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Singular stochastic partial differential equations

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Abstract

Singular stochastic partial differential equations (SSPDE) first appeared in rather special contexts like the stochastic quantization of field theories or in the problem of crystal growth, the well known KPZ equation. In the last decade these equations have been intensely studied giving rise to an important branch of mathematics possibly relevant for physics. This talk will review some aspects and open problems in the subject.

What is a stochastic equation?

$$dx_t = b(x_t)dt + \sigma(x_t)dw_t \tag{1}$$

w_t is a Gaussian process with independent increments and covariance

$$\mathsf{E}(w_t w_{t'}) = \min(t, t') \tag{2}$$

The typical trajectories of w_t are continuous but not absolutely continuous that is the length of any trajectory between t, t' is infinite. Furthermore they are not differentiable. Another way of writing (1) is as an integral equation

$$x_t = x_0 + \int_0^t b(x_s)ds + \int_0^t \sigma(x_s)dw_s \tag{3}$$

The last term, called a *stochastic integral*, requires some specification.

Stochastic integrals

The first idea is to interpret the expression $\int_0^t \sigma(x_s)dw_s$ as a Stieltjes integral but this does not work due to the non-absolutely continuous trajectories of the Wiener process. In fact any approximation by finite sums would depend on where we evaluate the integrand. There are two main notions of stochastic integration due to Ito and Stratonovich.

Ito:

$$\int_0^t g_s dw_s = \lim_{n\to\infty} \sum_1^n g_{s_k}(w_{s_{k+1}} - w_{s_k}) \tag{4}$$

Ito integral is very natural from a probabilistic standpoint but does not obey the usual rules of calculus. The integrand is supposed to depend only on the past history so it is independent of dw_t.

$$\int_0^t w_s dw_s = \frac{1}{2}({w_t}^2 - t) \tag{5}$$

Stratonovich:

$$\int_0^t g_s \circ dw_s = \lim_{n\to\infty} \sum_1^n \frac{1}{2}(g_{s_k} + g_{s_{k+1}})(w_{s_{k+1}} - w_{s_k}) \tag{6}$$

Stratonovich satisfies the usual rules so that

$$\int_0^t w_s \circ dw_s = \frac{1}{2} {w_t}^2 \tag{7}$$

but the integrand and the increment dw are not independent.

Girsanov formula

This is an important formula which allows to relate the evolutions associated to processes solutions of equations like (1) differing for the term $b(x)$. In particular if the noise is purely additive that is $\sigma = 1$, takes the simple form

$$\mathsf{E}(f(x_t)) = \mathsf{E}(f(w_t) \exp \zeta_t) \tag{8}$$

where

$$\zeta_t = \int_0^t b(w_s) dw_s - \frac{1}{2} \int_0^t b^2(w_s) ds \tag{9}$$

When we deal only with expectations like (8) we speak of weak solutions of (1)

The generator of a diffusion process

To an equation like (1) is associated a differential operator called the generator

$$\mathsf{L} = \sigma^2(x)\partial_x^2 + b(x)\partial_x \tag{10}$$

When instead of the trajectories we deal with the transition probabilities $p(s, x, t, y)$, their evolution equations can be expressed in terms of L and its formal adjoint. They are called the forward and the backward Kolmogorov equations.

$$\partial_t p = \partial_y^2(\sigma^2(y)p) - \partial_y(b(y)p) \tag{11}$$

This equation was known to physicists as the Fokker-Planck equation. The backward equation is

$$\partial_s p = -\mathsf{L}p = -\sigma^2(x)\partial_x^2 p - b(x)\partial_x p \tag{12}$$

Freidlin-Wentzell (F-W) theory

Given a stochastic ODE of the form, $b(x)$ a Lipschitz function,

$$dx_t = b(x_t)dt + \epsilon dw_t \tag{13}$$

and an absolutely continuous function ϕ_t, define the rate (or action) functional

$$S_{0T}(\phi) = \frac{1}{2}\int_0^T |\dot{\phi}_t - b(\phi_t)|^2 dt \tag{14}$$

Then the following estimates hold for $\epsilon \to 0$

I. For any $\delta, \gamma, K > 0$ there exists $\epsilon_0 > 0$ such that

$$P(\rho_{0T}(x^\epsilon, \phi) < \delta) \geq e^{-\epsilon^{-2}[S_{0T}(\phi)+\gamma]} \tag{15}$$

for $0 < \epsilon \leq \epsilon_0$, $T > 0$, $\phi_0 = x_0$ and $T + S_{0T}(\phi) \leq K$. ρ is the distance in the uniform norm.

Define

$$\Phi(s) = [\phi : \quad \phi_0 = x_0, \quad S_{0T}(\phi) \le s] \tag{16}$$

II. For any $\delta, \gamma, s_0 > 0$ there exists $\epsilon_0 > 0$ such that

$$P(\rho_{0T}(x^\epsilon, \Phi(s)) \ge \delta) \le e^{-\epsilon^{-2}[s-\gamma]} \tag{17}$$

for $0 < \epsilon \le \epsilon_0$, $s < s_0$

From estimates I. and II.

$$e^{-\epsilon^{-2}[S_{0T}(\phi)-\gamma]} \ge P(\rho_{0T}(x^\epsilon, \phi) < \delta) \ge e^{-\epsilon^{-2}[S_{0T}(\phi)+\gamma]} \tag{18}$$

for $0 < \epsilon \le \epsilon_0$.

Reformulation

Estimates I. and II. are equivalent to (Varadhan)

I'. For any open set A

$$\underline{\lim}_{\epsilon \to 0} \epsilon^2 \ln P(A) \geq -\inf[S_{0T}(\phi) : \phi \in A] \tag{19}$$

II'. For any closed set A

$$\overline{\lim}_{\epsilon \to 0} \epsilon^2 \ln P(A) \leq -\inf[S_{0T}(\phi) : \phi \in A] \tag{20}$$

In this formulation we say that the family of probability distributions P parametrized by ϵ satisfies a large deviation principle.

Some non-singular stochastic PDEs

Stochastic quantization in one space dimension

$$\partial_t u = \frac{\partial^2 u}{\partial^2 x} - V'(u) + \epsilon\xi \qquad (21)$$

where

$$\mathsf{E}(\xi(t,x)\xi(t',x')) = \delta(t-t')\delta(x-x')$$

with $\xi(t,x) = \partial_t\partial_x w(t,x)$, and $w(t,x)$ the Brownian sheet.

Equation (21) can be written as an integral equation *(mild form)*

$$u = G * u_0 - \int_0^t G * V'(u) + \epsilon w \qquad (22)$$

$G = (\partial_t - \frac{\partial^2}{\partial^2 x})^{-1}(t,x,t',x')$ is the fundamental solution of the heat equation.

The stochastic Burgers equation in one dimension

$$\partial_t u = \nu \Delta_x u - \frac{1}{2}\partial_x u^2 + \epsilon \xi \tag{23}$$

We can rewrite (23) in mild form

$$u = G * u_0 - \frac{1}{2}\int_0^t \partial_x G * u^2 + \epsilon \int_0^t \partial_x G * dw \tag{24}$$

Using the Hopf-Cole tranformation $u(t,x) = -2\nu\partial_x \ln \psi(t,x)$ we obtain

$$\partial_t \psi = \nu \partial_x^2 \psi - \frac{\epsilon}{2\nu}\psi \circ \partial_t w \tag{25}$$

KPZ equation

$$\partial_t h = -\lambda(\partial_x h)^2 + \nu\partial_x^2 h + D^{\frac{1}{2}}\xi \tag{26}$$

where ξ denotes space-time white noise which is the distribution valued Gaussian field with correlation function

$$\mathsf{E}(\xi(t,x)\xi(s,y)) = \delta(t-s)\delta(x-y) \tag{27}$$

By denoting $u = \partial_x h$ we obtain

$$\partial_t u = \nu\Delta_x u - \lambda\partial_x u^2 + D^{\frac{1}{2}}\partial_x\xi \tag{28}$$

that is a stochastic Burgers equation which has the form of a conservation law.

Equation (26) can be changed, introducing the Cole-Hopf tranformation

$$z(x,t) = \exp h(x,t) \tag{29}$$

into the stochastic heat equation with multiplicative noise

$$\partial_t z = \partial_x^2 z + z\xi \tag{30}$$

Stochastic quantization in 2 and 3 dimensions

$$\partial_t \phi = \Delta\phi - m^2\phi - g\phi^3 + \xi \tag{31}$$

The stochastic quantization equation provides a dynamical approach to the euclidean quantum field theory ϕ^4

$$d\mu(\phi) = \exp -V(\phi) d\mu_G(\phi) \tag{32}$$

where $d\mu_G(\phi)$ is the Gaussian measure of covariance $(-\Delta + m^2)^{-1}$ and $V(\phi)$ is the space integral of a fourth order monomial

$$V(\phi) = \frac{1}{4} \int dx \phi^4 \tag{33}$$

Scaling

Define

$$\bar{\phi}(t,x) = \lambda^{\frac{d}{2}-1}\phi(\lambda^2 t, \lambda x)$$

Then (31) can be written

$$\partial_t \bar{\phi} = \Delta\bar{\phi} - \lambda^2 m^2 \bar{\phi} - \lambda^{4-d} g \bar{\phi}^3 + \bar{\xi} \tag{34}$$

where $\bar{\xi}$ has the same law as ξ.

This form suggests that at small distances the linear part dominates for $d < 4$ and the non-linearity is a small perturbation.

Equations (31) and (32) cannot be taken literally as they involve powers of distributions. They have to be modified to become mathematically meaningful. This is the *renormalization problem*.

In general there is not a unique way to renormalize. One follows the experience gained with quantum field theory. For example in two space dimensions it is enough to replace powers of the field with the so called Wick products according to the rule

$$\phi^n \quad \rightarrow \quad :\phi^n: \quad = \quad C^{\frac{n}{2}} H_n(C^{-\frac{1}{2}}\phi) \tag{35}$$

where H_n is the Hermite polynomial of order n and $C = E(\phi^2)$, E is the expectation with respect to the Gaussian measure $d\mu_G$. The requirement of physics is that measurable quantities should not depend on the way you renormalize.

Distribution valued stochastic fields

The field $\phi(x)$ is not a good stochastic variable as its moments in space dimension $d > 2$ are infinite. However our problem requires to deal with powers of ϕ, in particular with $\int \phi^4 dx$. Luckily $:\phi^n(x):$ are good distribution valued stochastic variables.

They can be defined as follows. One regularizes ϕ by introducing a cut-off κ in the fourier integral representation and showing that the moments of $\phi(f)$ with respect to the measure $d\mu_G$ form a Cauchy sequence so that

$$|| : \phi^n(f) : - : \phi^n_\kappa(f) : ||_p \leq c_{f,p} \kappa^{-\epsilon}$$

Weak dynamics

We first transform (31) into a modified and renormalized equation

$$\partial_t \phi = -(-\Delta - 1)^\rho \phi + (-\Delta + 1)^{-1+\rho} : \phi^3 : + \xi \tag{36}$$

where ξ satisfies

$$\mathsf{E}(\xi(t,x)\xi(s,y)) = \delta(t-s)(-\Delta + 1)^{-1+\rho}(x,y) \tag{37}$$

with $0 < \rho < 1$. A mild version of (36) is

$$\phi_t = Z_t + \int_0^t ds \exp[-(t-s)C^{-\rho}]C^{1-\rho} * : \phi^3 : \tag{38}$$

where Z_t is the solution of

$$dZ_t = -C^{-\rho} Z_t + \xi \tag{39}$$

The weak dynamics is defined by

$$E_{\phi_0}(f(\phi_t)) \equiv E_{\phi_0}(f(Z_t)\exp\zeta_t) \tag{40}$$

with

$$\zeta_t = \int_0^t (: Z_s^3 : d\xi_s) \tag{41}$$
$$-\frac{1}{2}\int_0^t ds(: Z_s^3 : C^{1-\rho} : Z_s^3 :)$$

where ϕ_0 is the initial condition and $f(\phi)$ is a functional of ϕ.

Ref: J-L, Mitter, CMP (1985)

Strong dynamics

G. Da Prato, A. Debussche, The Annals of Probability, ***31****, 1900 (2003).*

These authors showed that (36) for $\rho = 1$ has strong solutions in an appropriate Besov space. The trick is to split the unknown into two parts: $\phi_t = Y_t + Z_t$ where Z_t is the stochastic convolution

$$Z_t = \int_{-\infty}^{t} e^{(t-s)C^{-1}} d\xi(s) \tag{42}$$

Then they observe that Y_t is smoother than ϕ_t and

$$:\phi^k := \sum_{l=0}^{k} C_k^l Y^l : Z^{k-l} : \tag{43}$$

in this way the non-linearity is continuous in Y and the equation takes the form

$$\partial_t Y = (\Delta - 1)Y - \sum_{l=0}^{3} C_3^l Y^l : Z^{3-l} : \tag{44}$$

Besov spaces

We recall their definition. For any $q \in N$ and given the canonical Fourier basis e_k in $A = [0, 2\pi]^2$ we set

$$\delta_q u = \sum_{2^{q-1} < |k| \leq 2^q} u_k e_k$$

then for $\sigma \in R, p \geq 1, r \geq 1$ we define the Besov space

$$\mathsf{B}^{\sigma}_{p,r}(A) = \{u : \sum_q 2^{rq\sigma} |\delta_q u|^r_{L_p(A)} < \infty\}$$

with norm

$$|u|_{\mathsf{B}^{\sigma}_{p,r}(A)} = (\sum_q 2^{rq\sigma} |\delta_q u|^r_{L_p}(A))^{1/r} \qquad (45)$$

Fluctuating hydrodynamics

The macroscopic dynamics of diffusive systems is described by hydrodynamic equations provided by conservation laws and constitutive equations, that is equations expressing the current in terms of the thermodynamic variables. More precisely on the basis of a local equilibrium assumption, at the macroscopic level the system is completely described by a local multicomponent density $\rho(t,x)$ and the corresponding local currents $j(t,x)$

$$\begin{cases} \partial_t \rho(t) + \nabla \cdot j(t) = 0 \\ j(t) = J(\rho(t)) \end{cases} \tag{46}$$

For diffusive systems the constitutive equation takes the form

$$J(\rho) = -D(\rho)\nabla\rho + \chi(\rho)\, E \tag{47}$$

where the *diffusion coefficient* $D(\rho)$ and the *mobility* $\chi(\rho)$ are $d \times d$ symmetric and positive definite matrices, E is an external field.

To study fluctuations, for example at constant temperature, one adds to the current a fluctuating term $j = J(\rho) + (2\kappa T_0\, \chi_{ij}(\rho))^{\frac{1}{2}} \xi$, ξ is a Gaussian random term with variance

$$\mathsf{E}(\xi_i(t,x), \xi_j(t',x')) = \delta(t-t')\delta(x-x')$$

κ is the Boltzmann constant and T_0 the temperature.

The hydrodynamic equation takes the form

$$\partial_t \rho + \nabla(J(\rho) + (2\kappa T_0\, \chi_{ij}(\rho))^{\frac{1}{2}} \xi) = 0 \tag{48}$$

The noise is therefore multiplicative and there is an extra space derivative.

Large deviations

Heuristically the following large deviation formula can be obtained

$$P \asymp \exp\left\{-\frac{1}{\kappa T_0}\frac{1}{4}\int dt \int dx \,\big(j - J(\rho)\big)\cdot \chi(\rho)^{-1}\big(j - J(\rho)\big)\right\}, \quad (49)$$

j is the actual value of the current fluctuation, which is connected to ρ by the continuity equation $\partial_t \rho + \nabla \cdot j = 0$, while $J(\rho)$ is the hydrodynamic current for the given value of ρ, and χ is the mobility.

This formula can be proven for several lattice gas models, that is for microscopic systems, and its application provides exact results in all the cases where its has been tested.

Large deviations for the invariant measure

From the previous formula the following variational expression of the stationary large deviation rate follows

$$V(\rho) = \inf_{\substack{\rho(t), j(t):\\ \nabla\cdot j = -\partial_t \rho\\ \rho(-\infty)=\bar\rho, \rho(0)=\rho}} \mathcal{I}_{[-\infty,0]}(\rho, j) \tag{50}$$

where

$$\mathcal{I}_{[T_0,T_1]}(\rho, j) = \frac{1}{4}\int_{T_0}^{T_1} dt \int_\Lambda dx\, [j - J(\rho)] \cdot \chi(\rho)^{-1}[j - J(\rho)] \tag{51}$$

and $\bar\rho$ is the stationary solution of the hydrodynamic equations.

The variational principle (50) can be solved exactly for some models in particular for the boundary driven simple exclusion process.

The result is the same one would obtain by applying formally the Freidlin-Wentzell theory to (48)

Ref: Bertini, De Sole, Gabrielli, J-L, Landim, J. Stat. Phys. (2002)

Large deviation functional for the density

The large deviation functional for the density can be obtained by projection. We fix a path $\rho = \rho(t, u)$, $(t, u) \in [0, T] \times \Lambda$. There are many possible trajectories $j = j(t, u)$, differing by divergence free vector fields, such that the continuity equation is satisfied. By minimizing $\mathcal{I}_{[0,T]}(\rho, j)$ over all such paths j

$$I_{[0,T]}(\rho) = \inf_{\substack{j: \\ \nabla \cdot j = -\partial_t \rho}} \mathcal{I}_{[0,T]}(j) \tag{52}$$

Let F be the external field which generates the current j according to

$$j = -D(\rho)\nabla\rho + \chi(\rho)(E + F) \,.$$

and minimize with respect to F. We show that the infimum above is obtained when the external perturbation F is a gradient vector field whose potential H solves

$$\partial_t \rho = \nabla \cdot \Big(D(\rho)\nabla\rho - \chi(\rho)\big[E + \nabla H\big]\Big) \tag{53}$$

which is a Poisson equation for H.

Write

$$F = \nabla H + \widetilde{F} \tag{54}$$

We get

$$\mathcal{I}_{[0,T]}(j) = \frac{1}{4}\int_0^T dt\,\left\{\langle \nabla H, \chi(\rho)\nabla H\rangle + \langle \widetilde{F}, \chi(\rho)\widetilde{F}\rangle\right\}$$

Therefore the infimum is obtained when $\widetilde{F} = 0$. Then $I_{[0,T]}(\rho)$ can be written

$$I_{[0,T]}(\rho) = \frac{1}{4}\int_0^T dt\,\big\langle \nabla H(t), \chi(\rho(t))\nabla H(t)\big\rangle \tag{55}$$

$$= \frac{1}{4}\int_{T_1}^{T_2} dt\,\Big\langle \big[\partial_t\rho + \nabla\cdot J(\rho)\big]\,K(\rho)^{-1}\big[\partial_t\rho + \nabla\cdot J(\rho)\big]\Big\rangle$$

where the positive operator $K(\hat{\rho})$ is defined on functions $u : \Lambda \to R$ vanishing at the boundary $\partial\Lambda$ by $K(\hat{\rho})u = -\nabla\cdot\big(\chi(\hat{\rho})\nabla u\big)$.

An example in one space dimension: the boundary driven simple exclusion process

The fluctuating hydrodynamics equation is the stochastic heat equation on the interval $[-1, 1]$

$$\partial_t \rho = \partial_x^2 \rho + \partial_x ((2\kappa T_0 \rho(1-\rho))^{\frac{1}{2}} \xi) \tag{56}$$

with

$$< \xi(t,x)\xi(s,y) >= \delta(t-s)\delta(x-y) \tag{57}$$

and $0 \leq \rho \leq 1$.

The space boundary conditions are $\rho(-1) = \rho_-, \quad \rho(1) = \rho_+$

The case of the simple exclusion

The large deviation functional $V(\rho)$ can be calculated explicitely simple exclusion process

$$V(\rho) = F(\rho) + \int_{[-1,1]} dx \left\{ (1-\rho)\phi + \log\left[\frac{\nabla\phi}{\nabla\bar{\rho}\,(1+e^{\phi})}\right]\right\} \tag{58}$$

when $\phi(x;\rho)$ solves

$$\begin{cases} \dfrac{\Delta\phi(x)}{[\nabla\phi(x)]^2} + \dfrac{1}{1+e^{\phi(x)}} = \rho(x) \quad x \in (-1,1)\,, \\ \\ \phi(\pm 1) = \log\rho(\pm 1)/[1-\rho(\pm 1)]\,. \end{cases} \tag{59}$$

and

$$F(\rho) = \int_{[-1,+1]} dx\{\rho\log\rho + (1-\rho)\log(1-\rho)\} \tag{60}$$

Ref: Derrida, Lebowitz, Speer 2002; Bertini, De Sole, Gabrielli, J-L, Landim 2002

The theory of weak or strong solutions developed so far apparently is not sufficient for equations like (56). An effort should be made to derive microscopically fluctuating hydrodynamics in a spirit similar to what Bertini and Giacomin did in 1997 for the KPZ equation. This should help to give a mathematical meaning to (56)

Assuming that a theory of (56) is possible including large deviation estimates, we would expect that the large deviation functional coincides with (58). The experience we have so far with weak or strong solutions of the stochastic quantization tells us that the large deviation functional does not depend on the way the equations are renormalized.

Ref: J-L, Mitter (1990); Hairer, Weber (2014)

A linear singular equation

Dorfman, J., T. Kirkpatrick, and J. Sengers, 1994, Annu. Rev. Phys. Chem. 45, 213.

$$\partial_t \phi = D\Delta\phi + \xi \tag{61}$$

where

$$< \xi(t,x)\xi(t',x') >= (\chi_\perp \Delta_\perp + \chi_\parallel \Delta_\parallel)\delta(x-x')\delta(t-t') \tag{62}$$

We allow spatial anisotropy by having a different magnitude of the noise in two different subspaces.

This equation has been proposed to model non-equilibrium long range correlations of thermodynamics variables.

Non-gradient examples

$$\partial_t \vec{\phi} = \Delta \vec{\phi} - \frac{\delta V}{\delta \vec{\phi}} + \vec{F}(\vec{\phi}) + \vec{\xi} \tag{63}$$

The theory of weak solutions applies to an equation of this form in $d = 2$ provided the gradient term dominates over the non-gradient part for large values of $|\vec{\phi}|$.

Ref: J-L, Seneor (1991)

A special case of equation (63)

Let $\vec{\phi}$ be a two component field and consider the equation

$$\partial_t \vec{\phi} = (\Delta - 1)\vec{\phi} + A(|\vec{\phi}|)\vec{\phi} + \epsilon \vec{\xi} \tag{64}$$

where $A(|\vec{\phi}|)$ is the 2×2 matrix

$$\begin{vmatrix} \lambda_1(1 - |\vec{\phi}|^2) & -\lambda_2(1 - |\vec{\phi}|^2) \\ \lambda_2(1 - |\vec{\phi}|^2) & \lambda_1(1 - |\vec{\phi}|^2) \end{vmatrix}$$

With this choice the gradient and non-gradient parts are orthogonal.

Renormalized equations

$$\partial_t \vec{\phi} = -(-\Delta + 1)^{\rho} \vec{\phi} + (-\Delta + 1)^{-1+\rho} : A(|\vec{\phi}|)\vec{\phi} : + \vec{\xi} \tag{65}$$

where

$$\mathsf{E}(\xi_i(t,x)\xi_j(s,y)) = \delta_{ij}\delta(t-s)(-\Delta+1)^{-1+\rho}(x-y) \tag{66}$$

with $\rho < 1$ and

$$: |\vec{\phi}|^2 \phi_1 := \phi_1^3 - 3c\phi_1 + \phi_1(\phi_2^2 - c) =: \phi_1^3 : + \phi_1 : \phi_2^2 : \tag{67}$$

$$: |\vec{\phi}|^2 \phi_2 := \phi_2^3 - 3c\phi_2 + \phi_2(\phi_1^2 - c) =: \phi_2^3 : + \phi_2 : \phi_1^2 : \tag{68}$$

$c = C(x,x) = (-\Delta + 1)^{-1}(x,x)$

Stability condition

The condition of dominance of the gradient part in this case is $(\rho = 1)$

$$-\int_0^T ds|| : A_D(Z_s)Z_s : ||^2 + \lambda \int_0^T ds|| : A_{ND}(Z_s)Z_s : ||^2 < M \tag{69}$$

for an appropriate choice of λ, M is a positive constant. A_D, A_{ND} are the diagonal and non-diagonal parts of the matrix A.

A natural question is whether constraints are required in the present theory of strong solutions.

Some questions about large deviations in SSPDEs

Large deviations principles provide the asymptotic behavior of a family of probability distributions when the intensity of the noise vanishes. However to be useful in concrete problems, e.g. in physics, one must estimate the intensity of the noise below which one is asymptotic and also the errors involved.

In the case of SSPDEs, since the solutions are distributions, a typical question one may ask is about the behavior of an average of the field over a region, e.g. a sphere, of linear dimension L, that is $\int_\Lambda dx \phi(x)$.

A major difference with respect to the theory of stochastically perturbed finite dimensional dynamical systems is that the intensity of the noise depends on the size of the region. Fluctuations are stronger on small scales. It will be important in applications to know how the maximal intensity and the error allowed in a problem scale with L.

Large deviations for SSPDEs have been studied by J-L and Mitter in the case of weak solutions (1990) and by Hairer and Weber for strong solutions (2014). In our work we introduced the scale over which the field has to be studied.

Stochastic lyapunov functions

In the theory of ordinary stochastic differential equations if a smooth function $V(x) \geq 0$ satisfies

$$LV \leq -c_1 V + c_2 \tag{70}$$

where L is the generator and $c_i \geq 0$, the solution is bounded in probability. This property is called *stochastic boundedness*.

Is there an equivalent for SSPED's ?

For the stochastic quantization with a regularized noise the L_2 norm

$$||\phi||_2^2 = \int dx \phi^2(x) \tag{71}$$

is a stochastic Lyapunov function which however does not make sense if ϕ is a distribution.

A simulation of the renormalized equation

R.Benzi, G. Jona-lasinio, A. Sutera, J. Stat. Phys. **55**, *505 (1989)*

We consider the equation

$$\partial_t \phi = \nu \Delta \phi - m\phi - g\phi^3 + 3gC\phi + \epsilon^{1/2}\xi \tag{72}$$

and its discretized version on a two-dimensional lattice

$$d\phi_{j,k} = (\nu/a^2(\Delta\phi)_{j,k}) - \mu\phi_{j,k} - g\phi^3_{j,k})dt + \epsilon^{1/2}/a dw_{j,k} \tag{73}$$

where $j, k = 1, ..., N$, a is the lattice step and

$$\mu = -m + 3gC \tag{74}$$

For fixed finite C and sufficiently large g the quadratic potential becomes a double well which is at the origin of a phase transition in infinite volume as proved long ago by Glimm, Jaffe and Spencer.

In finite volume we expect the process to jump between the two minima a fact that should be reflected by the invariant measure. However, and this is the effect of interest, the appearance of bimodality depends on the scale at which we observe the field. Due to ultraviolet divergences the fluctuations of the field smeared over a small space region will be so large as to conceal the double well.

Denote by

$$\psi(M) = \frac{1}{M^2} \sum_{j,k=1,2...,M} \phi_{j,k} \tag{75}$$

and by $P(M, g)$ the stationary probability distribution of $\psi(M)$ for a given value of g and $a = 1/16$.

For $M = N$, ψ is the space average of $\phi_{j,k}$, while for $M = 1$ is the value of the field in one point. We take periodic boundary conditions.

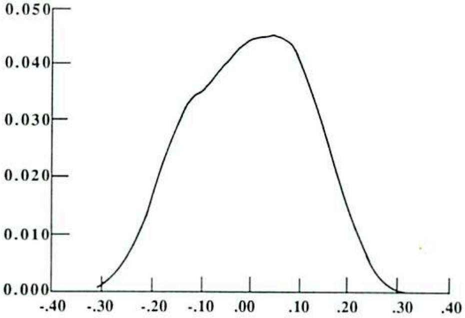

Fig. 1. Probability density distribution $P(16, g)$ for the numerical simulation of (5.1) at "small" $g = 10$. The probability density distributions plotted in this and the following figures are not normalized.

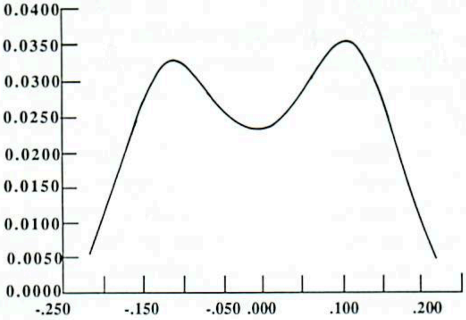

Fig. 2. Probability density distribution $P(16, g)$ for the numerical simulation of (5.1) at "large" $g = 140$.

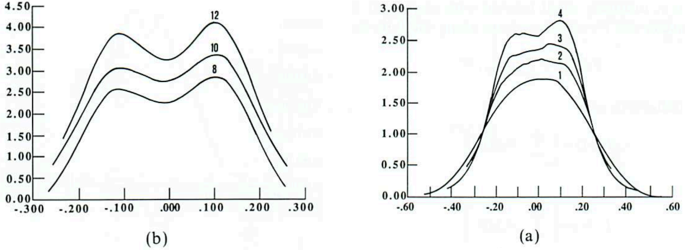

Fig. 3. Probability density distributions $P(M, 140)$ for the numerical simulation of (5.1) for (a) $M = 1, 2, 3, 4$, (b) $M = 8, 10, 12$. The numerical labels on the curves refer to the different values of M.

An interesting feature is the absence of a bimodal distribution at small scale. This property was proposed as a possible interpretation of certain geophysical data.

Cluster expansion in time

G. Jona-Lasinio, R. Seneor, J. Stat. Phys. **83**, *1109 (1996)*

We consider the model equation

$$\dot{\phi} = -\phi - \lambda : \phi^3 : +\dot{w} \tag{76}$$

We want to calculate the $T \to \infty$ limit of

$$\mathsf{E}_{\phi_0}(F(\phi_T)e^{\xi_T}) \tag{77}$$

Then the following holds. Let ϕ_0 be an arbitrary real number and F a polynomial. For λ small enough

1. (77) can be expressesd as a convergent series uniformly in T
2. There exists a stationary measure μ such that

$$\int d\mu F_1(\phi_{t_1}) \cdots F_n(\phi_{t_n}) = \lim_{T\to\infty} \mathsf{E}_{\phi_0}(F_1(\phi_{T+t_1}) \cdots F_n(\phi_{T+t_n})) \tag{78}$$

independently of ϕ_0

The expansion is defined in such a way that one tries to decouple intervals of time containing the final time T. To define the initial step one introduces an interpolation parameter $s \in [0, 1]$ and defines the interpolating covariance for the OU process

$$C(s,t,t') = [\chi_{[T-1,T]}(t)C(t,t')\chi_{[T-1,T]}(t') \\ +(1-\chi_{[T-1,T]})(t)C(t,t')(1-\chi_{[T-1,T]})(t')](1-s) + sC(t,t')$$

together with the interpolating "final condition"

$$\phi_T^4(s) = \phi_T^4 + (1-s)\phi_{T-1}^4$$

Then one introduces these interpolations in (77) obtaining $\mathsf{E}_{\phi_0}(s)$

$$\mathsf{E}_{\phi_0}(F(\phi_T)e^{\xi_T}) = \mathsf{E}_{\phi_0}(s=0) + \int_0^1 ds \frac{d}{ds}\mathsf{E}_{\phi_0}(s) \qquad (79)$$

If I recall correctly similar ideas were proposed by Bricmont and Kupiainen.

Some home work

Provide a foundation, microscopic and macroscopic, to fluctuating hydrodynamics

Extend the theory of strong solutions to non-gradient equations where the deterministic part has non trivial attractors. Is it possible to develop a notion of random attractor, in the sense of Crauel and Flandoli 1994, for singular stochastic PDEs?

Construct stochastic lyapunov functions for SSPDEs