RECONSTRUCTION OF SMALL INTERFACE CHANGES OF AN INCLUSION FROM MODAL MEASUREMENTS II: THE ELASTIC CASE

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Abstract. In order to reconstruct small changes in the interface of an elastic inclusion from modal measurements, we rigorously derive an asymptotic formula which is in some sense dual to the leading-order term in the asymptotic expansion of the perturbations in the eigenvalues due to interface changes of the inclusion. Based on this (dual) formula we propose an algorithm to reconstruct the interface perturbation. We also consider an optimal way of representing the interface change and the reconstruction problem using incomplete data. A discussion on resolution is included. Proposed algorithms are implemented numerically to show their viability.

1. Introduction

In our recent work [ABFKL], we have proposed an original and promising optimization approach for reconstructing interface changes of a conductivity inclusion from measurements of eigenvalues and eigenfunctions associated with the transmission problem for the Laplacian. The key identity, which naturally yields the formulation of the proposed optimization problem, is a formula in some sense dual to the leading-order expansion in the eigenvalue perturbations.

In this paper, we extend our approach to elasticity. We consider a soft elastic inclusion inside a background medium. We first derive in Theorem 2.1 the leading-order term in the perturbations in the eigenvalues of the Lamé system that are due to small changes in the interface of the inclusion. We call this formula the direct formula. Then, we provide in Theorem 3.1 an asymptotic formula which is in some sense dual to the direct one. Our derivations of the direct formula are based on fine gradient estimates together with Osborn's result on spectral approximation for compact operators. The dual formula follows from the direct formula by using again fine gradient estimates.

The dual formula can be used successfully to provide a representation of the changes in the shape of the inclusion by searching for such changes as linear combination of what we will call "optimally illuminated vectors". Our approach leads to a robust reconstruction of the shape deformation. Indeed, the resolution limit of our algorithm can be estimated. The viability of our reconstruction approach is documented by a variety of numerical results.

The paper is organized as follows. In the next section we derive an asymptotic formula for the eigenvalue perturbations due to shape deformation of the elastic

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inclusion. In section 3, we prove a key dual identity which naturally yields the formulation of the proposed optimization problem. We find in section 4 a functional whose minimizer yields the interface of the inclusion. We also provide optimal representation of the changes in terms of the optimally illuminated vectors and discuss the uniqueness of a solution to the minimization procedure and its robustness with respect to error measurements. The resolution limit of our algorithm is quantified. Note that our procedure is designed for a simple eigenvalue but the case of a multiple eigenvalue can be handled in exactly the same manner [AKL]. In section 5, we generalize our procedure to the case where the measurements are done only on an open part of the boundary. In section 6, we perform numerical experiments to test the viability of the algorithm.

Many applications of our results in this paper are expected, especially in structural vibration testing of elastic structures [S].

2. Direct asymptotic formula

Throughout this paper, let $\mathcal{C}^{k,\alpha}$ denote the Hölder space which consists of functions having derivatives up to order k and such that the kth derivative is Hölder continuous with exponent α, where $0 < \alpha \leq 1$.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $\mathcal{C}^{1,1}$ boundary representing the region occupied by an elastic material. Let D be an open subset of Ω such that $\operatorname{dist}(\partial\Omega, \partial D) \geq d_0 > 0$ representing an inclusion made of a different elastic material. The boundary ∂D of D is assumed to be of class $\mathcal{C}^{2,1}$. Let $\mathbb{C}_0$ and $\mathbb{C}_1$ be the elastic tensor fields in $\Omega \setminus \overline{D}$ and D, respectively.

We assume that both $\Omega \setminus \overline{D}$ and D are occupied by isotropic and homogeneous materials; *i.e.*, the elastic tensor fields $\mathbb{C}_0$ and $\mathbb{C}_1$ are of the following form:

$$(\mathbb{C}_m)_{ijlk} = \lambda_m \delta_{ij}\delta_{kl} + \mu_m(\delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}) \text{ for } i,j,k,l = 1,2, \;\; m = 0,1, \tag{2.1}$$

where (λ_0, μ_0) and (λ_1, μ_1) are the Lamé constants of $\Omega \setminus \overline{D}$ and D, respectively, and $(\lambda_0 - \lambda_1)^2 + (\mu_0 - \mu_1)^2 \neq 0$. There is another way of expressing the isotropic elastic tensor which will be useful later. Let $\mathbf{I}_4$ be the identity 4-tensor and $\mathbf{I}_2$ be the identity 2-tensor (the 2×2 identity matrix). Then $\mathbb{C}_m$ can be rewritten as

$$\mathbb{C}_m = \lambda_m \mathbf{I}_2 \otimes \mathbf{I}_2 + 2\mu_m \mathbf{I}_4, \quad m = 0,1. \tag{2.2}$$

We assume that there are two positive constants α_0 and β_0 such that

$$\min(\mu_0, \mu_1) \geq \alpha_0, \quad \min(2\lambda_0 + 2\mu_0, 2\lambda_1 + 2\mu_1) \geq \beta_0, \tag{2.3}$$

which guarantees the strong convexity of $\mathbb{C}_0$ and $\mathbb{C}_1$. Given two 2×2 matrices A and B we denote by $A : B$ the contraction, *i.e.*, $A : B = \sum_{ij} a_{ij} b_{ij}$.

Let $\mathbb{C}_D = \mathbb{C}_0 \chi_{\Omega\setminus D} + \mathbb{C}_1 \chi_D$ and $(u_0, \omega_0^2) \in H^1(\Omega) \times \mathbb{R}^+$ be the solution to the following eigenvalue problem

$$\left\{ \begin{array}{rcl} \nabla \cdot \left(\mathbb{C}_D \widehat{\nabla} u_0 \right) & = & -\omega_0^2 u_0 \quad \text{in} \quad \Omega, \\ u_0 & = & 0 \quad \text{on} \quad \partial\Omega, \\ ||u_0||_{L^2(\Omega)} & = & 1, \end{array} \right. \tag{2.4}$$

where $\widehat{\nabla} u_0 = \frac{1}{2}\left(\nabla u_0 + (\nabla u_0)^T\right)$ is the strain. Here and throughout the paper T denotes the transpose.

One can easily see from the equation in (2.4) that u_0 satisfies the transmission conditions along the interface ∂D:

$$\left\{ \begin{array}{rcl} u_0^i &=& u_0^e, \\ (\mathbb{C}_1 \widehat{\nabla} u_0^i)\nu &=& (\mathbb{C}_0 \widehat{\nabla} u_0^e)\nu, \end{array} \right. \tag{2.5}$$

where ν is the outer normal unit vector field to ∂D and

$$u_0^e = u_0|_{\Omega \setminus D} \quad \text{and} \quad u_0^i = u_0|_{\overline{D}}. \tag{2.6}$$

Let τ be the unit tangential vector field to ∂D. The first identity in (2.5) shows that

$$(\nabla u_0^i)\tau = (\nabla u_0^e)\tau \quad \text{on } \partial D,$$

and hence

$$\langle (\widehat{\nabla} u_0^i)\tau, \tau \rangle = \frac{1}{2}\left[\langle (\nabla u_0^i)\tau, \tau \rangle + \langle \tau, (\nabla u_0^i)\tau \rangle \right] = \langle (\widehat{\nabla} u_0^e)\tau, \tau \rangle \quad \text{on } \partial D.$$

Therefore, we have

$$\left\{ \begin{array}{rcl} \langle \widehat{\nabla} u_0^i \tau, \tau \rangle &=& \langle \widehat{\nabla} u_0^e \tau, \tau \rangle, \\ \lambda_1 (\nabla \cdot u_0^i) + 2\mu_1 \langle \widehat{\nabla} u_0^i \nu, \nu \rangle &=& \lambda_0 (\nabla \cdot u_0^e) + 2\mu_0 \langle \widehat{\nabla} u_0^e \nu, \nu \rangle \\ \mu_1 \langle \widehat{\nabla} u_0^i \nu, \tau \rangle &=& \mu_0 \langle \widehat{\nabla} u_0^e \nu, \tau \rangle. \end{array} \right. \tag{2.7}$$

Observe that

$$\nabla \cdot u_0^i = \operatorname{tr}(\widehat{\nabla} u_0^i) = \langle \widehat{\nabla} u_0^i \tau, \tau \rangle + \langle \widehat{\nabla} u_0^i \nu, \nu \rangle,$$

where $\operatorname{tr}(A)$ denotes the trace of the matrix A. It thus follows that

$$\nabla \cdot u_0^i = \frac{\lambda_0 + 2\mu_0}{\lambda_1 + 2\mu_1} \nabla \cdot u_0^e + \frac{2(\mu_1 - \mu_0)}{\lambda_1 + 2\mu_1} \langle \widehat{\nabla} u_0^e \tau, \tau \rangle. \tag{2.8}$$

We then obtain from (2.7) and (2.8) that

$$\begin{aligned} (\mathbb{C}_1 \widehat{\nabla} u_0^i)\tau &= \lambda_1 (\nabla \cdot u_0^i)\tau + 2\mu_1 \widehat{\nabla} u_0^i \tau \\ &= \lambda_1 (\nabla \cdot u_0^i)\tau + 2\mu_1 \langle \widehat{\nabla} u_0^i \tau, \tau \rangle \tau + 2\mu_1 \langle \widehat{\nabla} u_0^i \tau, \nu \rangle \nu \\ &= \frac{\lambda_1(\lambda_0 + 2\mu_0)}{\lambda_1 + 2\mu_1} (\nabla \cdot u_0^e)\tau + \frac{2\lambda_1(\mu_1 - \mu_0)}{\lambda_1 + 2\mu_1} \langle \widehat{\nabla} u_0^e \tau, \tau \rangle \tau \\ &\quad + 2\mu_1 \langle \widehat{\nabla} u_0^e \tau, \tau \rangle \tau + 2\mu_0 \langle \widehat{\nabla} u_0^e \tau, \nu \rangle \nu \\ &= p(\nabla \cdot u_0^e)\tau + 2\mu_0 \widehat{\nabla} u_0^e \tau + q \langle \widehat{\nabla} u_0^e \tau, \tau \rangle \tau, \end{aligned} \tag{2.9}$$

where

$$p := \frac{\lambda_1(\lambda_0 + 2\mu_0)}{\lambda_1 + 2\mu_1} \quad \text{and} \quad q := \frac{4(\mu_1 - \mu_0)(\lambda_1 + \mu_1)}{\lambda_1 + 2\mu_1}. \tag{2.10}$$

If we define a new 4-tensor $\mathbb{K}$ by

$$\mathbb{K} := p\mathbf{I}_2 \otimes \mathbf{I}_2 + 2\mu_0 \mathbf{I}_4 + q\mathbf{I}_2 \otimes (\tau \otimes \tau), \tag{2.11}$$

then (2.9) can rewritten in the following condensed form:

$$(\mathbb{C}_1 \widehat{\nabla} u_0^i)\tau = (\mathbb{K} \widehat{\nabla} u_0^e)\tau \quad \text{on } \partial D. \tag{2.12}$$

The ϵ-perturbation, denoted by D_ϵ, of the domain D is given by

$$\partial D_\epsilon = \Big\{ \tilde{x} : \tilde{x} = x + \epsilon h(x)\nu(x), \ x \in \partial D \Big\},$$

where, we assume, $h \in \mathcal{C}^{1,1}(\partial D)$ with $\|h\|_{\mathcal{C}^{1,1}} \leq H$ for some positive constant H and ϵ is a positive small parameter.

Let $\mathbb{C}_{D_\epsilon} = \mathbb{C}_0\chi_{\Omega\setminus D_\epsilon} + \mathbb{C}_1\chi_{D_\epsilon}$ and consider the solution $(u_\epsilon, \omega_\epsilon^2) \in H^1(\Omega) \times \mathbb{R}^+$ of the eigenvalue problem on the perturbed domain:

$$\left\{ \begin{array}{rcl} \nabla \cdot \left(\mathbb{C}_{D_\epsilon}\widehat{\nabla} u_\epsilon\right) &=& -\omega_\epsilon^2 u_\epsilon \quad \text{in} \quad \Omega, \\ u_\epsilon &=& 0 \quad \text{on} \quad \partial\Omega, \\ ||u_\epsilon||_{L^2(\Omega)} &=& 1. \end{array} \right. \tag{2.13}$$

The purpose of this section is to investigate the asymptotic behavior of the eigenvalue of (2.13) as ϵ tends to 0 and the main result is the following.

Theorem 2.1. *Let ω_0^2 be a simple eigenvalue of the problem (2.4). Then, there exists a simple eigenvalue of problem (2.13), denoted by ω_ϵ^2, such that $\omega_\epsilon^2 \to \omega_0^2$ as $\epsilon \to 0$ and*

$$\omega_\epsilon^2 - \omega_0^2 = \epsilon \int_{\partial D} h(x)\mathcal{M}[\widehat{\nabla} u_0^e](x) : \widehat{\nabla} u_0^e(x)\, d\sigma(x) + O(\epsilon^{1+\beta}), \tag{2.14}$$

for some positive β and where

$$\mathcal{M}[\widehat{\nabla} u_0^e] := (\mathbb{C}_1 - \mathbb{C}_0)\mathbb{C}_1^{-1}\left((\mathbb{K}\widehat{\nabla} u_0^e\tau) \otimes \tau + (\mathbb{C}_0\widehat{\nabla} u_0^e\nu) \otimes \nu\right). \tag{2.15}$$

Here, ν, τ are respectively the outward normal vector and the tangent vector to ∂D.

Before proving Theorem 2.1, let us express $\mathcal{M}[\widehat{\nabla} u_0^e]$ in more explicit forms. Put

$$\mathbb{C} := (\mathbb{C}_1 - \mathbb{C}_0)\mathbb{C}_1^{-1}$$

for convenience and set

$$\mathbf{\Lambda}_1 := \frac{1}{2}\mathbf{I}_2 \otimes \mathbf{I}_2, \quad \mathbf{\Lambda}_2 := \mathbf{I}_4 - \mathbf{\Lambda}_1. \tag{2.16}$$

Since for any 2×2 symmetric matrix A

$$\mathbf{I}_2 \otimes \mathbf{I}_2(A) = (A : \mathbf{I}_2)\,\mathbf{I}_2 = \mathrm{tr}(A)\,\mathbf{I}_2 \quad \text{and} \quad \mathbf{I}_4(A) = A,$$

one can immediately see that

$$\mathbf{\Lambda}_1\mathbf{\Lambda}_1 = \mathbf{\Lambda}_1, \quad \mathbf{\Lambda}_2\mathbf{\Lambda}_2 = \mathbf{\Lambda}_2, \quad \mathbf{\Lambda}_1\mathbf{\Lambda}_2 = \mathbf{\Lambda}_2\mathbf{\Lambda}_1 = 0.$$

With the notation (2.16), one can easily see that

$$\mathbb{C}_1^{-1} = \frac{1}{2(\lambda_1 + \mu_1)}\mathbf{\Lambda}_1 + \frac{1}{2\mu_1}\mathbf{\Lambda}_2,$$

which immediately yields

$$\mathbb{C} = \lambda\mathbf{I}_2 \otimes \mathbf{I}_2 + 2\mu\mathbf{I}_4,$$

where

$$\lambda = \frac{\lambda_1 - \lambda_0 + \mu_1 - \mu_0}{2(\lambda_1 + \mu_1)} - \frac{\mu_1 - \mu_0}{2\mu_1}, \quad \mu = \frac{\mu_1 - \mu_0}{2\mu_1}. \tag{2.17}$$

Straightforward computations yield

$$\begin{aligned} &(\mathbb{K}\widehat{\nabla} u_0^e\tau) \otimes \tau + (\mathbb{C}_0\widehat{\nabla} u_0^e\nu) \otimes \nu \\ &= p(\nabla \cdot u_0^e)\tau \otimes \tau + 2\mu_0(\widehat{\nabla} u_0^e\tau) \otimes \tau + q\langle\widehat{\nabla} u_0^e\tau,\ \tau\rangle\tau \otimes \tau \\ &\qquad + \lambda_0(\nabla \cdot u_0^e)\nu \otimes \nu + 2\mu_0(\widehat{\nabla} u_0^e\nu) \otimes \nu \\ &= p(\nabla \cdot u_0^e)\tau \otimes \tau + q\langle\widehat{\nabla} u_0^e\tau,\ \tau\rangle\tau \otimes \tau + \lambda_0(\nabla \cdot u_0^e)\nu \otimes \nu + 2\mu_0\widehat{\nabla} u_0^e, \end{aligned}$$

and hence

$$\begin{aligned}
&\mathbb{C}\left((\mathbb{K}\widehat{\nabla}u_0^e\tau)\otimes\tau + (\mathbb{C}_0\widehat{\nabla}u_0^e\nu)\otimes\nu\right)\\
&= \lambda(p+\lambda_0+2\mu_0)(\nabla\cdot u_0^e)\mathbf{I}_2 + \lambda q\langle\widehat{\nabla}u_0^e\tau,\ \tau\rangle\mathbf{I}_2\\
&\quad + 2\mu\left[p(\nabla\cdot u_0^e)\tau\otimes\tau + q\langle\widehat{\nabla}u_0^e\tau,\ \tau\rangle\tau\otimes\tau + \lambda_0(\nabla\cdot u_0^e)\nu\otimes\nu + 2\mu_0\widehat{\nabla}u_0^e\right]\ .
\end{aligned}$$

Therefore, as an operator, $\mathcal{M}$ can be expressed as

$$\begin{aligned}
\mathcal{M} &= \lambda(p+\lambda_0+2\mu_0)\mathbf{I}_2\otimes\mathbf{I}_2 + \lambda q\mathbf{I}_2\otimes(\tau\otimes\tau) + 2\mu p(\tau\otimes\tau)\otimes\mathbf{I}_2\\
&\quad + 2\mu q(\tau\otimes\tau)\otimes(\tau\otimes\tau) + 2\mu\lambda_0(\nu\otimes\nu)\otimes\mathbf{I}_2 + 4\mu\mu_0\mathbf{I}_4\ .
\end{aligned}\tag{2.18}$$

We will prove Theorem 2.1 using Osborn's result in [O] concerning estimates for the eigenvalues of a sequence of self-adjoint compact operators.

Let $T : L^2(\Omega) \to L^2(\Omega)$ be the operator given by $Tf = v_0$ where v_0 is the solution to

$$\begin{cases} \nabla\cdot\left(\mathbb{C}_D\widehat{\nabla}v_0\right) = f & \text{in} \quad \Omega,\\ v_0 = 0 & \text{on} \quad \partial\Omega, \end{cases}\tag{2.19}$$

and let $T_\epsilon : L^2(\Omega) \to L^2(\Omega)$ be the operator given by $T_\epsilon f = v_\epsilon$ where v_ϵ is the solution to

$$\begin{cases} \nabla\cdot\left(\mathbb{C}_{D_\epsilon}\widehat{\nabla}v_\epsilon\right) = f & \text{in} \quad \Omega,\\ v_\epsilon = 0 & \text{on} \quad \partial\Omega. \end{cases}\tag{2.20}$$

Clearly $T(:= T_0)$ and $\{T_\epsilon\}_{\epsilon>0}$ are linear and self-adjoint operators.

We claim that T_ϵ is a compact operator. In fact, by standard energy estimates based on Korn and Poincaré inequalities, we have that for all $\epsilon \geq 0$,

$$\|T_\epsilon f\|_{H^1(\Omega)} = \|v_\epsilon\|_{H^1(\Omega)} \leq C\|\nabla v_\epsilon\|_{L^2(\Omega)} \leq C\|\widehat{\nabla}v_\epsilon\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)},$$

where the constant C is independent of ϵ. Since the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ is compact, we conclude that T_ϵ is compact. Moreover, since the constant C is independent of ϵ, the sequence of operators $(T_\epsilon)_{\epsilon\geq 0}$ is collectively compact.

We now prove that $T_\epsilon f$ converges to Tf in $L^2(\Omega)$ for every $f \in L^2(\Omega)$. We first observe a simple relation

$$\int_\Omega \mathbb{C}_{D_\epsilon}\widehat{\nabla}(v_\epsilon - v_0) : \widehat{\nabla}(v_\epsilon - v_0) = \int_{D_\epsilon \triangle D}(\mathbb{C}_0 - \mathbb{C}_1)\widehat{\nabla}v_0 : \widehat{\nabla}(v_\epsilon - v_0).\tag{2.21}$$

The strong convexity assumption (2.3) on $\mathbb{C}_{D_\epsilon}$ and Korn's inequality yield

$$\int_\Omega \mathbb{C}_{D_\epsilon}\widehat{\nabla}(v_\epsilon - v_0) : \widehat{\nabla}(v_\epsilon - v_0) \geq C\int_\Omega |\widehat{\nabla}(v_\epsilon - v_0)|^2 \geq C\int_\Omega |\nabla(v_\epsilon - v_0)|^2,$$

where C depends only on α_0, β_0 and Ω. On the other hand, by Hölder's inequality, we get

$$\begin{aligned}
&\int_{D_\epsilon \triangle D}(\mathbb{C}_0 - \mathbb{C}_1)\,\widehat{\nabla}v_0 : \widehat{\nabla}(v_\epsilon - v_0)\,dx\\
&\leq \max\{2|\mu_0 - \mu_1|, |\lambda_0 - \lambda_1|\}\,\|\nabla v_0\|_{L^2(D_\epsilon \triangle D)}\|\nabla(v_\epsilon - v_0)\|_{L^2(\Omega)}.
\end{aligned}$$

We then obtain from the above two inequalities and (2.21) that

$$\|\nabla(v_\epsilon - v_0)\|_{L^2(\Omega)} \leq C\|\nabla v_0\|_{L^2(D_\epsilon \triangle D)}.$$

It then follows from Poincaré's inequality that

$$\|v_\epsilon - v_0\|_{H^1(\Omega)} \le C\|\nabla v_0\|_{L^2(D_\epsilon \triangle D)}. \tag{2.22}$$

Since $\nabla v_0 \in L^2(\Omega)$ and $|D_\epsilon \triangle D| \to 0$ as $\epsilon \to 0$, we get $\|v_\epsilon - v_0\|_{H^1(\Omega)} \to 0$ as $\epsilon \to 0$. In particular, $\|v_\epsilon - v_0\|_{L^2(\Omega)} = \|T_\epsilon f - Tf\|_{L^2(\Omega)} \to 0$ as $\epsilon \to 0$.

So, a theorem of Osborn [O] yields

$$\left| \frac{1}{\omega_\epsilon^2} - \frac{1}{\omega_0^2} + \langle (T - T_\epsilon)u_0, u_0\rangle \right| \le C\|(T - T_\epsilon)u_0\|^2_{L^2(\Omega)}, \tag{2.23}$$

where C is independent of ϵ and u_0 is the solution of (2.4). Furthermore, if u_ϵ is the solution to (2.13), then

$$\|u_\epsilon - u_0\|_{L^2(\Omega)} \le C\|(T - T_\epsilon)u_0\|_{L^2(\Omega)}. \tag{2.24}$$

Let us state some regularity results on u_ϵ and u_0 that will be used in the sequel: There is a constant C independent of ϵ such that

$$\|u_\epsilon\|_{\mathcal{C}^{1,\alpha}(\bar{D}_\epsilon)} + \|u_\epsilon\|_{\mathcal{C}^{1,\alpha}(\Omega_{d_0/2}\setminus D_\epsilon)} \le C, \tag{2.25}$$

for some $\alpha > 0$. This estimate extends the regularity results obtaind by De Giorgi and Nash in the scalar case (cf., for instance, [GT]) to the case of bidimensional elliptic systems.

Let $\Omega_{d_0/2} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > d_0/2\}$ for some $d_0 > 0$. Li and Nirenberg proved in [LN] that $u_\epsilon \in \mathcal{C}^{1,\alpha}(\bar{D}_\epsilon) \cap \mathcal{C}^{1,\alpha}(\Omega\setminus D_\epsilon)$ for some $\alpha \in (0,1)$, and there is a constant C depending on the ellipticity constants α_0 and β_0, d_0, and $\mathcal{C}^{1,1}$ norm of D_ϵ such that

$$\|u_\epsilon\|_{\mathcal{C}^{1,\alpha}(\bar{D}_\epsilon)} + \|u_\epsilon\|_{\mathcal{C}^{1,\alpha}(\Omega_{d_0/2}\setminus D_\epsilon)} \le C(\|u_\epsilon\|_{L^2(\Omega)} + \|u_\epsilon\|_{L^\infty(\Omega_{d_0/2})}). \tag{2.26}$$

Since $u_\epsilon \in H^1(\Omega)$ and its norm is bounded regardless of ϵ, it follows from the Sobolev embedding theorem that $u_\epsilon \in L^q(\Omega)$ for $q > 2$ independently of ϵ. Then, by Theorem A.1, it follows that $\nabla u_\epsilon \in L^{2+\eta}_{\text{loc}}(\Omega)$ for some $\eta > 0$. Again by Sobolev embedding theorem, this implies that $u_\epsilon \in \mathcal{C}^\gamma_{\text{loc}}(\Omega)$ with $\gamma = 1 - \frac{2}{2+\eta}$. Finally, recalling that $\|u_\epsilon\|_{L^2(\Omega)} = 1$, we obtain (2.25).

Let us now evaluate the right-hand side of (2.24). We know that $Tu_0 = -\frac{1}{\omega_0^2}u_0$ and $T_\epsilon u_0 = \tilde{v}_\epsilon$ where $\tilde{v}_\epsilon$ is the solution to

$$\left\{ \begin{array}{rcll} \nabla \cdot \left(\mathbb{C}_{D_\epsilon} \widehat{\nabla} \tilde{v}_\epsilon\right) & = & u_0 & \text{in} \quad \Omega, \\ \tilde{v}_\epsilon & = & 0 & \text{on} \quad \partial\Omega. \end{array} \right. \tag{2.27}$$

Let $\tilde{u}_0 = -\frac{1}{\omega_0^2}u_0$, then

$$\left\{ \begin{array}{rcll} \nabla \cdot \left(\mathbb{C}_{D} \widehat{\nabla} \tilde{u}_0\right) & = & u_0 & \text{in} \quad \Omega, \\ \tilde{u}_0 & = & 0 & \text{on} \quad \partial\Omega. \end{array} \right. \tag{2.28}$$

Hence, one can show in the same way as for (2.22) that

$$\|\tilde{v}_\epsilon - \tilde{u}_0\|^2_{H^1(\Omega)} \le C\|\nabla u_0\|^2_{L^2(D_\epsilon \triangle D)},$$

and by the regularity estimates (2.25)

$$\|\nabla u_0\|_{L^2(D_\epsilon \triangle D)} \le C|D_\epsilon \triangle D|^{1/2},$$

which implies

$$\|\tilde{v}_\epsilon - \tilde{u}_0\|_{H^1(\Omega)} \le C|D_\epsilon \triangle D|^{1/2} \tag{2.29}$$

for some constant C independent of ϵ.

We now prove the following estimate

$$\|\tilde{v}_\epsilon - \tilde{u}_0\|_{L^2(\Omega)} \leq C|D_\epsilon \triangle D|^{1/2+\eta} \tag{2.30}$$

for $\eta > 0$. To this end, we need the following lemma whose proof will be given in Appendix A.

Lemma 2.2. *Let $\mathbb{C} = (C_{ijkl})$ be an $L^\infty(\Omega)$ strongly convex elliptic tensor field, $F \in L^\infty(\omega)$ 2×2 matrix-valued function, where $\omega \subset \Omega$ is a measurable set. Let φ be a solution to*

$$\left\{ \begin{array}{rcl} \nabla \cdot \left(\mathbb{C} \widehat{\nabla} \varphi \right) & = & \nabla \cdot (\chi_\omega F) \quad \text{in} \quad \Omega, \\ \varphi & = & 0 \quad \text{on} \quad \partial\Omega. \end{array} \right. \tag{2.31}$$

Then,

$$\|\varphi\|_{L^2(\Omega)} \leq C|\omega|^{1/2+\eta} \|F\|_{L^\infty(\omega)}, \tag{2.32}$$

where $\eta > 0$.

We apply the above lemma to the function $\tilde{v}_\epsilon - \tilde{u}_0$. Observe that $\tilde{v}_\epsilon - \tilde{u}_0$ satisfies

$$\left\{ \begin{array}{rcl} \nabla \cdot \left(\mathbb{C} \widehat{\nabla} (\tilde{v}_\epsilon - \tilde{u}_0) \right) & = & \nabla \cdot ((\mathbb{C}_D - \mathbb{C}_\epsilon) \widehat{\nabla} \tilde{v}_\epsilon) \quad \text{in} \quad \Omega, \\ \tilde{v}_\epsilon - \tilde{u}_0 & = & 0 \quad \text{on} \quad \partial\Omega, \end{array} \right.$$

and hence we get

$$\|\tilde{v}_\epsilon - \tilde{u}_0\|_{L^2(\Omega)} \leq C|D_\epsilon \triangle D|^{1/2+\eta} \|\nabla \tilde{v}_\epsilon\|_{L^\infty(\omega)}. \tag{2.33}$$

Furthermore, according to (2.26), we have

$$\|\tilde{v}_\epsilon\|_{\mathcal{C}^{1,\alpha}(\bar{D}_\epsilon)} + \|\tilde{v}_\epsilon\|_{\mathcal{C}^{1,\alpha}(\Omega_{d_0/2} \setminus D_\epsilon)} \leq C(\|\tilde{v}_\epsilon\|_{L^2(\Omega)} + \|u_0\|_{L^\infty(\Omega_{d_0/2})}). \tag{2.34}$$

Since $\|\tilde{v}_\epsilon\|_{H^1(\Omega)} \leq C\|u_0\|_{L^2(\Omega)} \leq C$, it follows from (2.25) that

$$\|\tilde{v}_\epsilon\|_{\mathcal{C}^{1,\alpha}(\bar{D}_\epsilon)} + \|\tilde{v}_\epsilon\|_{\mathcal{C}^{1,\alpha}(\Omega_{d_0/2} \setminus D_\epsilon)} \leq C. \tag{2.35}$$

The desired estimate (2.30) now follows from (2.33), (2.34), and (2.35), and we conclude that

$$\|(T_\epsilon - T)u_0\|_{L^2(\Omega)} = \|\tilde{v}_\epsilon - \tilde{u}_0\|_{L^2(\Omega)} \leq C\epsilon^{1/2+\eta}. \tag{2.36}$$

It also follows from (2.24) that

$$\|u_{\epsilon -} u_0\|_{L^2(\Omega)} \leq C\epsilon^{1/2+\eta}. \tag{2.37}$$

The following lemma holds.

Lemma 2.3. *There exists a constant C independent of ϵ such that*

$$\|\nabla(\tilde{v}_\epsilon - \tilde{u}_0)\|_{L^\infty(\partial D_\epsilon \setminus D)} + \|\nabla(\tilde{v}_\epsilon - \tilde{u}_0)\|_{L^\infty(\partial D_\epsilon \cap \overline{D})} \leq C\epsilon^{\frac{\alpha}{2(\alpha+2)}}. \tag{2.38}$$

Proof. To prove (2.38) we make use of a mean value property for biharmonic functions (see [BF, Theorem 4.1]).

Let $2\epsilon < d < d_0/2$ and let

$$\Omega_d^\epsilon := \{\, x \in \Omega \setminus (D \cup D_\epsilon) : \operatorname{dist}(x, \partial(\Omega \setminus (D \cup D_\epsilon))) > d \,\}. \tag{2.39}$$

Since $\nabla(\tilde{v}_\epsilon - \tilde{u}_0)$ is biharmonic in $\Omega\backslash(D\cup D_\epsilon)$, we may apply the mean value theorem at points $y \in \Omega_d^\epsilon$:

$$\nabla(\tilde{v}_\epsilon - \tilde{u}_0)(y) = \frac{12}{\pi}\left[\frac{4}{d^4}\int_{B_{\frac{d}{2}}(y)} (\tilde{v}_\epsilon - \tilde{u}_0)\otimes \underline{r}\, dx - \frac{1}{d^4}\int_{B_{\frac{d}{2}}(y)} r^2 \nabla(\tilde{v}_\epsilon - \tilde{u}_0)\, dx\right],$$

where $\underline{r}(x) = x - y$ and $r = |\underline{r}|$. It then follows from the Hölder inequality and (2.29) that

$$\|\nabla(\tilde{v}_\epsilon - \tilde{u}_0)\|_{L^\infty(\Omega_d^\epsilon)} \le C d^{-2}\epsilon^{\frac{1}{2}}, \tag{2.40}$$

where C is independent of ϵ.

Set

$$\tilde{v}_\epsilon^e = \tilde{v}_\epsilon|_{\Omega\backslash D} \quad\text{and}\quad \tilde{v}_\epsilon^i = \tilde{v}_\epsilon|_{\overline{D}},$$

as in (2.6). For $y \in \partial D_\epsilon\backslash D$, let y_d denote the closest point to y in the set $\overline{\Omega_d^\epsilon}$. By (2.35), we obtain

$$|\nabla\tilde{v}_\epsilon^e(y) - \nabla\tilde{v}_\epsilon^e(y_d)| \le C d^\alpha.$$

Likewise, we have

$$|\nabla\tilde{u}_0(y) - \nabla\tilde{u}_0(y_d)| \le C d^\alpha.$$

It then follows from (2.40) that

$$\begin{aligned}|\nabla(\tilde{v}_\epsilon^e - \tilde{u}_0^e)(y)| &\le |\nabla\tilde{v}_\epsilon^e(y) - \nabla\tilde{v}_\epsilon^e(y_d)| + |\nabla\tilde{v}_\epsilon^e(y_d) - \nabla\tilde{u}_0^e(y_d)| \\ &\quad + |\nabla\tilde{u}_0^e(y_d) - \nabla\tilde{u}_0^e(y)| \\ &\le C(d^\alpha + d^{-2}\epsilon^{1/2}).\end{aligned}$$

Minimizing the right-hand side of the above inequality with respect to d, we get

$$\|\nabla(\tilde{v}_\epsilon^e - \tilde{u}_0^e)\|_{L^\infty(\partial D_\epsilon\backslash D)} \le C\epsilon^{\frac{\alpha}{2(\alpha+2)}}.$$

In a similar way one can prove that

$$\|\nabla(\tilde{v}_\epsilon^i - \tilde{u}_0^i)\|_{L^\infty(\partial D_\epsilon\cap D)} \le C\epsilon^{\frac{\alpha}{2(\alpha+2)}}$$

to complete the proof. □

Proof of Theorem 2.1. We begin by computing the term $\langle (T - T_\epsilon)u_0, u_0\rangle$ appearing in (2.24). In view of (2.27) and (2.28), we have

$$\begin{aligned}&\langle (T - T_\epsilon)u_0, u_0\rangle = \langle \tilde{u}_0 - \tilde{v}_\epsilon, u_0\rangle \\ &= -\frac{1}{\omega_0^2}\int_\Omega u_0^2 - \int_\Omega u_0\tilde{v}_\epsilon \\ &= \frac{1}{\omega_0^2}\int_\Omega (\mathbb{C}_{D_\epsilon} - \mathbb{C}_D)\widehat{\nabla}\tilde{v}_\epsilon : \widehat{\nabla}u_0 \\ &= \frac{1}{\omega_0^2}\int_{D_\epsilon\backslash D}(\mathbb{C}_1 - \mathbb{C}_0)\widehat{\nabla}\tilde{v}_\epsilon^i : \widehat{\nabla}u_0^e - \frac{1}{\omega_0^2}\int_{D\backslash D_\epsilon}(\mathbb{C}_1 - \mathbb{C}_0)\widehat{\nabla}\tilde{v}_\epsilon^e : \widehat{\nabla}u_0^i.\end{aligned}$$

Let $x_t := x + th(x)\nu(x)$ for $x \in \partial D$ and $t \in [0, \epsilon]$. We get, for ϵ small enough,

$$\begin{aligned}&\frac{1}{\omega_0^2}\int_{D_\epsilon\backslash D}(\mathbb{C}_1 - \mathbb{C}_0)\widehat{\nabla}\tilde{v}_\epsilon^i : \widehat{\nabla}u_0^e dx \\ &= \frac{1}{\omega_0^2}\int_0^\epsilon\int_{\partial D\cap\{h>0\}} h(x)(\mathbb{C}_1 - \mathbb{C}_0)\widehat{\nabla}\tilde{v}_\epsilon^i(x_t) : \widehat{\nabla}u_0^e(x_t)\, d\sigma(x)\, dt + O(\epsilon^2),\end{aligned} \tag{2.41}$$

and

$$\begin{aligned} &-\frac{1}{\omega_0^2}\int_{D\setminus D_\epsilon}(\mathbb{C}_1-\mathbb{C}_0)\widehat{\nabla}\tilde{v}_\epsilon^e : \widehat{\nabla}u_0^i\,dx \\ &= \frac{1}{\omega_0^2}\int_0^\epsilon\int_{\partial D\cap\{h<0\}} h(x)(\mathbb{C}_1-\mathbb{C}_0)\widehat{\nabla}\tilde{v}_\epsilon^e(x_t) : \widehat{\nabla}u_0^i(x_t)\,d\sigma(x)\,dt + O(\epsilon^2). \end{aligned} \tag{2.42}$$

Using the gradient estimates (2.34) and (2.25) for $\tilde{v}_\epsilon$ and u_0, we can approximate

$$(\mathbb{C}_1-\mathbb{C}_0)\widehat{\nabla}\tilde{v}_\epsilon^i(x_t) : \widehat{\nabla}u_0^e(x_t) = (\mathbb{C}_1-\mathbb{C}_0)\widehat{\nabla}\tilde{v}_\epsilon^i(x_\epsilon) : \widehat{\nabla}u_0^e(x_\epsilon) + O(\epsilon^\alpha)$$

for ϵ sufficiently small. It thus follows from the transmission conditions (2.5) and (2.12) for the function $\tilde{v}_\epsilon$ that

$$\begin{aligned} \widehat{\nabla}\tilde{v}_\epsilon^i(x_\epsilon) &= \mathbb{C}_1^{-1}\left((\mathbb{C}_1\widehat{\nabla}\tilde{v}_\epsilon^i(x_\epsilon)\tau)\otimes\tau + (\mathbb{C}_1\widehat{\nabla}\tilde{v}_\epsilon^i(x_\epsilon)\nu)\otimes\nu\right) \\ &= \mathbb{C}_1^{-1}\left((\mathbb{K}\widehat{\nabla}\tilde{v}_\epsilon^e(x_\epsilon)\tau)\otimes\tau + (\mathbb{C}_0\widehat{\nabla}\tilde{v}_\epsilon^e(x_\epsilon)\nu)\otimes\nu\right). \end{aligned}$$

We then get using Lemma 2.3 that

$$\widehat{\nabla}\tilde{v}_\epsilon^i(x_\epsilon) = \frac{1}{\omega_0^2}\mathbb{C}_1^{-1}\left((\mathbb{K}\widehat{\nabla}u_0^e(x_\epsilon)\tau)\otimes\tau + (\mathbb{C}_0\widehat{\nabla}u_0^e(x_\epsilon)\nu)\otimes\nu\right) + O(\epsilon^{\frac{\alpha}{2(\alpha+2)}})$$

for some $\gamma > 0$ and hence

$$\widehat{\nabla}\tilde{v}_\epsilon^i(x_\epsilon) = \frac{1}{\omega_0^2}\mathbb{C}_1^{-1}\left((\mathbb{K}\widehat{\nabla}u_0^e(x)\tau)\otimes\tau + (\mathbb{C}_0\widehat{\nabla}u_0^e(x)\nu)\otimes\nu\right) + O(\epsilon^{\frac{\alpha}{2(\alpha+2)}}).$$

Thus we get

$$\begin{aligned} &\frac{1}{\omega_0^2}\int_{D_\epsilon\setminus D}(\mathbb{C}_1-\mathbb{C}_0)\widehat{\nabla}\tilde{v}_\epsilon^i : \widehat{\nabla}u_0^e dx \\ &= \frac{\epsilon}{\omega_0^4}\int_{\partial D\cap\{h>0\}} h(x)\mathcal{M}[\widehat{\nabla}u_0^e](x) : \widehat{\nabla}u_0^e(x)\,d\sigma(x) + O(\epsilon^{1+\frac{\alpha}{2(\alpha+2)}}), \end{aligned}$$

for $\alpha > 0$, where $\mathcal{M}[\widehat{\nabla}u_0^e]$ is given by (2.15).

Similarly, we get

$$\begin{aligned} &-\frac{1}{\omega_0^2}\int_{D\setminus D_\epsilon}(\mathbb{C}_1-\mathbb{C}_0)\widehat{\nabla}\tilde{v}_\epsilon^i : \widehat{\nabla}u_0^e dx \\ &= \frac{\epsilon}{\omega_0^4}\int_{\partial D\cap\{h<0\}} h(x)\mathcal{M}[\widehat{\nabla}u_0^e](x) : \widehat{\nabla}u_0^e(x)\,d\sigma(x) + O(\epsilon^{1+\frac{\alpha}{2(\alpha+2)}}). \end{aligned}$$

We finally conclude that

$$\langle(T-T_\epsilon)u_0, u_0\rangle = \frac{\epsilon}{\omega_0^4}\int_{\partial D} h(x)\mathcal{M}[\widehat{\nabla}u_0^e](x) : \widehat{\nabla}u_0^e(x)\,d\sigma(x) + O(\epsilon^{1+\frac{\alpha}{2(\alpha+2)}}),$$

which together with (2.23) yields Theorem 2.1. This completes the proof. $\square$

3. Dual asymptotic formula

Let (u_0, ω_0^2) be the solution to (2.4). For $g \in L^2(\partial\Omega)$ such that $\int_{\partial\Omega} g \cdot (\mathbb{C}_D \widehat{\nabla} u_0)\nu = 0$, let w_g be a solution to

$$\left\{ \begin{array}{rcll} \nabla \cdot (\mathbb{C}_D \nabla w_g) & = & \omega_0^2 w_g & \text{in} \quad \Omega, \\ w_g & = & g & \text{on} \quad \partial\Omega. \end{array} \right. \tag{3.1}$$

Multiplying the first equation in (3.1) by u_ϵ and integrating over Ω we get

$$\omega_0^2 \int_\Omega w_g \cdot u_\epsilon = \int_\Omega \mathbb{C}_D \widehat{\nabla} u_\epsilon : \widehat{\nabla} w_g.$$

Since $\int_{\partial\Omega} g \cdot (\mathbb{C}_D \widehat{\nabla} u_0)\nu = 0$ and

$$\omega_\epsilon^2 \int_\Omega w_g \cdot u_\epsilon = \int_\Omega \mathbb{C}_{D_\epsilon} \widehat{\nabla} u_\epsilon : \widehat{\nabla} w_g - \int_{\partial\Omega} g \cdot \mathbb{C}_0 (\widehat{\nabla} u_\epsilon - \widehat{\nabla} u_0)\nu,$$

we obtain

$$\int_{\partial\Omega} g \cdot \mathbb{C}_0 (\widehat{\nabla} u_\epsilon - \widehat{\nabla} u_0)\nu + (\omega_\epsilon^2 - \omega_0^2) \int_\Omega w_g \cdot u_\epsilon = \int_\Omega (\mathbb{C}_{D_\epsilon} - \mathbb{C}_D) \widehat{\nabla} u_\epsilon : \widehat{\nabla} w_g.$$

Since $\omega_\epsilon^2 - \omega_0^2 = O(\epsilon)$ and $\|u_{\epsilon -} u_0\|_{L^2(\Omega)} \leq C\epsilon^{1/2+\eta}$, we get, for ϵ small enough,

$$\begin{aligned} &\int_{\partial\Omega} g \cdot \mathbb{C}_0 (\widehat{\nabla} u_\epsilon - \widehat{\nabla} u_0)\nu + (\omega_\epsilon^2 - \omega_0^2) \int_\Omega w_g \cdot u_0 \\ &= \int_\Omega (\mathbb{C}_{D_\epsilon} - \mathbb{C}_D) \widehat{\nabla} u_\epsilon : \widehat{\nabla} w_g + O(\epsilon^{1+\beta}), \end{aligned} \tag{3.2}$$

for some $\beta > 0$.

We now prove the following theorem. The asymptotic formula in this theorem can be regarded as a dual formula to that of $\omega_\epsilon^2 - \omega_0^2$ in (2.13). It plays a key role in our reconstruction procedure in later sections.

Theorem 3.1. *The following asymptotic formula holds as* $\epsilon \to 0$*:*

$$\begin{aligned} &\int_{\partial\Omega} g \cdot \mathbb{C}_0 (\widehat{\nabla} u_\epsilon - \widehat{\nabla} u_0)\nu + (\omega_\epsilon^2 - \omega_0^2) \int_\Omega w_g \cdot u_0 \\ &= \epsilon \int_{\partial D} h(x) \mathcal{M}[\widehat{\nabla} u_0^e](x) : \widehat{\nabla} w_g^e(x) d\sigma(x) + O(\epsilon^{1+\beta}) \end{aligned} \tag{3.3}$$

for some $\beta > 0$.

To prove (3.3), it suffices, thanks to (3.2), to show that

$$\begin{aligned} &-\int_\Omega (\mathbb{C}_{D_\epsilon} - \mathbb{C}_D) \widehat{\nabla} u_\epsilon : \widehat{\nabla} w_g \\ &= -\epsilon \int_{\partial D} h(x) \mathcal{M}[\widehat{\nabla} u_0^e](x) : \widehat{\nabla} w_g^e(x) d\sigma(x) + O(\epsilon^{1+\beta}). \end{aligned}$$

This can be proved following the same lines of the proof of Theorem 2.1 in the previous section, as long as we have proper estimates for u_ϵ and w_g. The required estimates are

$$\|w_g\|_{\mathcal{C}^{1,\alpha}(\bar{D})} + \|w_g\|_{\mathcal{C}^{1,\alpha}(\Omega_{d_0/2} \setminus D)} \leq C \tag{3.4}$$

and

$$\|\nabla(u_\epsilon^e - u_0^e)\|_{L^\infty(\partial D_\epsilon \setminus D)} + \|\nabla(u_\epsilon^i - u_0^i)\|_{L^\infty(\partial D_\epsilon \cap \overline{D})} \leq C\epsilon^\gamma \tag{3.5}$$

for some constant C independent of ϵ and $\gamma > 0$. The rest of this section is devoted to proving (3.4) and (3.5).

The estimate (3.4) holds since $\nabla\cdot(\mathbb{C}_D\widehat{\nabla})+\omega_0^2$ with Dirichlet boundary conditions is well posed on the subspace of $H^1(\Omega)$ orthogonal to u_0 and, on the other hand, u_0 itself satisfies such an estimate.

In order to prove (3.5), let $2\epsilon < d < d_0/2$ and Ω_d^ϵ be defined as in (2.39). Clearly, the function $\phi_\epsilon := \nabla(u_\epsilon - u_0)$ is a solution to the following equation in $\Omega\backslash D \cup D_\epsilon$:

$$\nabla\cdot\left(\mathbb{C}_0\widehat{\nabla}\phi_\epsilon\right) + \omega_\epsilon^2\phi_\epsilon = (\omega_0^2 - \omega_\epsilon^2)\nabla u_0.$$

By standard regularity results for elliptic systems with constant coefficients, ∇u_0 and ϕ_ϵ belong to $L^{2+\eta}_{\mathrm{loc}}$ for some $\eta > 0$. Now, from a generalization of Meyer's theorem to systems (see Appendix A) we have

$$\|\nabla\phi_\epsilon\|_{L^{2+\eta}(\Omega_d^\epsilon)} \le C\left(d^{-1+\frac{2}{2+\eta}}\|\nabla\phi_\epsilon\|_{L^2(\Omega^\epsilon_{d/2})} + \left|\omega_0^2 - \omega_\epsilon^2\right|\|u_0\|_{H^1(\Omega^\epsilon_{d/2})}\right). \tag{3.6}$$

We now apply Caccioppoli's inequality on ϕ_ϵ to have

$$\|\nabla\phi_\epsilon\|_{L^2(\Omega^\epsilon_{d/2})} \le C\left(d^{-2}\|\phi_\epsilon\|_{L^2(\Omega^\epsilon_{d/3})} + \left|\omega_0^2 - \omega_\epsilon^2\right|\|\nabla u_0\|_{L^2(\Omega^\epsilon_{d/3})}\right).$$

Since $\left|\omega_0^2 - \omega_\epsilon^2\right| \le C\epsilon$ and $\|\phi_\epsilon\|_{L^2(\Omega^\epsilon_{d/3})} \le C\sqrt{\epsilon}$, we have

$$\|\nabla\phi_\epsilon\|_{L^2(\Omega^\epsilon_{d/2})} \le C\left(d^{-2}\sqrt{\epsilon} + \epsilon\right). \tag{3.7}$$

Inserting (3.7) into (3.6), we obtain

$$\|\nabla\phi_\epsilon\|_{L^{2+\eta}(\Omega_d^\epsilon)} \le C\left(d^{-3+\frac{2}{2+\eta}}\sqrt{\epsilon} + \epsilon\right) \le Cd^{-3+\frac{2}{2+\eta}}\sqrt{\epsilon}. \tag{3.8}$$

On the other hand, since $\|\phi_\epsilon\|_{L^2(\Omega^\epsilon_{d/2})} \le C\sqrt{\epsilon}$, we have from the Sobolev embedding theorem and (3.7) that

$$\|\phi_\epsilon\|_{L^{2+\eta}(\Omega_d^\epsilon)} \le C\|\phi_\epsilon\|_{H^1(\Omega^\epsilon_{d/2})} \le Cd^{-2}\sqrt{\epsilon}. \tag{3.9}$$

Using the Sobolev imbedding theorem again, it follows from (3.9) and (3.8) that

$$\|\phi_\epsilon\|_{L^\infty(\Omega_d^\epsilon)} \le Cd^{-3+\frac{2}{2+\eta}}\sqrt{\epsilon}.$$

Now, let $y \in \partial D_\epsilon\backslash D$ and let y_d denote the closest point to y in the set Ω_d^ϵ. From the gradient estimates for u_ϵ and u_0, we have

$$|\nabla u_\epsilon^e(y) - \nabla u_\epsilon^e(y_d)| \le Cd^\alpha, \tag{3.10}$$

which yields

$$\begin{aligned}|\nabla(u_\epsilon^e - u_0^e)(y)| &\le |\nabla u_\epsilon^e(y) - \nabla u_\epsilon^e(y_d)| + |\nabla u_\epsilon^e(y_d) - \nabla u_0^e(y_d)| \\ &\quad + |\nabla u_0^e(y_d) - \nabla u_0^e(y)| \\ &\le C(d^\alpha + d^{-3+\frac{2}{2+\eta}}\epsilon^{1/2}).\end{aligned}$$

Choosing $d = \epsilon^{\frac{1}{2(3+\alpha-\frac{2}{2+\eta})}}$, we get

$$|\nabla(u_\epsilon^e - u_0^e)(y)| \le C\epsilon^\gamma,$$

where $\gamma = \frac{\alpha}{2\left(3+\alpha-\frac{2}{2+\eta}\right)}$, and hence

$$\|\nabla(u_\epsilon^e - u_0^e)\|_{L^\infty(\partial D_\epsilon\backslash D)} \le C\epsilon^\gamma\ .$$

In a similar way, one can show that

$$\|\nabla(u^i_\epsilon - u^i_0)\|_{L^\infty(\partial D_\epsilon \cap \overline{D})} \leq C\epsilon^\gamma \ .$$

4. Reconstruction procedure

The inverse problem we consider in this section is to recover some information about h from the variations of the modal parameters $(\omega_\epsilon - \omega_0, \mathbb{C}_0(\widehat{\nabla} u_\epsilon - \widehat{\nabla} u_0)\nu|_{\partial\Omega})$ associated with the eigenvalue problem (2.13).

The dual asymptotic formula can be used to reconstruct some information about h from measurements of $\omega_\epsilon^2 - \omega_0^2$ and $\mathbb{C}_0(\widehat{\nabla} u_\epsilon - \widehat{\nabla} u_0)\nu$ on $\partial\Omega$. In fact, we minimize over h the functional

$$\begin{aligned}\sum_{l=1}^{L} \bigg| \int_{\partial\Omega} g_l \cdot \mathbb{C}_0(\widehat{\nabla} u_\epsilon - \widehat{\nabla} u_0)\nu + (\omega_\epsilon^2 - \omega_0^2) \int_\Omega w_{g_l} \cdot u_0 \\ -\epsilon \int_{\partial D} h(x) \mathcal{M}[\widehat{\nabla} u^e_0](x) : \widehat{\nabla} w^e_{g_l}(x) d\sigma(x) \bigg|^2\end{aligned} \tag{4.1}$$

for functions $g_l \in L^2(\partial\Omega)$ satisfying $\int_{\partial\Omega} g_l \cdot (\mathbb{C}_D \widehat{\nabla} u_0)\nu = 0$ for $l = 1, \ldots, L$.

The best choice of $g_1, \ldots, g_L$ is such that the functions

$$\mathcal{M}[\widehat{\nabla} u^e_0] : \widehat{\nabla} w^e_{g_l} \quad \text{on } \partial D$$

are highly oscillating. Let

$$\mathcal{V} := \left\{ g \in L^2(\partial\Omega) : \int_{\partial\Omega} g \cdot (\mathbb{C}_D \widehat{\nabla} u_0)\nu = 0 \right\}$$

and define $\Lambda : \mathcal{V} \to L^2(\partial D)$ by

$$\Lambda(g) := \mathcal{M}[\widehat{\nabla} u^e_0] : \widehat{\nabla} w^e_g \quad \text{on } \partial D, \tag{4.2}$$

where w_g is the solution to (3.1). The best choice of $\{g_1, \ldots, g_L\}$ is then to take them as a basis of the image space of $\Lambda^*\Lambda$, where $\Lambda^* : L^2(\partial D) \to \mathcal{V}(\partial\Omega)$ is the adjoint of Λ. Moreover, one should look for the changes h as a linear combination of $\mathcal{M}[\widehat{\nabla} u^e_0] : \widehat{\nabla} w^e_g|_{\partial D}$ for $g \in \text{Image}(\Lambda^*\Lambda)$:

$$h(x) = \sum_{l=1}^{L} \alpha_l v_{g_l},$$

where

$$v_{g_l} := \mathcal{M}[\widehat{\nabla} u^e_0] : \widehat{\nabla} w^e_{g_l} \quad \text{on } \partial D, \quad l = 1, \ldots, L, \tag{4.3}$$

L is the dimension of Image$(\Lambda^*\Lambda)$, and g_l are the significant singular vectors of Λ. We call the vectors v_{g_l}, $l = 1, \ldots, L$, the *optimally illuminated vectors*. The minimization procedure reduces then to

$$\begin{aligned}\min_{\alpha_{l'}, l'=1,\ldots,L} \sum_{l=1}^{L} \bigg| \int_{\partial\Omega} g_l \cdot \mathbb{C}_0(\widehat{\nabla} u_\epsilon - \widehat{\nabla} u_0)\nu + (\omega_\epsilon^2 - \omega_0^2) \int_\Omega w_{g_l} \cdot u_0 \\ -\epsilon \sum_{l'}^{L} \alpha_{l'} \int_{\partial D} v_{g_{l'}}(x) v_{g_l}(x) \bigg|^2 .\end{aligned} \tag{4.4}$$

This quadratic minimization problem has a unique solution which is stable with respect to the measurements vector given by

$$\Big(\int_{\partial\Omega} g_1 \cdot \mathbb{C}_0(\widehat{\nabla} u_\epsilon - \widehat{\nabla} u_0)\nu, \ldots, \int_{\partial\Omega} g_L \cdot \mathbb{C}_0(\widehat{\nabla} u_\epsilon - \widehat{\nabla} u_0)\nu\Big).$$

This implies that if h is a linear combination of the optimally illuminated vectors, then it can be uniquely reconstructed from the measurements in a robust way. Moreover, the resolution limit in reconstructing the changes h is given by

$$\delta = \frac{1}{\max_l(||\partial w_{g_l}/\partial\tau||_{L^2(\partial D)}/||w_{g_l}||_{L^2(\partial D)})}. \tag{4.5}$$

See [AGJK].

5. Incomplete measurements

Suppose that $\mathbb{C}_0(\widehat{\nabla} u_\epsilon - \widehat{\nabla} u_0)\nu$ is measured only in an open part Γ_1 of the boundary $\partial\Omega$. For $g \in L^2(\partial\Omega)$ such that $g = 0$ on Γ_2 and $\int_{\Gamma_1} g \cdot (\mathbb{C}_D \widehat{\nabla} u_0)\nu = 0$, let w_g be the solution to (3.1). As in Theorem 3.1, we can prove that the following asymptotic formula holds as $\epsilon \to 0$:

$$\begin{aligned}&\int_{\Gamma_1} g \cdot \mathbb{C}_0(\widehat{\nabla} u_\epsilon - \widehat{\nabla} u_0)\nu + (\omega_\epsilon^2 - \omega_0^2)\int_\Omega w_g \cdot u_0 \\ &= \epsilon \int_{\partial D} h(x)\mathcal{M}[\widehat{\nabla} u_0^e](x) : \widehat{\nabla} w_g^e(x) d\sigma(x) + O(\epsilon^{1+\beta})\end{aligned} \tag{5.1}$$

for some $\beta > 0$. Define

$$\mathcal{V}_{\rm loc} := \Big\{ g \in L^2(\partial\Omega) : g = 0 \text{ on } \Gamma_2 \text{ and } \int_{\Gamma_1} g \cdot (\mathbb{C}_D \widehat{\nabla} u_0)\nu = 0 \Big\}.$$

Consider $\Lambda_{\rm loc} : \mathcal{V}_{\rm loc} \to L^2(\partial D)$ given by

$$\Lambda_{\rm loc}(g) := \mathcal{M}[\widehat{\nabla} u_0^e] : \widehat{\nabla} w_g^e \quad \text{on } \partial D,$$

where w_g is the solution to (3.1).

In the case of incomplete measurements, the optimally illuminated vectors are given by (4.3) for g significant (right) singular vector of $\Lambda_{\rm loc}$. The minimization procedure follows the one with complete measurements. However, the resolution in reconstructing h is not uniform. The 'illuminated region' would be better reconstructed than the non-illuminated one.

6. Numerical results

We present several examples of the interface reconstruction. For computations, the background domain Ω is assumed to be the unit disk centered at the origin, and the inclusion D is a disk centered at $(0, 0.1)$ with the radius 0.4. The Lamé constants of $\Omega \setminus \overline{D_\epsilon}$ and D_ϵ are given by $(\lambda_0, \mu_0) = (1, 1)$ and $(\lambda_1, \mu_1) = (1.5, 2)$, respectively.

We represent the perturbation function h as

$$h = \sum_{p=0}^{18} a_p \Phi(\theta),$$

where

$$\Phi_0(\theta) = 1, \Phi_{2p-1}(\theta) = \cos p\theta,\ \Phi_{2p}(\theta) = \sin p\theta, \quad p = 1, \ldots, 9. \tag{6.1}$$

We use the first eigenvalue and the corresponding (two) eigenfunctions of D and D_ϵ, which are denoted by $u_{0,j}$ and $u_{\epsilon,j}$ $(j = 1, 2)$, respectively. The eigenvalue, eigenfunctions, and $w_{g_{il}}$ in the following are simulated using the PDE Toolbox of MATLAB. Numerical computation reveals that the first eigenvalue has multiplicity two, which may be two very close simple eigenvalues. Even though the theory developed in previous sections is for simple eigenvalues, this does not cause any trouble. We simply superpose the algebraic systems to minimize the functional (4.1) (see below).

For the test function w_g, which is a solution to (3.1), we use

$$g_{il} = (c_{il}, d_{il}) + \begin{cases} (\cos l\theta, 0) & \text{for } i = 1, \\ (0, \cos l\theta) & \text{for } i = 2, \\ (\sin l\theta, 0) & \text{for } i = 3, \\ (0, \sin l\theta) & \text{for } i = 4, \end{cases} \qquad l = 1, \ldots, L(= 5), \tag{6.2}$$

and corresponding solutions are denoted by $w_{g_{il}}$. They are such that $\int_\Omega w_{g_{il}} \cdot u_{0,j} \neq 0$. Moreover, the constants (c_{il}, d_{il}) are chosen to fulfil the orthogonality conditions

$$\int_{\partial\Omega} g_{il} \cdot (\mathbb{C}_D \widehat{\nabla} u_{0,j})\nu = 0, \quad j = 1, 2.$$

In order to minimize the functional (4.1), we construct a 40×19 matrix M as

$$M\Big(20(j-1) + 4(l-1) + i, p\Big) = \epsilon \int_{\partial D} \Phi_p(x) \mathcal{M}[\widehat{\nabla} u^e_{0,j}](x) : \widehat{\nabla} w^e_{g_{il}}(x) d\sigma(x),$$

where $1 \le j \le 2$, $1 \le l \le 5$, $1 \le i \le 4$, and $0 \le p \le 18$. The measurements vector B is 40-dimensional vector given by

$$B(20(j-1)+4(l-1)+i) = \int_{\partial\Omega} g_{il} \cdot \mathbb{C}_0(\widehat{\nabla} u_{\epsilon,j} - \widehat{\nabla} u_{0,j})\nu + (\omega_0^2 - \omega_\epsilon^2) \int_\Omega w_{g_{il}} \cdot u_{0,j} \, .$$

We then compute the coefficients a_p's of h using the formula

$$(a_0, \ldots, a_{18}) = \big(M^T M + \delta I_{19}\big)^{-1} M^T B, \tag{6.3}$$

where I_{19} is the 19×19 identity matrix and δ is the regularization parameter.

Example 1. In this example, $h(\theta) = 1 + 2\cos p\theta$, $p = 0, 3, 6, 9$, and $\epsilon = 0.03$. Here and in the examples that follow, we assume that ϵ is known and reconstruct h. The regularization parameter δ is set to be $10^{-3}, 10^{-3}, 10^{-5}, 2 \cdot 10^{-6}$ for each $p = 0, 3, 6, 9$. Figure 1 shows results of reconstruction with well chosen δ. It shows that the reconstruction algorithm works pretty well if the perturbation h is not highly oscillating. Even when h is highly oscillating, the reconstructed interface $\partial\widetilde{D}_\epsilon$ reveals general information of the shape of the interface. Table 1 shows the ratio of symmetric differences $|\widetilde{D}_\epsilon \triangle D|$ and $|D_\epsilon \triangle D|$ for $\epsilon = 0.02, 0.03, 0.04$ with various regularization parameters δ, where $\widetilde{D}$ is the reconstructed inclusion. It shows that the ratio is close to 1 for well-chosen δ.

The next example is to show the result of minimizing the functional (4.4) using the optimally illuminated vectors. To compute the significant eigenvalues and eigenvectors, we use the basis given in (6.2). To make the index simpler, we denote

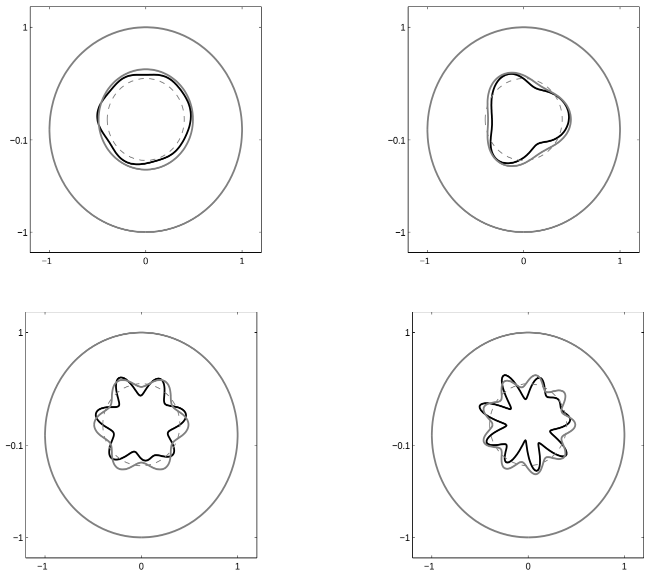

FIGURE 1. The solid grey curves represent the interfaces, which are perturbations of disks, given by the dashed grey curves. The perturbation is given by ϵh where $\epsilon = 0.03$. The black curves are the reconstructed interfaces.

g_{il} as g_p, $p = 1, \dots, 20$. For $j = 1, 2$, let Λ_j be the operator defined in (4.2) using $u_{0,j}$, which is one of two eigenfunctions corresponding the first eigenvalue, and let

$$\Lambda_j^* \Lambda_j(g_p) = \sum_{l=1}^{20} d_{pq}^{(j)} g_q \quad \text{for } p = 1, \dots, 20,$$

We then compute $(d_{pq}^{(j)})$ by solving the matrix equation

$$\Big(\int_{\partial D} \Lambda_j^*(g_p) \Lambda_j(g_q) d\sigma \Big) = (d_{pq}^{(j)}) \cdot \Big(\int_{\partial \Omega} g_p g_q d\sigma \Big). \tag{6.4}$$

It turns out that, for each $j = 1, 2$, $(d_{pq}^{(j)})$ has six significant eigenvalues counting multiplicities as shown in Figure 2.

Let $c^{(j,i)} = (c_p^{(j,i)})_{p=1}^{20}$, $i = 1, \dots, 6$, be significant eigenvectors of $(d_{pq}^{(j)})$, and define

$$\phi_i^{(j)} = \sum_{l=1}^{20} c_p^{(j,i)} g_p(x), \quad j = 1, 2, \ i = 1, \dots, 6.$$

We note that $\phi_i^{(j)}$, $i = 1, \dots, 6$, are significant eigenvectors of $\Lambda_j^* \Lambda_j$, $j = 1, 2$.

In example 2, we look for h as a linear combination of $\Lambda_j(\phi_i^{(j)})$, $j = 1, 2$, $i = 1, \dots, 6$.

Example 2 [Minimization using significant eigenvectors] . In this example, we look for h as the linear combination of $\Lambda_j(\phi_i^{(j)})$, $j = 1, 2$, $1 \le i \le 6$. The actual

p	δ	$\frac{\lvert\widetilde{D}_\epsilon \triangle D\rvert}{\lvert D_\epsilon \triangle D\rvert}$		
		$\epsilon = 0.02$	$\epsilon = 0.03$	$\epsilon = 0.04$
0	10^{-2}	0.8835	0.8411	0.8127
	10^{-3}	0.5622	0.4130	0.3447
	10^{-4}	0.4527	0.5210	0.6647
	10^{-5}	0.8558	1.1803	1.4565
3	10^{-2}	0.7667	0.7244	0.7821
	10^{-3}	0.6484	0.7769	1.0457
	10^{-4}	0.6371	0.8967	1.3637
	10^{-5}	1.1516	1.6356	2.2430
6	10^{-2}	0.9977	1.0196	1.0577
	10^{-3}	0.9950	1.1380	1.4119
	10^{-4}	0.9137	1.1642	1.6217
	10^{-5}	1.0286	1.3878	1.9081
9	10^{-2}	1.0103	1.0419	1.0928
	10^{-3}	1.0741	1.2865	1.6192
	10^{-4}	1.1330	1.4803	1.9743
	10^{-5}	1.1339	1.5083	1.9957

TABLE 1. For $h(\theta) = 1 + 2\cos p\theta$, $p = 0, 3, 6, 9$, the area difference ratio $\frac{\lvert\widetilde{D}_\epsilon \triangle D\rvert}{\lvert D_\epsilon \triangle D\rvert}$ is presented, where $\widetilde{D}_\epsilon$ is the reconstructed inclusion.

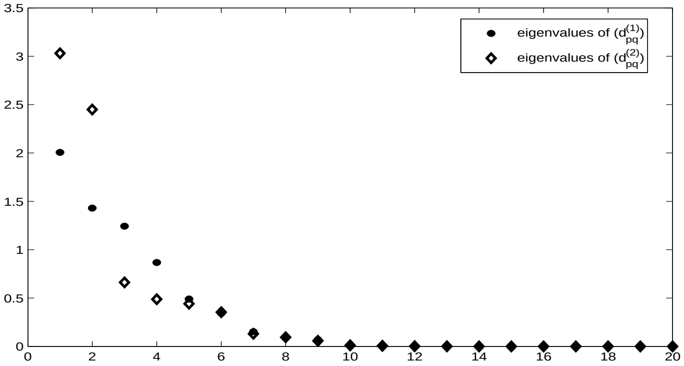

FIGURE 2. Significant eigenvalues of $\Lambda_j^* \Lambda_j$, $j = 1, 2$. There are 6 such eigenvalues.

perturbation is given by $h = \Lambda_1(\phi_3^{(1)})$ and $h = 2\Lambda_1(\phi_2^{(1)}) - \Lambda_2(\phi_1^{(2)})$. The example in Figure 3 shows the reconstruction of the inclusion. It shows that the minimization using the optimally illuminated vectors is as effective as that using (4.1) or (6.3) (see also Example 4). We emphasize that in this reconstruction h is represented using only 12 basis functions $\Lambda_j(\phi_i^{(j)})$, while in the previous reconstruction 19 functions

(Φ_p) are used. Moreover, representing h in terms of the optimally illuminated vectors avoids to compute a basis for functions defined on the boundary of the unperturbed inclusion.

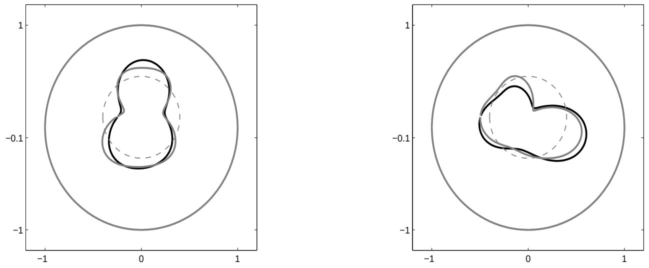

FIGURE 3. Reconstruction in the case where h is expressed in terms of the significant eigenvectors of $\Lambda_j^*\Lambda_j$, $j = 1, 2$.

Example 3 [Incomplete measurements]. In this example, we use the data only measured on the part of $\partial\Omega$, that is $\{e^{i\theta} : \theta \in [0, \pi]\}$. We look for h as the linear combination of $\Lambda_j(\phi_i^{(j)})$, $j = 1, 2$, $1 \leq i \leq 6$. Here the domain of Λ_j is restricted to the functions supported on $\{e^{i\theta} : \theta \in [0, \pi]\}$. The example in Figure 4 shows the reconstruction of the inclusion, which is given by $h = \Lambda_1(\phi_3^{(1)})$ and $h = 2\Lambda_1(\phi_2^{(1)}) - \Lambda_2(\phi_1^{(2)})$. Even with incomplete data the reconstructions are pretty accurate. See the next example for reconstruction of more general shapes.

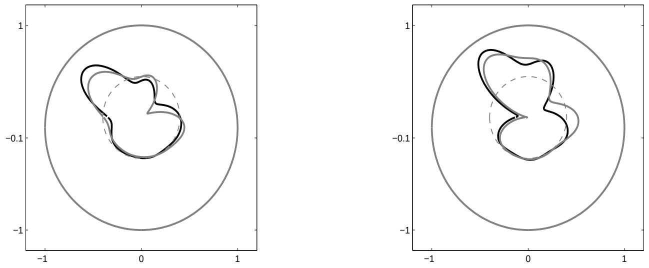

FIGURE 4. Reconstruction from incomplete measurements.

Example 4. Figure 5 shows the reconstruction of an inclusion which is given by $\epsilon h = 0.04(1 + 2\cos 3\theta)$ (the first row), shifted to the top by 0.2 (the second row), and an ellipse (the third row). The left column is the results obtained using (6.3), the middle one by using significant eigenfunctions of $\Lambda_j^*\Lambda_j$, $j = 1, 2$, and the right column is obtained using the incomplete measurements on $\{e^{i\theta} : \theta \in [0, \pi]\}$. In this example, the left and middle column give similar results, and the reconstructed images are very close to the real ones. The incomplete measurement gives worse images, but upper part which is the illuminated region is better reconstructed.

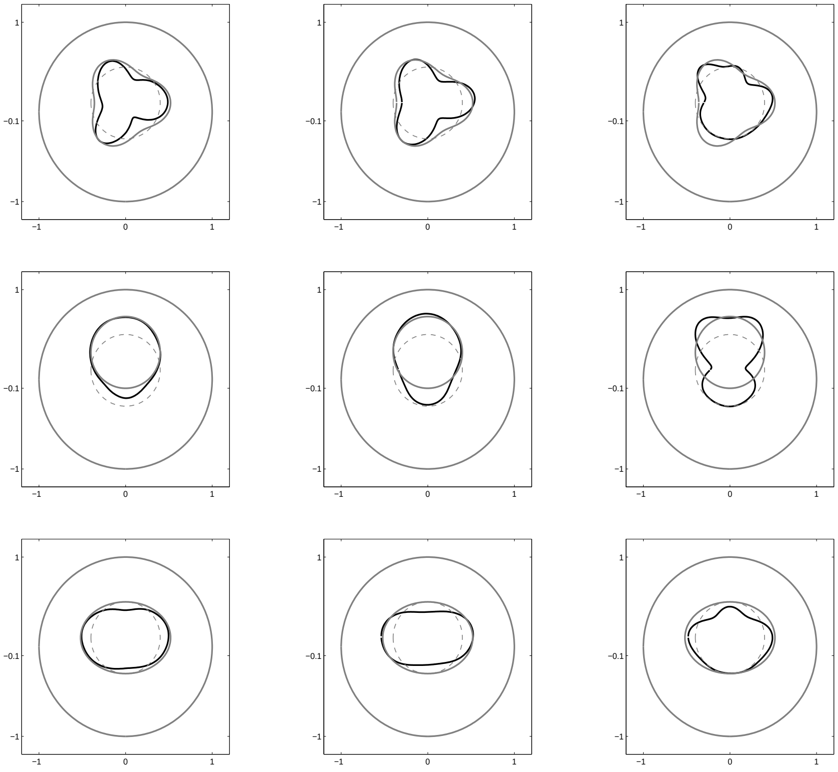

FIGURE 5. The left column is obtained using (6.3), the middle one by using the significant eigenfunctions of $\Lambda_j^*\Lambda_j$, $j = 1, 2$. In the right column we use incomplete measurements.

7. CONCLUSION

In this paper we have first derived the leading-order term in the asymptotic formula for the eigenvalue perturbation due to small changes of the interface in an elastic body. The derivation is rigorous and based on fine estimates of the gradient of the solution to the transmission problem of the Lamé system. We then derived a dual asymptotic formula for the eigenvalue perturbation. We have also considered an optimal way of representing the interface perturbation using optimally illuminated vectors. Our representation is optimal: following [AGJK] one can easily prove that one has uniqueness and Lipschitz stability for the reconstruction of the changes spanned by the optimally illuminated vectors. Based on the dual asymptotic formula, we have proposed optimization approaches for reconstructing the interface changes from either complete or incomplete data. We have performed numerical experiments to test the viability of the proposed algorithms. The presented results clearly exhibit their effectiveness.

Appendix A. Useful estimates

We state without proof a generalization of Meyer's theorem concerning the regularity of solutions to systems with bounded coefficients. For $\eta > 0$, define $H^{1,2+\eta}(\Omega)$ by

$$H^{1,2+\eta}(\Omega) := \left\{ u \in L^{2+\eta}(\Omega), \nabla u \in L^{2+\eta}(\Omega) \right\}$$

and let $H^{-1,2+\eta}(\Omega)$ be its dual. Introduce

$$H^{1,2+\eta}_{\mathrm{loc}}(\Omega) := \left\{ u \in H^{1,2+\eta}(K) \ \forall K \subset\subset \Omega \right\}.$$

Theorem A.1. *There exists $\eta > 0$ such that if $u \in H^1(\Omega)$ is solution to*

$$\nabla \cdot \left(\mathbb{C} \widehat{\nabla} u \right) = f \quad in \quad \Omega,$$

where $\mathbb{C} \in L^\infty(\Omega)$ is a strongly convex tensor and $f \in H^{-1,2+\eta}(\Omega)$ then $u \in H^{1,2+\eta}_{loc}(\Omega)$ and for any two disks $B_\rho \subset B_{2\rho} \subset \Omega$

$$\|\nabla u\|_{L^{2+\eta}(B_\rho)} \leq C(\|f\|_{H^{-1,2+\eta}(B_{2\rho})} + \rho^{\frac{2}{2+\eta}} \|\nabla u\|_{L^2(B_{2\rho})}).$$

The above theorem has been proved by Campanato in [C] in the case of strongly elliptic systems but it is possible to extend it to more general systems. See [LN]. In [BFM] a detailed proof of Theorem A.1 is given, which extends the proof contained in [C] to strongly convex systems.

Proof of Lemma 2.2. We have

$$\int_\Omega \mathbb{C} \widehat{\nabla} \varphi : \widehat{\nabla} \varphi = \int_\Omega \chi_\omega F : \widehat{\nabla} \varphi.$$

Hence by the Cauchy–Schwarz inequality and Korn's inequality we immediately get

$$\|\nabla \varphi\|_{L^2(\Omega)} \leq \|F\|_{L^\infty(\omega)} |\omega|^{1/2}$$

and therefore,

$$\|\varphi\|_{H^1(\Omega)} \leq \|F\|_{L^\infty(\omega)} |\omega|^{1/2}.$$

Let ψ be the unique solution to

$$\text{(A.1)} \qquad \left\{ \begin{array}{rcll} \nabla \cdot \left(\mathbb{C} \widehat{\nabla} \psi \right) & = & \varphi & \text{in} \quad \Omega, \\ \psi & = & 0 & \text{on} \quad \partial\Omega. \end{array} \right.$$

We have

$$\text{(A.2)} \qquad \|\nabla \psi\|_{L^2(\Omega)} \leq \|\varphi\|_{H^1(\Omega)}.$$

By Theorem A.1, since $\varphi \in H^1(\Omega)$ there exists $\eta > 0$ such that

$$\|\nabla \psi\|_{L^{2+\eta}(\omega)} \leq C(\|\nabla \psi\|_{L^2(\omega')} + \|\varphi\|_{L^{2+\eta}(\omega')}),$$

where $\omega \subset \omega' \subset \Omega$. Finally, inserting (A.2) into the last inequality and using Sobolev immersion theorem we readily get

$$\|\nabla \psi\|_{L^{2+\eta}(\omega)} \leq C \|\varphi\|_{L^{2+\eta}(\Omega)}.$$

By the Gagliardo-Nirenberg inequality, we have that

$$\|\varphi\|_{L^{2+\eta}(\Omega)} \leq C \|\nabla \varphi\|^{1-\alpha}_{L^2(\Omega)} \|\varphi\|^{\alpha}_{L^2(\Omega)}$$

with $\alpha = \frac{\eta}{\eta+2}$. Hence

$$\|\varphi\|_{L^{2+\eta}(\Omega)} \leq C|\omega|^{\frac{1}{\eta+2}} \|\varphi\|_{L^2(\Omega)}^{\frac{\eta}{\eta+2}}.$$

Multiplying the equation for ψ by φ, integrating by parts and applying Hölder's inequality, we obtain

$$\int_\Omega \varphi^2 dx = -\int_\Omega \mathbb{C}\widehat{\nabla}\varphi \cdot \widehat{\nabla}\psi = \int_\Omega \chi_\omega F \cdot \widehat{\nabla}\psi$$

and consequently,

$$\begin{aligned}\int_\Omega \varphi^2 dx &\leq \|F\|_{L^\infty(\omega)} \|\nabla\psi\|_{L^{2+\eta}(\omega)} |\omega|^{\frac{\eta+1}{\eta+2}} \\ &\leq C|\omega| \|\varphi\|_{L^2(\Omega)}^{\frac{\eta}{\eta+2}}.\end{aligned}$$

Hence, we get

$$\|\varphi\|_{L^2(\Omega)} \leq C|\omega|^{\frac{\eta+2}{\eta+4}},$$

which shows that

$$\|\varphi\|_{L^2(\Omega)} \leq C|\omega|^{1/2+\gamma},$$

where $\gamma = \frac{\eta}{2(\eta+4)}$. This completes the proof. $\square$.

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