Infinite disorder renormalization fixed point: the big picture and one specific result

Giambattista Giacomin

Université Paris Diderot and Laboratoire Probabilités, Statistiques et Modélisation

November 23^{rd} 2018

Second part (on the board!) is work in Collaboration with:

- Quentin Berger (Sorbonne Université)
- Hubert Lacoin (IMPA)

(Hostorical) overview

- In 1944 Lars Onsager published the solution of the 2d ferromagnetic Ising model (square lattice, nearest neighbor interactions, no external field). Explicit formula for the free energy as function of the temperature: it is analytic except at one value of the temperature, where the second derivative has a (logarithmic) divergence.

(Hostorical) overview

- In 1944 Lars Onsager published the solution of the 2d ferromagnetic Ising model (square lattice, nearest neighbor interactions, no external field). Explicit formula for the free energy as function of the temperature: it is analytic except at one value of the temperature, where the second derivative has a (logarithmic) divergence.
- Soon after the issue of the stability of such a result under introduction of impurities was raised: bond disorder, for example "dilution".

(Hostorical) overview

- In 1944 Lars Onsager published the solution of the 2d ferromagnetic Ising model (square lattice, nearest neighbor interactions, no external field). Explicit formula for the free energy as function of the temperature: it is analytic except at one value of the temperature, where the second derivative has a (logarithmic) divergence.
- Soon after the issue of the stability of such a result under introduction of impurities was raised: bond disorder, for example "dilution".
- And for a while even the existence of a transition was put in question (disorder smooths).

(Hostorical) overview

- In 1944 Lars Onsager published the solution of the 2d ferromagnetic Ising model (square lattice, nearest neighbor interactions, no external field). Explicit formula for the free energy as function of the temperature: it is analytic except at one value of the temperature, where the second derivative has a (logarithmic) divergence.
- Soon after the issue of the stability of such a result under introduction of impurities was raised: bond disorder, for example "dilution".
- And for a while even the existence of a transition was put in question (disorder smooths).
- But by the end of the 60s confidence on the existence of the transition was installed and the question was rather: is the critical behavior in presence of impurities the same as in the pure case?

(Hostorical) overview

- In 1944 Lars Onsager published the solution of the 2d ferromagnetic Ising model (square lattice, nearest neighbor interactions, no external field). Explicit formula for the free energy as function of the temperature: it is analytic except at one value of the temperature, where the second derivative has a (logarithmic) divergence.
- Soon after the issue of the stability of such a result under introduction of impurities was raised: bond disorder, for example "dilution".
- And for a while even the existence of a transition was put in question (disorder smooths).
- But by the end of the 60s confidence on the existence of the transition was installed and the question was rather: is the critical behavior in presence of impurities the same as in the pure case?
- In 1974 A. B. Harris came up with an argument based on the idea that one should be able to predict whether introducing impurities changes (or not) the critical behavior just in terms of properties of the pure model (perturbation theory)

Harris criterion

Harris' result (claim?) is very (or deceivingly) simple to state.

Harris criterion

Harris' result (claim?) is very (or deceivingly) simple to state.
We just need a notion of correlation length $\ell(T)$ for the pure system at temperature T and to know that $\ell(T) \approx |T - T_c|^{-\nu}$ for T close to T_c.

Harris criterion

Harris' result (claim?) is very (or deceivingly) simple to state.
We just need a notion of correlation length $\ell(T)$ for the pure system at temperature T and to know that $\ell(T) \approx |T - T_c|^{-\nu}$ for T close to T_c.

The Harris criterion in dimension d

If $\nu d > 2$ the disorder is irrelevant, meaning that (a moderate amount of) impurities will not change the critical behavior (i.e. the critical exponents).

Harris criterion

Harris' result (claim?) is very (or deceivingly) simple to state.
We just need a notion of correlation length $\ell(T)$ for the pure system at temperature T and to know that $\ell(T) \approx |T - T_c|^{-\nu}$ for T close to T_c.

The Harris criterion in dimension d

If $\nu d > 2$ the disorder is irrelevant, meaning that (a moderate amount of) impurities will not change the critical behavior (i.e. the critical exponents).

Harris' arguments are based on renormalization group ideas and ultimately $\nu d > 2$ can be seen as a contraction criterion for the size of the disorder under renormalization.

Harris criterion

Harris' result (claim?) is very (or deceivingly) simple to state.
We just need a notion of correlation length $\ell(T)$ for the pure system at temperature T and to know that $\ell(T) \approx |T - T_c|^{-\nu}$ for T close to T_c.

The Harris criterion in dimension d

If $\nu d > 2$ the disorder is irrelevant, meaning that (a moderate amount of) impurities will not change the critical behavior (i.e. the critical exponents).

Harris' arguments are based on renormalization group ideas and ultimately $\nu d > 2$ can be seen as a contraction criterion for the size of the disorder under renormalization.

$\nu d < 2$ is therefore an expansion criterion, strongly suggesting change of critical behavior: disorder is relevant.

Harris criterion

Harris' result (claim?) is very (or deceivingly) simple to state.
We just need a notion of correlation length $\ell(T)$ for the pure system at temperature T and to know that $\ell(T) \approx |T - T_c|^{-\nu}$ for T close to T_c.

The Harris criterion in dimension d

If $\nu d > 2$ the disorder is irrelevant, meaning that (a moderate amount of) impurities will not change the critical behavior (i.e. the critical exponents).

Harris' arguments are based on renormalization group ideas and ultimately $\nu d > 2$ can be seen as a contraction criterion for the size of the disorder under renormalization.

$\nu d < 2$ is therefore an expansion criterion, strongly suggesting change of critical behavior: disorder is relevant.

$\nu d = 2$ is called "marginal case".

Harris criterion

Harris' result (claim?) is very (or deceivingly) simple to state.
We just need a notion of correlation length $\ell(T)$ for the pure system at temperature T and to know that $\ell(T) \approx |T - T_c|^{-\nu}$ for T close to T_c.

The Harris criterion in dimension d

If $\nu d > 2$ the disorder is irrelevant, meaning that (a moderate amount of) impurities will not change the critical behavior (i.e. the critical exponents).

Harris' arguments are based on renormalization group ideas and ultimately $\nu d > 2$ can be seen as a contraction criterion for the size of the disorder under renormalization.

$\nu d < 2$ is therefore an expansion criterion, strongly suggesting change of critical behavior: disorder is relevant.

$\nu d = 2$ is called "marginal case".

This appealing picture turns out to be difficult to be made into theorems

Getting down to business: pinning models

Two (probabilistically independent) ingredients:

1. Basic choice: $\{S_n\}_{n=0,1,\ldots}$ is a simple symmetric lazy RW (law $\mathbf{P}$)
2. The disorder: $\{\omega_n\}_{n=1,2,\ldots}$ IID sequence. We set $\lambda(s) := \mathbb{E}[\exp(s\omega_1)$ and assume $\lambda(s) < \infty$ at least for $|s|$ small. Without loss of generality $\mathbb{E}[\omega_1] = 0$ and $\mathbb{E}[\omega_1^2] = 1$.

Getting down to business: pinning models

Two (probabilistically independent) ingredients:

1. Basic choice: $\{S_n\}_{n=0,1,\ldots}$ is a simple symmetric lazy RW (law $\mathbf{P}$)
2. The disorder: $\{\omega_n\}_{n=1,2,\ldots}$ IID sequence. We set $\lambda(s) := \mathbb{E}[\exp(s\omega_1)$ and assume $\lambda(s) < \infty$ at least for $|s|$ small. Without loss of generality $\mathbb{E}[\omega_1] = 0$ and $\mathbb{E}[\omega_1^2] = 1$.

The model is defined for $\beta \geq 0$, $h \in \mathbb{R}$, $N \in \mathbb{N}$

$$\mathbf{P}_{N,\omega,\beta,h}(S_1, \ldots, S_N) = \frac{\exp\left(\sum_{n=1}^{N-1}(\beta\omega_n + h)\delta_n\right)}{Z_{N,\omega,\beta,h}} \mathbf{1}_{S_N=0}\mathbf{P}(S_1, \ldots, S_N)$$

where $Z_{N,\omega,\beta,h}$ is the normalization and

1. Contact pinning: $\delta_n := \mathbf{1}_{S_n=0}$

Getting down to business: pinning models

Two (probabilistically independent) ingredients:

1. Basic choice: $\{S_n\}_{n=0,1,\ldots}$ is a simple symmetric lazy RW (law $\mathbf{P}$)
2. The disorder: $\{\omega_n\}_{n=1,2,\ldots}$ IID sequence. We set $\lambda(s) := \mathbb{E}[\exp(s\omega_1)$ and assume $\lambda(s) < \infty$ at least for $|s|$ small. Without loss of generality $\mathbb{E}[\omega_1] = 0$ and $\mathbb{E}[\omega_1^2] = 1$.

The model is defined for $\beta \geq 0$, $h \in \mathbb{R}$, $N \in \mathbb{N}$

$$\mathbf{P}_{N,\omega,\beta,h}(S_1,\ldots,S_N) = \frac{\exp\left(\sum_{n=1}^{N-1}(\beta\omega_n + h)\delta_n\right)}{Z_{N,\omega,\beta,h}} \mathbf{1}_{S_N=0}\mathbf{P}(S_1,\ldots,S_N)$$

where $Z_{N,\omega,\beta,h}$ is the normalization and

1. Contact pinning: $\delta_n := \mathbf{1}_{S_n=0}$
2. Copolymer pinning: $\delta_n := \mathbf{1}_{S_n<0}$

Pinning models: the partition function

The partition function of the model

$$Z_{N,\omega,\beta,h} = \mathbf{E}\left[\exp\left(\sum_{n=1}^{N}(\beta\omega_n + h)\delta_n\right); S_N = 0\right]$$

with $\delta_n = \mathbf{1}_{S_n=0}$ (contact) or $\delta_n = \mathbf{1}_{S_n<0}$ (copolymer), contains a lot of information: it is a generating function.

Pinning models: the partition function

The partition function of the model

$$Z_{N,\omega,\beta,h} = \mathbf{E}\left[\exp\left(\sum_{n=1}^{N}(\beta\omega_n + h)\delta_n\right); S_N = 0\right]$$

with $\delta_n = \mathbf{1}_{S_n=0}$ (contact) or $\delta_n = \mathbf{1}_{S_n<0}$ (copolymer), contains a lot of information: it is a generating function.

Note for example that:

$$\partial_h \frac{1}{N} \log Z_{N,\omega,\beta,h} = \mathbf{E}_{N,\omega,\beta,h}\left[\frac{1}{N}\sum_{n=1}^{N}\delta_n\right]$$

Pinning models: the partition function

The partition function of the model

$$Z_{N,\omega,\beta,h} = \mathbf{E}\left[\exp\left(\sum_{n=1}^{N}(\beta\omega_n + h)\delta_n\right); S_N = 0\right]$$

with $\delta_n = \mathbf{1}_{S_n=0}$ (contact) or $\delta_n = \mathbf{1}_{S_n<0}$ (copolymer), contains a lot of information: it is a generating function.

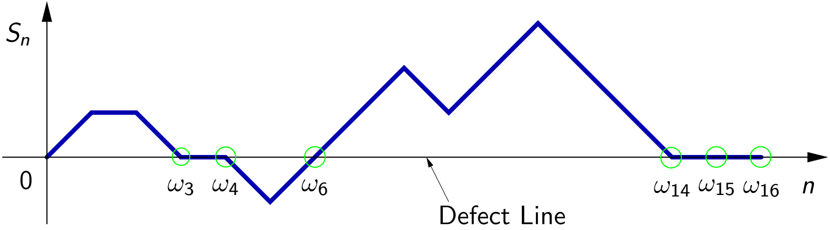
Note for example that:

$$\partial_h \frac{1}{N} \log Z_{N,\omega,\beta,h} = \mathbf{E}_{N,\omega,\beta,h}\left[\frac{1}{N}\sum_{n=1}^{N}\delta_n\right]$$

So we understand the relevance of the free energy (density):

$$\mathrm{F}(\beta, h) := \lim_{N\to\infty} \frac{1}{N}\mathbb{E} \log Z_{N,\omega,\beta,h}$$

Contact pinning



All that matters of S for $Z_{N,\omega,\beta,h}$ is the zero level set τ of the RW!

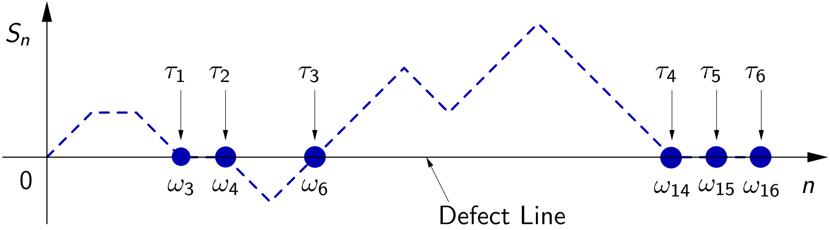
$$\tau_0 = 0, \quad \tau_{j+1} = \inf\{n > \tau_j : \ S_n = 0\}$$

$$Z_{N,\omega,\beta,h} = \mathbf{E}\left[\exp\left(\sum_{n=1}^{N}(\beta\omega_n + h)\delta_n\right); N \in \tau\right]$$

with $\delta_n = \mathbf{1}_{n\in\tau}$.

τ is a renewal process: ω targeting strategy?

Contact pinning



All that matters of S for $Z_{N,\omega,\beta,h}$ is the zero level set τ of the RW!

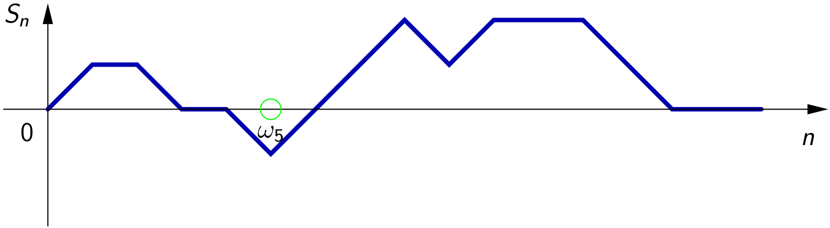
$$\tau_0 = 0, \quad \tau_{j+1} = \inf\{n > \tau_j : \ S_n = 0\}$$

$$Z_{N,\omega,\beta,h} = \mathbf{E}\left[\exp\left(\sum_{n=1}^{N}(\beta\omega_n + h)\delta_n\right); N \in \tau\right]$$

with $\delta_n = \mathbf{1}_{n\in\tau}$.

τ is a renewal process: ω targeting strategy?

Copolymer pinning



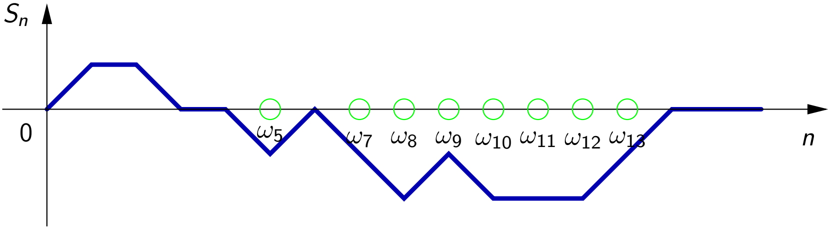
All that matters of S for $Z_{N,\omega,\beta,h}$ is the zero level set τ and the (up or down) position of the excursions!

$$Z_{N,\omega,\beta,h} = \mathbf{E}\left[\exp\left(\sum_{n=1}^{N}(\beta\omega_n + h)\delta_n\right); N \in \tau\right]$$

with $\delta_n = \mathbf{1}_{n\in\tau\cap A_N}$, $A_N = \cup_{j:s_j=1}(\tau_{j-1}, \tau_j) \cap \{1, \ldots, N\}$ and $s_j \sim B(1/2)$.

Strategy: ω targeting strategy by τ and/or excursion up/down switch?

Copolymer pinning



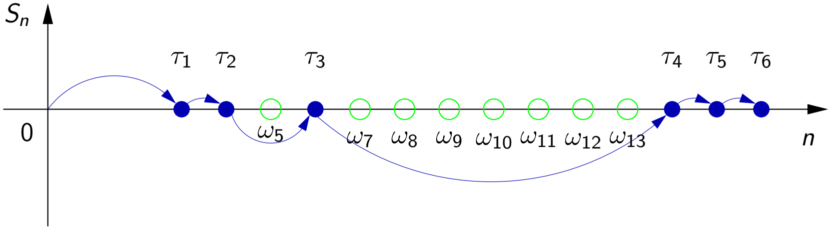
All that matters of S for $Z_{N,\omega,\beta,h}$ is the zero level set τ and the (up or down) position of the excursions!

$$Z_{N,\omega,\beta,h} = \mathbf{E}\left[\exp\left(\sum_{n=1}^{N}(\beta\omega_n + h)\delta_n\right); N \in \tau\right]$$

with $\delta_n = \mathbf{1}_{n\in\tau\cap A_N}$, $A_N = \cup_{j:s_j=1}(\tau_{j-1}, \tau_j) \cap \{1, \ldots, N\}$ and $s_j \sim B(1/2)$.

Strategy: ω targeting strategy by τ and/or excursion up/down switch?

Copolymer pinning



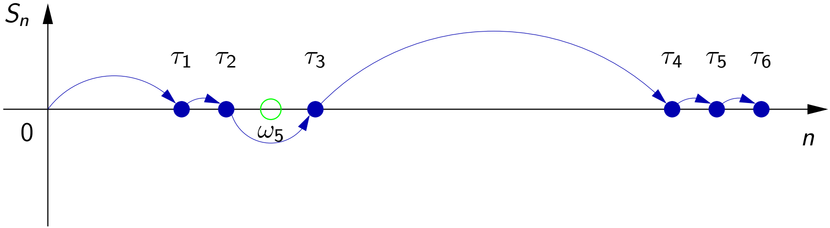
All that matters of S for $Z_{N,\omega,\beta,h}$ is the zero level set τ and the (up or down) position of the excursions!

$$Z_{N,\omega,\beta,h} = \mathbf{E}\left[\exp\left(\sum_{n=1}^{N}(\beta\omega_n + h)\delta_n\right); N \in \tau\right]$$

with $\delta_n = \mathbf{1}_{n\in\tau\cap A_N}$, $A_N = \cup_{j:s_j=1}(\tau_{j-1},\tau_j) \cap \{1,\ldots,N\}$ and $s_j \sim B(1/2)$.

Strategy: ω targeting strategy by τ and/or excursion up/down switch?

Copolymer pinning



All that matters of S for $Z_{N,\omega,\beta,h}$ is the zero level set τ and the (up or down) position of the excursions!

$$Z_{N,\omega,\beta,h} = \mathbf{E}\left[\exp\left(\sum_{n=1}^{N}(\beta\omega_n + h)\delta_n\right); N \in \tau\right]$$

with $\delta_n = \mathbf{1}_{n \in \tau \cap A_N}$, $A_N = \cup_{j: s_j = 1}(\tau_{j-1}, \tau_j) \cap \{1, \ldots, N\}$ and $s_j \sim B(1/2)$.

Strategy: ω targeting strategy by τ and/or excursion up/down switch?

Free energy density and phase transition

$$\mathrm{F}(\beta, h) := \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z_{N,\omega,\beta,h}$$

$\mathrm{F}(\cdot)$ is convex (hence C^0), non decreasing and $\mathrm{F}(\beta, h) \geq 0$

Free energy density and phase transition

$$\mathrm{F}(\beta, h) := \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z_{N,\omega,\beta,h}$$

$\mathrm{F}(\cdot)$ is convex (hence C^0), non decreasing and $\mathrm{F}(\beta, h) \geq 0$:

$$\mathrm{F}(\beta, h) \geq \limsup_{N\to\infty} \frac{1}{N} \mathbb{E} \log \mathbf{E}\left[\exp\left(\sum_{n=1}^{N-1} (\beta\omega_n + h)\delta_n\right); \tau_1 = N, s_1 = 0\right]$$

$$= \limsup_{N\to\infty} \frac{1}{N} \log \mathbf{P}\left(\tau_1 = N, s_1 = 0\right) = \lim_{N\to\infty} \frac{\log(cN^{-3/2})}{N} = 0$$

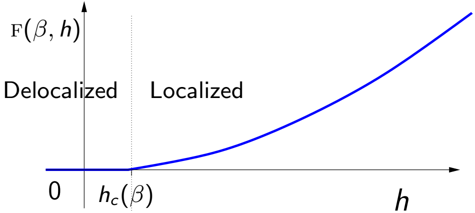
Free energy density and phase transition

$$\mathrm{F}(\beta, h) := \lim_{N\to\infty} \frac{1}{N} \mathbb{E} \log Z_{N,\omega,\beta,h}$$

$\mathrm{F}(\cdot)$ is convex (hence C^0), non decreasing and $\mathrm{F}(\beta, h) \geq 0$:

$$\mathrm{F}(\beta, h) \geq \limsup_{N\to\infty} \frac{1}{N} \mathbb{E} \log \mathbf{E} \left[\exp\left(\sum_{n=1}^{N-1} (\beta\omega_n + h)\delta_n \right); \tau_1 = N, s_1 = 0 \right]$$

$$= \limsup_{N\to\infty} \frac{1}{N} \log \mathbf{P}(\tau_1 = N, s_1 = 0) = \lim_{N\to\infty} \frac{\log(cN^{-3/2})}{N} = 0$$



Beyond the RW case

The RW dependence of the model is ultimately encoded by just by $K(n) := \mathbf{P}(\tau_1 = n)$ and for symmetric (lazy) walks

$$K(n) \overset{n\to\infty}{\sim} \frac{c}{n^{1+\frac{1}{2}}}$$

Beyond the RW case

The RW dependence of the model is ultimately encoded by just by $K(n) := \mathbf{P}(\tau_1 = n)$ and for symmetric (lazy) walks

$$K(n) \overset{n\to\infty}{\sim} \frac{c}{n^{1+\frac{1}{2}}}$$

Generalized model:

$$K(n) \overset{n\to\infty}{\sim} \frac{L(n)}{n^{1+\alpha}}$$

with $\alpha \geq 0$ and $L(\cdot)$ slowly varying. Without loss of generality: $\sum_n K(n) = 1$.

Beyond the RW case

The RW dependence of the model is ultimately encoded by just by $K(n) := \mathbf{P}(\tau_1 = n)$ and for symmetric (lazy) walks

$$K(n) \overset{n\to\infty}{\sim} \frac{c}{n^{1+\frac{1}{2}}}$$

Generalized model:

$$K(n) \overset{n\to\infty}{\sim} \frac{L(n)}{n^{1+\alpha}}$$

with $\alpha \geq 0$ and $L(\cdot)$ slowly varying. Without loss of generality: $\sum_n K(n) = 1$.

Vast amount of (mostly) physics literature:

- [M. Fisher 84], [Derrida, Hakim, Vannimenus 92]
- [Garel, Huse, Leibler, Orland 89], [Sinai 93], [Bolthausen-den Hollander 97]
- ...

The pure model: $\beta = 0$ (contact pinning case)

$$Z_{N,h} = \mathbf{E}\left[\exp\left(hL_N\right); N \in \tau\right] \quad \text{with} \quad L_N := \sum_{n=1}^{N} \delta_n$$

Contact pinning case: L_N is the local time at the origin.

The pure model: $\beta = 0$ (contact pinning case)

$$Z_{N,h} = \mathbf{E}\left[\exp\left(hL_N\right); N \in \tau\right] \quad \text{with} \quad L_N := \sum_{n=1}^{N} \delta_n$$

Contact pinning case: L_N is the local time at the origin. Summary:

1. By a simple algebraic manipulation one finds a new renewal process $\widetilde{\tau}$ such that

$$Z_{N,h} = \exp(\mathrm{F}(h)N)\mathbf{P}(N \in \widetilde{\tau})$$

so everything is reduced to renewal questions [Feller, Erdos, Pollard, Garsia, Lamperti,...] and $\mathrm{F}(h)$ is the unique solution of

$$\sum_n K(n)e^{h-n\mathrm{F}(h)} = 1$$

when such a solution exists, otherwise $\mathrm{F}(h) = 0$

The pure model: $\beta = 0$ (contact pinning case)

$$Z_{N,h} = \mathbf{E}\left[\exp\left(hL_N\right); N \in \tau\right] \quad \text{with} \quad L_N := \sum_{n=1}^{N} \delta_n$$

Contact pinning case: L_N is the local time at the origin. Summary:

1. By a simple algebraic manipulation one finds a new renewal process $\widetilde{\tau}$ such that

$$Z_{N,h} = \exp(\mathrm{F}(h)N)\mathbf{P}(N \in \widetilde{\tau})$$

so everything is reduced to renewal questions [Feller, Erdos, Pollard, Garsia, Lamperti,...] and $\mathrm{F}(h)$ is the unique solution of

$$\sum_n K(n) e^{h - n\mathrm{F}(h)} = 1$$

when such a solution exists, otherwise $\mathrm{F}(h) = 0$

2. In particular $\mathrm{F}(h) = 0$ for $h \leq 0$ and for $h \searrow 0$

$$\mathrm{F}(h) \sim L_\alpha(h) h^{\max(1/\alpha, 1)}$$

The pure model: $\beta = 0$ (contact pinning case)

$$Z_{N,h} = \mathbf{E}\left[\exp\left(hL_N\right); N \in \tau\right] \quad \text{with} \quad L_N := \sum_{n=1}^{N} \delta_n$$

Contact pinning case: L_N is the local time at the origin. Summary:

1. By a simple algebraic manipulation one finds a new renewal process $\widetilde{\tau}$ such that
$$Z_{N,h} = \exp(\mathrm{F}(h)N)\mathbf{P}(N \in \widetilde{\tau})$$
so everything is reduced to renewal questions [Feller, Erdos, Pollard, Garsia, Lamperti,...] and $\mathrm{F}(h)$ is the unique solution of
$$\sum_n K(n)e^{h-n\mathrm{F}(h)} = 1$$
when such a solution exists, otherwise $\mathrm{F}(h) = 0$
2. In particular $\mathrm{F}(h) = 0$ for $h \leq 0$ and for $h \searrow 0$
$$\mathrm{F}(h) \sim L_\alpha(h)h^{\max(1/\alpha,1)}$$

Obs.: tuning $\alpha \geq 0$ we find all possible critical behavior

The pure model: $\beta = 0$ (copolymer pinning case)

$$Z_{N,h} = \mathbf{E}\left[\exp\left(hL_N\right); N \in \tau\right] \quad \text{with} \quad L_N := \sum_{n=1}^{N-1} \delta_n$$

Copolymer pinning case: L_N is the time spent below level zero.

The pure model: $\beta = 0$ (copolymer pinning case)

$$Z_{N,h} = \mathbf{E}\left[\exp\left(hL_N\right); N \in \tau\right] \quad \text{with} \quad L_N := \sum_{n=1}^{N-1} \delta_n$$

Copolymer pinning case: L_N is the time spent below level zero.

Much simpler now: of course

$$Z_{N,h} \leq \exp(\max(0,h)N)$$

so $\mathrm{F}(h) \leq \max(0,h)$ ($\Longrightarrow \mathrm{F}(h) = 0$ for $h \leq 0$)

The pure model: $\beta = 0$ (copolymer pinning case)

$$Z_{N,h} = \mathbf{E}\left[\exp\left(hL_N\right); N \in \tau\right] \quad \text{with} \quad L_N := \sum_{n=1}^{N-1} \delta_n$$

Copolymer pinning case: L_N is the time spent below level zero.

Much simpler now: of course

$$Z_{N,h} \leq \exp(\max(0, h)N)$$

so $\mathrm{F}(h) \leq \max(0, h)$ ($\Longrightarrow \mathrm{F}(h) = 0$ for $h \leq 0$) and for $h > 0$

$$Z_{N,h} \geq Z_{N,h}(\tau_1 = N, s = +1) = \frac{1}{2} e^{h(N-1)} \mathbf{P}(\tau_1 = N)$$

so $\mathrm{F}(h) \geq h$ for $h > 0$.

The pure model: $\beta = 0$ (copolymer pinning case)

$$Z_{N,h} = \mathbf{E}\left[\exp\left(hL_N\right); N \in \tau\right] \quad \text{with} \quad L_N := \sum_{n=1}^{N-1} \delta_n$$

Copolymer pinning case: L_N is the time spent below level zero.

Much simpler now: of course

$$Z_{N,h} \le \exp(\max(0,h)N)$$

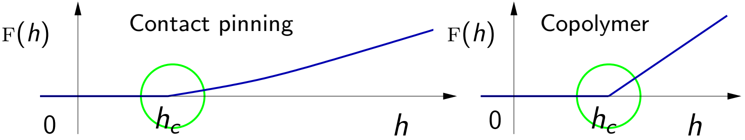
so $\mathrm{F}(h) \le \max(0,h)$ ($\Longrightarrow \mathrm{F}(h) = 0$ for $h \le 0$) and for $h > 0$

$$Z_{N,h} \ge Z_{N,h}(\tau_1 = N, s = +1) = \frac{1}{2} e^{h(N-1)} \mathbf{P}(\tau_1 = N)$$

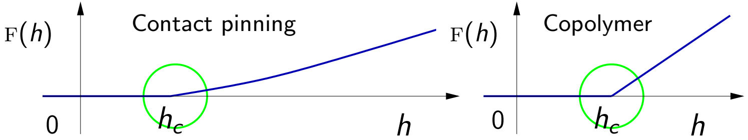
so $\mathrm{F}(h) \ge h$ for $h > 0$. Hence

$$\mathrm{F}(h) = \max(0,h)$$

The correlation length



The correlation length

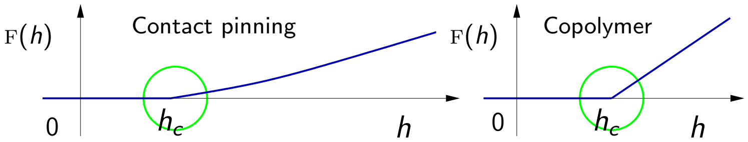


Other element: correlation length in these models

$$\ell(h) = 1/\mathrm{F}(h) \qquad (\text{or } \ell(h) = Const./\mathrm{F}(h))$$

"because" $Z_{N,h} \approx \exp(\mathrm{F}(h)N)$ [...].

The correlation length



Other element: correlation length in these models

$$\ell(h) = 1/\mathrm{F}(h) \qquad (\text{or } \ell(h) = Const./\mathrm{F}(h))$$

"because" $Z_{N,h} \approx \exp(\mathrm{F}(h)N)$ [. . .].

Hence $\ell(h) \sim h^{-\nu}$ for $h \searrow 0$ with

$$\begin{aligned} \nu &= \max(1, 1/\alpha) && \text{for contact pinning} \\ \nu &= 1 && \text{for copolymer pinning} \end{aligned}$$

Ready for testing the Harris criterion

Now we would like to switch the disorder on: $\beta > 0$.

What is the Harris criterion for disorder irrelevance telling us?

$$\nu d = \nu > 2 \implies \text{irrelevant if } \begin{cases} \alpha \in [0, 1/2) & \text{for contact pinning} \\ \alpha \in \emptyset & \text{for copolymer pinning} \end{cases}$$

Ready for testing the Harris criterion

Now we would like to switch the disorder on: $\beta > 0$.

What is the Harris criterion for disorder irrelevance telling us?

$$\nu d = \nu > 2 \implies \text{irrelevant if } \begin{cases} \alpha \in [0, 1/2) & \text{for contact pinning} \\ \alpha \in \emptyset & \text{for copolymer pinning} \end{cases}$$

Temptation: the irrelevant case should be easy!

Ready for testing the Harris criterion

Now we would like to switch the disorder on: $\beta > 0$.

What is the Harris criterion for disorder irrelevance telling us?

$$\nu d = \nu > 2 \implies \text{irrelevant if } \begin{cases} \alpha \in [0, 1/2) & \text{for contact pinning} \\ \alpha \in \emptyset & \text{for copolymer pinning} \end{cases}$$

Temptation: the irrelevant case should be easy!
And in fact [K. Alexander 08, Toninelli 08, Lacoin 10] showed that disorder for contact pinning is irrelevant (if $\beta \in (0, \beta_0]$), but **(WARNING!)** contact pinning is the only class of models under control.

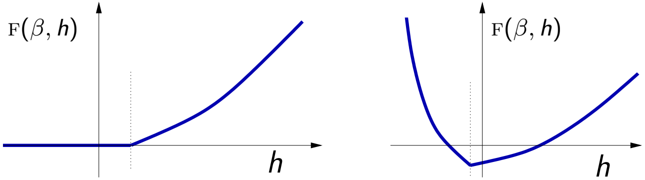
Ready for testing the Harris criterion

Now we would like to switch the disorder on: $\beta > 0$.

What is the Harris criterion for disorder irrelevance telling us?

$$\nu d = \nu > 2 \implies \text{irrelevant if } \begin{cases} \alpha \in [0, 1/2) & \text{for contact pinning} \\ \alpha \in \emptyset & \text{for copolymer pinning} \end{cases}$$

Temptation: the irrelevant case should be easy!
And in fact [K. Alexander 08, Toninelli 08, Lacoin 10] showed that disorder for contact pinning is irrelevant (if $\beta \in (0, \beta_0]$), but **(WARNING!)** contact pinning is the only class of models under control.



Smoothing inequality and disorder relevance

In [G., Toninelli 06] (also [Caravenna, den Hollander 13]): for $\beta > 0$ there exists $c_\beta > 0$ such that for every $\alpha \geq 0$

$$F(\beta, h) \overset{h \geq h_c(\beta)}{\leq} c_\beta (h - h_c(\beta))^2 \,,$$

and $c_\beta \sim const.\beta^{-2}$ for $\beta \searrow 0$.

Smoothing inequality and disorder relevance

In [G., Toninelli 06] (also [Caravenna, den Hollander 13]): for $\beta > 0$ there exists $c_\beta > 0$ such that for every $\alpha \geq 0$

$$F(\beta, h) \overset{h \geq h_c(\beta)}{\leq} c_\beta (h - h_c(\beta))^2 \,,$$

and $c_\beta \sim const. \beta^{-2}$ for $\beta \searrow 0$.

Note that this inequality

- is empty for contact pinning if $\alpha < 1/2$ and $\beta \leq \beta_0$ (the irrelevant disorder results show that $F(\beta, h)$ is smaller than that approaching criticality)

Smoothing inequality and disorder relevance

In [G., Toninelli 06] (also [Caravenna, den Hollander 13]): for $\beta > 0$ there exists $c_\beta > 0$ such that for every $\alpha \geq 0$

$$F(\beta, h) \overset{h \geq h_c(\beta)}{\leq} c_\beta (h - h_c(\beta))^2 \, ,$$

and $c_\beta \sim const. \beta^{-2}$ for $\beta \searrow 0$.

Note that this inequality

- is empty for contact pinning if $\alpha < 1/2$ and $\beta \leq \beta_0$ (the irrelevant disorder results show that $F(\beta, h)$ is smaller than that approaching criticality)
- shows disorder relevance for contact pinning if $\alpha > 1/2$ and for the copolymer case (any α)

Smoothing inequality and disorder relevance

In [G., Toninelli 06] (also [Caravenna, den Hollander 13]): for $\beta > 0$ there exists $c_\beta > 0$ such that for every $\alpha \ge 0$

$$F(\beta, h) \stackrel{h \ge h_c(\beta)}{\le} c_\beta (h - h_c(\beta))^2 \,,$$

and $c_\beta \sim const.\beta^{-2}$ for $\beta \searrow 0$.

Note that this inequality

- is empty for contact pinning if $\alpha < 1/2$ and $\beta \le \beta_0$ (the irrelevant disorder results show that $F(\beta, h)$ is smaller than that approaching criticality)
- shows disorder relevance for contact pinning if $\alpha > 1/2$ and for the copolymer case (any α)
- $\alpha = 1/2$ is marginally relevant (but only in a weak sense): [Derrida, Hakim, Vannimenus 92] . . . [G., Lacoin, Toninelli 10, 12], [Berger, Lacoin 18]

Challenging question: what happens if disorder is relevant?

Several physical predictions. . ., but two somewhat converging lines:

Challenging question: what happens if disorder is relevant?

Several physical predictions..., but two somewhat converging lines:

1. - [D. Fisher 92, 95] developed (starting from some ideas of Ma and Dasgupta) a renormalization procedure for systems with one dimensional disorder structure (quantum Ising chain with random transversal field). Non rigorous procedure expected to give exact results (∞ disorder renormalization fixed point)

Challenging question: what happens if disorder is relevant?

Several physical predictions…, but two somewhat converging lines:

1.
   - [D. Fisher 92, 95] developed (starting from some ideas of Ma and Dasgupta) a renormalization procedure for systems with one dimensional disorder structure (quantum Ising chain with random transversal field). Non rigorous procedure expected to give exact results (∞ disorder renormalization fixed point)
   - Fisher's idea have been developed by several authors and applied to several systems: 1d RW in RE,…, pinning models $\leftarrow$ infinite order transition [Le Doussal, Monthus, Vojta,…]:

$$\mathrm{F}(\beta, h_c(\beta) + \Delta) \stackrel{\Delta \searrow 0}{\approx} \exp(-1/\Delta)$$

Challenging question: what happens if disorder is relevant?

Several physical predictions..., but two somewhat converging lines:

1. - [D. Fisher 92, 95] developed (starting from some ideas of Ma and Dasgupta) a renormalization procedure for systems with one dimensional disorder structure (quantum Ising chain with random transversal field). Non rigorous procedure expected to give exact results (∞ disorder renormalization fixed point)
   - Fisher's idea have been developed by several authors and applied to several systems: 1d RW in RE,..., pinning models $\leftarrow$ infinite order transition [Le Doussal, Monthus, Vojta,...]:

$$\mathrm{F}(\beta, h_c(\beta) + \Delta) \overset{\Delta \searrow 0}{\approx} \exp(-1/\Delta)$$

2. Mysterious paper [Tang, Chaté 00] and [Derrida, Retaux 14]:

$$\mathrm{F}(\beta, h_c(\beta) + \Delta) \approx \exp(-1/\Delta^{1/2})$$

Challenging question: what happens if disorder is relevant?

Several physical predictions..., but two somewhat converging lines:

1.
   - [D. Fisher 92, 95] developed (starting from some ideas of Ma and Dasgupta) a renormalization procedure for systems with one dimensional disorder structure (quantum Ising chain with random transversal field). Non rigorous procedure expected to give exact results (∞ disorder renormalization fixed point)
   - Fisher's idea have been developed by several authors and applied to several systems: 1d RW in RE,..., pinning models $\leftarrow$ infinite order transition [Le Doussal, Monthus, Vojta,...]:

$$\mathrm{F}(\beta, h_c(\beta) + \Delta) \overset{\Delta \searrow 0}{\approx} \exp(-1/\Delta)$$

2. Mysterious paper [Tang, Chaté 00] and [Derrida, Retaux 14]:

$$\mathrm{F}(\beta, h_c(\beta) + \Delta) \approx \exp(-1/\Delta^{1/2})$$

[DR14] is about a simplified pinning model ([Chen,Hu,Lifshits,Shi]) for which one can compute exactly the critical point (for $\beta > 0$) and then arguments that lead to the Kosterlitz-Thouless ODE system.

What I am going to tell you next (why can't we do more?)

Substantial limit for the moment: no idea on how to capture the critical behavior without knowing the critical point of the disordered system
(intermediate disorder? [Alberts, Khanin, Quastel 14], [Caravenna,Sun, Zygouras 17])

What I am going to tell you next (why can't we do more?)

Substantial limit for the moment: no idea on how to capture the critical behavior without knowing the critical point of the disordered system (intermediate disorder? [Alberts, Khanin, Quastel 14], [Caravenna,Sun, Zygouras 17])

On the other hand, knowing the critical point (for pinning models!!!) is an excellent starting point: just do upper and lower bounds...

What I am going to tell you next (why can't we do more?)

Substantial limit for the moment: no idea on how to capture the critical behavior without knowing the critical point of the disordered system (intermediate disorder? [Alberts, Khanin, Quastel 14], [Caravenna,Sun, Zygouras 17])

On the other hand, knowing the critical point (for pinning models!!!) is an excellent starting point: just do upper and lower bounds...

It turns out that there is one pinning case for which we know the critical point [Bodineau, G. 03]: the copolymer pinning with $\alpha = 0$.
That is $K(n) = L(n)/n$ and (for example) $L(n) \sim 1/(\log n)^u$ with $u > 1$ because $\sum_n K(n) = 1$.

What I am going to tell you next (why can't we do more?)

Substantial limit for the moment: no idea on how to capture the critical behavior without knowing the critical point of the disordered system (intermediate disorder? [Alberts, Khanin, Quastel 14], [Caravenna,Sun, Zygouras 17])

On the other hand, knowing the critical point (for pinning models!!!) is an excellent starting point: just do upper and lower bounds...

It turns out that there is one pinning case for which we know the critical point [Bodineau, G. 03]: the copolymer pinning with $\alpha = 0$.
That is $K(n) = L(n)/n$ and (for example) $L(n) \sim 1/(\log n)^u$ with $u > 1$ because $\sum_n K(n) = 1$.

Our result, very informally: according to the choice of $L(\cdot)$, we find for $\mathrm{F}(\beta, h_c(\beta) + \Delta)$ the $\Delta \searrow 0$ behaviors

$$\exp(-\log(1/\Delta)/\Delta) \quad \text{and} \quad \exp(-1/\Delta^{1+b}) \quad \text{with } b > 0$$

and this is what I am going to talk next [Berger, G., Lacoin 18]