issue: efficient characterization of μ.

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First approach: Transition probabilities

Machine-learning approach:

- Use first part of the train to develop “rules” to predict rest
- By recurrence: enough to predict *next* bit given “history”

That is, estimate the conditional probabilities w.r.t. past

$$P(X_n \mid X_{n-1}, X_{n-2}, \ldots)$$

through its law, defined by a function g such that

$$P\big(X_0 = \omega_0 \mid X_{-\infty}^{-1} = \omega_{-\infty}^{-1}\big) \;=\; g\big(\omega_0 \mid \omega_{-\infty}^{-1}\big)$$

Look for μ determined by (consistent with) this g:

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Regular g-measures

Relevant transitions expected to be insensitive to farther past:

g is a **regular g-function** if $\forall \epsilon > 0 \, \exists n \geq 0$ such that

$$\sup_{\omega,\sigma} \Big| g\big(\omega_0 \,\big|\, \sigma_{-1}^{-n} \omega_{-\infty}^{-n-1}\big) - g\big(\omega_0 \,\big|\, \sigma_{-\infty}^{-1}\big) \Big| < \epsilon \qquad (1)$$

- (1) is equivalent to $g(\omega_0 \,|\, \cdot\,)$ continuous in product topology
- Additional, not very relevant, non-nullness condition

A probability measure μ is a **regular g-measure** if it is consistent with some regular g-function

Signal μ thought as a process: past determines future (causality)

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Fields point of view

If the full train is available, why use only the past?

Learn to predict a bit using past *and future*!

X_n determined by finite-window probabilities

$$\mathbb{P}(X_n \mid X_{n-1}, X_{n-2}, \ldots ; X_{n+1}, X_{n+2}, \ldots)$$

through conditional laws determined by a function γ s.t.

$$P\big(X_0 = \omega_0 \mid X_{\{0\}^c} = \omega_{\{0\}^c}\big) \;=\; \gamma\big(\omega_0 \mid \omega_{\{0\}^c}\big)$$

Specification: γ satisfying certain compatibility condition
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Quasilocal measures

A specification γ is **quasilocal** if $\forall \epsilon > 0 \, \exists \, n, m \geq 0$

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for every σ, ω

- (2) is equivalent to $\gamma(\omega_0 \,|\, \cdot\,)$ continuous in product topology
- **Gibbs** specifications are, in addition, strongly non-null

A probability measure μ is a **quasilocal** (**Gibbs**) **measure** if it is consistent with some quasilocal (Gibbs) specification

Signal μ thought as non-causal or with anticipation

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Signals best described as processes or as Gibbs?

Both setups give complementary information:

- Processes: ergodicity, coupling, renewal, perfect simulation
- Fields: Gibbs theory

Are these setups mathematically equivalent?

Is every regular g-measure Gibbs and viceversa?

What is more efficient: One or two-side conditioning?

Efficiency vs interpretation?

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Prehistory

- Onicescu-Mihoc (1935): **chains with complete connections**
  - Existence of limit measures in non-nul cases
  - → *random systems with complete connections* (book by Iosifescu and Grigorescu, Cambridge 1990)
- Doeblin-Fortet (1937):
  - Taxonomy: A or B, dep. on continuity and non-nullness
  - Existence of invariant measures
  - Suggested: uniqueness of invariant measures (coupling!). Completed by Iosifescu (1992)
- Harris (1955): **chains of infinite order**
  - Framework of D-ary expansions
  - Weaker uniqueness condition
  - Cut-and-paste coupling

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More recent history

- Keane (1972): *g*-**measures** (*g*-functions), existence and uniqueness
- Ledrapier (1974): variational principle
- Walters (1975): relation with transfer operator theory
- Lalley (1986): **list processes**, regeneration, uniqueness
- Berbee (1987): uniqueness
- Kalikow (1990):
  - **random Markov processes**
  - **uniform martingales**
- Berger, Bramson, Bressaud, Comets, Dooley, F, Ferrari, Galves, Grigorescu, Hoffman, Hulse, Iosifescu, Johansson, Lacroix, Maillard, Öberg, Pollicott, Quas, Stanflo, Sidoravicius, Theodorescu, . . .

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Differences with Markov: Invariance

- Invariant measures: on space of trajectories (not just on $\mathcal{A}$)

$$\mu(x_0) = \sum_y g\big(x_0 \mid y\big)\, \mu(y)$$

$$\longrightarrow \quad \mu(x_0) = \int g\big(x_0 \mid x_{-\infty}^{-1}\big)\, \mu(dx_{-\infty}^{-1})$$

- Conditioning is over measure zero events: $\{X_{-\infty}^{-1} = x_{-\infty}^{-1}\}$
  - Importance of "μ-almost surely"
  - Properties must be *essential* = survive measure-zero changes

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Differences with Markov: Phase diagrams

There may be several invariant measures

- Not due to lack of ergodicity (non-null transitions)
- Different histories can lead to different invariant measures
- Analogous to statistical mechanics:

  Many invariant measures = 1st order phase transitions

Issues are, then, similar to those of stat mech:

- How many invariant measures? (= phase diagrams)
- Properties of measures? (mixing, extremality, ergodicity)
- Uniqueness criteria
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Transition probabilities

Basic structure:

- Space $\mathcal{A}^{\mathbb{Z}}$ with product σ-algebra $\mathcal{F}$ (and product topo)
- For $\Lambda \subset \mathbb{Z}$, $\mathcal{F}_\Lambda = \{\text{events depending on } \omega_\Lambda\} \subset \mathcal{F}$

Definition

(i) A *family of transition probabilities* is a measurable function

$$g(\cdot \mid \cdot) : \mathcal{A} \times \mathcal{A}_{-\infty}^{n-1} \longrightarrow [0,1]$$

such that $\sum_{x_0 \in \mathcal{A}} g(x_0 \mid x_{-\infty}^{-1}) = 1$

(ii) μ is a *process consistent* with $g(\cdot \mid \cdot)$ if

$$\mu(\{x_0\}) = \int g(x_0 \mid y_{-\infty}^{-1})\, \mu(dy)$$

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such that $\sum_{x_0 \in \mathcal{A}} g\big(x_0 \mid x_{-\infty}^{-1}\big) = 1$

(ii) μ is a *process consistent* with $g\big(\cdot \mid \cdot\big)$ if

$$\mu(\{x_0\}) = \int g\big(x_0 \mid y_{-\infty}^{-1}\big)\, \mu(dy)$$

General results (no hypotheses on g)

Let

- $\mathcal{G}(g) = \{\mu \text{ consistent with } g\}$
- $\mathcal{F}_{-\infty} := \bigcap_{k\in\mathbb{Z}} \mathcal{F}_{(-\infty,k]}$ *(tail σ-algebra)*

Theorem

(a) *$\mathcal{G}(g)$ is a convex set*

(b) *μ is extreme in $\mathcal{G}(g)$ iff μ is trivial on $\mathcal{F}_{-\infty}$*
($\mu(A) = 0, 1$ for $A \in \mathcal{F}_{-\infty}$)

(c) *μ is extreme in $\mathcal{G}(g)$ iff*

$$\lim_{\Lambda\uparrow\mathbb{Z}} \sup_{B\in\mathcal{F}_{\Lambda_-}} \big|\mu(A\cap B) - \mu(A)\mu(B)\big| = 0\,, \quad \forall A \in \mathcal{F}$$

(d) *Each $\mu \in \mathcal{G}(g)$ is determined by its restriction to $\mathcal{F}_{-\infty}$*

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Construction through limits

Let $P_{[m,n]}$ be the "window transition probabilities"

$$g_{[m,n]}\big(x_m^n \mid x_{-\infty}^{m-1}\big) \;:=\; g\big(x_n \mid x_{-\infty}^{n-1}\big)\, g\big(x_{n-1} \mid x_{-\infty}^{n-2}\big) \cdots g\big(x_m \mid x_{-\infty}^{m-1}\big)$$

Theorem

If μ is extreme on $\mathcal{G}(g)$, then for μ-almost all $y \in \mathcal{A}^{\mathbb{Z}}$,

$$g_{[-\ell,\ell]}\big(x_m^n \mid y_{-\infty}^{-\ell-1}\big) \xrightarrow[\ell\to\infty]{} \mu\big(\{x_m^n\}\big)$$

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Regular g-measures

Definition

A measure μ on $\mathcal{A}^{\mathbb{Z}}$ is **regular** (continuous) if it is consistent with regular transition probabilities

Theorem (Palmer, Parry and Walters (1977))

μ is a regular g-measure if and only if the sequence $\mu(\omega_0 \mid \omega_{-n}^{-1})$ converges uniformly in ω as $n \to \infty$

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Continuity rates

Uniqueness conditions: continuity and non-nulness hypotheses

- The **continuity rate** of g:

$$\mathrm{var}_k(g) \;:=\; \sup_{x,y}\Big|g\big(x_0 \mid x_{-\infty}^{-1}\big) - g\big(x_0 \mid x_{-1}^{-k}\, y_{-\infty}^{-k-1}\big)\Big|$$

- The **log-continuity rate** of g:

$$\mathrm{var}_k(\log g) \;:=\; \sup_{x,y} \log \frac{g\big(x_0 \mid x_{-\infty}^{-1}\big)}{g\big(x_0 \mid x_{-1}^{-k}\, y_{-\infty}^{-k-1}\big)}$$

- The **Δ-rate** of g:

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Non-nullness hypotheses

- g is **weakly non-null** if

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[Doeblin-Fortet:

- *Chain of type A*: for g continuous and weakly non-null
- *Chain of type B*: for g log-continuous and non-null]

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Uniqueness criteria (selected)

- Doeblin-Fortet (1937 + Iosifescu, 1992): g non-null and

$$\sum_k \mathrm{var}_k(g) < \infty$$

- Harris (1955): g weakly non-null and

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Leaving non-nullness aside, criteria are not fully comparable

Rough comparison:

- Doeblin-Fortet: $\mathrm{var}_k \sim 1/k^{1+\delta}$
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Criterion of a different species

Let

$$\operatorname{osc}_j(g) \;:=\; \sup_{x=y \text{ off } j} \Big| g\big(x_0 \mid x_{-\infty}^{-1}\big) - g\big(x_0 \mid y_{-\infty}^{-1}\big) \Big|$$

Then (F-Maillard, 2005) there is a unique consistent chain if

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Examples of non-uniqueness

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$$\mathrm{var}_k(g) \geq C/\log|k|$$

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Gibbs measures: Historic highlights

Prehistory:

- Boltzmann, Maxwell (kinetic theory): Probability weights
- Gibbs: Geometry of phase diagrams

History:

- Dobrushin (1968), Lanford and Ruelle (1969): Conditional expectations
- Preston (1973): Specifications
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Issue: Given microscopic behavior in finite regions, determine the macroscopic behavior

Basic tenets:

(i) Equilibrium = probability measure

(ii) Finite regions = finite parts of an infinite system

(iii) Exterior of a finite region = frozen external condition

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Equilibrium = Probability kernels

Set up: Product space $\Omega = \mathcal{A}^{\mathbb{L}}$

System in $\Lambda \Subset \mathbb{L}$ described by a probability kernel $\gamma_\Lambda(\,\cdot \mid \cdot\,)$

$$\gamma_\Lambda(f \mid \omega) \;=\; \text{equilibrium value of } f \text{ when the configuration outside } \Lambda \text{ is } \omega$$

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A **specification** is a family $\gamma = \{\gamma_\Lambda : \Lambda \Subset \mathbb{L}\}$ of probability kernels $\gamma_\Lambda : \mathcal{F} \times \Omega \longrightarrow [0,1]$ such that

(i) *External dependence:* $\gamma_\Lambda(f \mid \cdot)$ is $\mathcal{F}_{\Lambda^c}$-measurable

(ii) *Frozen external conditions:* Each γ_Λ is *proper*,

$$\gamma_\Lambda(h\, f \mid \omega) \;=\; h(\omega)\, \gamma_\Lambda(f \mid \omega)$$

if h depends only on ω_{Λ^c}

(iii) *Equilibrium in finite regions:* The family γ is *consistent*

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(i) *External dependence:* $\gamma_\Lambda(f \mid \cdot)$ is $\mathcal{F}_{\Lambda^c}$-measurable

(ii) *Frozen external conditions:* Each γ_Λ is *proper*,

$$\gamma_\Lambda(h\, f \mid \omega) \;=\; h(\omega)\, \gamma_\Lambda(f \mid \omega)$$

if h depends only on ω_{Λ^c}

(iii) *Equilibrium in finite regions:* The family γ is *consistent*

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Consistency

Definition

A probability measure μ on Ω is **consistent** with γ if

$$\mu \gamma_\Lambda = \mu \qquad \text{for each } \Lambda \Subset \mathbb{L}$$

(DLR equations = equilibrium in infinite regions)

Remarks

- Several consistent measures = first-order phase transition
- Specification $\sim$ system of regular conditional probabilities
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General results (no hypotheses on γ)

Let

- $\mathcal{G}(\gamma) = \{\mu \text{ consistent with } \gamma\}$
- $\mathcal{F}_\infty := \bigcap_{\Lambda \Subset \mathbb{L}} \mathcal{F}_{\Lambda^c}$ (*σ-algebra at infinity*)

Theorem

(a) *$\mathcal{G}(\gamma)$ is a convex set*

(b) *μ is extreme in $\mathcal{G}(\gamma)$ iff μ is trivial on $\mathcal{F}_\infty$ ($\mu(A) = 0, 1$ for $A \in \mathcal{F}_\infty$)*

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(d) *Each $\mu \in \mathcal{G}(\gamma)$ is determined by its restriction to $\mathcal{F}_\infty$*

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Construction through limits

Theorem

If μ is extreme on $\mathcal{G}(\gamma)$, then for μ-almost all $\sigma \in \Omega$,

$$\gamma_\Delta\big(\omega_\Lambda \,\big|\, \sigma_{\Delta^c}\big) \xrightarrow[\Delta\to\mathbb{L}]{} \mu\big(\{\omega_\Lambda\}\big)$$

for all $\omega \in \Omega$ (no hypotheses on γ)

Quasilocality

Definition

A measure μ on $\mathcal{A}^{\mathbb{L}}$ is **quasilocal** (continuous) if it is consistent with a quasilocal specification

Theorem

μ is quasilocal if and only if the sequence $\mu\big(\omega_0 \,\big|\, \omega_{-n}^{-1}\,\omega_1^m\big)$ converges uniformly in ω as $n, m \to \infty$

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Link with statistical mechanics

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A specification γ is

- **non-null** if $\inf_\sigma \gamma_\Lambda\left(\omega_\Lambda \,\middle|\, \sigma_{\Lambda^c}\right) > 0$ for $\omega \in \Omega, \Lambda \Subset \mathbb{L}$
- **Gibbs** if it is quasilocal and non-null

Theorem (Kozlov)

A specification is Gibbsian iff it has the Boltzmann form

$$\gamma\left(\omega_\Lambda \,\middle|\, \omega_{\Lambda^c}\right) = \exp\Big\{-\sum_{A\cap\Lambda\neq\emptyset} \phi_A(\omega_A)\Big\}/Norm. \,,$$

where $\{\phi_A\}$ (interaction) satisfy

$$\sum_{A\ni 0} \|\phi_A\|_\infty < \infty \,.$$

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Uniqueness and non-uniqueness

Uniqueness results

- Berbee: $\sum_{n\geq 1} \exp\Big(-\sum_{k=1}^{n} \mathrm{var}_k(\log\gamma)\Big) = +\infty$
- Dobrushin: $\sum_{j<0} \delta_j(g) \ < \ 1$

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Signal description: Process or Gibbs? II. Relation between approaches

Contributors: S. Berghout (Leiden)
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Florence in May, 2017

The issues

(I) Given a measure μ on $\mathcal{A}^{\mathbb{Z}}$

- Is it always both a g and a Gibbs measure?
- If yes, which are the pros and cons of each point of view?

(II) Are g-functions and specifications in correspondance?

- Same uniqueness regions?
- Same phase diagrams?

(III) Can theoretical aspects be "imported"?

- Variational approach
- Large deviations

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Mathematical formalization

Mathematically, there are three natural questions:

(Q1) Is there a map $b : g \longrightarrow \gamma^g$ such that $\mathcal{G}(g) = \mathcal{G}(\gamma^g)$?

(Q2) Is there a map $c : \gamma \longrightarrow g^\gamma$ such that $\mathcal{G}(\gamma) = \mathcal{G}(g^\gamma)$?

(Q3) If so, are these map mutual inverses:

$$bc = \mathrm{id} = cb \qquad \left[\gamma^{g^\gamma} = \gamma\ ,\ g^{\gamma^g} = g\right]?$$

True for Markov ($\mathcal{A}$ finite)
[Georgii, Chapter 3, uses eigenvalues]

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Construction of the map b

How would you construct a map $b : g \longrightarrow \gamma^g$?

Natural answer:

$$\gamma^g_{[k,\ell]}\big(\omega_k^\ell \bigm| \sigma\big) \;=\; \lim_{n\to\infty} \frac{g_{[k,n]}\big(\omega_k^\ell\,\sigma_{\ell+1}^n \bigm| \sigma_{-\infty}^{k-1}\big)}{g_{[k,n]}\big(\sigma_{\ell+1}^n \bigm| \sigma_{-\infty}^{k-1}\big)}$$

Need to guarantee that the limit exists for all σ

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A g function has **good future** if

- g is non-null and
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Positive answer to (Q1)

Denote

- $\Theta_{\mathrm{GF}} := \{g \text{ has GF}\}$
- $\Pi := \{\gamma \text{ quasilocal}\}$
- $\Pi_1 := \{\gamma : |\mathcal{G}(\gamma)| = 1\}$

Theorem ($g \rightsquigarrow$ specification)

The previous prescription defines a map

$$\begin{aligned} b : \Theta_{\mathrm{GF}} &\to \gamma \\ g &\mapsto \gamma^g \end{aligned}$$

which satisfies

(a) $\mathcal{G}(g) \subset \mathcal{G}(\gamma^g)$

(b) *b restricted to $b^{-1}(\Pi_1)$ is one-to-one.*

Thus, if $g \in b^{-1}(\Pi_1)$,

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provided that, for each σ,

- the limit exists and
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Theorem (specification $\rightsquigarrow g$)

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(a) $\mathcal{G}(f^{\gamma}) = \mathcal{G}(\gamma) = \{\mu^{\gamma}\}$

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Invertibility of the maps

Proofs of previous theorems yield bounds on $\delta_j(\gamma^g)$ and $\delta_j(g^\gamma)$

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(a) *$b \circ c = \mathrm{Id}$ over $c^{-1}(\Theta_{\mathrm{GF}})$, and $\mathcal{G}(g^\gamma) = \mathcal{G}(\gamma) = \{\mu^\gamma\}$*

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Proofs of previous theorems yield bounds on $\delta_j(\gamma^g)$ and $\delta_j(g^\gamma)$

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- $\Theta_{\mathrm{EXP}} = \{g : \exists\, a > 1 \text{ s.t. } \lim_{j\to-\infty} a^{|j|}\,\delta_j(g) = 0\}$
- $\Pi_{\mathrm{EXP}} = \{\gamma : \exists\, a > 1 \text{ s.t. } \lim_{j\to\infty} a^{j}\,\delta_j(\gamma) = 0\}$

Theorem (LIS $\leftrightsquigarrow$ specification)

(a) $b \circ c = \mathrm{Id}$ *over* $c^{-1}(\Theta_{\mathrm{GF}})$, *and* $\mathcal{G}(g^\gamma) = \mathcal{G}(\gamma) = \{\mu^\gamma\}$

(b) $c \circ b = \mathrm{Id}$ *over* $b^{-1}(\Pi_{\mathrm{HUC}})$ *and* $\mathcal{G}(\gamma^f) = \mathcal{G}(f) = \{\mu^f\}$

(c) *b and c establish a one-to-one correspondence between Θ_{EXP} and Π_{EXP} that preserves the consistent measure.*

A regular g that is not Gibbs

$\mathcal{A} = \{0, 1\}$; denote $\underline{\omega} = \omega_{-\infty}^{-1}$

Consider g-functions of the form

$$g(1 \,|\, \underline{\omega}) \;=\; p_{\ell(\underline{\omega})}$$

where

- $\ell(\underline{\omega})$ = number of 0's before first 1 looking backwards:

$$\ell(\underline{\omega}) \;=\; \min\{j \geq 0 \colon \omega_{-j-1} = 1\}$$

- $\{p_i\}_{i\geq 0} \in (0, 1)$ satisfy

$$\inf_{i\geq 0} p_i = \epsilon > 0 \quad , \qquad p_\infty = \lim_{i\to\infty} p_i \; .$$

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Regularity

Non-nullness: $g(\cdot \mid \cdot) \geq \epsilon \wedge 1 - \epsilon$

Continuity:

$$\begin{aligned}
&\sup_{\omega_{-k}^{-1}=\sigma_{-k}^{-1}} \Big| g\big(1 \mid \underline{\omega}\big) - g\big(1 \mid \underline{\sigma}\big)\Big| \\
&= \sup \Big| g\big(1 \mid 0_{-k}^{-1}\omega_{-\infty}^{-k-1}\big) - g\big(1 \mid 0_{-k}^{-1}\sigma_{-\infty}^{-k-1}\big)\Big| \\
&= \sup_{l,m\geq k} \big|p_l - p_m\big| \\
&\xrightarrow[k]{} 0
\end{aligned}$$

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$$= \sup_{l,m\geq k} |p_l - p_m|$$

$$\underset{k}{\longrightarrow} 0$$

Properties of the process

For all choices of sequences p_i as above

- There exists a unique stationary chain μ compatible with g
- μ is supported on infinitely many 1's with intervals of 0's
- μ is a renewal chain with visible renewals
- μ can be perfectly simulated

For all practical purposes, chains are as regular as they can be
Nevertheless, for some choices of p_i the chains are not Gibbsian.

Cause: problem when conditioning on "all 0"

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Main result

Theorem

There exist choices of $\{p_i\}_{i\geq 0}$ *as above for which the sequences*

$$\Big[\mu\big(X_0 = \omega_0 \mid X_{-i-1} = 1, X_{-i}^{-1} = 0_{-i}^{j}, X_1^j = 0_1^j, X_{j+1} = 1\big)\Big]_{i,j\geq 1}$$

does not converge as $i, j \to \infty$.

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It is based on the following

Claim

$$\mu\bigl(X_0 = \omega_0 \bigm| X_{-i-1} = 1, X_{-i}^{j} = 0_{-i}^{j}, X_{j+1} = 1\bigr)$$

is determined by the ratio

$$\prod_{k=0}^{j-1} \frac{1-p_k}{1-p_{k+i}}.$$

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Proof (cont.)

Economical way: Define $p_k = 1 - (1-p_\infty)\xi^{v_k}$ so that

$$\prod_{k=0}^{j-1} \frac{1-p_k}{1-p_{k+i}} = \xi^{\sum_{k=0}^{j-1}(v_k - v_{k+i})}$$

Choose $v_k \to 0$, but such that $\sum_{k=0}^{j} v_k$ oscillates

Example: $\xi \in (1, (1-p_\infty)^{-2})$ and

$$v_k = \frac{(-1)^{r_k}}{r_k} \quad \text{with} \quad r_k = \inf\Big\{ i \geq 1 : \sum_{j=1}^{i} j \geq k+1 \Big\}$$

First terms:

$$-1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots$$

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Proof of the claim

$$\mu\big(X_{-i-1}=1, X^{j}_{-i}=0^{j}_{-i}, X_{j+1}=1\big)$$
$$= \mu\big(X_{-i-1}=1\big)\mu\big(X^{j-1}_{-i}=0^{j+1}_{-i}, X_j=1 \mid X_{-i-1}=1\big)$$
$$= \mu\big(X_{-i-1}=1\big)\prod_{k=0}^{i+j}(1-p_k)\,p_{i+j+1}$$

Analogously

$$\mu\big(X_{-i-1}=1, X^{-1}_{-i}=0^{-1}_{-i}, X_0=1, X^{j-1}_{1}=0^{j-1}_{1}, X_{j+1}=1\big)$$
$$= \mu\big(X_{-i-1}=1\big)\left(\prod_{k=0}^{i-1}(1-p_k)p_i\right)\left(\prod_{k=0}^{j-1}(1-p_k)p_j\right)$$

Proof of the claim (cont.)

Hence

$$\mu\big(X_0 = 0 \mid X_{-i-1} = 1, X_{-i}^{j} = 0_{-i}^{j}, X_{j+1} = 1\big)$$

$$= \frac{\prod_{k=0}^{i+j}(1-p_k)p_{i+j+1}}{\prod_{k=0}^{i-1}(1-p_k)p_i \prod_{k=0}^{j-1}(1-p_k)p_j + \prod_{k=0}^{i+j}(1-p_k)p_{i+j+1}}$$

$$= \left[1 + \frac{p_i\, p_j}{(1-p_{i+j})\, p_{i+j+1}} \prod_{k=0}^{j-1} \frac{1-p_k}{1-p_{k+i}}\right]^{-1}$$

$$\sim \left[1 + \frac{p_\infty}{(1-p_\infty)} \prod_{k=0}^{j-1} \frac{1-p_k}{1-p_{k+i}}\right]^{-1}$$

A Gibbs that is not regular g

[Bissacot, Endo, van Enter and Le Ny (2017)]

Consider *Dyson models:*

- $\mathcal{A} = \{-1, 1\}$, $\mathbb{L} = \mathbb{Z}$
- Specification defined by

$$\gamma_{\{0\}}\big(\sigma_0 \mid \sigma_{\{0\}^c}\big) \;=\; \exp\Big[\beta \sum_{j\in\mathbb{Z}_{\neq 0}} \frac{\sigma_0\,\sigma_j}{|j|^\alpha}\Big]\Big/\text{Norm.}$$

for $1 < \alpha < 2$

At low temperature there is a phase transition:

$$\mathcal{G}(\gamma) = \{\mu^+, \mu_-\} \text{ with } \mu^\pm \;=\; \lim_{n\to\infty} \gamma_{[-n,n]}(\cdot \mid \pm)$$

Theorem

Let $\alpha^* = 3 - \frac{\log 3}{\log 2} \in (1,2)$. Then, for each $\alpha \in (\alpha^*, 2)$ the measures $\mu^\pm$ are not regular g at low enough temperatures.

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First ingredient of the argument: Interfaces

Crucial! [Cassandro, Merola, Picco and Rozikov (2014)]

Argument for μ^+: Let $\alpha^* < \alpha < 2$ and T low enough

Under Dobrushin boundary conditions:

$$\sigma_i = \begin{cases} -1 & i \leq -1 \\ +1 & i \geq L+1 \end{cases}$$

an interface develops at $I^* \sim L/2$ such that

- Mostly "-1" in $[0, I^*)$ and "$+1$" on $(I^*, L]$
- Probability of displacing interface $\sim \mathrm{e}^{-cL^{2-\alpha}}$

$$\gamma_{[0,L]}\big(|I^* - (L/2)| > \epsilon L \,\big|\, -+\big) \;\leq\; f(\epsilon)\, L\, \mathrm{e}^{-cL^{2-\alpha}} \tag{1}$$

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Second ingredient: Wetting

Flipping the left “−” beyond $-N$ has an energy cost of at most

$$\sum_{\substack{i \in [0,L] \\ j \leq -N}} \frac{1}{|i-j|} \sim \frac{L}{N^{\alpha-1}}$$

negligible w.r.t. RHS of (1) if N is grows superlinearly with L:

$$\frac{L}{N^{\alpha-1}} = o(1) \tag{2}$$

Consequence: $\exists\, \delta > 0$ s.t. for each ϵ

$$\mu^+\left(\omega_i \,\middle|\, (-1)_{-N}^{-1}\right) \leq -\delta \quad , \quad i \in [0, (1-\epsilon)L/2] \tag{3}$$

for L large enough and N as in (2)

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Third ingredient: Energy cost of alternating

Alternating spins in $[-L_1, 0]$ have a L_1-independent energy cost

$$\max_{\omega} \sum_{\substack{i\in[-L_1,-1]\\ j\notin[-L_1,-1]}} \frac{(1)^i}{|i-j|^{\alpha}}\,\omega_j \;\le\; c \tag{4}$$

with c independent of L_1. From (1), (3) and (4):

$$\mu^+\Big(\omega_0 \,\Big|\, (\omega^{\text{alt}})_{-L_1}^{-1}(-1)_{-N-L_1}^{-L_1-1}\Big) \;\le\; -\delta \tag{5}$$

for L large enough as long as $L/N^{\alpha-1} = o(1)$ and $L_1 = o(L)$.

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Conclusion

Analogously, conditioning on "+" in $[-N,-1]$:

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Hence, for L large enough

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Left-conditioning is not quasilocal (discontinuous w.r.t. past)

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Review of additional issues and results
I. When a regular g is Gibbs

Theorem

A regular g-measure is Gibbs iff the sequence

$$\prod_{i=1}^{n} \frac{g\left(\omega_i \mid \omega_1^{i-1}\, \sigma_0\, \omega_{-\infty}^{-1}\right)}{g\left(\omega_i \mid \omega_1^{i-1}\, \eta_0\, \omega_{-\infty}^{-1}\right)}$$

converges, $\forall\, \sigma_0, \eta_0$, *uniformly on* ω, *as* $n \to \infty$

II. Reversibility

Relation between left- and right-conditioning?

Definition

A regular g-measure is **reversible** if it is continuous w.r.t. the *future*:

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Overview of examples

- ∃ non-reversible measures (example is also non-Gibbs)
- ∃ reversible g-measures with different left and right continuity rates
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Transitions vs kernels

Asymmetry in conditional kernels:

- g-measures determined by single-time transitions $g\big(\cdot \;\big|\; \omega_{-\infty}^{-1}\big)$
- Gibbs measures determined by full specifications $\big\{\gamma_\Lambda\big(\cdot \;\big|\; \omega_{\Lambda^c}\big) : \Lambda \Subset \mathbb{Z}\big\}$

To put approaches on a common ground

- $g \longrightarrow$ left-interval specifications (LIS)
- specifications $\longrightarrow \gamma_{\{0\}}$ plus *order-consistency*

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Left-interval specifications

g-functions admit a specification-like framework. Denote

- $\mathcal{J}$ = set of bounded intervals in $\mathbb{Z}$
- If $[a, b] \in \mathcal{J}$, $m_\Lambda := b$,
  - $\mathcal{F}_{\leq\Lambda} := \mathcal{F}_{(-\infty, b]}$
  - $\mathcal{F}_{\Lambda_-} := \mathcal{F}_{(-\infty, a-1]}$

The iterated-conditioning formula

$$g_{[m,n]}\big(\omega_m^n \mid \omega_{-\infty}^{n-1}\big) = g\big(\omega_m \mid \omega_{-\infty}^{m-1}\big)\, g\big(\omega_{m-1} \mid \omega_{-\infty}^{m-2}\big) \cdots g\big(\omega_n \mid \omega_{-\infty}^{n-1}\big)$$

defines a family of probability kernels $G = \{g_\Lambda : \Lambda \in \mathcal{J}\}$ s.t.

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Definition of LIS

(i) *Increasing measurability:* $g_\Lambda : \mathcal{F}_{\le m_\Lambda} \times \Omega \longrightarrow [0,1]$

(ii) *Dependence on past:* $g_\Lambda(f \mid \cdot)$ is $\mathcal{F}_{\Lambda_-}$-measurable

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if h depends only on ω_{Λ_-}

(iv) *Consistency:* For $\Delta, \Lambda \in \mathcal{J} : \Delta \supset \Lambda$,

$$g_\Delta\, g_\Lambda \;=\; g_\Delta \quad \text{over } \mathcal{F}_{\le m_\Lambda}$$

Properties (i)–(iv): **left interval-specification (LIS)**

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Comments

Knowledge of the LIS G is equivalent to knowledge of g

In particular $\mathcal{G}(G) = \mathcal{G}(g)$:

$$\mu\, g_\Lambda = \mu \;\forall\, \Lambda \in \mathcal{J} \quad \Leftrightarrow \quad \mu\, g = \mu$$

Observations:

- Unlike specifications, kernels apply to *different* σ-algebras
- Kernels *only* for intervals
- Nevertheless the theory for specifications can be adapted
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From singletons to specifications (general $\mathbb{L}$)

Would like to generate kernels from the singletons $\gamma_{\{i\}}$

However, not any family of singletons is admissible

Choice of internal regions lead to *compatibility conditions*

Let us start with two sites:

- The consistency $\gamma_{\{i,j\}} = \gamma_{\{i,j\}}\gamma_{\{i\}}$ implies

$$\gamma_{\{i,j\}}(\sigma_i\sigma_j \mid \omega) = \gamma_{\{i\}}(\sigma_i \mid \sigma_j\,\omega_{\{j\}^c})\,\gamma_{\{i,j\}}(\sigma_j \mid \omega) \quad (7)$$

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Using, instead, (8) we similarly arrive to the $i \leftrightarrow j$ expression:

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RHS of (9) = RHS of (10) $\Longrightarrow$ **order-consistency condition**:

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The reconstruction theorem

Further compatibility conditions from other $\Lambda \Subset \mathbb{L}$?

Miracle! (11) is enough

Theorem

If (11) hold for all $i, j \in \mathbb{L}, \omega \in \Omega$ (denominators ¿ 0!), then

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- Consistency condition (11) are automatically satisfied if
  - Singletons come from a specification. Hence theorem shows that a specification is uniquely defined by singletons [Georgii's Theorem 1.33]
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$$\gamma_i\big(\omega_i \,\big|\, \omega\big) \;=\; \lim_{n\to\infty} \frac{\mu\big(\omega_{V_n}\big)}{\mu\big(\omega_{V_n\setminus\{i\}}\big)}$$

    for an exhausting sequence of volumes $\{V_n\}$
- Dachian and Nahapetian (2001) provided alternative construction (weaker non-nullness, stronger order-consistency)
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The general mathematical framework is clear enough:

- Gibbs and g have comparable but not identical theories
- General theory: partially ordered specifications

What about practical considerations?

- In some cases one theory is applicable but not the other
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