Monotone cellular automata

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(Based on joint work with Paul Balister, József Balogh, Béla Bollobás, Hugo Duminil-Copin, Ivailo Hartarsky, Fabio Martinelli, Paul Smith, and Cristina Toninelli.)

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Note that this is a random variable, and is also a function of the initial state, and of p. An interesting particular case is when the initial state is chosen randomly (e.g., with density p of empty sites), and $p \to 0$.

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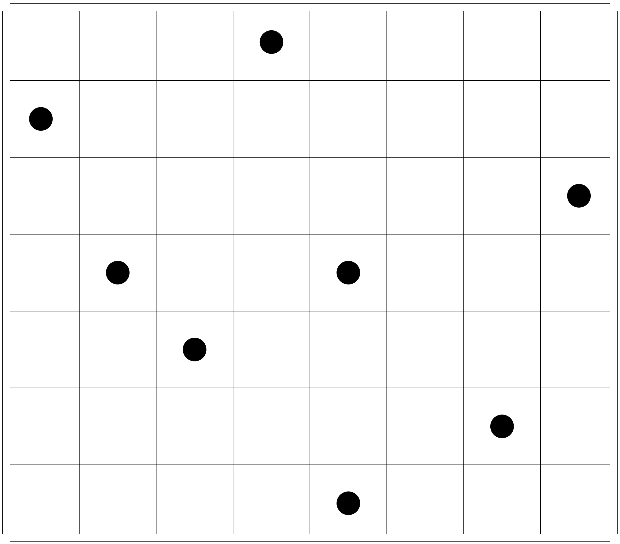

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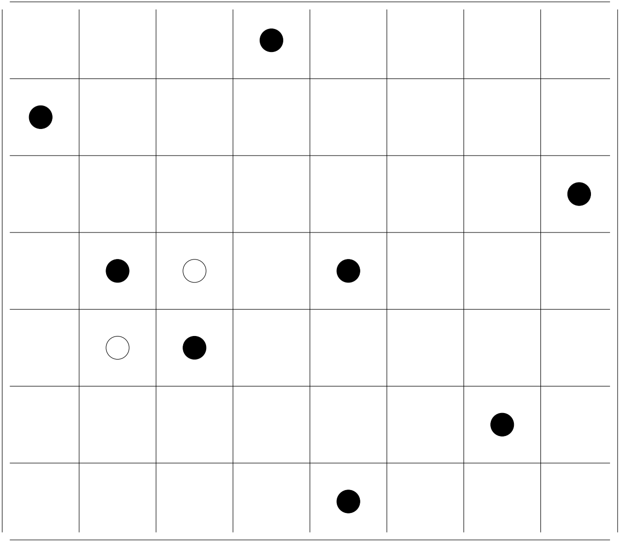

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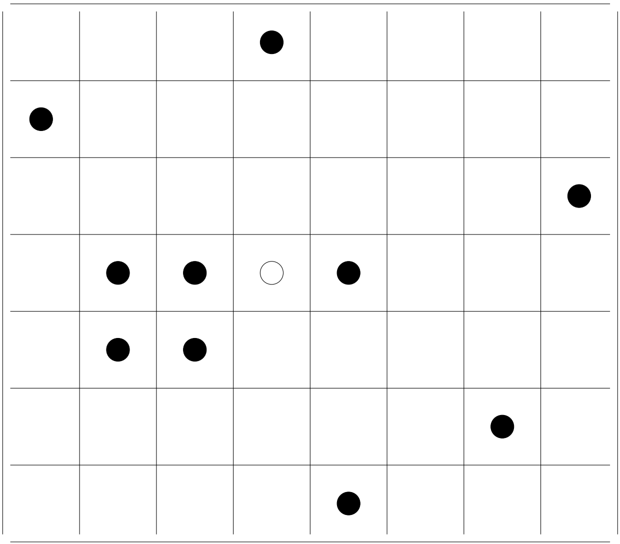

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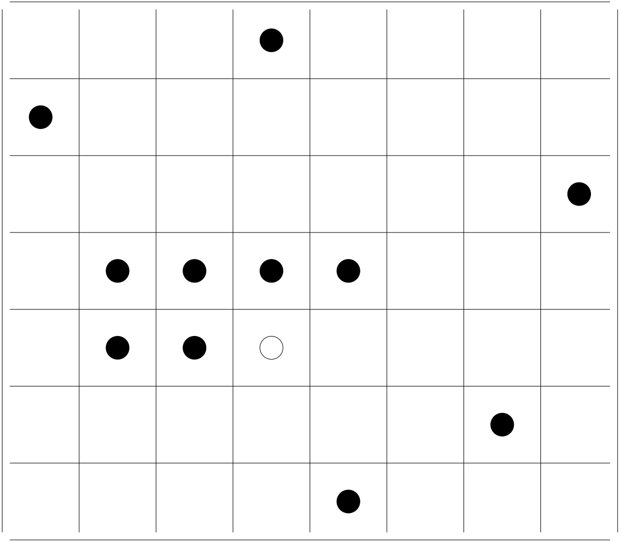

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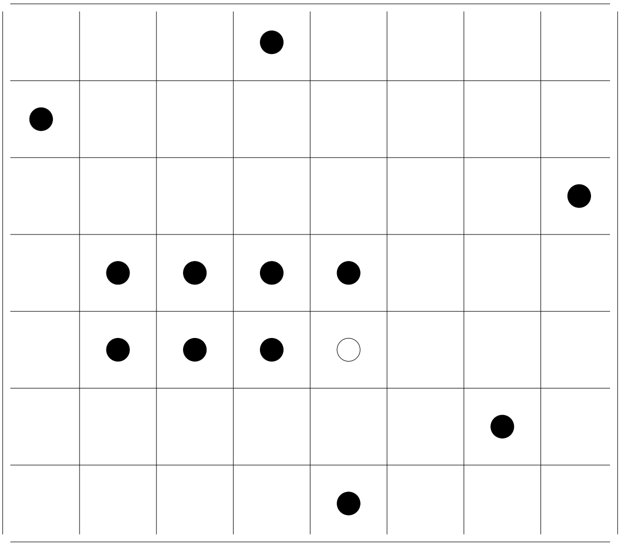

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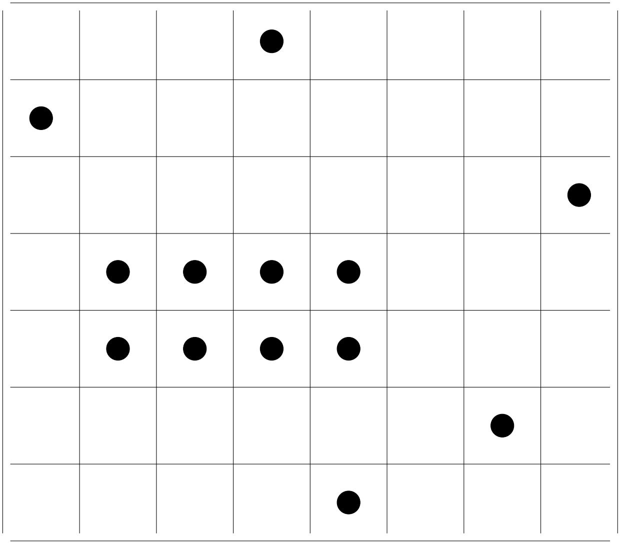

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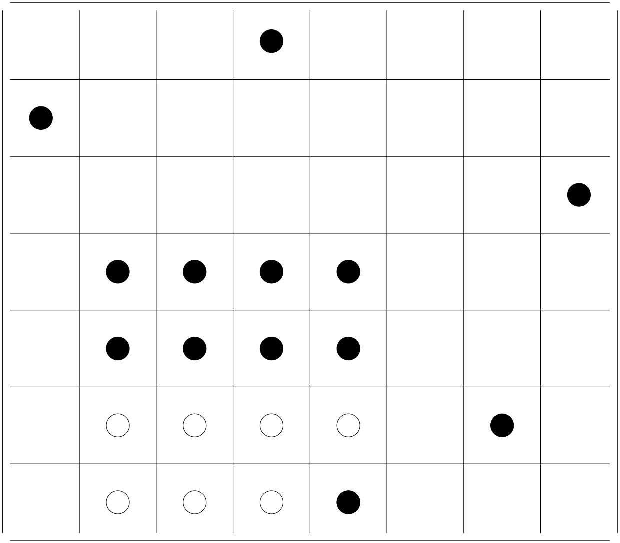

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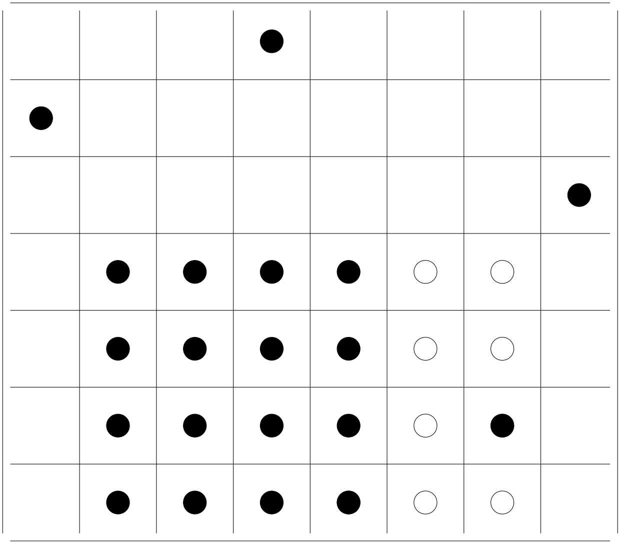

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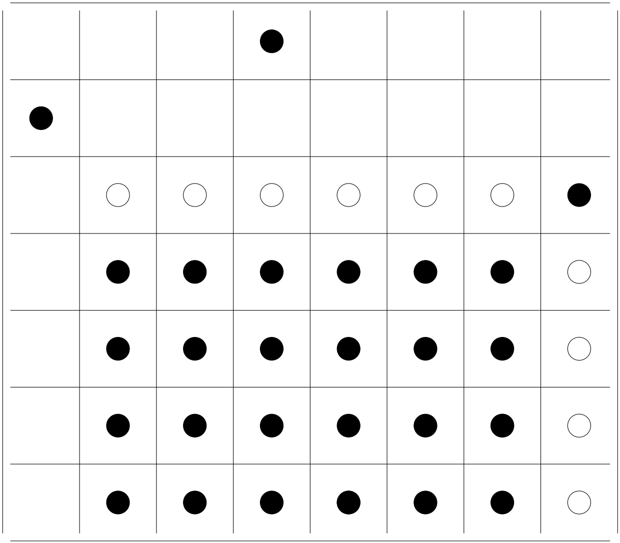

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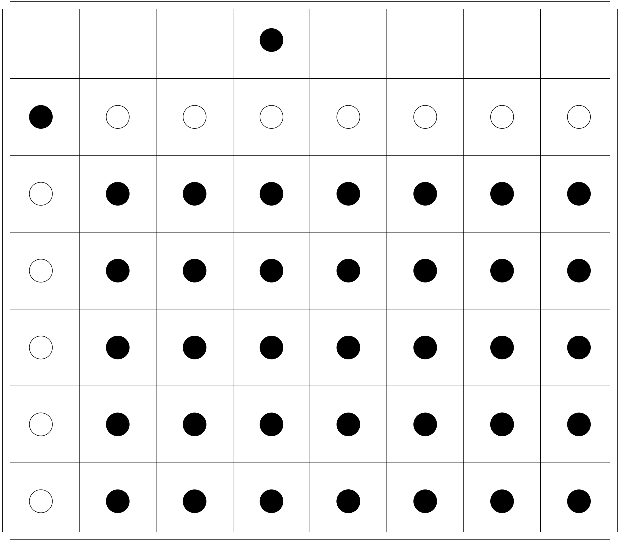

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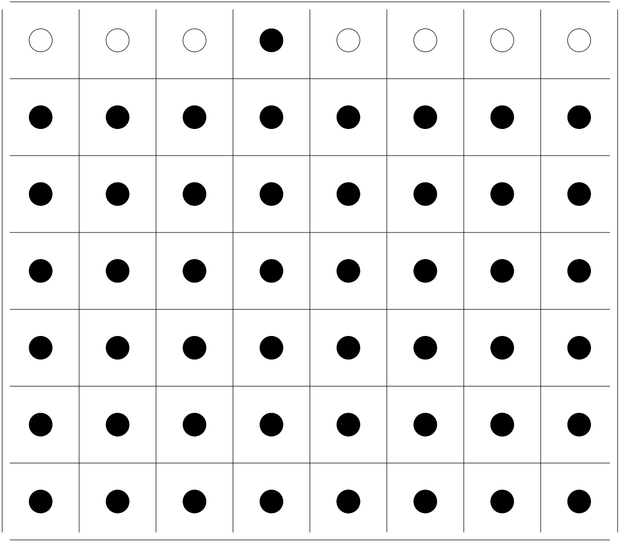

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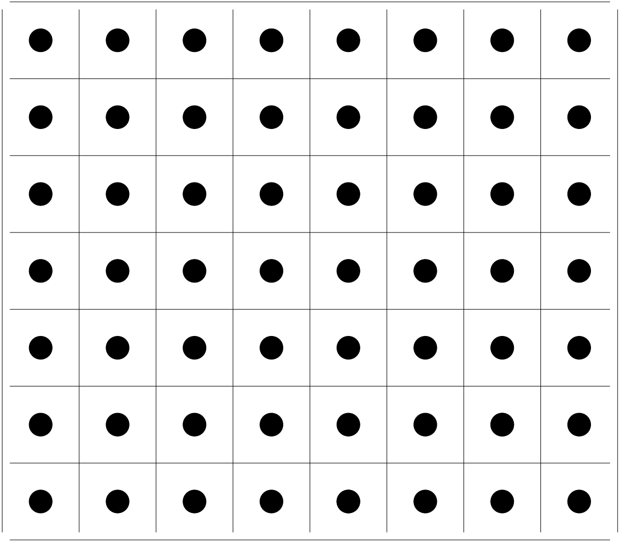

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Answer: $p_c(\mathbb{Z}^2, 2) = 0$ (!!)

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Answer: $p_c(\mathbb{Z}^2, 2) = 0$ (van Enter, 1987)

van Enter's proof that $p_c(\mathbb{Z}^2, 2) = 0$ (sketch)

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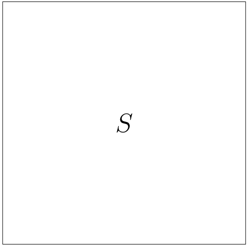
With probability 1, there exists a very large completely infected square S (a *critical droplet*) somewhere in $\mathbb{Z}^2$:



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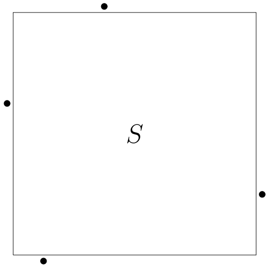

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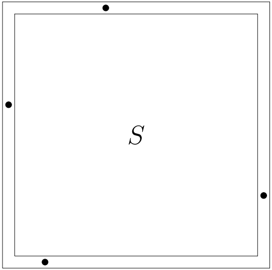

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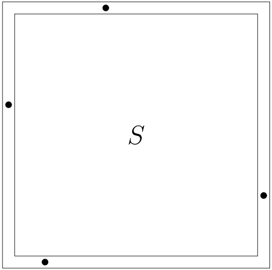

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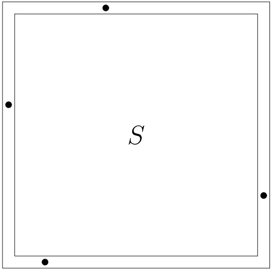

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We define the *critical probability* on an $n \times n$ torus to be

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The upper bound follows from a more careful analysis of van Enter's argument, so we will instead focus on the (more interesting) lower bound.

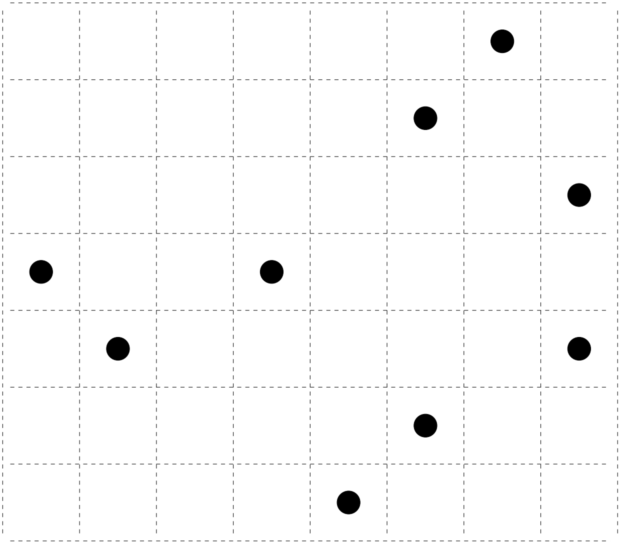
The key idea is to control the growth of critical droplets using an algorithm called the *rectangles process*.

The rectangles process:

We begin with a collection of $|A|$ rectangles, each consisting of a single site of A. At each step of the process, we choose two rectangles that lie within distance 2 of one another, and combine them to form a larger (entirely infected) rectangle. We stop when we can no longer find such a pair of rectangles.

- The union of the final collection of rectangles is equal to $[A]$.
- Every rectangle R that appears at some point in the rectangles process is *internally filled* by A, i.e., $[A \cap R] = R$.

The rectangles process



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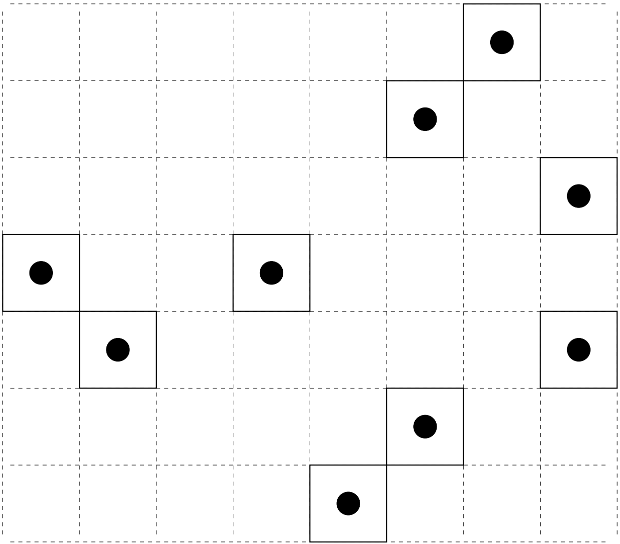

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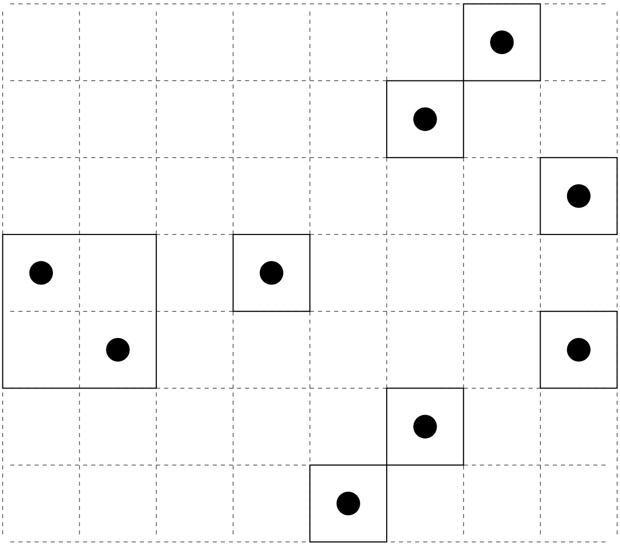

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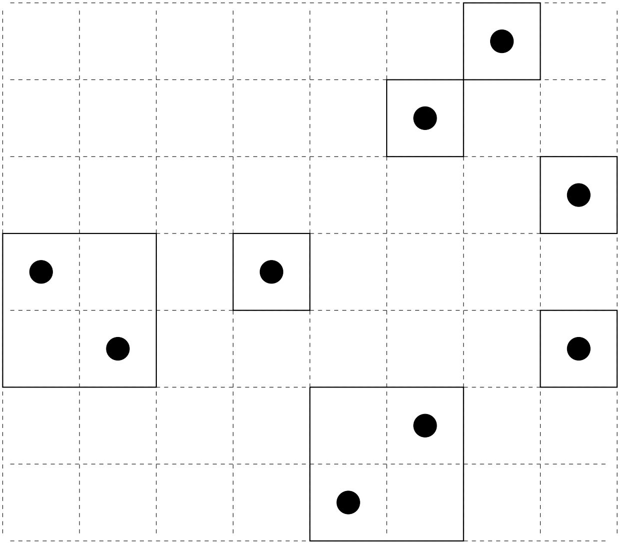

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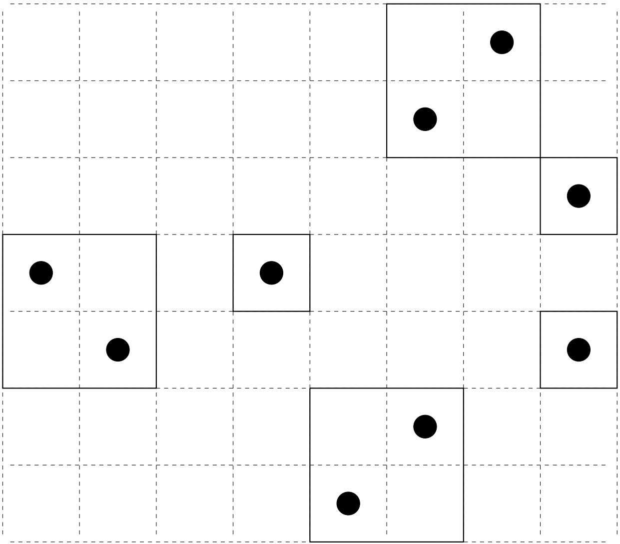

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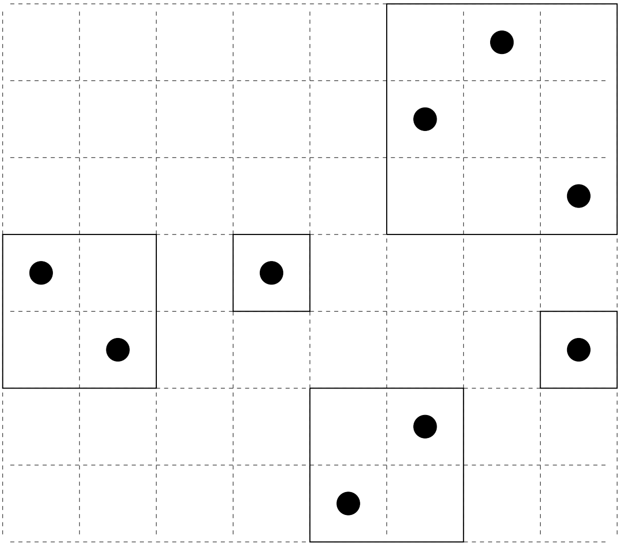

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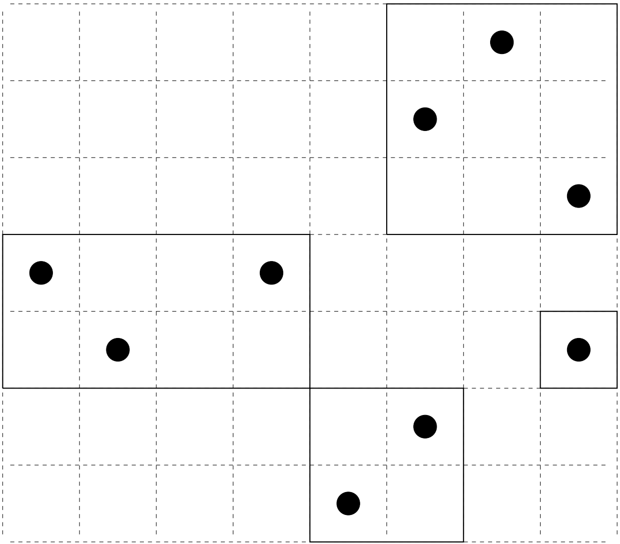

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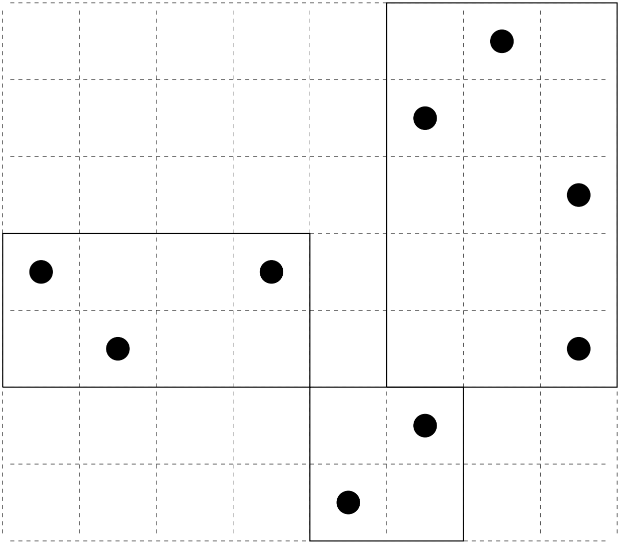

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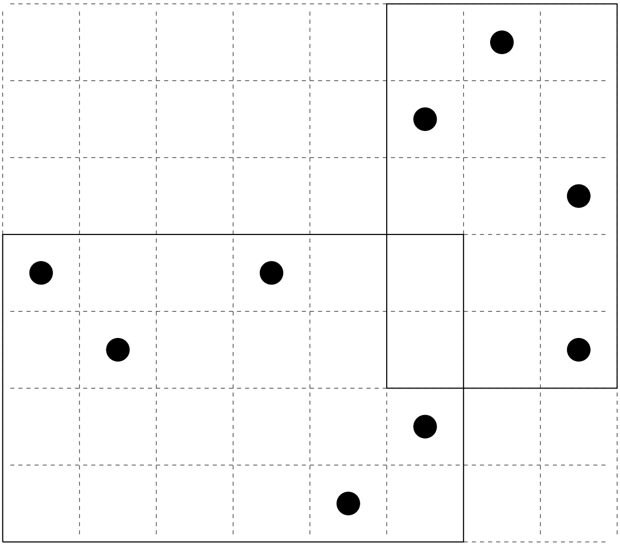

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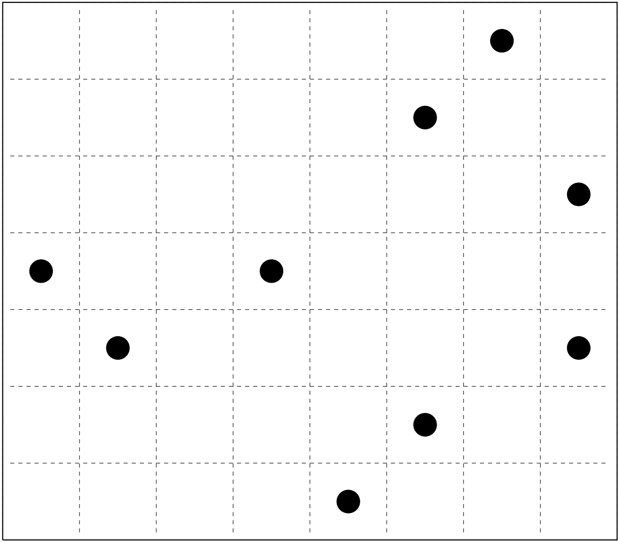

Aizenman and Lebowitz's proof (sketch, continued)

Using the rectangles process, we can prove the following key lemma.

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The Aizenman–Lebowitz Lemma

If A percolates in $\mathbb{Z}_n^2$, then there exists a rectangle $R \subset \mathbb{Z}_n^2$, with

$$\log n \leqslant \mathsf{long}(R) \leqslant 2 \log n,$$

that is "internally filled", i.e., $[A \cap R] = R$.

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Proof: Run the rectangles process until a rectangle with $\text{long}(R) \geqslant \log n$ appears for the first time. This rectangle is internally filled, by the definition of the process. Moreover, it was obtained from two rectangles with $\text{long}(R) < \log n$, so we have $\text{long}(R) \leqslant 2 \log n$, as required.

Aizenman and Lebowitz's proof (sketch, final calculation)

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To finish the proof, we simply bound the expected number of such rectangles. To do so, note that if R is internally filled then it must contain at least one element of A in each pair of consecutive rows or columns. If

$$p = \frac{\varepsilon}{\log n}$$

for some small constant $\varepsilon > 0$, then we obtain

$$\mathbb{P}\left([A \cap R] = R\right) \leqslant \left(4p \log n\right)^{\log n/2} \leqslant (4\varepsilon)^{\log n/2} \leqslant \frac{1}{n^3}.$$

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There are $n^3 (\log n)^{O(1)}$ choices for R, so by Markov's inequality

$$\mathbb{P}\left(A \text{ percolates}\right) \to 0$$

as $n \to \infty$, as required.

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Conjecture (Folklore)

If $p > 1/2$ then the system fixates.

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Theorem (Fontes, Schonmann and Sidoravicius, 2002)

If $p > 1 - 10^{-10}$ *then the system fixates.*

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Combining the proof of this theorem with some more advanced techniques from bootstrap percolation (see Balogh, Bollobás and M., 2009) one can prove the following result in high dimensions.

Theorem (M., 2011)

If $p > \frac{1}{2}$ and $d \geqslant d_0(p)$, then on $\mathbb{Z}^d$ the system fixates.

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Theorem (Martinelli and Toninelli, 2017+)

There exist constants $C > c > 0$ such that

$$\exp\left(\frac{c}{p}\right) \leqslant \tau(\mathbb{Z}^2, 2) \leqslant \exp\left(\frac{\big(\log(1/p)\big)^C}{p}\right)$$

with high probability as $p \to 0$.

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Sharp thresholds and higher dimensions

For the 2-neighbour bootstrap model, much more precise bounds are known.

Sharp thresholds and higher dimensions

For the 2-neighbour bootstrap model, much more precise bounds are known. Recall the Aizenman–Lebowitz theorem:

Theorem (Aizenman and Lebowitz, 1988)

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Sharp thresholds and higher dimensions

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Theorem (Holroyd, 2003)

$$p_c(\mathbb{Z}_n^2, 2) = \left(\frac{\pi^2}{18} + o(1) \right) \frac{1}{\log n}.$$

Sharp thresholds and higher dimensions

For the 2-neighbour bootstrap model, much more precise bounds are known. Gravner and Holroyd later refined the upper bound argument, and together with them we proved an almost matching lower bound:

Theorem (Gravner–Holroyd, 2008; Gravner–Holroyd–M., 2012)

There exist constants $C > c > 0$ *such that*

$$\frac{\pi^2}{18 \log n} - \frac{C(\log \log n)^3}{(\log n)^{3/2}} \leqslant p_c(\mathbb{Z}_n^2, 2) \leqslant \frac{\pi^2}{18 \log n} - \frac{c}{(\log n)^{3/2}}$$

for every sufficiently large $n \in \mathbb{N}$.

Sharp thresholds and higher dimensions

For the 2-neighbour bootstrap model, much more precise bounds are known. Finally, with Hartarsky, we have managed to determine the order of the second term:

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The proof of Aizenman and Lebowitz also works in higher dimensions, but only for the 2-neighbour model:

Theorem (Aizenman and Lebowitz, 1988)

$$p_c(\mathbb{Z}_n^d, 2) = \left(\frac{\Theta(1)}{\log n} \right)^{d-1}.$$

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For the 3-neighbour model in three dimensions, the threshold was determined up to a constant factor by Cerf and Cirillo:

Theorem (Cerf and Cirillo, 1999)

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Theorem (Balogh, Bollobás and M., 2009)

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For the r-neighbour model in d dimensions, the threshold was determined up to a constant factor by Cerf and Manzo:

Theorem (Cerf and Manzo, 2002)

$$p_c(\mathbb{Z}_n^d, r) = \left(\frac{\Theta(1)}{\log_{(r-1)} n} \right)^{d-r+1}.$$

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For the r-neighbour model in d dimensions, the threshold was determined up to a constant factor by Cerf and Manzo, and we determined the sharp threshold with Balogh, Bollobás and Duminil-Copin:

Theorem (Balogh, Bollobás, Duminil-Copin and M., 2012)

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The Bollobás–Smith–Uzzell model

We now turn our attention to some dramatic recent developments in the study of bootstrap percolation, which were initiated a few years ago in a remarkable paper of Béla Bollobás, Paul Smith, and Andrew Uzzell. They studied the following large family of models:

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Definition (The $\mathcal{U}$-bootstrap process)

Let $\mathcal{U} = \{X_1, \ldots, X_m\}$ be an arbitrary finite collection of finite subsets of $\mathbb{Z}^2$, and let $A \subset \mathbb{Z}_n^2$. Set $A_0 = A$, and define, for each $t \geqslant 0$,

$$A_{t+1} = A_t \cup \Big\{ x \in \mathbb{Z}_n^2 \, : \, x + X \subset A_t \text{ for some } X \in \mathcal{U} \Big\}.$$

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One of the key insights of Bollobás, Smith and Uzzell was that the typical global behaviour of the $\mathcal{U}$-bootstrap process (with random initial set) should be determined by the action of the process on discrete half-spaces.

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Define $\mathcal{S} = \mathcal{S}(\mathcal{U}) \subseteq S^1$, the collection of *stable directions*, to be the set

$$\mathcal{S}(\mathcal{U}) := \big\{ u \in S^1 : [\mathbb{H}_u]_{\mathcal{U}} = \mathbb{H}_u \big\},$$

where

$$\mathbb{H}_u := \big\{ x \in \mathbb{Z}^2 : \langle x, u \rangle < 0 \big\}$$

denotes the discrete half-plane whose boundary is perpendicular to u.

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Let $\mathcal{C}$ denote the collection of open semicircles in S^1. The following key definition is due to Bollobás, Smith and Uzzell:

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We say that a two-dimensional update family $\mathcal{U}$ is:

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- *critical* if there exists $C \in \mathcal{C}$ that has finite intersection with $\mathcal{S}$, and if every $C \in \mathcal{C}$ has non-empty intersection with $\mathcal{S}$,

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$$A_{t+1} = A_t \cup \Big\{ x \in \mathbb{Z}_n^2 \,:\, x + X \subset A_t \text{ for some } X \in \mathcal{U} \Big\}.$$

Let $\mathcal{C}$ denote the collection of open semicircles in S^1.

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We say that a two-dimensional update family $\mathcal{U}$ is:

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The first two parts of the following theorem were proved by Bollobás, Smith and Uzzell; the proof for subcritical families was obtained slightly later by Balister, Bollobás, Przykucki and Smith.

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With Bollobás, Duminil-Copin and Smith, we proved the following much more precise bounds for critical update families:

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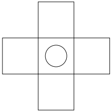
The 2-neighbour model: $\mathcal{U}$ consists of the 2-subsets of $\mathcal{N}$, where

$\mathcal{N} =$ 

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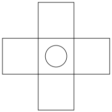
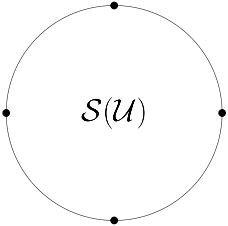
$\mathcal{N} =$ 



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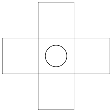
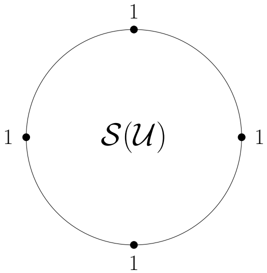
$\mathcal{N} =$ 



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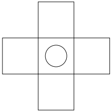
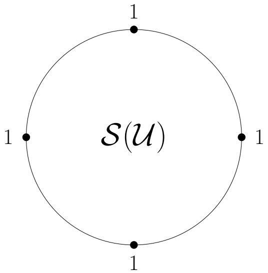
$\mathcal{N} =$ 



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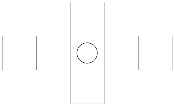
$\mathcal{N} =$ 

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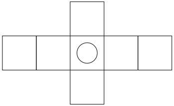
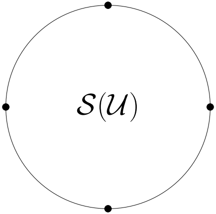
$\mathcal{N} =$ 



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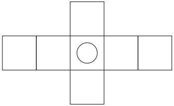
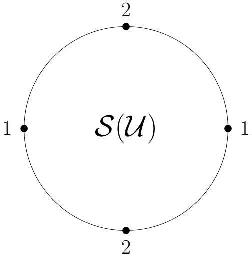
$\mathcal{N} =$ 



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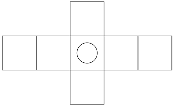
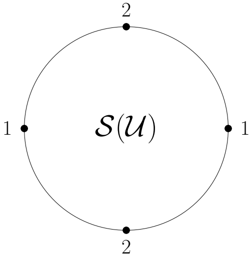
$\mathcal{N} =$ 



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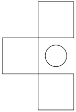
$\mathcal{N} =$ 

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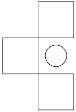
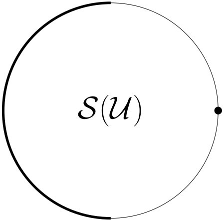
$\mathcal{N} =$ 



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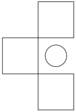
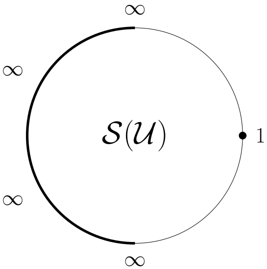
$\mathcal{N} =$ 



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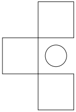
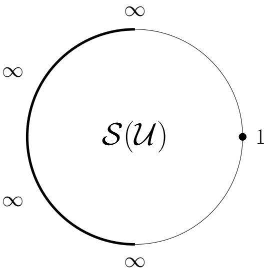
$\mathcal{N} =$ 



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Theorem (Martinelli, M. and Toninelli, 2017+)

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Conjecture (Martinelli, M. and Toninelli, 2017+)

For every critical rooted update family $\mathcal{U}$, there exists $\beta > \alpha$ such that

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Thank you!

Universality for higher dimensions

Theorem (Balister–Bollobás–M.–Smith, 2017+)

Let $\mathcal{U}$ be a d-dimensional update family.

(a) *If $\mathcal{U}$ is supercritical then $p_c(\mathbb{Z}_n^d, \mathcal{U}) = n^{-\Theta(1)}$,*

(b) *If $\mathcal{U}$ is critical then there exists $r = r(\mathcal{U}) \in \{2, \ldots, d\}$ such that*

$$p_c(\mathbb{Z}_n^d, \mathcal{U}) = \left(\frac{1}{\log_{(r-1)} n} \right)^{\Theta(1)},$$

(c) *If $\mathcal{U}$ is subcritical then $p_c(\mathbb{Z}^d, \mathcal{U}) > 0$.*

When $r < d$, the constant in the power is in general uncomputable (!!)