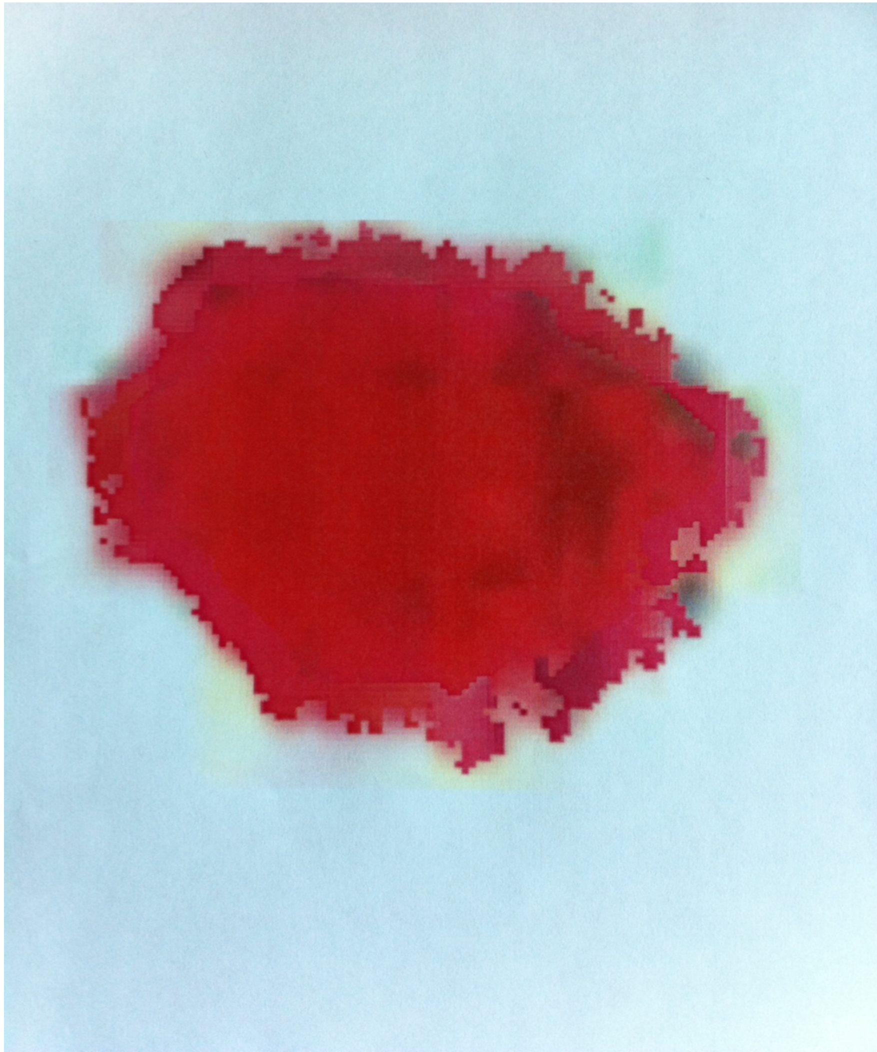
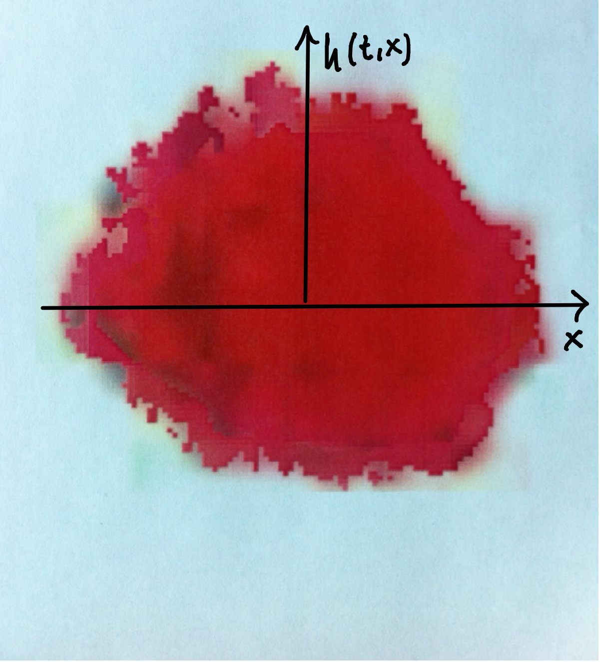


Combinatorial Structures in KPZ stochastic models

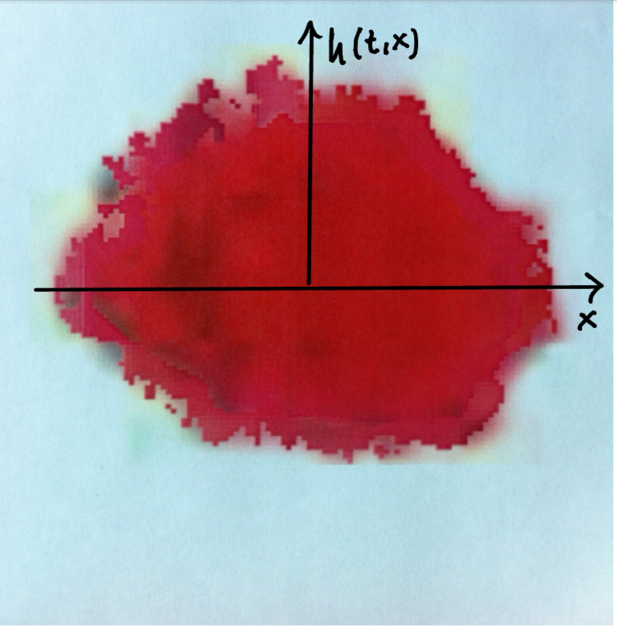
Nikos Zygouras



What is KPZ ?

$$\frac{\partial h}{\partial t} = \frac{1}{2}\Delta h + \frac{1}{2}|\nabla h|^2 + \underbrace{\dot{W}(t,x)}_{\text{space-time white noise}}$$

Kardar-Parisi-Zhang: "Dynamic scaling of growing interfaces" '86



What is KPZ ?

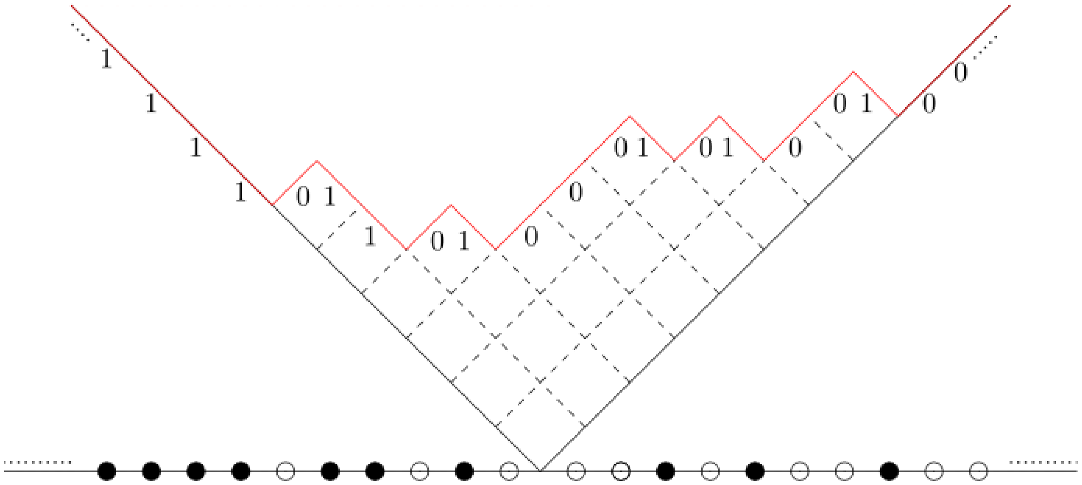
$$\frac{\partial h}{\partial t} = \frac{1}{2}\Delta h + \frac{1}{2}|\nabla h|^2 + \underbrace{\dot{W}(t,x)}_{\text{space-time white noise}}$$

Airy conjecture: $\left\{ \frac{h(t, t^{2/3}x) - tf}{t^{1/3}\sigma} \right\}_{x\in\mathbb{R}} \xrightarrow{d} \left\{ \mathcal{A}i(x) - x^2 \right\}_{x\in\mathbb{R}}$

$\{\mathcal{A}i(\cdot)\}_{x\in\mathbb{R}}$: Airy process [Prähofer-Spohn '02]

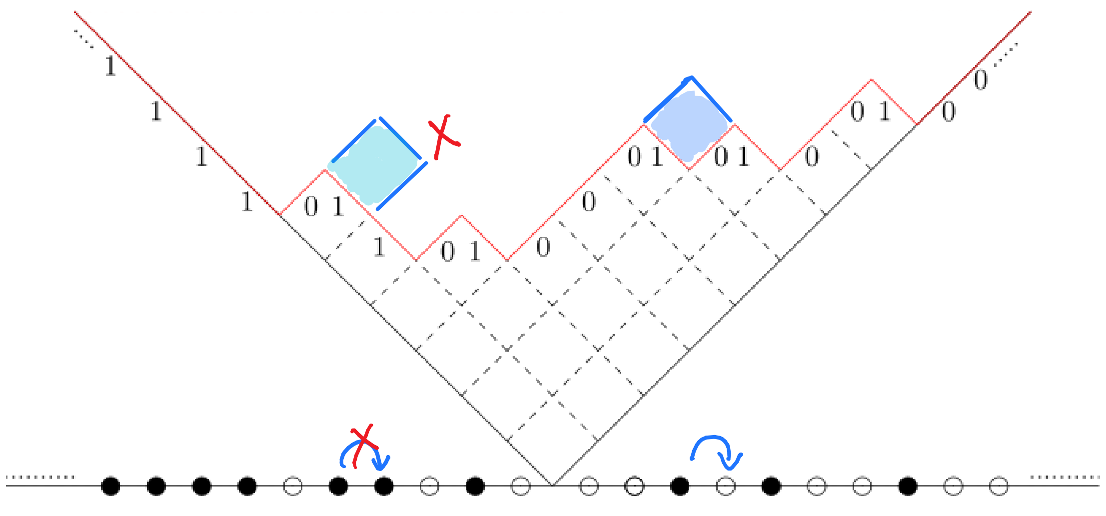
Universality Conjecture: This limit behaviour is universal for a large class of stochastic systems.

What are (some) models in the KPZ class ?



Interface / growth models

Interacting particle systems



Corner growth model

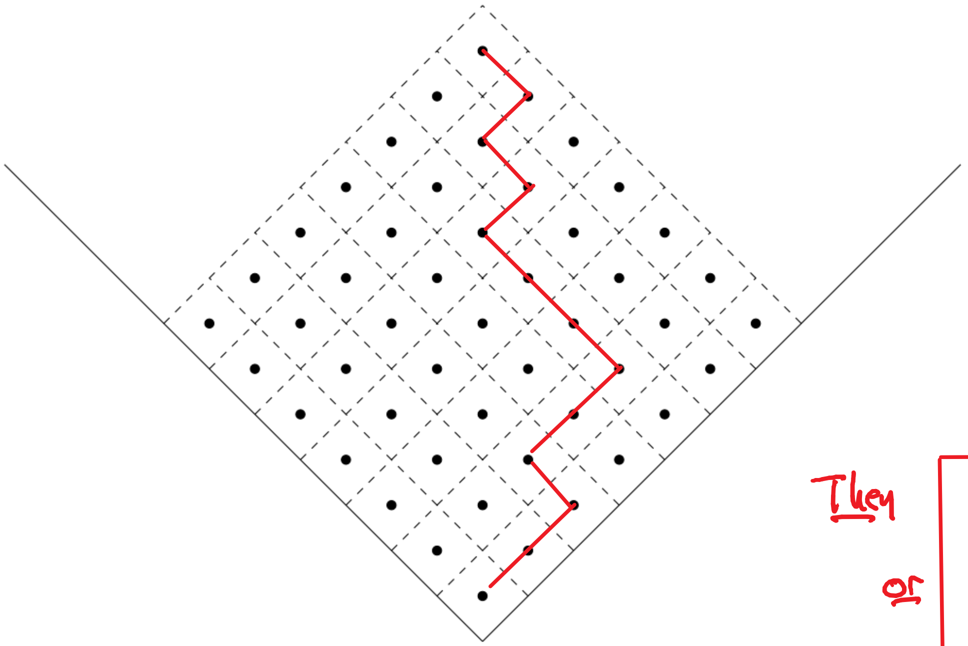
Boson - Fermion correspondence

Totally asymmetric exclusion process

After a random (e.g. exponential) time w_{ij}, a square appears assuming that it sits on a corner.

Or the particle that sits below (i,j) jumps to the right assuming no particle occupies the target site.

Last passage percolation



Q: How much time will it take until a square is filled in the corner growth?

Denote: $\tau_{N,M}$ the time that square (N,M) is filled.

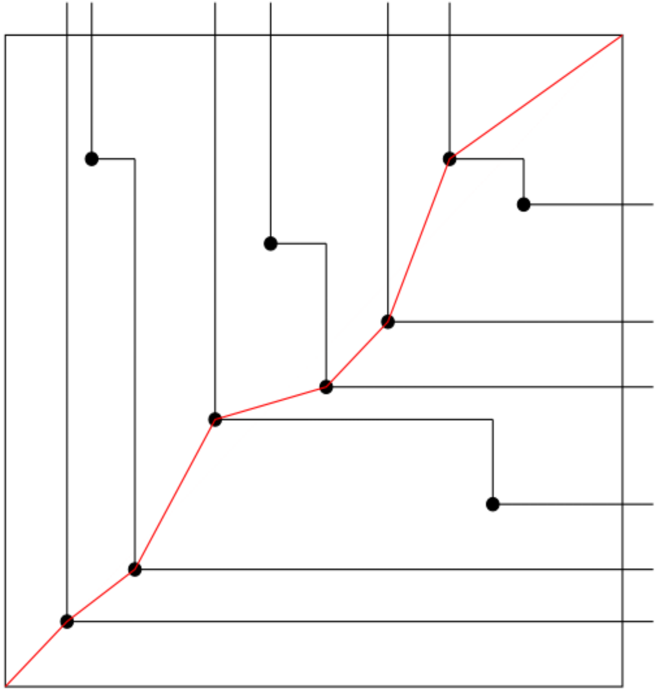
$w_{i,j}$ the (random) time that it takes for a square to fill (i,j) assuming that (i,j) is a corner site

Then

$$\boxed{\tau_{N,M} = w_{N,M} + \max\{\tau_{N-1,M}, \tau_{N,M-1}\}}$$

or

$$\boxed{\tau_{N,M} = \max_{\pi:(1,1)\to(N,M)} \sum_{(i,j)\in\pi} w_{ij}}$$

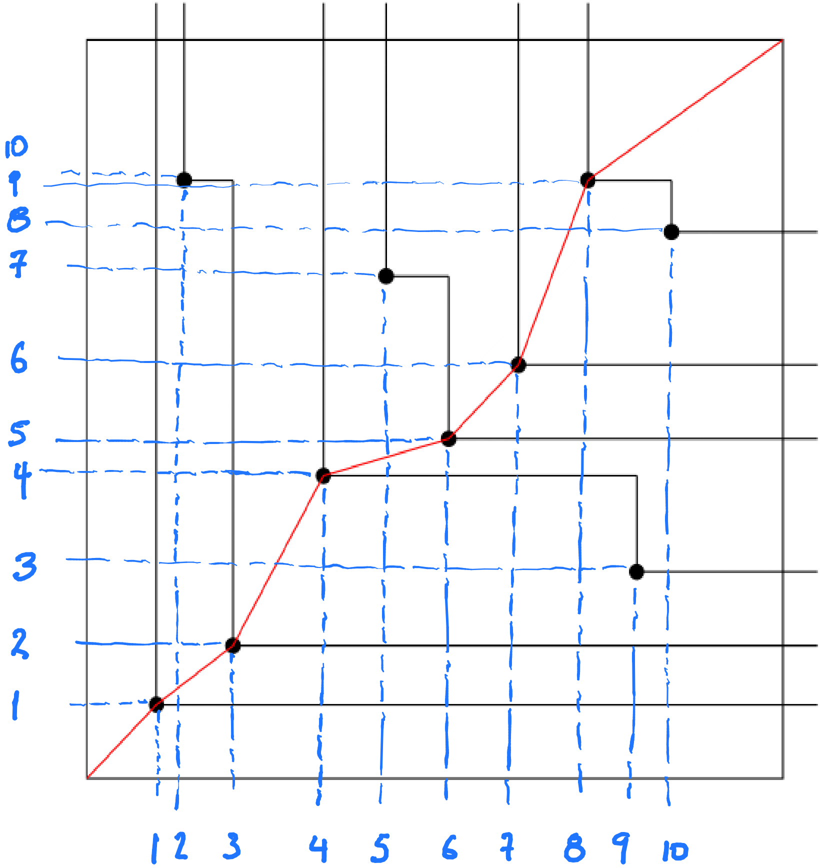


HAMMERSLEY'S PROCESS

A Poisson Point Process with intensity N is considered inside a unit square.

Q What is the "length" of the longest up-right path through the Poisson points ?

- A particle system version of Hammersley was studied by Aldous-Diaconis '95



From Hammersley to longest increasing subsequence

We can encode the Poisson points via a random permutation:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 10 & 2 & 4 & 7 & 5 & 6 & 9 & 3 & 8 \end{pmatrix}$$

Then the length of longest path equals the length of the longest subsequence in the permutation (Ulam's problem)

From KPZ to Stochastic Heat Equation & Random Polymers

- $\dfrac{\partial h}{\partial t} = \dfrac{1}{2}\Delta h + \dfrac{1}{2}|\nabla h|^2 + \dot{W}$ (KPZ)

  $\Downarrow$ ($u = \log h$ [Hopf-Cole])

- $\dfrac{\partial u}{\partial t} = \dfrac{1}{2}\Delta u + \dot{W}u$ (SHE)

  $\Downarrow$ (Feynman-Kac)

- $u(t,x) = E_0\left[\exp\left\{\int_0^t \dot{W}(s,\beta(s))\,ds\right\}\delta_x(\beta(t))\right]$

  $\Downarrow$ $\beta(\cdot)$ is Brownian Motion, $E_0[\cdot]$ with respect to $\beta(\cdot)$

- $Z_N^\omega = E\left[\exp\left\{\sum_{n=1}^{N}\beta\,\omega(n,S_n)\right\}\right]$

  $\{S_n\}_{n\geq}$ Simple Random Walk, $E[\cdot]$ with respect to $\{S_n\}$

  $\{\omega(n,x)\}_{n\in\mathbb{N},\,x\in\mathbb{Z}}$ i.i.d.

(Formal !)

$$T_N := \begin{cases} \text{Last passage percolation with EXPONENTIAL weights} \\ \text{\# of particles that have crossed origin in TASEP starting from wedge initial configuration} \\ \text{Length of longest increasing subsequence in a permutation of } N^2 \text{ numbers} \end{cases}$$

Thm [Baik-Deift-Johansson '99, Johansson '01, Prähofer-Spohn '02 & more ...]

$$\boxed{\frac{T_N - Nf}{\sigma N^{1/3}} \xrightarrow[N\to\infty]{(d)} T.W._{GUE}}$$

$T.W._{GUE}$ is the Tracy-Widom asymptotic law of the largest eigenvalue of a GUE matrix ensemble

f, σ : model specific constants

Remark: The more general Airy process asymptotic has been proven in this setting.

KEY TOOL : Robinson-Schensted-Knuth (RSK) correspondence

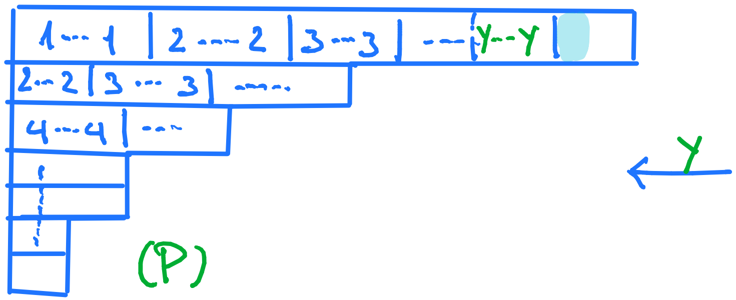
RS correspondence : A bijection between permutations & YOUNG TABLEAUX
clasifying irreducible representation of the symmetric group.

YOUNG TABLEAU : Array filled with numbers (weakly) increasing along rows
strongly increasing along columns

1	1	2	2	2	3	4	4
2	2	3	3				
3	3						
4	5						
5							

Example of Robinson-Schensted

Suppose you have mapped some of the letters of the permutation (word) to a pair (P, Q)



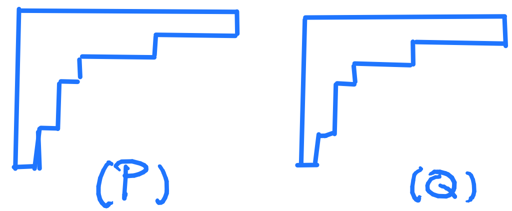
Insert $\binom{x}{y}$ as follows:
- start at the first row
- take the place of the first number $> y$
- the number removed (bumped) from the first row is inserted in the second row in the sammer manner
- if inserted number does not bump any other, then it is entered at the end of the row & procedure STOPS
- letter x is entered at the "same" position in the Q tableau.

Knuth Correspondence

Bijection between integer valued matrices &

pair of Young tableaux (P, Q)

$$\begin{pmatrix} w_{ij} \end{pmatrix}_{n\times m} \longmapsto$$



Think of the rows of the matrix as words &

$$w_{ij} = \#\{\text{letters } j \text{ in word } i\}$$

So what ?

- The length λ_1 of the first row
  $=$ length of longest increasing subsequence
- The law of the Young tableaux pair (P, Q)
  is tractable
- The marginal law of λ_1 can be computed

Schur Functions

$$s_\lambda(x) := \sum_{sh(Y)=\lambda} \prod_{i=1}^{N} x_i^{\#\{i\text{'s in } YT\}}$$

(generating function of YT)

$$\lambda = (\lambda_1, \lambda_2, \dots) \quad , \quad \lambda_1 \geq \lambda_2 \geq \cdots$$

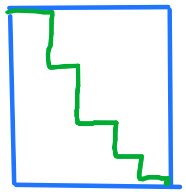
$$s_\lambda(x) = \frac{\det\left(\lambda_i^{x_j+n-j}\right)_{i,j}}{\det\left(\lambda_i^{n-j}\right)_{i,j}}$$

(Schur's original definition
Weyl character formula)

Remark: Schur functions are determinantal & this leads to determinantal processes (cf. Okounkov, Okounkov-Reshetikhin, Borodin ...)

Thm (Johansson '00, '03)

Let $\tau_N = \max_{\pi} \sum_{(i,j)\in\pi} w_{ij}$ with paths π :



with $\mathbb{P}(w_{ij}) = (1-p_iq_j)(p_iq_j)^{w_{ij}}$ & $\{w_{ij}\}$ independent

then $$\mathbb{P}(\tau_N \le x) = \prod_{i,j}(1-p_iq_j) \sum_{\lambda:\lambda_1\le x} s_\lambda(p)\, s_\lambda(q)$$

(determinantal processes)

$$= \det(I+K_N)_{L^2(x+N,\infty)}$$

with $$K_N(s,t) = \frac{1}{(2\pi i)^2}\oint_{\gamma_1} d\zeta \oint_{\gamma_2} d\eta\, \frac{\eta^s \zeta^t}{1-\zeta\eta} \prod_{j=1}^{N} \frac{1-\eta q_j}{\eta-p_j} \prod_{i=1}^{N} \frac{1-p_i\zeta}{\zeta-q_i}$$

(steepest descent analysis)

$$\lim_{N\to\infty} \mathbb{P}(\tau_N \le N\mu + N^{1/3}x) = F_2(x) = \det(I+K_{Ai})_{L^2(x,\infty)}$$

with $$K_{Ai}(z,z') = \int_0^\infty Ai(z+\lambda)\, Ai(z'+\lambda)\, d\lambda$$

End of PART I

Domino tiling made by Dan Betea.
Exhibited at Inst. Henri Poincaré.

Combinatorial (and integrable) structures in KPZ stochastic models

Part II (2008-today)

Some of the main boost ups

Tracy-Widom '08 [solved Asymmetric Exclusion Process via Bethe Ansatz]

T.W.-GUE asymptotics for KPZ itself [Sasamoto-Spohn, Calabrese-Le Doussal, Dotsenko, Amir-Corwin-Quastel '10]

Quantum Toda Hamiltonian [O'Connell '10]

Log-gamma Polymer [Seppäläinen '10]

Links to "TROPICAL" Combinatorics & geometric RSK [Corwin-O'Connell-Seppäläinen-Zygouras
O'Connell-Seppäläinen-Zygouras
Nguyen-Zygouras]

Macdonald Processes (Borodin-Corwin)

Links to 6-vertex model & higher integrable structures [Borodin, Corwin, Sasamoto, Gorin, Petrov ....]

Kirillov's RSK

Def: Gelfand-Tsetlin (GT) pattern

$$\begin{array}{ccccccc}
 & & & z_{11} & & & \\
 & & z_{22} & & z_{21} & & \\
 & z_{33} & & z_{32} & & z_{31} & \\
z_{44} & & z_{43} & & z_{42} & & z_{41} \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

In the standard RSK (representation theoretic) setting a GT array is ordered

$$z_{i+1,j+1} \le z_{ij} \le z_{i+1,j}$$

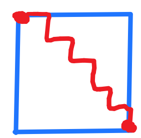
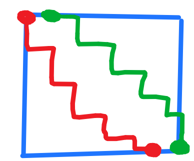
Think as $z_{ij} = \sum_{a \leq i} \#\{a\text{'s in } j \text{ rows of YT}\}$

Kirillov's (tropical / geometric) RSK is a bijective mapping

$$\left(w_{ij} \right)_{N\times N} \longrightarrow (Z, Z')$$

Properties:

1) $(z_{N1}, z_{N2}, \dots, z_{N,N}) = (z'_{N1}, z'_{N2}, \dots, z'_{N,N})$

2) $z_{N1} = \sum_{\pi}$  , $z_{N1} z_{N2} = \sum_{\pi_1 \cap \pi_2 = \emptyset}$  , etc.

3) $$\prod_{i=1}^{N} w_{ij} = \frac{z_{j1} z_{j2} \cdots z_{jj}}{z_{j-1,1} z_{j-1,2} \cdots z_{j-1,j-1}}$$ (this encodes the total # of j's "inserted")

$$\prod_{j=1}^{N} w_{ij} = \frac{z_{i1} z_{i2} \cdots z_{ii}}{z_{i-1,1} z_{i-1,2} \cdots z_{i-1,i-1}}$$

4) [OSZ '14]
$$\sum_{i,j} \frac{1}{w_{ij}} = \frac{1}{z_{NN}} + \sum_{i,j} \left(\frac{z_{ij}}{z_{i+1,j}} + \frac{z_{i+1,j+1}}{z_{ij}} \right) + \sum_{i,j} \left(\frac{z'_{ij}}{z'_{i+1,j}} + \frac{z'_{i+1,j+1}}{z'_{ij}} \right)$$

5) [OSZ '14]
$$\left| \frac{\partial \{ \log z_{ij}, \log z'_{ij} : 1 \le j \le i \le N \}}{\partial \{ \log w_{ij} : 1 \le i,j \le N \}} \right| = 1$$

(Integrable) Probability : Log-gamma measure

$$\mathbb{P}(\{w_{ij}\}) = \frac{1}{\prod \Gamma(\alpha_i+\beta_j)} \prod_{i,j} w_{ij}^{-\alpha_i-\beta_j} e^{-\frac{1}{w_{ij}}} \frac{dw_{ij}}{w_{ij}}$$

Using combinatorial properties of gRSK compute Laplace transform of

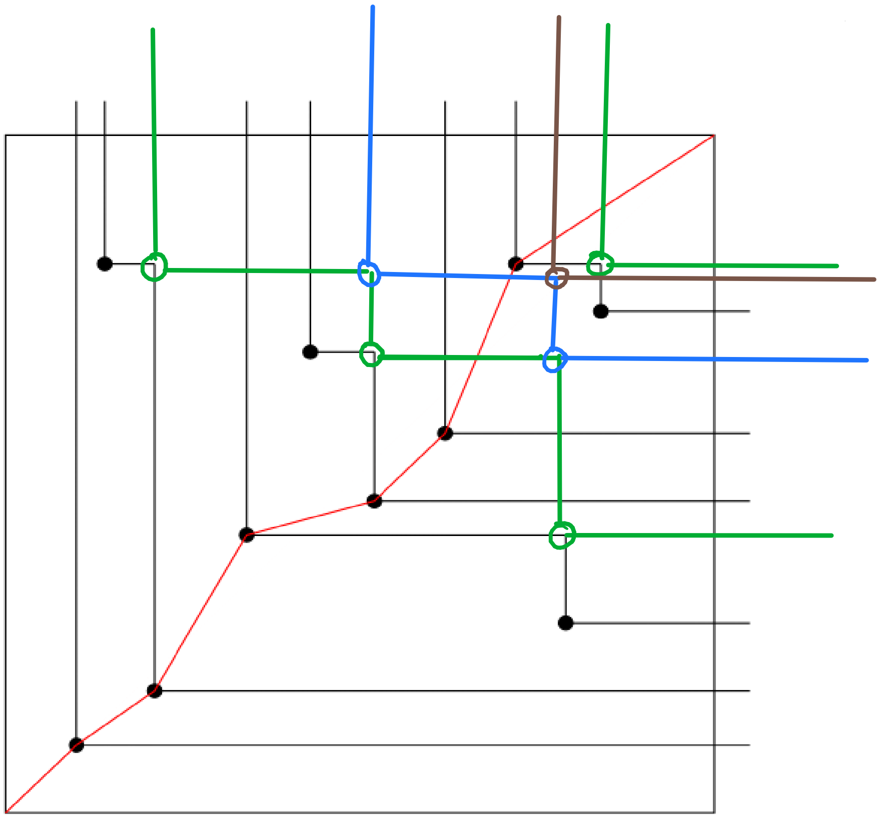
$$Z_N^{\omega} = \sum_{\pi:(1,1)\to(N,N)} \prod_{i,j\in\pi} w_{ij}$$

Recall $Z_N^{\omega} = z_{N1}(w)$

QUESTION: Why do we then need the whole RSK structure?

There are lots of cancellations (coloured rays)

RSK encodes all these!



$$\mathbb{E}\, e^{-u Z_N^{\omega}} = \int_{\mathbb{R}^{N^2}} e^{-u Z_{N1}(\{w\})}\, d\mathbb{P}(\{w\}) =$$

$$= \frac{1}{\prod_{i,j} \Gamma(\alpha_i+\beta_j)} \int_{\mathbb{R}^{N^2}} e^{-u Z_{N1}(\{w\})} \prod_i \Big(\prod_j w_{ij}\Big)^{-\alpha_i} \prod_j \Big(\prod_i w_{ij}\Big)^{-\beta_j} e^{-\sum_{i,j} \frac{1}{w_{ij}}} \prod_{i,j} \frac{dw_{ij}}{w_{ij}}$$

$$\overset{\text{change variables}}{\underset{\{w\}\mapsto\{Z,Z'\}}{=}} \frac{1}{\prod_{i,j}\Gamma(\alpha_i+\beta_j)} \int_{\mathbb{R}^{N^2}} e^{-u Z_{N1}} \prod_i \left(\frac{\prod_{j=1}^{i} Z_{ij}}{\prod_{j=1}^{i-1} Z_{i-1,j}}\right)^{-\alpha_i} \prod_j \left(\frac{\prod_{i=1}^{j} Z'_{ji}}{\prod_{i=1}^{j-1} Z'_{j-1,i}}\right)^{-\beta_j}$$

$$\cdot\, e^{-\frac{1}{Z_{NN}}} \exp\left\{-\sum_{i,j} \frac{Z_{ij}}{Z_{i+1,j}} + \frac{Z_{i+1,j+1}}{Z_{i,j}}\right\} \exp\left\{-\sum_{i,j} \frac{Z'_{ij}}{Z'_{i+1,j}} + \frac{Z'_{i+1,j+1}}{Z'_{i,j}}\right\}$$

$$\prod_{i,j} \frac{dZ_{ij}}{Z_{ij}} \prod_{i,j} \frac{dZ'_{ij}}{Z'_{ij}}$$

$$\overset{\text{first integrate}}{\underset{\text{all variables except the shape}}{=}} \frac{1}{\prod_{i,j}\Gamma(\alpha_i+\beta_j)} \int_{\mathbb{R}^{N}} e^{-u x_1 - \frac{1}{x_N}}\, \Psi^{gl_N}_{\alpha_1\cdots\alpha_N}(x_1,\dots,x_N)\, \Psi^{gl_N}_{\beta_1\cdots\beta_N}(x_1,\dots,x_N) \prod_{i=1}^{N} \frac{dx_i}{x_i}$$

Whittaker functions

Special functions with many incarnations (Number Th., Mirror Symmetry, Integrable systems ....)

One incarnation: eigenfunctions of QUANTUM TODA HAMILTONIAN:

For a group g, $\varphi^{g}_{\alpha}(x)$ are eigenfunctions of

$$\Delta - 2\sum_{a\in S} e^{-\langle a, x\rangle}$$

with eigenvalue $\sum \alpha_i^2$. S : the set of positive roots of g

GL_n : $$\Delta - 2\sum_{j=1}^{n} e^{-x_i+x_{i+1}}$$

SO_{2n+1} : $$\Delta - 2\sum_{j=1}^{n} e^{-x_i+x_{i+1}} - 2e^{-x_n}$$

FOURIER ANALYSIS

For $\lambda, \mu \in (i\mathbb{R})^N$,

$$\int_{i\mathbb{R}^N} \Psi_\lambda^{gl_N}(x)\, \Psi_\mu^{gl_N}(x)\, \frac{dx}{x} = \sum_{\sigma \in S_N} \delta(\lambda - \sigma(\mu)) \cdot S_N(\lambda)$$

$$S_N(\lambda) = \frac{1}{N!\,(2\pi i)^N} \prod_{i \neq j} \frac{1}{\Gamma(\lambda_i - \lambda_j)}$$

Fourier analysis + Whittaker integrals + combinatorics [OSZ'14]

$$\Downarrow$$

$$\mathbb{E}\, e^{-s Z_N^\omega} = \int_{(i\mathbb{R})^N} d\lambda\, s_N(\lambda) \prod_{i,j \le N} \Gamma(\alpha_i - \lambda_j) \prod_{i=1}^N \frac{s^{-\lambda_i} \prod_{j=1}^N \Gamma(\lambda_i + \beta_j)}{s^{-\alpha_i} \prod_{j=1}^N \Gamma(\alpha_i + \beta_j)} \qquad (*)$$

Fredholm determinant [Borodin - Corwin - Remenik '14]

$$(*) = \det\left(I + K_N\right)_{L^2(C_\delta)}$$

with
$$K_N(v, v') = \frac{1}{2\pi i} \int_\gamma \frac{dw}{v' - w} \cdot \frac{\pi}{\sin(\pi(w - v))} \frac{F(w)}{F(v)} \cdot \prod_{j=1}^N \frac{\Gamma(v - \alpha_j)}{\Gamma(w - \alpha_j)}$$

SUMMARY OF STEPS

Step 1. Combinatorics: Analyse & encode the structure of the dynamics/model

Step 2. Integrable probability: Determine the solvable probabilities / stochastic dynamic that are tractable

Step 3. Algebraic Structures / Harmonic Analysis: Relate to special, e.g. symmetric functions eigenfunction of symmetric operators & use the background harmonic analysis (or BETHE STATES).

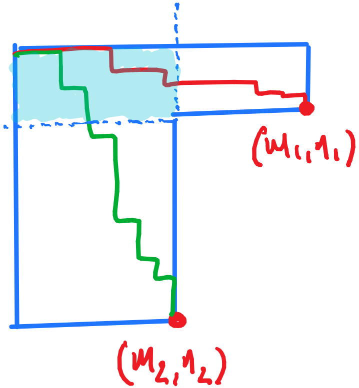
STEP 4. Asymptotic analysis: Typically via steepest descent

Correlations

Thm (Nguyen-Z '17)

$$\mathbb{E}\, e^{-u_1 Z_{n_1, m_1} - u_2 Z_{n_2 m_2}} =$$

$$= \int d\lambda\, s_{m_1}(\lambda) \prod_{1 \le i,j \le m_1} \Gamma(\lambda_i - \alpha_j) \prod_{i=1}^{m_1} \frac{u_1^{-\lambda_i} \prod \Gamma(\lambda_i + \beta_j)}{\{\lambda_i \to \alpha_i\}}$$

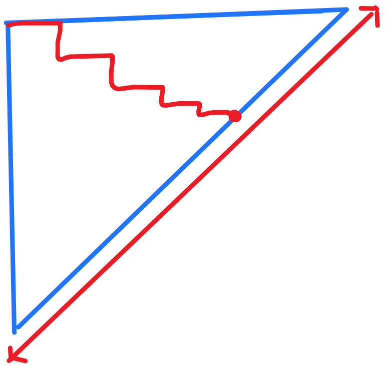


$$\int d\mu\, s_{n_1}(\mu) \prod_{i,j \le n_1} \Gamma(\mu_i - \beta_j) \prod_{j=1}^{n_2} \frac{u_2^{-\mu_j} \prod \Gamma(\mu_j + \alpha_i)}{\{\mu_j \to \beta_j\}}$$

$$\cdot \prod \frac{\Gamma(\lambda_i + \mu_j)}{\Gamma(\alpha_i + \beta_j)}$$

Other geometries / initial conditions

Thm (Bisi - Z '17)



$$\mathbb{E}\, e^{-s Z_N^{flat}} = \int_{\mathbb{R}^N} e^{-s x_1}\, \psi_\alpha^{SO_{2N+1}}(x)\, \psi_\beta^{SO_{2N+1}}(x)\, \frac{dx}{x}$$

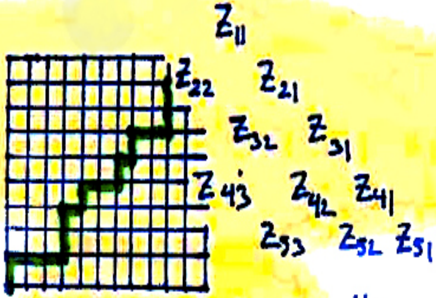
Some Questions

Determinantal Structure: Explain why Fredholm determinants appear without any (obvious) underlying determinantal process.

Advance current understanding of correlations.

General geometries & symmetries: [BZ'17] hints that there might be some deeper connection between KPZ models & integrability relating to solvable groups & integrable hamiltonians.

UNIVERSALITY: Use integrable structures to move out of them!



$$\log \sum_{\pi} \prod_{(i,j)\in\pi} w_{ij} = Nf + O(N^{1/3})$$

Thank you!