BUSEMANN FUNCTIONS AND JULIA-WOLFF-CARATHÉODORY THEOREM FOR POLYDISCS

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Abstract. The classical Julia-Wolff-Carathéodory Theorem is one of the main tools to study the boundary behavior of holomorphic self-maps of the unit disc of \( \mathbb{C} \). In this paper we prove a Julia-Wolff-Carathéodory’s type theorem in the case of the polydisc of \( \mathbb{C}^n \). The Busemann functions are used to define a class of “generalized horospheres” for the polydisc and to extend the notion of non-tangential limit. With these new tools we give a generalization of the classical Julia’s Lemma and of the Lindelöf Theorem, which the new Julia-Wolff-Carathéodory Theorem relies upon.

1. Introduction

The Julia-Wolff-Carathéodory Theorem and its variants are powerful tools for investigating the boundary behavior of holomorphic self-maps of the unit disc \( \Delta \subset \mathbb{C} \) (see, e.g., [3], [8], [10], [21], [30], [31]). The importance of this classical theorem (JWC’s Theorem for short) in different contexts such as the study of dynamics, extension of biholomorphisms, composition operators, semigroups of holomorphic maps, is well known and justifies several generalizations to higher and infinite dimensions due to various authors. We cite here Rudin [32] for the case of the unit ball in \( \mathbb{C}^n \), Abate ([4], [5]) for strongly convex and strongly pseudoconvex domains, and Reich and Shoikhet [28] for the infinite dimensional case. The argument used to prove the classical JWC’s Theorem inspires its generalizations: let \( f \) be a holomorphic self-map of a domain \( D \) and \( p \in \partial D \). If the distance of \( f(z) \) from the boundary of the domain, \( \text{dist}(f(z), \partial D) \), is “comparable” to \( \text{dist}(z, \partial D) \) as \( z \to p \) (no matter along which direction), then \( f \) together with its normal derivatives has a limit at \( p \) along some admissible directions. We point out that to find the class of these admissible directions is one of the main efforts to state and prove all JWC’s-type theorems.

In this paper we prove a JWC’s Theorem in the case of the polydisc \( \Delta^n \) of \( \mathbb{C}^n \) which generalizes the one obtained by Abate [2] (see also Jafari [20]). In order to achieve this result we first prove generalizations of Julia’s Lemma and the Lindelöf Theorem. Our main issue is the use of Busemann sublevel sets [6] for the polydisc as “generalized horospheres” in Julia’s Lemma. The Busemann sublevel sets are also used to define the analogous of the Koranyi regions. In contrast with what happens for the existing generalizations of the JWC’s Theorem, in our statement
the class of admissible directions at a point, \( p \), of the boundary of the polydisc depends upon an entire family of complex geodesics \([3]\) “passing through” \( p \).

In order to avoid technical complications and give a more geometric approach, we deal only with the bidisc \( \Delta^2 \) and use the following terminology. The symbol \( K_{\Delta^2} \) will denote the Kobayashi distance on \( \Delta^2 \) (see \([3]\) pag. 194). A map \( \Psi \in \text{Hol}(\Delta, \Delta^2) \) is called a complex geodesic if it is an isometry between the Poincaré distance \( \omega \) of \( \Delta \) and the Kobayashi distance of \( \Delta^2 \). It is well known that if \( \Psi \) is a complex geodesic, then at least one component of \( \Psi \) is an automorphism of \( \Delta \) (see \([3]\) Proposition 2.6.10)). Thus, up to re-parametrization, we can assume that \( \Psi \) is complex geodesic, then at least one component of \( \Psi \) is an automorphism of \( \Delta \) (see \([3]\) Proposition 2.6.10)).

Thus, \( \Delta \) centered at \( R > 0 \) given by \( \Delta(\Omega, z, g) \) and \( \Delta(\Omega, z, h) \) for some \( g, h \in \text{Hol}(\Delta, \Delta) \). In what follows we will always consider complex geodesics \( \Psi \) of type \( \Delta \supset z \rightarrow (z, g(z)) \) and \( \Delta \supset z \rightarrow (h(z), z) \), for some \( g, h \in \text{Hol}(\Delta, \Delta) \). We will say that \( \Psi \) is a complex geodesic passing through \( y = (y_1, y_2) \in \partial \Delta \times \Delta \) if the radial limit of \( g \) at \( y_1 \) is \( y_2 \). In this case we will assume that \( |y_1| = 1 \). Notice that we will distinguish the key cases in which the boundary dilation coefficient \( \lambda_y(y_1) := \lambda_y \) of \( g \) at \( y_1 \) (see section 2) is finite (and hence \( g \) is ”regular” at \( y_1 \)) thanks to the classical JWC theorem) from the cases in which \( \lambda_y \) is infinite (see e.g. Remark 7.)

Since the function \( [K_{\Delta^2}(x, \Psi(r)) - K_{\Delta^2}(\Psi(0), \Psi(r))] \) is not increasing in \( r \) and bounded from below, the Busseman function \( B_{\Psi}(x) \) associated to the geodesic \( \Psi \), passing through \( y = (y_1, y_2) \in \partial \Delta^2 \) can be defined by

\[
B_{\Psi}(x) := \lim_{r \to 1} [K_{\Delta^2}(x, \Psi(ry_1)) - K_{\Delta^2}(\Psi(0), \Psi(ry_1))]
\]

(see e.g. \([6]\) pag. 23). The Busseman sublevel set of center \( y \in \partial \Delta^2 \) and radius \( R > 0 \) of the function \( B_{\Psi}(x) \) will be the set

\[
B_{\Psi}(y, R) := \{ x \in \Delta^2 : B_{\Psi}(x) \leq \frac{1}{2} \log R \}.
\]

To begin with we study the geometry of the sets \( B_{\Psi}(y, R) \) (see Proposition 6). It turns out that for any point \( y = (y_1, y_2) \in (\partial \Delta) \times (\partial \Delta) \) on the Silov boundary of the bidisc, given any \( R > 0 \) and given any complex geodesic \( \Psi \) passing through \( y \), we have a continuous family of Busseman sublevel sets of radius \( R \) of the form

\[
E_\Delta(y_1, R) \times E_\Delta(y_2, S)
\]

where \( E_\Delta(y_1, R) := \{ z \in \Delta : \lim_{w \to y_1} [\omega(z, w) - \omega(0, w)] < \frac{1}{2} \log R \} \) is a horocycle centered at \( y_1 \) of radius \( R \) and \( S = \lambda_y(y_1)R \). Notice that every product of horocycles can be seen as a Busseman sublevel set in at least two different ways:

\[
E_\Delta(y_1, R) \times E_\Delta(y_2, S) = B_{\varphi_{\Delta}}(y, R)
\]

with \( \varphi_{\Delta}(z) = (z, g(z)) \) and \( \lambda_y(y_1) = \frac{S}{R} \) and

\[
E_\Delta(y_1, R) \times E_\Delta(y_2, S) = B_{\varphi_{\Delta}}(y, S)
\]

with \( \varphi_{\Delta}(z) = (h(z), z) \) and \( \lambda_h(y_2) = \frac{R}{S} \). From now on we will simply write

\[
B(y, R) = E_\Delta(y_1, R) \times E_\Delta(y_2, R)
\]

to denote the Busseman sublevel set associated to a complex geodesic \( \varphi_{\Delta} \) passing through \( y \) such that \( \lambda_y = 1 \). Instead, for the points on the flat components of the boundary of \( \Delta^2 \) there is only one Busseman sublevel set of a given radius \( R > 0 \) of the form

\[
\Delta \times E_\Delta(y_2, S) \text{ if } (y_1, y_2) \in \Delta \times \partial \Delta
\]
Let the curve

\[ E_\Delta(y_1, R) \times \Delta \] if \((y_1, y_2) \in \partial \Delta \times \Delta.\]

In the sequel, we will denote by \( B_{(\lambda_1, \lambda_2)}(y, R) \) (with \( \lambda_1, \lambda_2 > 0 \), possibly +\( \infty \)) the Busemann sublevel set given by \( E_\Delta(y_1, \lambda_1 R) \times E_\Delta(y_2, \lambda_2 R) \), with the convention that \( E_\Delta(y_1, \lambda_1 R) = \Delta \) if either \( y_1 \in \Delta \) or \( y_1 \in \partial \Delta \) and \( \lambda_1 = +\infty \).

Our first result is the following version of Julia's lemma:

**Theorem 1.** Let \( f = (f_1, f_2) \in \text{Hol}(\Delta^2, \Delta^2) \). Let \( x = (x_1, x_2) \in \partial (\Delta \times \Delta) = \partial \Delta^2 \) and let \( \varphi_g(z) = (z, g(z)) \) be a complex geodesic passing through \( x \), assuming \(|x_1| = 1 \).

Set

\[ \frac{1}{2} \log \lambda_j := \lim_{t \to 1^-} [K_{\Delta^2}(0, \varphi_g(tx_1)) - \omega(0, f_j(\varphi_g(tx_1)))] \text{ for } j = 1, 2 \]

Suppose that both \( \lambda_1 < \infty \) and \( \lambda_2 < \infty \). Then there exists a point \( y = (y_1, y_2) \in \partial \Delta^2 \) such that for all \( R > 0 \)

\[ f(B_{(\lambda_1, \lambda_2)}(x, R)) \subseteq B_{(\lambda_1, \lambda_2)}(y, R). \]

A second achievement is a generalization of the Lindelöf Theorem which is based on the definition of admissible limits. Let \( x \in \partial \Delta^2 \). A continuous curve \( \sigma(t) \subset \Delta^2 \) converging to \( x \) as \( t \to 1^- \) is called a \( x \)-curve. Let \( \varphi_g : \Delta \to \Delta^2 \) be a complex geodesic passing through \( z \) and parameterized by \( z \mapsto (z, g(z)) \) with \( g \in \text{Hol}(\Delta, \Delta) \). A holomorphic function \( \tilde{\pi}_g : \Delta^2 \to \Delta \) such that: \( \tilde{\pi}_g \circ \varphi_g = \text{id}_\Delta \) is called a \( g \)-left inverse of \( \varphi_g \). The composition \( \pi_g := \varphi_g \circ \tilde{\pi}_g : \Delta^2 \to \Delta^2, \) (such that \( \pi_g \circ \varphi_g = \varphi_g, \) and \( \tilde{\pi}_g \circ \varphi_g = \tilde{\pi}_g \)) is called a \( g \)-holomorphic retraction. The pair \( (\varphi_g, \pi_g) \) is a \( g \)-projection device. Existence of \( g \)-projection devices, also known as Lempert’s projection devices, in convex domains is established in [29] (see also [27], [3]). In strongly convex domains Lempert’s projection devices are essentially unique (see [7]) while in the bidisc various holomorphic retractions with different “fibers” may correspond to a given complex geodesic. The following definitions will be used in the statements of the generalizations of the Lindelöf Theorem and JWC’s Theorem.

**Definition 2.** Let \( x \in \partial \Delta^2 \) and \( M > 1 \). The g-Koranyi region \( H_{\varphi_g}(x, M) \), of vertex \( x \) and amplitude \( M \) is:

\[ H_{\varphi_g}(x, M) := \{ z \in \Delta^2 : \lim_{r \to 1^-} [K_{\Delta^2}(z, \varphi_g(r)) - K_{\Delta^2}(\varphi_g(0), \varphi_g(r))] + K_{\Delta^2}(\varphi_g(0), z) < \log M \}. \]

A holomorphic function \( f \in \text{Hol}(\Delta^2, \Delta) \) has \( K_g \)-limit \( L \in \mathbb{C} \) at \( x \) if \( f(z) \to L \) as \( z \to x \) inside any g-Koranyi region.

The function \( f \) is \( K_g \)-bounded if for every \( M \) there exists a constant \( C_M > 0 \) such that \(|f(z)| < C_M \) for all \( z \in H_{\varphi_g}(x, M) \).

**Definition 3.** Let \( \sigma(t) \subset \Delta^2 \) be a \( x \)-curve.

- the curve \( \sigma(t) \) is \((g, \pi_g)\)-special if \( K_{\Delta^2}(\sigma(t), \pi_g(\sigma(t))) \to 0 \) as \( t \to 1^- \).
- the curve \( \sigma(t) \) is \((g, \pi_g)\)-restricted if \( \tilde{\pi}_g(\sigma(t)) \to \tilde{\pi}_g(x) \) non-tangentially as \( t \to 1^- \).

Moreover, if \( h : \Delta^2 \to \mathbb{C} \) is holomorphic we say that \( h \) has \( K_g \)-limit equal to \( L \in \mathbb{C} \) if \( h \) has limit \( L \) along any curve which is \((g, \pi_g)\)-special and \((g, \pi_g)\)-restricted, and we write

\[ K_{g \rightarrow 2 \rightarrow} h(z) = L. \]
Let given by \( \tilde{\phi} \) and \( y = (f, g, \pi) \) a projection device with \( \pi g(z) = (z_1, g(z_1)) \). If \( \sigma_0 \) is a \((g, \pi g)\)-special and \((g, \pi g)\)-restricted \( x \)-curve such that
\[
\lim_{t \to 1^-} f(\sigma_0(t)) = L
\]
then \( f \) admits restricted \( K_\gamma \)-limit equal to \( L \) at \( x \).

The above result plays a key role in the proof of our main result:

**Theorem 4.** Let \( f \in \text{Hol}(\Delta^2, \Delta) \) be a holomorphic function. Given \( x = (x_1, x_2) \in \partial \Delta^2 \), let \( \varphi_g(z) = (z, g(z)) \) be a complex geodesic passing through \( x \), assuming \( |x_1| = 1 \). Let \((g, \pi g)\) a projection device with \( \pi g(z) = (z_1, g(z_1)) \). If \( \sigma_0 \) is a \((g, \pi g)\)-special and \((g, \pi g)\)-restricted \( x \)-curve such that
\[
\lim_{t \to 1^-} f(\sigma_0(t)) = L
\]
then \( f \) admits restricted \( K_\gamma \)-limit equal to \( L \) at \( x \).

The paper is organized as follows: in Section 2 we study in detail the geometry the Busemann sublevel sets. In Section 3 we discuss of special and restricted curves. In Section 4 we introduce a new extension of the notion of non-tangential limits and in Section 5 we prove a new version of the Lindelöf Theorem. In Section 6 we give our extension of the classical Julia’s Lemma. In Section 7 we prove our generalization of the Julia-Wolff-Carathéodory Theorem. We end the paper in Section 8 with an application of our results to the study of the dynamics of fixed points free holomorphic self-maps of the bidisc. In fact in this section we give a geometrical interpretation of a result due to Hervé [19] in terms of the set of generalized Wolff points of a fixed point free \( f \in \text{Hol}(\Delta^2, \Delta^2) \).

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## 2. Busemann Functions and a Family of Horospheres

The aim of this section is to study in detail the Busemann sublevel sets and their relations with the horospheres of the polydisc. To begin with we fix the notation. Let \( \varphi_h \in \text{Hol}(\Delta, \Delta^2) \) be a complex geodesic, passing through a point \( y = (y_1, y_2) \in \partial \Delta^2 \), parameterized as \( z \to (\theta(z), h(z)) \), where \( \theta \) is an automorphism.
of $\Delta$, $\theta \in \text{Aut}(\Delta)$, and $h \in \text{Hol}(\Delta, \Delta)$. Let us denote by $\lambda_h$ the boundary dilation coefficient of $h$ at $y_1$, which is

$$\lambda_h := \liminf_{z \to y_1} \frac{1 - |h(z)|}{1 - |z|}.$$  

This coefficient plays a key role in the study of the boundary behavior of holomorphic functions in the unit disc, $\Delta$ of $\mathbb{C}$. By the classical JWC Theorem, if $\lambda_h < \infty$ then $h(z) \to y_2$ non-tangentially as $z \to y_1$. It means that $h(z)$ converges to $y_2$ as $z$ tends to $y_1$ within any angular region $H_{\Delta}(y_1, M)$, of vertex $y_1$ and amplitude $M > 1$, called Stolz region and defined as follows:

$$H_{\Delta}(y_1, M) := \left\{ z \in \Delta : \frac{|y_1 - z|}{1 - |z|} < M \right\}.$$  

Let $E(y, R)$ be the small horosphere of center $y \in \Delta^n$ and radius $R > 0$ given by

$$E(y, R) = \left\{ z \in \Delta^n : \limsup_{w \to y} [K_{\Delta^n}(z, w) - K_{\Delta^n}(0, w)] < \frac{1}{2} \log R \right\},$$  

and let $F(y, R)$ be the big horosphere of center $y$ and radius $R$ given by

$$F(y, R) = \left\{ z \in \Delta^n : \liminf_{w \to y} [K_{\Delta^n}(z, w) - K_{\Delta^n}(0, w)] < \frac{1}{2} \log R \right\}.$$

In strongly convex domains of $\mathbb{C}^n$ the definition of Busemann sublevel set is equivalent to the definition of horosphere, but in the bidisc this is no longer true. In particular if $n = 1$ then $F(y, R) \equiv E(y, R) \equiv E_{\Delta}(y, R) \subset \Delta$, the horocycle centered in $y$ with radius $R$. If $n > 1$ then the following proposition holds:

**Proposition 6.** Let $\varphi_h(z) = (\theta(z), h(z))$ be a complex geodesic in $\Delta^2$ passing through a point $y = (y_1, y_2) \in \partial \Delta^2$, where $\theta \in \text{Aut}(\Delta)$ and $h \in \text{Hol}(\Delta, \Delta)$. If $y \in (\partial \Delta)^2$ and the boundary dilation coefficient of the map $h$ at $y_1$, $\lambda_h$, is finite then

$$B^{\varphi_h}(y, R) = E_{\Delta}(y_1, \lambda_h R) \times E_{\Delta}(y_2, \lambda_h R)$$  

where $\lambda_0$ is the boundary dilation coefficient of $\theta$ at $y_1$. If $y \in [\partial \Delta^2 \setminus (\partial \Delta)^2]$ then

$$B^{\varphi_h}(y, R) \equiv E(y, R) \equiv F(y, R).$$  

**Proof.** If $y \in (\partial \Delta)^2$ then, up to composition with automorphisms, we can suppose $y = (1, 1)$. Let us remark that $\theta(1) = 1$. Suppose that $x \in B^{\varphi_h}((1, 1), R)$, then, by definition of Busemann sublevel sets, $\lim_{r \to 1} [\max_{\omega \in \partial \Delta} \omega(x_1, \theta(r)), \omega(x_2, h(r))] - \omega(0, r)] \leq \frac{1}{2} \log R$ and there are the two following possibilities:

- a) there exists a sequence $\{r_k\}_{k \in \mathbb{N}} \subseteq (0, 1)$ such that $r_k \to 1^-$ as $k \to \infty$ and $\max_{\omega \in \partial \Delta} \omega(x_1, \theta(r_k)), \omega(x_2, h(r_k))] = \omega(x_1, \theta(r_k)), \omega(x_2, h(r_k))]$;
- b) there exists a sequence $\{r_k\}_{k \in \mathbb{N}} \subseteq (0, 1)$ such that $r_k \to 1^-$ as $k \to \infty$ and $\max_{\omega \in \partial \Delta} \omega(x_1, \theta(r_k)), \omega(x_2, h(r_k))] = \omega(x_2, h(r_k))]$.  


In case $a$ we have that
\begin{align*}
\frac{1}{2} \log R &\geq \lim_{r \to 1} \left[ \max\{\omega(x_1, \theta(r)), \omega(x_2, h(r))\} - \omega(0, r) \right] \\
&= \lim_{k \to \infty} [\omega(x_1, \theta(r_k)) - \omega(0, r_k)] \\
&= \lim_{k \to \infty} [\omega(x_1, \theta(r_k)) - \omega(0, \theta(r_k)) + \omega(0, \theta(r_k)) - \omega(0, r_k)] \\
&\geq \lim_{r \to 1^{-}} [\omega(x_1, \theta(r)) - \omega(0, \theta(r)) + \omega(0, \theta(r)) - \omega(0, r)] \\
&= \lim_{w \to 1^{-}} [\omega(x_1, w) - \omega(0, w)] + \frac{1}{2} \log \frac{1}{\lambda_0},
\end{align*}
and it follows that $x_1 \in E_{\Delta}(1, \lambda_0 R)$. Moreover
\begin{align*}
\frac{1}{2} \log R &\geq \lim_{k \to \infty} [\omega(x_1, \theta(r_k)) - \omega(0, r_k)] \\
&\geq \lim_{k \to \infty} [\omega(x_2, h(r_k)) - \omega(0, r_k)] \\
&= \lim_{k \to \infty} [\omega(x_2, h(r_k)) - \omega(0, h(r_k))] + \lim_{k \to \infty} [\omega(0, h(r_k)) - \omega(0, r_k)] \\
&\geq \lim_{r \to 1^{-}} [\omega(x_2, h(r)) - \omega(0, h(r))] + \lim_{r \to 1^{-}} [\omega(0, h(r)) - \omega(0, r)].
\end{align*}
Let us notice that
\begin{align*}
0 < \lambda_h &:= \liminf_{w \to 1^{-}} \frac{1 - |h(w)|}{1 - |w|} \leq \lim_{r \to 1^{-}} \frac{1 - |h(r)|}{1 - r} \\
&\leq \lim_{r \to 1^{-}} \frac{|1 - h(r)|}{1 - r},
\end{align*}
and, by the classical Julia-Wolf-Carathéodory Theorem ([3]) we have that:
\begin{align*}
\lim_{r \to 1^{-}} \frac{|1 - h(r)|}{1 - r} &= \lambda_h.
\end{align*}
This implies that
\begin{align*}
\lim_{r \to 1^{-}} [\omega(x_2, h(r)) - \omega(0, h(r))] + \lim_{r \to 1^{-}} [\omega(0, h(r)) - \omega(0, r)] &= \\
\lim_{w \to 1^{-}} [\omega(x_2, w) - \omega(0, w)] + \frac{1}{2} \log \frac{1}{\lambda_0}
\end{align*}
and $x_2 \in E_{\Delta}(1, \lambda_0 R)$. Thus, in case $a$, the first inclusion is proved and $B_{\rho_h}(1, 1) \subseteq E_{\Delta}(1, \lambda_0 R) \times E_{\Delta}(1, \lambda_0 R)$.

In case $b$ we notice that
\begin{align*}
\frac{1}{2} \log R &\geq \lim_{r \to 1^{-}} \left[ \max\{\omega(x_1, \theta(r)), \omega(x_2, h(r))\} - \omega(0, r) \right] \\
&= \lim_{k \to \infty} [\omega(x_2, h(r_k)) - \omega(0, r_k)] \\
&= \lim_{r \to 1^{-}} [\omega(x_2, h(r)) - \omega(0, r_k)] \\
&\geq \lim_{k \to \infty} [\omega(x_1, \theta(r_k)) - \omega(0, r_k)],
\end{align*}
then proceeding as in case $a$ it follows that $x_1 \in E_{\Delta}(1, \lambda_0 R)$ and $x_2 \in E_{\Delta}(1, \lambda_0 R)$. We conclude that, also in this case, $B_{\rho_h}(1, 1) \subseteq E_{\Delta}(1, \lambda_0 R) \times E_{\Delta}(1, \lambda_0 R)$.

On the other hand, if a point $x \in E_{\Delta}(1, \lambda_0 R) \times E_{\Delta}(1, \lambda_0 R)$ then by definition of horocycle
\begin{align*}
\lim_{w \to 1^{-}} [\omega(x_1, w) - \omega(0, w)] &= \lim_{r \to 1^{-}} [\omega(x_1, \theta(r)) - \omega(0, \theta(r))] \leq \frac{1}{2} \log \lambda_0 R
\end{align*}
Let us suppose that

\[ \{x_2(h(r)) - \omega(0, h(r)) \leq \frac{1}{2} \log \lambda_h R. \]

Thus

\[ \lim_{r \to 1^-} [\omega(x_2, h(r)) - \omega(0, r)] = \lim_{r \to 1^-} [\omega(x_2, h(r)) - \omega(0, r) + \omega(0, h(r)) - \omega(0, r)] \]

\[ \leq \frac{1}{2} \log \lambda_h R \frac{1}{\lambda_h} = \frac{1}{2} \log R. \]

Swapping \( h \) with \( \theta \), arguing as above, we have

\[ \lim_{r \to 1^-} [\omega(x_1, \theta(r)) - \omega(0, r)] \leq \frac{1}{2} \log R. \]

We conclude that

\[ (2.1) \quad \lim_{r \to 1^-} \max \{\omega(x_1, \theta(r)), \omega(x_2, h(r))\} - \omega(0, r) \leq \frac{1}{2} \log R \]

and \( \mathbb{B}^{\phi_h}((1, 1), R) = E_\Delta(1, \lambda_0 R) \times E_\Delta(1, \lambda_0 R). \)

If \( y \) is a point of the flat component of \( \partial \Delta^2 \), then \( E(y, R) \equiv F(y, R) \) (see [3]) and therefore the limit that defines the small and big horosphere exists. Thus it immediately follows that \( \mathbb{B}^{\phi}(y, R) \equiv E(y, R) \equiv F(y, R) \) for all geodesic \( \phi \) and for all \( R > 0 \).

**Remark 7.** Let us suppose that \( \lambda_h(y_1) = +\infty \). If \( x = (x_1, x_2) \in \mathbb{B}^{\phi_h}((1, 1), R) \) then, proceeding as before, \( x_1 \in E_\Delta(1, \lambda_0 R) \) and \( x_2 \in \Delta \). It follows that \( \mathbb{B}^{\phi_h}((1, 1), R) \subseteq E_\Delta(1, \lambda_0 R) \times \Delta = E_\Delta(1, \lambda_0 R) \times E_\Delta(1, \lambda_0 R) \) with the convention (explained in the Introduction) that \( E_\Delta(1, \lambda_0 R) = \Delta \) when \( \lambda_h = \infty \). On the other hand if \( x = (x_1, x_2) \in E_\Delta(1, \lambda_0 R) \times \Delta \) then (as above) \( \lim_{r \to 1^-} [\omega(x_1, \theta(r)) - \omega(0, r)] \leq \frac{1}{2} \log R. \)

Moreover

\[ \lim_{r \to 1^-} [\omega(x_2, h(r)) - \omega(0, r)] \leq \lim_{r \to 1^-} [\omega(x_2, 0) + \omega(0, h(r)) - \omega(0, r)] \]

and since \( \lambda_h = +\infty \) then \( \lim_{r \to 1^-} [\omega(0, h(r)) - \omega(0, r)] = -\infty \). Thus

\[ \lim_{r \to 1^-} [\omega(x_2, h(r)) - \omega(0, r)] \leq \lim_{r \to 1^-} [\omega(x_2, 0) + \omega(0, h(r)) - \omega(0, r)] < \frac{1}{2} \log R \]

for every \( R > 0 \). We conclude that \( \mathbb{B}^{\phi_h}((1, 1), R) = E_\Delta(1, \lambda_0 R) \times E_\Delta(1, \lambda_0 R) = E_\Delta(1, \lambda_0 R) \times \Delta. \)

Using the same arguments of the proof of Proposition 6, it follows that if we consider the re-parametrization of \( \varphi_h(z) = (\theta(z), h(z)) \) given by \( \varphi_\theta(z) = (z, g(z)) \), with \( g := h \circ \theta^{-1} \), we have that

\[ \mathbb{B}^{\varphi_\theta}((1, 1), R) = E_\Delta(1, R) \times E_\Delta(1, \lambda_0 R), \]

where \( \lambda_0 = \frac{\lambda_0}{\lambda_0} \). From now on, we consider only parametrization of the type \( z \to (z, g(z)) \) (respectively \( z \to (g(z), z) \)) (see also the Introduction).

For later use we now compute explicitly the Busemann sublevel sets using the above notation. As Proposition 6 states, the Busemann sublevel sets centered in a flat boundary point of \( \Delta^2 \), say \( y \), coincide with the small and the big horospheres that have been already explicitly described by Abate in [2]. Therefore let us consider a point \( y = (y_1, y_2) \in (\partial \Delta)^2 \). Without loss of generality we can suppose that
The holomorphic function 

Assume first that \( z = (z_1, z_2) \in \Delta^2 \) is such that

\[
\max_{j=1,2} \left\{ \frac{|1 - z_1|^2}{(1 - |z_1|^2)} \cdot \frac{|1 - z_2|^2}{(1 - |z_2|^2)} \lambda_g \right\} \leq R.
\]

There are two possibilities:

i) \( \max_{j=1,2} \left\{ \frac{|1 - z_1|^2}{(1 - |z_1|^2)} \cdot \frac{|1 - z_2|^2}{(1 - |z_2|^2)} \frac{1}{\lambda_g} \right\} = \frac{|1 - z_1|^2}{(1 - |z_1|^2)} \frac{1}{\lambda_g} \); in which case \( \frac{|1 - z_2|^2}{(1 - |z_2|^2)} \frac{1}{\lambda_g} \leq \frac{|1 - z_1|^2}{(1 - |z_1|^2)} \frac{1}{\lambda_g} \leq R \) and, by definition of horocycles, \( z_1 \in E_\Delta(1, R) \) and \( z_2 \in E_\Delta(1, \lambda_y R) \);

ii) \( \max_{j=1,2} \left\{ \frac{|1 - z_1|^2}{(1 - |z_1|^2)} \cdot \frac{|1 - z_2|^2}{(1 - |z_2|^2)} \frac{1}{\lambda_g} \right\} = \frac{|1 - z_2|^2}{(1 - |z_2|^2)} \frac{1}{\lambda_g} \); in which case \( \frac{|1 - z_1|^2}{(1 - |z_1|^2)} \frac{1}{\lambda_g} \leq \frac{|1 - z_2|^2}{(1 - |z_2|^2)} \frac{1}{\lambda_g} \leq R \) and, by definition of horocycles, \( z_1 \in E_\Delta(1, R) \) and \( z_2 \in E_\Delta(1, \lambda_y R) \). Namely \( z \in E_\Delta(1, R) \times E_\Delta(1, \lambda_y R) \).

Conversely, let \( z = (z_1, z_2) \in E_\Delta(1, R) \times E_\Delta(1, \lambda_y R) = \mathbb{B}^\Delta((1, 1), R) \), with \( \varphi_g(z) = (z, g(z)) \). By definition of horocycles, it follows that both \( \frac{|1 - z_1|^2}{(1 - |z_1|^2)} \leq R \) and \( \frac{|1 - z_2|^2}{(1 - |z_2|^2)} \frac{1}{\lambda_g} \leq R \), hence

\[
\max_{j=1,2} \left\{ \frac{|1 - z_1|^2}{(1 - |z_1|^2)} \cdot \frac{|1 - z_2|^2}{(1 - |z_2|^2)} \frac{1}{\lambda_g} \right\} \leq R.
\]

By the very definition of sublevel sets of Busemann functions, \( z = (z_1, z_2) \in \mathbb{B}^\Delta((1, 1), R) \), proving the claim.

3. Special and restricted curves

Let \( x = (x_1, x_2) \in \partial \Delta^2 \) and \( \varphi_x : \Delta \to \Delta^2 \) be the complex geodesic passing through \( x \), defined by \( \varphi_x(z) = zx := (zx_1, zx_2) \). Let us denote by \( d_x \) the Šilov degree of \( x \), that is the number of components of \( x \) with absolute value equal to 1, and by \( \hat{x} := (\hat{x}_1, \hat{x}_2) \) the Šilov part of \( x \), defined by

\[
\hat{x}_j = \left\{ \begin{array}{ll} x_j & \text{if } |x_j| = 1, \\ 0 & \text{if } |x_j| < 1. \end{array} \right.
\]

In this setting Abate [2] gave the following definition:

**Definition 8.** The holomorphic function \( \tilde{p}_x : \Delta^2 \to \Delta \) given by

\[
\tilde{p}_x(z) := \frac{1}{d_x} \langle z, \hat{x} \rangle,
\]

such that \( \tilde{p}_x \circ \varphi_x = id_{\Delta^*} \), is a left inverse of \( \varphi_x \).
The holomorphic function \( p_x : \Delta^n \to \Delta^n \) given by \( p_x(z) := \varphi_x \circ \tilde{p}_x \), such that \( p_x \circ p_x = p_x \) and \( p_x \circ \varphi_x = \varphi_x \) is a holomorphic retraction.

A \( x \)-curve \( \sigma(t) \subset \Delta^n \) is \( A \)-special if \( K_{\Delta^2}(\sigma(t), p_x(\sigma(t))) \to 0 \), as \( t \to 1^- \).

A \( x \)-curve \( \sigma(t) \subset \Delta^n \) is \( A \)-restricted if \( p_x(\sigma(t)) \) converges to \( p_x(x) \) non-tangentially. The pair \((p_x, \varphi_x)\) is called an \( A \)-projection device.

We notice that the \( A \)-projection device due to Abate is not unique, that is, given the complex geodesic \( \varphi_x = zz \), the left inverse \( \tilde{p}_x \) and the holomorphic retraction \( p_x \) are not unique. For instance, if \( x = (1,1), \pi'(z) = (z_1, z_1) \) and \( \pi''(z) = (z_2, z_2) \) are not unique. For instance, if \( x = (1,1), \pi'(z) = (z_1, z_1) \) and \( \pi''(z) = (z_2, z_2) \) are honest holomorphic projections, as well. Moreover \( \varphi_x \) is not the unique complex geodesic passing through \( x \). Thus we are led to give the following definitions:

**Definition 9.** Let \( \varphi_x : \Delta \to \Delta^2 \) be a complex geodesic passing through \( x = (x_1, x_2) \) and parameterized by \( z \mapsto (z, g(z)) \); with \( g \in \text{Hol}(\Delta, \Delta) \) and assuming \(|x_1| = 1\).

A holomorphic function \( \pi_g : \Delta^2 \to \Delta \) such that \( \pi_g \circ \varphi_g = \text{id}_\Delta \) is called a \( g \)-left inverse function of \( \varphi_g \).

A holomorphic function \( \pi_g : \Delta^2 \to \Delta^2 \) such that \( \pi_g \circ \varphi_g = \varphi_g \), and \( \pi_g \circ \pi_g = \pi_g \) is called a \( g \)-holomorphic retraction.

The pair \((\varphi_g, \pi_g)\) is a \( g \)-projection device.

Let us recall that (see Definition 3) given a \( x \)-curve \( \sigma(t) \subset \Delta^2 \), we say that \( \sigma(t) \) is \((g, \pi_g)\)-special if \( K_{\Delta^2}(\sigma(t), \pi_g(\sigma(t))) \to 0 \) as \( t \to 1^- \), and \( \sigma(t) \) is \((g, \pi_g)\)-restricted if \( \pi_g(\sigma(t)) \to \pi_g(x) \) non-tangentially as \( t \to 1^- \).

For sake of simplicity, as a matter of notation, when we refer to the geodesic parameterized by \( \varphi(z) = (z, z) \), we omit the index \( g \), since \( g = \text{id}_\Delta \). In addition we denote by \( p \) the \( A \)-holomorphic retraction given in Definition 8.

In this setting, a \( x \)-curve \( \gamma \) is \( A \)-special and \( A \)-restricted if \( K_{\Delta^2}(\gamma(t), p(\gamma(t))) \to 0 \) as \( t \to 1^- \) and \( \tilde{p}(\gamma(t)) \) approaches the point \( \tilde{p}(x) \) non tangentially.

**Remark 10.** Let \( x = (x_1, x_2) \) be a point on a flat component of \( \partial \Delta^2 \), and let \((\varphi_g, \pi_g)\) be a \( g \)-projection device with \( \varphi_g \) a complex geodesic passing through \( x \) parameterized as \( \Delta \ni w \mapsto (w, g(w)) \) (as usually we suppose \(|x_1| = 1\) and \( \pi_g(z_1, z_2) = (z_1, g(z_1)) \)). For sake of simplicity set \( x = (1,0) \). We notice that every \( x \)-curve \( \sigma \) is \((g, \pi_g)\)-special for every \( g \in \text{Hol}(\Delta, \Delta) \). Indeed \( \sigma \) is \((g, \pi_g)\)-special if and only if \( K_{\Delta^2}(\sigma(t), \pi_g(\sigma(t))) \to 0 \) as \( t \to 1^- \) and by definition of Kobayashi distance on polydiscs it is true if and only if \( \omega(\sigma(t), g(\sigma(t))) \to 0 \) as \( t \to 1^- \).

Since \( \varphi_g \) passes through \((1,0)\) and \( \sigma \) is a \((1,0)\)-curve it follows that both \( \sigma_2(t) \) and \( g(\sigma_1(t)) \) tend to 0 as \( t \to 1^- \). Thus, actually, \( \omega(\sigma_2(t), g(\sigma_1(t))) \to 0 \) as \( t \to 1^- \) for every \( g \in \text{Hol}(\Delta, \Delta) \) and \( \sigma \) is \((g, \pi_g)\)-special for every \( g \in \text{Hol}(\Delta, \Delta) \). In particular it follows that every \( x \)-curve \( \sigma \) is \( A \)-special. Moreover it is clear by definition that \( \sigma \) is \((g, \pi_g)\)-restricted if and only if it is \( A \)-restricted. Indeed, in this setting, \( \tilde{\pi}(z_1, z_2) = z_1 = \tilde{p}(1,0)(z_1, z_2) \).

On the other hand, if \( x \in (\partial \Delta)^2 \), we have the following characterization:

**Proposition 11.** Let us denote by \( \varphi_x(z) \) the complex geodesic passing through the point \( x = (x_1, x_2) \in (\partial \Delta)^2 \) parameterized by \( z \mapsto (zx) \) and let \( \pi_g : \Delta^2 \to \varphi_x(\Delta) \) be any linear holomorphic retraction on the image of the complex geodesic \( \varphi_x \). Let \( \gamma(t) = (\gamma_1(t), \gamma_2(t)) \) be a \((g, \pi_g)\)-restricted \( x \)-curve in \( \Delta^2 \). Then \( \gamma \) is \((g, \pi_g)\)-special if and only if \( \frac{\gamma_2(t)}{\gamma_1(t)} \to 1 \) as \( t \to 1^- \).

**Proof.** Without loss of generality we suppose \( x = (1,1) \), therefore \( \varphi_x(z) = \varphi(z) = (z, z) \). Let us consider a linear projection \( \pi_g(z_1, z_2) = (az_1 + bz_2, az_1 + bz_2) \) with
By definition of holomorphic retraction we have that \( \pi_g(z, z) = (az + bz, az + bz) = (z, z) \), and \( b = 1 - a \). We know that \( \gamma \) is \( (g, \pi_g) \)-special if and only if
\[
K_{\Delta^2}(\gamma(t), \pi_g(\gamma(t))) =
\]
\[
= \max \{ \omega(\gamma_1(t), a\gamma_1(t) + (1 - a)\gamma_2(t)), \omega(\gamma_2(t), a\gamma_1(t) + (1 - a)\gamma_2(t)) \} \to 0
\]
as \( t \to 1^− \). Let us suppose, without loss of generality that,
\[
\max \{ \omega(\gamma_1(t), a\gamma_1(t) + (1 - a)\gamma_2(t)), \omega(\gamma_2(t), a\gamma_1(t) + (1 - a)\gamma_2(t)) \} = \omega(\gamma_1(t), a\gamma_1(t) + (1 - a)\gamma_2(t)).
\]
It follows that either \( a \neq 1 \) or \( \gamma_2 \equiv \gamma_1 \) (but in the second case the thesis follows at once). Moreover by the very definition of the Kobayashi distance, equation (3.1) is equivalent to
\[
\lim_{t \to 1^-} \left| \frac{\gamma_1(t) - a\gamma_1(t) - (1 - a)\gamma_2(t)}{1 - \gamma_1(t)[a\gamma_1(t) + (1 - a)\gamma_2(t)]} \right| = 0.
\]
By an easy calculation we get:
\[
\left| \frac{\gamma_1(t) - a\gamma_1(t) - (1 - a)\gamma_2(t)}{1 - \gamma_1(t)[a\gamma_1(t) + (1 - a)\gamma_2(t)]} \right| = \left( 1 - a \right) \left| \frac{\gamma_1(t) - \gamma_2(t)}{1 - \gamma_1(t)[a\gamma_1(t) + (1 - a)\gamma_2(t)]} \right|.
\]
We claim that if the curve \( \gamma \) is \( (g, \pi_g) \)-special we necessarily have that \( (1 - \gamma_2(t))/(1 - \gamma_1(t)) \to 1 \) as \( t \to 1^- \). For this to be true we show that
\[
\left| \frac{1 - \gamma_1(t)[a\gamma_1(t) + (1 - a)\gamma_2(t)]}{1 - \gamma_1(t)} \right| \leq C < +\infty.
\]
Notice that
\[
\left| \frac{1 - \gamma_1(t)[a\gamma_1(t) + (1 - a)\gamma_2(t)]}{1 - \gamma_1(t)} \right| = \left| 1 - \gamma_1(t)^2 \left( \frac{\gamma_2 - \gamma_1}{1 - \gamma_1} \right) \right| = \left| 1 - \frac{\gamma_1(t)^2}{1 - \gamma_1} - (1 - a)\gamma_1 \gamma_2 \left( \frac{\gamma_2 - \gamma_1}{1 - \gamma_1} \right) \right| = \left| 1 - \frac{\gamma_1(t)^2}{1 - \gamma_1} - (1 - a)\gamma_1 \left( \frac{\gamma_2 - \gamma_1}{1 - \gamma_1} \right) \right|.
\]
Letting \( \theta = \frac{\gamma_2 - \gamma_1}{1 - \gamma_1} \), if \( |\theta| \) is bounded as \( t \to 1^- \) then the previous expression is bounded and we are done. Otherwise, assuming \( |\theta| \to \infty \) and denoting by \( c = \) constant, the formula (3.2) becomes
\[
\left| \frac{(1 - a)(1 - \theta)}{\theta(\frac{\theta}{\theta - 1} - \frac{1}{a}(\frac{1}{\theta} - 1))} \right| = \left| \frac{\theta(1 - a)(1 - \frac{\theta}{\theta - 1})}{\theta(\frac{\theta}{\theta - 1} - \frac{1}{a}(\frac{1}{\theta} - 1))} \right| \to \left| \frac{1 - a}{1 - \frac{\theta}{\theta - 1}} \right| = 1
\]
but such an expression has to tend to zero by hypothesis, contradiction. Thus the claim is proved. On the other hand, taking into account that \( \gamma \) is \( (g, \pi_g) \)-restricted, if
\[
\lim_{t \to 1^-} \frac{1 - \gamma_2(t)}{1 - \gamma_1(t)} = 1,
\]

(3.3)
then
\[
\frac{1 - \gamma_1(t)(a\gamma_1(t) + (1 - a)\gamma_2(t))}{1 - \gamma_1(t)} \geq \frac{1 - |a\gamma_1(t) + (1 - a)\gamma_2(t)|}{1 - \gamma_1(t)}
\]
\[
\geq \frac{|1 - a\gamma_1(t) - (1 - a)\gamma_2(t)|}{M[1 - \gamma_1(t)]} \rightarrow \frac{1}{M}
\]
as \( t \to 1^- \), and condition (3.3) is also sufficient. Notice that in case \( a = 1 \) then
\( K_{\Delta^2}(\gamma(t), \pi_\varphi(\gamma(t))) = \omega(\gamma_2(t), \gamma_1(t) + (1 - a)\gamma_2(t)) \) and using the same arguments as above you prove the statement. \( \square \)

It is worth noticing that the Abate projection \( p \) is a special linear projection with \( a = \frac{1}{2} \):

**Proposition 12.** Let \( x = (x_1, x_2) \in (\partial \Delta)^2 \) and let \( \gamma(t) = (\gamma_1(t), \gamma_2(t)) \) be a \((g, \pi_g)\)-restricted \( x \)-curve in \( \Delta^2 \). Let \( \varphi_x(z) \) be the complex geodesic, parameterized by \( z \to (zz) \), passing through the point \( x \). Let \( \pi_g \) be any linear projection on \( \varphi_x \). Then \( \gamma \) is \((g, \pi_g)\)-special and \((g, \pi_g)\)-restricted if and only if \( \gamma \) is \( A \)-special and \( A \)-restricted.

**Proof.** Without loss of generality we suppose \( x = (1, 1) \), therefore \( \varphi_x(z) = \varphi(z) = (z, z) \). As in the proof of Proposition 11 we can state that \( \pi_g(z_1, z_2) = (az_1 + (1 - a)z_2, az_1 + (1 - a)z_2) \), with \( a \in \mathbb{C} \).

We first claim that \( \omega(\pi_g(\gamma(t)), \varphi(\gamma(t))) \to 0 \) (as \( t \to 1^- \)) whenever \( \frac{1 - \gamma_2(t)}{1 - \gamma_1(t)} \to 1 \) and \( \gamma \) is either \((g, \pi_g)\)-restricted or \( A \)-restricted. By the very definition of Poincaré metric

\[
\omega(\pi_g(\gamma(t)), \varphi(\gamma(t))) = \frac{|\varphi(\gamma(t)) - \varphi(x)|}{|1 - \varphi(\gamma(t))\pi_g(\gamma(t))|}
\]

First we notice that:

\[
\frac{|\varphi(\gamma(t)) - \varphi(x)|}{|1 - \varphi(\gamma(t))\pi_g(\gamma(t))|} = \frac{|\varphi(\gamma(t)) - 1 + \varphi(x)|}{|1 - \varphi(\gamma(t))\pi_g(\gamma(t))|}
= \left| \frac{1 - \varphi(x)\pi_g(\gamma(t))}{\frac{1 - \varphi(\gamma(t))\pi_g(\gamma(t))}{1 - \varphi(\gamma(t))}} - 1 \right|
\]

Moreover, by definition of \( A \)-projection device,

\[
\frac{1 - \varphi(x)\pi_g(\gamma(t))}{\frac{1 - \varphi(\gamma(t))\pi_g(\gamma(t))}{1 - \varphi(\gamma(t))}} = \frac{1 - a\gamma_1 - (1 - a)\gamma_2}{1 - \frac{1}{2}(\gamma_1 + \gamma_2)} = 2 \frac{1 - a\gamma_1 - (1 - a)\gamma_2}{2 - \gamma_1 - \gamma_2}
= \frac{a + (1 - a) - a\gamma_1 - (1 - a)\gamma_2}{1 - \gamma_1 + 1 - \gamma_2}
= \frac{a(1 - \gamma_1) + (1 - a)(1 - \gamma_2)}{1 - \gamma_1 + 1 - \gamma_2} = \frac{a + (1 - a)\frac{1 - \gamma_2}{1 - \gamma_1}}{1 + \frac{1 - \gamma_2}{1 - \gamma_1}},
\]

and by Proposition 11 we get

\[
\lim_{t \to 1^-} \frac{2}{1 + \frac{1 - \gamma_2}{1 - \gamma_1}} = 1.
\]
Let suppose that the curve $\gamma$ is $A$–restricted, then there exists $M > 1$ such that
\[
\frac{|1 - \gamma(t)|}{1 - |\gamma(t)|} < M,
\]
and in particular
\[
\left| \frac{1 - \tilde{p}(\gamma(t))\tilde{\sigma}_g(\gamma(t))}{1 - \tilde{p}(\gamma(t))} \right| \geq \left| \frac{1 - |\tilde{p}(\gamma(t))|\tilde{\sigma}_g(\gamma(t))}{1 - |\tilde{p}(\gamma(t))|} \right| \geq \frac{1 - |\tilde{p}(\gamma(t))|}{1 - |\tilde{p}(\gamma(t))|} > \frac{1}{M}.
\]
Then we conclude that
\[
\lim_{t \to 1^-} \omega(\tilde{\sigma}_g(\gamma(t)), \tilde{p}(\gamma(t))) = 0
\]
Moreover by equation (3.4) we notice that
\[
\left| \frac{1 - \tilde{\sigma}_g(\gamma(t))}{1 - \tilde{\sigma}_g(\gamma(t))} \right| = \left| \frac{1 - \tilde{\sigma}_g(\gamma(t))}{1 - \tilde{p}(\gamma(t))} \right| \geq \frac{1 - |\tilde{p}(\gamma(t))|}{1 - |\tilde{p}(\gamma(t))|} < 4M,
\]
and then the curve $\gamma$ is $(g, \pi_g)$–restricted. Let us notice that if we interchange the hypothesis of $A$–restriction with the one of $(g, \pi_g)$–restriction, proceeding as above we get the initial claim.

Finally by Proposition 11 we get that $\gamma$ is $A$–special (or $\gamma$ is $(g, \pi_g)$–special) if and only if $\frac{1 - |\tilde{\sigma}_g(\gamma(t))|}{1 - |\tilde{\sigma}_g(\gamma(t))|} \to 1$ as $t \to 1$ and it concludes the first part of the proof.

The last step consists in proving the “$\Rightarrow$” implication of the theorem. To do this it is sufficient to interchange the Abate’s projection $p$ with the linear projection $\pi_g$ in the proof above and the thesis easily follows. \qed

**Remark 13.** By Proposition 12 it follows the Abate’s Julia-Wolf-Carathéodory theorem for linear projections.

The next question is what happens if we consider another geodesic passing through the point $x \in (\partial \Delta)^2$. Arguing as in Proposition 12 we have:

**Proposition 14.** Let $(\varphi_g, \pi_g)$ be a projection device. Let us assume that the geodesic $\varphi_g$ passes through a point $x \in (\partial \Delta)^2$ and set $\lambda_g := \lim_{z \to x} \frac{1 - |\varphi_g(z)|}{1 - |z|} < \infty$.

Let $\gamma := (\gamma_1, \gamma_2)$ be a $(g, \pi_g)$–restricted $x$–curve in $\Delta^2$. Then $\gamma$ is $(g, \pi_g)$–special if and only if
\[
\lim_{t \to 1^-} \frac{x_2 - \gamma_2(t)}{x_1 - \gamma_1(t)} = \lambda_g.
\]

4. The non-tangential limit

In one complex variable the notion of non-tangential limit can be defined in two equivalent ways. We can say that a function $f \in \text{Hol}(\Delta, \Delta)$ has non-tangential limit $L \in \mathbb{C}$ at a point $y \in \partial \Delta$ if $f(z) \to L$ as $z \to y$ within any Stolz region $H_\Delta(y, M)$, or we can equivalently say that $f \in \text{Hol}(\Delta, \Delta)$ has non-tangential limit $L \in \mathbb{C}$ at a point $y \in \partial \Delta$ if $f(\sigma(t)) \to L$ as $t \to 1$, along any curve $\sigma : [0, 1) \to \Delta$ such that $\sigma(t) \to y$ non-tangentially as $t \to 1^-$. In [2] (see also [4]) Abate generalizes the Stolz region giving the following definition of (small) Koranyi region (of vertex $y \in \partial \Delta^2$ and amplitude $M$),
\[
H(y, M) := \{ z \in \Delta^2 : \limsup_{w \to y} [K_{\Delta^2}(z, w) - K_{\Delta^2}(0, w)] + K_{\Delta^2}(0, z) < \log M \}.
\]

Thus an extension of the first definition of non-tangential limit is (see [2]):
Definition 15. A map \( f : \Delta^2 \to \mathbb{C}^m \) has \( K \)-limit \( L \in \mathbb{C}^m \) at \( y \in \partial \Delta^2 \) if \( f(z) \to L \) as \( z \to y \) inside any Koranyi region.

On the other hand, by means of \( A \)-special and \( A \)-restricted curves, Abate (in [2]) says that a holomorphic function \( f : \Delta^2 \to \mathbb{C}^m \) has restricted \( K \)-limit \( L \) at \( x \) if \( f(\sigma(t)) \to L \) for any \( A \)-special and \( A \)-restricted \( x \)-curve \( \sigma(t) \subset \Delta^2 \), and he writes

\[
\tilde{K}_{\lim} f(z) = L.
\]

It is worthwhile observing that the definitions of \( K \)-limit and restricted \( K \)-limit are no more equivalent. More precisely if \( f \) has \( K \)-limit \( L \) at \( y \in \partial \Delta^2 \) then it has restricted \( K \)-limit too. The converse is in general false even for bounded holomorphic functions. For example (see [2]) let \( f : \Delta^2 \to \mathbb{C} \) be defined as

\[
f(z_1, z_2) = \frac{(1-z_1)^2 - (1-z_2)^2}{(1-z_1)^2 + (1-z_2)^2}.
\]

Such a function has restricted \( K \)-limit 0 at \((1, 1)\) but \( f \) has not \( K \)-limit at \((1, 1)\).

We extend these definitions by means of Busemann functions. The first step consists in giving the following extension of the notion of Stolz region, as done in Definition 2: let \( x \in \partial \Delta^2 \) and \( M > 1 \), the \( g \)-Koranyi region \( H_{\varphi_g}(x, M) \), of vertex \( x \) and amplitude \( M \) is:

\[
H_{\varphi_g}(x, M) := \{ z \in \Delta^2 : \lim_{r \to 1^-} [K_{\Delta^2}(z, \varphi_g(r))-K_{\Delta^2}(\varphi_g(0), \varphi_g(r))] + K_{\Delta^2}(\varphi_g(0), z) < \log M \}.
\]

And then we, naturally, say that

Definition 16. A holomorphic function \( f \in \text{Hol}(\Delta^2, \Delta^2) \) has \( K_g \)-limit \( L \in \mathbb{C} \) if \( f \) approaches \( L \) inside any \( g \)-Koranyi region.

If we consider the complex geodesic \( \varphi(z) = (z, z) \) then the Koranyi region \( H((1, 1), M) \) coincide with the \( g \)-Koranyi region \( H_{\varphi_g}((1, 1), M) \).

Moreover let \( (\varphi_g, \pi_g) \) be a \( g \)-projection device as in Definition 9:

Definition 17. A holomorphic function \( h : \Delta^2 \to \mathbb{C} \) is said to have restricted \( K_g \)-limit \( L \) if \( h \) has limit \( L \) along any curve which is \((g, \pi_g)\)-special and \((g, \pi_g)\)-restricted, and we write

\[
\tilde{K}_g \lim h(z) = L.
\]

Obviously Definition 16 and Definition 17 are not equivalent but again the \( K_g \)-limit implies the restricted \( \tilde{K}_g \)-limit.

5. Lindelöf Theorems

The one-variable Lindelöf principle has been proved by Lindelöf [24], [25]. The first important several variables version is due to Cirka [12]. A different generalization is due to Dovbush [13] and his approach has been further pursued in Cima and Krantz [11] (see also [17], [18]).

The classical Lindelöf principle implies that if \( f \in \text{Hol}(\Delta, \Delta) \) has limit \( L \) along any given 1-curve, then \( L \) is the non-tangential limit of \( f \) at 1. The first step to generalize this theorem to several complex variables consists in detecting a correct class of curves. Let \((\varphi_g, \pi_g)\) a \( g \)-projection device. The idea is to consider the \((g, \pi_g)\)-special and \((g, \pi_g)\)-restricted curves.

In this setting we prove the following first generalization of the Lindelöf principle:
Theorem 18. Let $f \in \text{Hol}(\Delta^2, \mathbb{C})$ be a bounded holomorphic function. Let $x \in \partial \Delta^2$. Assume there exists a $(g, \pi_g)$–special $x$–curve $\sigma_0$ such that
\[
\lim_{t \to 1^-} f(\sigma_0(t)) = L \in \mathbb{C}.
\]
Then $f$ has restricted $K_g$–limit $L$ at $x$.

Proof. This proof is inspired by the one in [2] (see Theorem 2.1). We can assume that $f(\Delta^2) \subset \Delta$. We first observe that, given $\sigma$ a $(g, \pi_g)$–special $x$–curve,
\[
0 \leq \omega(f(\sigma(t)), f(\pi_g(\sigma(t)))) \leq K_{\Delta^2}(\sigma(t), \pi_g(\sigma(t))) \to 0
\]
as $t \to 1^-$. Therefore the limit of $f(\pi_g(\sigma(t)))$ exists, as $t \to 1^-$, if and only if the limit of $f(\sigma(t))$ (as $t \to 1^-$) does, and the two limits are equal. In particular $f(\pi_g(\sigma_0(t))) \to L$ as $t \to 1^-$ and, by classical Lindelöf principle, $f(\pi_g(\sigma(t))) \to L$ for any $(g, \pi_g)$–restricted $x$–curve. By inequality (5.1) it follows that $f(\sigma(t)) \to L$ for any $(g, \pi_g)$–restricted and $(g, \pi_g)$–special $x$–curve $\sigma$.

Since in the Julia-Wolff-Carathéodory theorem the functions we deal with are incremental ratios, a stronger result than Theorem 18 is needed. It is worthwhile to introduce some definitions and preliminary results.

Definition 19. Let $f \in \text{Hol}(\Delta^2, \mathbb{C})$. We say that $f$ is $K_g$–bounded if \forall M there exists a constant $C_M > 0$ such that $|f(z)| < C_M$ for all $z \in H_{\varphi_g}(x, M)$.

Lemma 20. Let $x \in \partial \Delta^2$ and let $(\varphi_g, \pi_g)$ a $g$–projection device. Then $H_{\varphi_g}(x, M) \cap \varphi_g(\Delta) = \varphi_g(H(1, M))$ for all $M > 1$.

Proof. The proof follows by the definition of $g$–Koranj region and of complex geodesic. Indeed $z \in H_{\varphi_g}(x, M) \cap \varphi_g(\Delta)$ if and only if $z = (\alpha, g(\alpha))$, with $\alpha \in \Delta$, and
\[
\lim_{t \to 1^-} K_{\Delta_2}((\varphi_g(\alpha), \varphi_g(t)) - \omega(0, t) + K_{\Delta_2}(\varphi_g(0), \varphi_g(\alpha)) < \frac{1}{2} \log M.
\]
Since $\varphi_g$ is a complex geodesic it follows that
\[
\frac{1}{2} \log M > \lim_{t \to 1^-} K_{\Delta_2}(\varphi_g(\alpha), \varphi_g(t)) - \omega(0, t) + K_{\Delta_2}(\varphi_g(0), \varphi_g(\alpha)) = \lim_{t \to 1^-} \omega(\alpha, t) - \omega(0, t) + \omega(0, \alpha)
\]
and then $\alpha \in H(1, M)$ and $z = (\alpha, g(\alpha)) \in \varphi_g(H(1, M))$. \hfill \Box

Remark 21. Let $x \in \partial \Delta^2$ and let $(\varphi_g, \pi_g)$ a $g$–projection device. By Lemma 20 it follows that if $\sigma$ is an $x$–curve, then $\sigma$ is $(g, \pi_g)$–restricted if and only if $\pi(\sigma(t)) \in H_{\varphi_g}(x, M)$ as $t \approx 1$.

Remark 22. Consider $\sigma$ a $(g, \pi_g)$–special $x$–curve. We notice that it is possible to write $\sigma(t) := (\sigma_1(t), \sigma_2(t)) = \pi_g(\sigma(t)) + \alpha(t)$ with $\alpha(t) := (\alpha_1(t), \alpha_2(t)) \to (0, 0)$ as $t \to 1^-$. By definition of the projection $\pi_g$ we get that $\alpha_1(t) \equiv 0$ and $\alpha_2(t) \to 0$, as $t \to 1^-$.

Lemma 23. Let $x \in \partial \Delta^2$ and $(\varphi_g, \pi_g)$ a $g$–projection device, with $\varphi_g$ a complex geodesic parameterized as $\Delta \ni w \to (w, g(w))$ (assuming $|x_1| = 1$) and $\pi_g(z_1, z_2) = (z_1, g(z_1))$. Let $\sigma$ be an $x$–curve. Write $\sigma(t) = \pi_g(\sigma(t)) + \alpha(t)$ with $\alpha(t) \to 0$ as $t \to 1^-$. Then $\sigma$ is $(g, \pi_g)$–special if and only if
\[
\lim_{t \to 1^-} \frac{|\alpha_2(t)|}{1 - |g(\sigma_1(t))|} = 0.
\]
Proof. Assume first that
\[
\lim_{t \to 1^{-}} \frac{|\alpha_2(t)|}{1 - |g(\sigma_1(t))|} = 0.
\]
By the triangular inequality and by definition of Kobayashi distance in the bidisc, we have that
\[
K_{\Delta^2}(\sigma(t), \pi_g(\sigma(t))) = \max\{\omega(\sigma_1(t), \sigma_1(t)); \omega(\sigma_2(t), g(\sigma_1(t)))\}
\]
\[
= \frac{1}{2} \log \frac{1 + |\sigma_2(t) - g(\sigma_1(t))|}{1 - |\sigma_2(t) - g(\sigma_1(t))|} \leq \frac{1}{2} \log \frac{1 + |\alpha_2(t)|}{1 - |\alpha_2(t)|} \to 0
\]
as \(t \to 1^{-}\). Thus the curve \(\sigma\) is \((g, \pi_g)\)-special and the first implication has been proved. On the other hand, let us suppose that \(\sigma\) is \((g, \pi_g)\)-special.

If, by contradiction, \(\lim_{t \to 1^{-}} \frac{|\alpha_2(t)|}{1 - |g(\sigma_1(t))|} \neq 0\) then there exists \(\varepsilon > 0\) such that
\[
T := \frac{|\alpha_2(t)|}{1 - |g(\sigma_1(t))|^2} > \varepsilon > 0
\]
for \(t \approx 1\). Furthermore,
\[
\frac{1 - g(\sigma_1(t))\sigma_2(t)}{1 - |g(\sigma_1(t))|^2} = \frac{1 - g(\sigma_1(t))(\sigma_1(t) + \alpha_2(t))}{1 - |g(\sigma_1(t))|^2} = 1 - \frac{g(\sigma_1(t))\alpha_2(t)}{1 - |g(\sigma_1(t))|^2}
\]
\[
\leq 1 + \frac{|g(\sigma_1(t))\alpha_2(t)|}{1 - |g(\sigma_1(t))|^2} = 1 + |g(\sigma_1(t))|\frac{|\alpha_2(t)|}{1 - |g(\sigma_1(t))|^2} \leq (T + 1),
\]
and since \(T \to \frac{T}{1 + T}\) is a growing function we have that
\[
\frac{|\alpha_2(t)|}{1 - g(\sigma_1(t))\sigma_2(t)} \geq \frac{|\alpha_2(t)|}{1 - |g(\sigma_1(t))|^2} \frac{1 - |g(\sigma_1(t))|^2}{1 - g(\sigma_1(t))\sigma_2(t)} \geq \frac{T}{1 + T} \geq \frac{\varepsilon}{1 + \varepsilon} > 0,
\]
which contradicts the hypothesis of \((g, \pi_g)\)-speciality. \(\square\)

We have now the following result of Lindelöf type for Busemann functions:

**Theorem 24.** Let \(f \in \text{Hol}(\Delta^2, \Delta)\) be a holomorphic function. Given \(x = (x_1, x_2) \in \partial \Delta^2\) such that \(|x_1| = 1\), let \(\varphi_f\) be a complex geodesic passing through \(x\), parameterized as \(\varphi_f(z) = (z, g(z))\), and let \((\varphi_f, \pi_f)\) be a \(g\)-projection device, with \(\pi_f(z_1, z_2) = (z_1, g(z_1))\). Assume that \(f\) is \(K_g\)-bounded. If \(\sigma_0\) is a \((g, \pi_g)\)-special and \((g, \pi_g)\)-restricted \(x\)-curve such that
\[
\lim_{t \to 1^{-}} f(\sigma_0(t)) = L
\]
then \(f\) admits restricted \(K_g\)-limit equal to \(L\) at \(x\).

**Proof.** Let us consider a \((g, \pi_g)\)-special and \((g, \pi_g)\)-restricted \(x\)-curve \(\sigma\). By definition there exists a constant \(M > 1\) such that \(\pi_g(\sigma(t))\) approaches \(x_1\) inside a Stolz region \(H_{\Delta}(x_1, M)\). We claim that
\[
(5.3) \quad \forall M_1 > M, \ K_{H_{\pi_g}(x, M_1)}(\sigma(t), \pi_g(\sigma(t))) \to 0 \text{ as } t \to 1^{-}.
\]
For any \(t \in [0, 1)\) let us consider the map \(\phi_t : \mathbb{C} \to \mathbb{C}^2\) given by
\[
\phi_t(z) = \pi_g(\sigma(t)) + z[\sigma(t) - \pi_g(\sigma(t))].
\]
Let us notice that \( \phi_t(0) = \pi_g(\sigma(t)) \) and \( \phi_t(1) = \sigma(t) \). We claim that the following statement is true:

\[
\forall R > 0 \exists t_0 = t_0(R) \in [0,1) \text{ such that } \forall t \in [0,1) : t > t_0(R) \quad \phi_t(\Delta_t) \subset H_{\varphi_g}(x, M_1).
\]

(5.4)

Assuming (5.4) we get that

\[
R(t) := \sup\{r > 0 : \phi_t(\Delta_t) \subset H_{\varphi_g}(x, M_1)\} \to \infty
\]
as \( t \to 1^- \). Moreover (by the very definition)

\[
K_{H_{\varphi_g}(x, M_1)}(\sigma(t), \pi_g(\sigma(t))) \leq \\
\leq \inf\left\{ \frac{1}{R} : \exists \varphi_g \in \text{Hol}(\Delta_R, H_{\varphi_g}(x, M_1)) : \varphi_g(0) = \pi_g(\sigma(t)) \text{ and } \varphi_g(1) = \sigma(t) \right\}.
\]

Then equation (5.3) follows from equation (5.5) and statement (5.4). Therefore we are left to prove (5.4). Assume by contradiction that (5.4) is false. Then there exist \( M_1 > M \) and \( R_0 > 1 \) such that for any \( t_0 \in [0,1) \) there are \( t' = t'(t_0) \in (t_0, 1) \) and \( z_0 = z_0(t_0) \in \Delta_{R_0} \) such that \( \phi_{t'}(z_0) \notin H_{\varphi_g}(x, M_1) \). Moreover, by Proposition 20, \( \pi_g(\sigma(t)) \in H_{\varphi_g}(x, M_1) \) eventually, and in particular we can choose \( t_0 \in (0,1) \) such that \( \pi_g(\sigma(t)) \in H_{\varphi_g}(x, M_1) \) for all \( t > t_0 \). Being \( H_{\varphi_g}(x, M_1) \) open, we can also assume that \( \phi_{t'}(z_0) \in \partial H_{\varphi_g}(x, M_1) \), but \( \phi_{t'}(z) \in H_{\varphi_g}(x, M_1) \) for all \( z \in \Delta_{z_0} \). If \( z \) were such that \( \phi_{t'}(z) \in \partial \Delta^2 \) eventually for \( t_0 \to 1 \), then there would exist \( t_0^* \) and \( z_0^* \in \Delta_{R_0} \) with \( t_0^* > t_0 \) and \( \phi_{t_0^*}(z_0^*) \in \partial H_{\varphi_g}(x, M_1) \cap \partial \Delta^2 \). Since

\[
\phi_{t_0^*}(z_0^*) = \pi_g(\sigma(t_0^*)) + z_0^*[\sigma(t_0^*) - \pi_g(\sigma(t_0^*))] = (\sigma(t_0^*), g(\sigma_1(t_0^*)) + z_0^*\alpha_2(t_0^*))
\]

where \( \alpha_2 \) is defined as in Lemma 23, and then necessarily

\[
|g(\sigma_1(t_0^*)) + \alpha_2(t_0^*)| = 1.
\]

In particular, by Lemma 23

\[
0 = \frac{1 - |g(\sigma_1(t_0^*)) + z_0^*\alpha_2(t_0^*)|}{1 - |g(\sigma_1(t_0^*))|} \geq \frac{1 - |g(\sigma_1(t_0^*))| - |z_0^*\alpha_2(t_0^*)|}{1 - |g(\sigma_1(t_0^*))|} = 1 - \frac{|z_0^*\alpha_2(t_0^*)|}{1 - |g(\sigma_1(t_0^*))|} \geq 1 - \frac{R_0|\alpha_2(t_0^*)|}{1 - |g(\sigma_1(t_0^*))|} \to 1^-
\]
as \( t \to 1^- \), thus leading to a contradiction. Therefore \( \phi_{t'}(z_0) \in \Delta^2 \cap \partial H_{\varphi_g}(x, M_1) \) eventually for all \( t' \to 1^- \). By definition of \( g \)-Koranyi region, we can write

\[
(5.6) \quad \log M_1 = \lim_{s \to 1^-} K_{\Delta^2}(\phi_{t'}(z_0), \varphi_g(s)) - \omega(0, s) + K_{\Delta^2}(\phi_{t'}(z_0), \varphi_g(0)).
\]

Furthermore, for any \( z \in \Delta_{R_0} \),

\[
(5.7) \quad K_{\Delta^2}(\phi_{t'}(z), \varphi_g(s)) - \omega(0, s) + K_{\Delta^2}(\phi_{t'}(z), \varphi_g(0)) = \\
= \max\{\omega(\sigma_1(t'), s); \omega(g(\sigma_1(t')) + \alpha_2(t')z, g(s))\} - \omega(0, s) + \\
+ \max\{\omega(\sigma_1(t'), 0); \omega(g(\sigma_1(t')) + \alpha_2(t')z, g(0))\} \leq \\
\leq \omega(g(\sigma_1(t')) + \alpha_2(t')z, g(\sigma_1(t'))) + \omega(\sigma_1(t'), s) - \omega(0, s) + \\
+ \omega(\sigma_1(t'), 0; g(\sigma_1(t')) + \alpha_2(t')z, g(\sigma_1(t'))) = \\
= \omega(\sigma_1(t'), s) - \omega(0, s) + \omega(\sigma_1(t'), 0) + 2\omega(g(\sigma_1(t')) + \alpha_2(t')z, g(\sigma_1(t')))
Let 
\[ \omega(t', s) \leq \omega(g(t'), s) + \alpha_2(t')z, g(t'), \omega(t', s) \]
being
\[ \omega(g(t'), s) + \alpha_2(t')z, g(t') \geq 0 \]
and
\[ \omega(g(t'), s) + \alpha_2(t')z, g(t') \leq \omega(g(t'), s) + \alpha_2(t')z, g(t') + \omega(t', s) \]
by the triangular inequality. Let us observe that
\[ \lim_{t' \to 1^-} \omega(g(t'), s) + \alpha_2(t')z, g(t')) = 0 \]
uniformly for \( z \in \Delta_{R_0} \). Indeed by definition of Poincaré distance, we have
\[ \lim_{t' \to 1^-} \omega(g(t'), s) + \alpha_2(t')z, g(t')) = \lim_{t' \to 1^-} \frac{1}{2} \log \left| \frac{\alpha_2(t')z}{1 - g(t')(g(t') + \alpha_2(t')z)} \right| \]
and the argument of this logarithm tends to 1 since
\[ \frac{\alpha_2(t')z}{1 - g(t')(g(t') + \alpha_2(t')z)} \leq \frac{|\alpha_2(t')z|}{1 - |g(t')(g(t') + \alpha_2(t')z)|} \leq \frac{|\alpha_2(t')z|}{1 - |g(t')(g(t') + \alpha_2(t')z)|} \]
\[ \leq \frac{1}{1 - |g(t')(g(t') + \alpha_2(t')z)|} \to 0 \text{ as } t \to 1^- \]
by Lemma 23. Thus equation (5.8) is proved. In particular it is true for \( z = z_0 \), and then by equations (5.6) and (5.7) we get
\[ \log M_1 \leq \lim_{s \to 1^-} \omega(\sigma_1(t'), s) - \omega(0, s) + \omega(\sigma_1(t'), 0) + 2\omega(g(\sigma_1(t')) + \alpha_2(t')z, g(\sigma_1(t'))) \]
In particular, for \( t' \approx 1 \),
\[ M < M_1 \leq \frac{1 - \sigma_1(t')}{1 - |\sigma_1(t')|} \]
but it is a contradiction since the \( K \)-region in the disc is given by \( \{ w \in \Delta : \frac{1 - w}{1 - |w|} < M \} \) and \( \sigma \) is \((g, \pi_g)\)-restricted. This concludes the proof of (5.4). Since \( f \) is a \( K_g \)-bounded function, there exists \( c > 0 \) such that
\[ K_{\Delta\sigma(M_1)}(f(\sigma_0(t)), f(\pi_g(\sigma_0(t)))) \leq K_{\Delta\pi_g}(\sigma_0(t), \pi_g(\sigma_0(t))) \to 0 \text{ as } t \to 1^- \]
and proceeding as in Theorem 18 we complete the proof. \( \square \)

**Remark 25.** Let us remark that in in case of flat boundary points Theorem 18 can be rephrased in the following way:

**Theorem 26.** Let \( f \in \text{Hol}(\Delta^2, \mathbb{C}) \) be a bounded holomorphic function. Let \( x \in [\partial \Delta^2 \setminus (\partial \Delta)^2] \). Assume there exists a \( x \)-curve \( \sigma_0 \) such that
\[ \lim_{t \to 1^-} f(\sigma_0(t)) = L \in \mathbb{C}. \]
Then \( f \) has restricted \( K_g \)-limit \( L \) at \( x \).
It is worth noticing that this result confirm the one stated in the Lindelöf-Lehto-Virtanen theorem (see [26], [13]) since in case of points of the flat components of the boundary (as in the Lindelöf-Lehto-Virtanen theorem) there are not restriction on the curves (see remark 10).

6. Julia’s Lemma

The main idea in generalizing the classical Julia’s lemma is to consider the rate of approach of \( f \) along particular directions given by geodesics passing through \( x \in \partial \Delta^2 \).

**Definition 27.** Let \( f \in \text{Hol}(\Delta^2, \Delta) \) and \( x \in \partial \Delta^2 \). Let us consider a complex geodesic \( \varphi_g \in \text{Hol}(\Delta, \Delta^2) \) passing through \( x \). Let \( \lambda_g \) be the boundary dilation coefficient of \( g \) at \( x_1 \). The number \( \lambda_{\varphi_g}(f) \) defined by

\[
\frac{1}{2} \log \lambda_{\varphi_g}(f) := \lim_{t \to 1^-} K_{\Delta^2}(0, \varphi_g(tx_1)) - \omega(0, f(\varphi_g(tx_1)))
\]

is the \( \varphi_g \)-boundary dilation coefficient of \( f \) at \( x \).

**Remark 28.** This boundary dilation coefficient, \( \lambda_{\varphi_g}(f) \), is well defined. Indeed let \( \theta \) be an automorphism of \( \Delta \). Then

\[
\lim_{t \to 1^-} K_{\Delta^2}(0, \varphi_g(\theta(tx_1))) - \omega(0, f(\varphi_g(\theta(tx_1)))) = \lim_{t \to 1^-} K_{\Delta^2}(\varphi(0), \varphi_g(\theta(tx_1))) - \omega(0, f(\varphi_g(\theta(tx_1)))) + K_{\Delta^2}(0, \varphi(\theta(tx_1))) - K_{\Delta^2}(0, \varphi_g(\theta(tx_1))) = \lim_{t \to 1^-} \omega(0, tx_1) - \omega(0, f(\varphi_g(\theta(tx_1)))) + \max[\omega(0, \theta(tx_1)), \omega(0, g(tx_1))] - \omega(0, \theta(tx_1))
\]

\[
= \frac{1}{2} \log \lambda_{\varphi \circ \varphi_g} + \max\{0, -\frac{1}{2} \log \lambda_g\}
\]

where \( \lambda_{\varphi \circ \varphi_g} \) is the boundary dilation coefficient of \( f \circ \varphi \) at \( \theta(1) \) (which we may assume equal to 1). This proves at once both that the limit (6.1) exists and that it is invariant under parametrization of the geodesic.

Let us notice that if \( \varphi_g(z) = \varphi(z) = (z, z) \) then \( \lambda_{\varphi_g}(f) \equiv \alpha_f \) where \( \alpha_f \) is the boundary dilation coefficient of \( f \) at \( x_1 \), defined by Abate in [2], as

\[
\frac{1}{2} \log \alpha_f := \liminf_{w \to x} [K_{\Delta^2}(0, w) - \omega(0, f(w))].
\]

Indeed Abate showed that the following property holds (see [2] for the proof)

\[
\frac{1}{2} \log \alpha_f = \limsup_{t \to 1} [K_{\Delta^2}(0, tx) - \omega(0, f(tx))].
\]

In this setting Julia’s Lemma is:

**Theorem 29.** Let \( f = (f_1, f_2) \in \text{Hol}(\Delta^2, \Delta^2) \). Let \( x = (x_1, x_2) \in \partial(\Delta \times \Delta) = \partial \Delta^2 \) and let \( \varphi_g = (z, g(z)) \) be a complex geodesic passing through \( x \) assuming \( |x_1| = 1 \).

Let

\[
\frac{1}{2} \log \lambda_j := \lim_{t \to 2^-} [K_{\Delta^2}(0, \varphi_g(tx_1)) - \omega(0, f_j(\varphi_g(tx_1)))] \text{ with } j = 1, 2
\]

Suppose that both \( \lambda_1 < \infty \) and \( \lambda_2 < \infty \). Then there exists a point \( y = (y_1, y_2) \in \partial \Delta^2 \) such that for all \( R > 0 \)

\[
f(B(1, \lambda_j)(x, R)) \subseteq B(\lambda_1, \lambda_2)(y, R).
\]
Proof. Without loss of generality we suppose $x_1 = 1$. Since $1/2 \log \lambda_j < \infty$ it follows, by remark 28, that also $\lambda_{f_j \circ \varphi_g}$ is finite and by the classical Julia-Wolff-Carathéodory theorem we have that there exists a unique point $y_j^{\varphi_g} \in \partial \Delta$ such that $f_j \circ \varphi_g : \Delta \to \Delta$ has non-tangential limit $y_j^{\varphi_g}$ at 1, for $j = 1, 2$. Fix $z \in \mathcal{B}_{(1, \lambda_g)}(x, R)$. We have, for $j = 1, 2$, that

$$\lim_{w \to y_j} \omega(f_j(z), w) - \omega(0, w) = \lim_{s \to 1} \omega(f_j(z), f_j(\varphi_g(s))) - \omega(0, f_j(\varphi_g(s)))$$

$$\leq \lim_{s \to 1} K_{\Delta^2}(z, \varphi_g(s)) - \omega(0, f_j(\varphi_g(s)))$$

$$= \lim_{s \to 1} K_{\Delta^2}(z, \varphi_g(s)) - \omega(0, s) + \omega(0, s) - K_{\Delta^2}(0, \varphi_g(s)) +$$

$$+ K_{\Delta^2}(0, \varphi_g(s)) - \omega(0, f_j(\varphi_g(s)))$$

$$\leq \lim_{s \to 1} K_{\Delta^2}(z, \varphi_g(s)) - \omega(0, s) + \lim_{s \to 1} K_{\Delta^2}(0, \varphi_g(s)) - \omega(0, f_j(\varphi_g(s))) \leq \frac{1}{2} \log \lambda_j R.$$

Then $\forall R > 0$

$$f(\mathcal{B}_{(1, \lambda_g)}(x, R)) = f(E(x_1, R) \times E(x_2, \lambda_g R)) \subseteq E(y_1^{\varphi_g}, \lambda_1 R) \times E(y_2^{\varphi_g}, \lambda_2 R).$$

We claim that actually the point $y_j^{\varphi_g}$ does not depend on the geodesic $\varphi_g$. For sake of simplicity we consider the case $j = 1$. Let $\varphi$ be the complex geodesic parameterized as $\varphi(z) = (z, z x_2)$. Since $\lambda_1 < \infty$, by the above definition of $\alpha_{f_1}$ we get that also $\alpha_{f_1}$ is finite and proceeding as above we have that there exists a unique point $\tau_1^{\varphi} \in \partial \Delta$ such that $f_1 \circ \varphi$ has non-tangential limit $\tau_1^{\varphi}$ at 1. Moreover

$$f(\mathcal{B}_{(1, 1)}(x, R)) = f(E(x_1, R) \times E(x_2, R)) \subseteq E(\tau_1^{\varphi}, \lambda_1 R) \times E(\tau_2^{\varphi}, \lambda_2 R).$$

Let us suppose that $y_1^{\varphi_g} \neq \tau_1^{\varphi}$ then we can choose $R > 0$ small enough such that $\mathcal{B}_{(1, 1)}(x, R) \cap \mathcal{B}_{(1, \lambda_g)}(x, R) \neq \emptyset$ and $E(y_1^{\varphi_g}, \lambda_1 R) \cap E(\tau_1^{\varphi}, \lambda_1 R) = \emptyset$ and it is a contradiction. Thus $y_j^{\varphi_g}$ does not depend on the geodesic $\varphi_g$ and we can denote such a point with $y_j$. Thus taking $y = (y_1, y_2)$ the statement is proved.

Moreover the following proposition holds:

**Proposition 30.** For all complex geodesic $\varphi_g$ passing through $x \in \partial \Delta^2$ such that the coefficient $\lambda_g < \infty$, $\lambda_{\varphi_g}(f)$ is finite if and only if $\alpha_f$ is finite.

**Proof.** It is clear that $\lambda_{\varphi_g}(f) \geq \alpha_f$ and thus if $\lambda_{\varphi_g}(f)$ is finite then $\alpha_f$ is finite too. On the other hand by remark 28 we have that

$$\frac{1}{2} \log \lambda_{\varphi_g}(f) = \lim_{t \to 1} K_{\Delta^2}(0, \varphi_g(t)) - \omega(0, f(\varphi_g(t)))$$

$$= \frac{1}{2} \log \lambda_{f \circ \varphi} \max\{1, \frac{1}{\lambda_g}\}$$

(6.2)
From this equation we have

\[
\frac{1}{2} \log \lambda_{f \circ \varphi} = \lim_{t \to 1} \left[ \omega(0, t) - \omega(0, f(\varphi(t))) \right] \\
\leq \lim_{t \to 1} \left[ \omega(0, t) - \omega(0, f(t, t)) \right] + \limsup_{t \to 1} \omega(0, f(t, t)) - \omega(0, f(\varphi(t))) \\
\leq \frac{1}{2} \log \alpha_f + \limsup_{t \to 1} \omega(f(t, t), f(\varphi(t))) \\
= \frac{1}{2} \log \alpha_f + \limsup_{t \to 1} \omega(\varphi(t)) = \frac{1}{2} \log \alpha_f \pm \frac{1}{2} \log \lambda_g
\]

(6.3)

where + is taken if \( \lambda_g \geq 1 \) and - otherwise. Hence, for instance for \( \lambda_g \geq 1 \), (swapping the role of \( f \circ \varphi \) and \( f(t, t) \) above to obtain the second inequality)

\[
\frac{\lambda_{f \circ \varphi}}{\lambda_g} \leq \alpha_f \leq \lambda_{f \circ \varphi} \lambda_g
\]

which concludes the proof. \( \square \)

7. The Julia-Wolff-Carathéodory theorem

We are finally ready to state and prove our generalization of the Julia-Wolff-Carathéodory theorem obtained using Busemann functions.

**Theorem 31.** Let \( f \in \text{Hol}(\Delta^2, \Delta^2) \) and \( x \in \partial \Delta^2 \). Let \( \varphi_g \) be any complex geodesic passing through \( x = (x_1, x_2) \), \( |x_1| = 1 \), and parameterized by \( \varphi_g(z) = (z, g(z)) \), with \( g \in \text{Hol}(\Delta, \Delta) \) such that

\[
\frac{1}{2} \log \lambda_j = \lim_{t \to 1} K_{\Delta^2}(0, \varphi_g(tx_1)) - \omega(0, f_j(\varphi_g(tx_1))) < \infty.
\]

for \( j = 1, 2 \). Let \( \pi_g(z) : \Delta^2 \to \Delta \) be the \( g \)-left-inverse of \( \varphi_g \) given by \( \pi_g(z_1, z_2) = z_1 \). Then there exists a point \( y = (y_1, y_2) \in \partial \Delta^2 \) such that

\[
\tilde{K}_g \lim_{z \to x} \frac{y_j - f_j(z)}{1 - \pi_g(z)} = \frac{\lambda_j}{\max\{1, 1/\pi_g\}} \text{ and } \\
\tilde{K}_g \lim_{z \to x} \frac{y_j - f_j(z)}{1 - z_2} = \frac{\lambda_j}{\max\{1, 1/\pi_g\}} \lambda_g.
\]

To prove this theorem we need first the following two lemmas:

**Lemma 32.** Let \( f \in \text{Hol}(\Delta^2, \Delta^2) \) and \( x \in \partial \Delta^2 \). Suppose \( |x_1| = 1 \). Suppose there exists a complex geodesic \( \varphi_g \) passing through \( x \) and parameterized by \( \varphi_g(z) = (z, g(z)) \), with \( g \in \text{Hol}(\Delta, \Delta) \) such that

\[
\frac{1}{2} \log \lambda_j = \lim_{t \to 1} K_{\Delta^2}(0, \varphi_g(tx_1)) - \omega(0, f_j(\varphi_g(tx_1))) < \infty \text{ with } j = 1, 2.
\]

Let \( \pi_g(z) : \Delta^2 \to \Delta \) be the \( g \)-left-inverse of \( \varphi_g \) given by \( \pi_g(z_1, z_2) = z_1 \). Then there exists a point \( y = (y_1, y_2) \in \partial \Delta^2 \) and a constant, say \( c_g > 0 \), depending on \( g \), such that, given \( M > 1 \), for all \( z \in H_{\varphi_g}(x, M) \)

\[
\left| \frac{y_j - f_j(z)}{1 - \pi_g(z)} \right| \leq 2\lambda_1 M^2 c_g \text{ and } \left| \frac{y_j - f_j(z)}{1 - z_2} \right| \leq 2\lambda_1 M^2 c_g.
\]
Proof. Without loss of generality let us suppose that \( x_1 = 1 \). Let \( z \in H_{\varphi_g}(x, M) \) and set \( \frac{1}{2} \log R := \log M - K_{\Delta^2}(\varphi_g(0), z) \). Thus

\[
\lim_{s \to 1} K_{\Delta^2}(z, \varphi_g(s)) - K_{\Delta^2}(\varphi_g(0), \varphi_g(s)) < \frac{1}{2} \log R,
\]

which implies \( z \in \mathbb{B}_{(1, \lambda_j)}(x, R) \). By Theorem 29 there exists a point \( y \in \partial \Delta^2 \) and a complex geodesic \( \varphi_y \) passing through \( y \), parameterized by \( \varphi_y(z) = (y, \bar{g}(z)) \) (with \( \bar{g} \in \text{Hol}(\Delta, \Delta) \) and \( \lambda_y = \frac{\lambda_j}{c_g} \)) such that \( f(z) \in \mathbb{B}_{(1, \lambda_j)}(y, \lambda_i R) \). Without loss of generality let us suppose that \( y = (1, 1) \). In particular, by the very definition of Busemann sublevel sets, we have

\[
\frac{1}{2} \log \lambda_i R \geq \lim_{s \to 1} K_{\Delta^2}(f(z), \varphi_y(s)) - \omega(0, s)
\]

for \( j = 1, 2 \). Moreover let us notice that

\[
-\omega(0, f_j(z)) \leq \omega(f_j(z), s) - \omega(0, s) \forall s \in (0, 1).
\]

For sake of cleanness we argue for \( j = 1 \). Thus we have that

\[
\lim_{s \to 1} \omega(f_1(z), s) - \omega(0, s) - \omega(0, f_1(z)) \leq \log \lambda_i R
\]

and then

\[
\frac{|1 - f_1(z)|^2}{1 - |f_1(z)|^2} \frac{1 - |f_1(z)|}{1 + |f_1(z)|} \leq \left( \frac{1 - f_1(z)}{1 + |f_1(z)|} \right)^2 \leq (\lambda_i R)^2.
\]

Furthermore we know that

\[
-2K_{\Delta^2}(\varphi_g(0), z) \leq 2K_{\Delta^2}(\varphi_g(0), 0) - 2K_{\Delta^2}(0, z)
\]

\[
= \log \left( \frac{1 + ||\varphi_g(0)||}{1 - ||\varphi_g(0)||} \right) \frac{1 - ||z||}{1 + ||z||} = \log \frac{1 + |g(0)|}{1 - |g(0)|} \frac{1 - ||z||}{1 + ||z||}
\]

with \( ||z|| = \max_j \{ z_j \} \). By definition of \( R \), setting \( c_g := \frac{1 + |g(0)|}{1 - |g(0)|} \), we get

\[
\frac{|1 - f_i(z)|}{1 + |f_i(z)|} \leq \lambda_i M^2 c_g \frac{1 - ||z||}{1 + ||z||} \leq \lambda_i M^2 c_g \frac{1 - |z_i|}{1 + |z_i|} \text{ for } i = 1, 2.
\]

If \( i = 1 \), being \( \pi(z_1, z_2) = z_1 \), then

\[
\frac{|1 - f_1(z)|}{1 - |\pi(z)|} \leq \frac{|1 - f_1(z)|}{1 - |\pi(z)|} \leq \lambda_i M^2 c_g \frac{1 + |f_1(z)|}{1 + |z_i|} \leq 2\lambda_i M^2 c_g
\]

and if \( i = 2 \)

\[
\frac{|1 - f_1(z)|}{1 - |z_2|} \leq \frac{|1 - f_1(z)|}{1 - |z_2|} \leq 2\lambda_i M^2 c_g.
\]

With the same techniques we proved the statement for the second component \( f_2 \).

\[\square\]

Lemma 33. Let \( f \in \text{Hol}(\Delta^2, \Delta^2) \) be a holomorphic function and \( x \in \partial \Delta^2 \). Suppose there exists a complex geodesic \( \varphi_g \) passing through \( x = (x_1, x_2) \), \( |x_1| = 1 \), and parameterized by \( \varphi_g(z) = (z, g(z)) \), with \( g \in \text{Hol}(\Delta, \Delta) \) such that

\[
\frac{1}{2} \log \lambda_j = \lim_{t \to 1} K_{\Delta^2}(0, \varphi_g(tx_1)) - \omega(0, f_j(\varphi_g(tx_1))) < \infty \text{ for } j = 1, 2.
\]
Let \( \varphi_{g}(z) : \Delta^{2} \to \Delta \) be a \( g \)-left inverse of \( \varphi_{g} \) given by \( \varphi_{g}(z_{1}, z_{2}) = z_{1} \). Then there exists a point \( y = (y_{1}, y_{2}) \in \partial\Delta^{2} \) such that, for \( j = 1, 2 \),

\[
\lim_{s \to 1} \frac{1 - f_{j}(\varphi_{g}(sx_{1}))}{1 - sx_{1}} = \frac{\lambda_{j}}{\max\{1, \frac{1}{\lambda_{g}}\}} \quad \text{and} \quad \frac{1 - f_{j}(\varphi_{g}(sx_{1}))}{1 - g(sx_{1})} = \frac{\lambda_{j}}{\lambda_{g} \max\{1, \frac{1}{\lambda_{g}}\}}.
\]

(7.1)

(7.2)

**Proof.** Let us suppose that \( x_{1} = 1 \). Let \( y \) the point given in Theorem 29 and without loss of generality let us suppose that \( y = (1, 1) \). By equation (6.2) we get

\[ \lambda_{j} = \lambda_{f \circ g} \max\{1, \frac{1}{\lambda_{g}}\} \]

thus, for the classical Julia-Wolff-Caratheodory Theorem in the unit disc.

\[ \square \]

And now we are ready to prove Theorem 31.

**Proof of Theorem 31.** Let us suppose \( x_{1} = 1 \). By Lemma 33 we have

\[ \lim_{s \to 1} \frac{1 - |f_{j}(\varphi_{g}(s))|}{1 - s} = \frac{\lambda_{j}}{\max\{1, \frac{1}{\lambda_{g}}\}} \]

and by Lemma 32 we know that the function \( \left| \frac{1 - f_{j}(z)}{1 - \varphi_{g}(z)} \right| \) is \( K_{g} \)-bounded. Then, since the curve \( \varphi_{g}(s) \) is \( (g, \pi_{g}) \)-special and \( (g, \pi_{g}) \)-restricted, the conclusion of the proof follows by Theorem 24.

\[ \square \]

8. Application to the dynamics

Let \( f \in \text{Hol}(\Delta, \Delta) \) be without fixed points in \( \Delta \). The classical Wolff lemma ensures the existence of a unique point \( \tau \in \partial\Delta \) such that every horocycle centered in \( \tau \) is sent in itself by \( f \). The point \( \tau \) is called the Wolff point of \( f \).

Let \( n \in \mathbb{N} \), and set \( f^{n} = f \circ \cdots \circ f \) the composition of \( f \) with itself \( n \)-times. We say that \( \{f^{n}\}_{n \in \mathbb{N}} \) is the sequence of iterates of \( f \). The Wolff-Denjoy lemma says that \( \{f^{n}\} \) converges uniformly on compacta to the Wolff point \( \tau \).

If we call target set, \( T(f) \), the set of the limit points of the sequence of the iterates and we denote by \( W(f) \) the set of the Wolff points of \( f \), then in one complex variable we have that \( T(f) = W(f) = \{\tau\} \).

In [16] we considered \( f \in \text{Hol}(\Delta^{2}, \Delta^{2}) \) without fixed points in \( \Delta^{2} \) and we defined the Wolff points of \( f \) using the small and big horospheres:

**Definition 34.** Let \( f \in \text{Hol}(\Delta^{2}, \Delta^{2}) \) be without fixed points in \( \Delta^{2} \). A point \( \tau \in \partial\Delta^{2} \) is a Wolff point for \( f \) if \( f(E(\tau, R)) \subseteq E(\tau, R) \), for all \( R > 0 \).

In this setting, in [16] (see also [15]) we characterized the set of the Wolff points, \( W(f) \), for a holomorphic self map \( f \) of the bidisc without fixed points. As a spinning result (see also Hervé [19]), we find a relation between \( W(f) \) and \( T(f) \), where \( T(f) \) is the target set of \( f \) defined as follows ([15],[16]):

\[ T(f) := \{x \in \Delta^{2} : \exists \{k_{n}\} \subset \mathbb{N}, \exists z \in \Delta^{2} \text{ such that } f^{k_{n}}(z) \to x \text{ as } n \to \infty\}. \]

It turns out that this result can be improved using the Busemann functions.
Let us observe that there exists a Wolff point, there exists a holomorphic function. Let there exists a Wolff point, first type. The holomorphic map, there exists a Wolff point, third type. there exists a holomorphic function, second type.

Remark 36. Let us observe that $W_G(f)$ is arcwise connected. The proof is the same of proposition 3.14 in [16] (see also [15]).

In order to state the result which characterizes the set $W_G(f)$, we need to introduce some definitions and results. Hervé proved the following useful theorem (see [19] Theorem 1):

**Theorem 37.** Let $f : \Delta^2 \to \Delta^2$ be a holomorphic map, without interior fixed points in $\Delta^2$, whose components are $f_1, f_2$. Then either

1. there exists a Wolff point, $e^{\theta_1}$, of $f_1(\cdot, y)$, which does not depend on $y$ or
2. there exists a holomorphic function $F_1 : \Delta \to \Delta$, such that $f_1(F_1(y), y) = F_1(y), \forall y \in \Delta$. In this case $f_1(x, y) = x \Rightarrow x = F_1(y)$.

Let us remark that, if $f \neq id_{\Delta^2}$, then cases i) and ii) cannot hold at the same time. Motivated by the last mentioned result of Hervé we give the following definition ([15], [16]):

**Definition 38.** The holomorphic map $f : \Delta^2 \to \Delta^2$, whose components are $f_1, f_2$, is called of:

1. **first type** if:
   - there exists a holomorphic function $F_1 : \Delta \to \Delta$, such that $f_1(F_1(y), y) = F_1(y), \forall y \in \Delta$ and
   - there exists a holomorphic function $F_2 : \Delta \to \Delta$, such that $f_2(x, F_2(x)) = F_2(x), \forall x \in \Delta$.
2. **second type** if (up to switching $f_1$ with $f_2$):
   - there exists a Wolff point, $e^{\theta_1}$, of $f_1(\cdot, y)$ (necessarily independent of $y$), and
   - there exists a holomorphic function $F_2 : \Delta \to \Delta$, such that $f_2(x, F_2(x)) = F_2(x), \forall x \in \Delta$.
3. **third type** if:
   - there exists a Wolff point, $e^{\gamma_1}$, of $f_1(\cdot, y)$ (independent of $y$), and
   - there exists a Wolff point, $e^{\gamma_2}$, of $f_2(x, \cdot)$ (independent of $x$).

In case $f$ is of **first type** and without interior fixed points in $\Delta^2$, then it turns out that $F_1 \circ F_2$ and $F_2 \circ F_1$ have a Wolff point (see Lemma 3.10 in [16] and also [15]). Let $e^{\theta_1}$ (respectively $e^{\theta_2}$) be the Wolff point of $F_1 \circ F_2$ (respectively $F_2 \circ F_1$). We also let $\lambda_1$ and $\lambda_2$ be, respectively, the boundary dilation coefficients of $F_1$ at $e^{\theta_1}$ and of $F_2$ at $e^{\theta_2}$ (see Lemma 3.10 in [16] and also [15]). In case $f$ is of **second type** we denote by $e^{\alpha_1}$ the Wolff point of $f_1(\cdot, y)$, by $e^{\alpha_2}$ the non-tangential limit of $F_2$ at $e^{\alpha_1}$ and $k_2 := \lim_{x \to e^{\alpha_1}} |F_2'(x)|$. In case $f$ is of **third type**, we set $e^{\gamma_1}$ and $e^{\gamma_2}$ to be, respectively, the Wolff points of $f_1(\cdot, y)$ and $f_2(x, \cdot)$. Let $\pi_j : \Delta^2 \to \Delta (j = 1, 2)$ be the projection on the $j$–th component. Finally, without loss of generalization, we suppose that $e^{\theta_1} = e^{\theta_2} = e^{\lambda_1} = e^{\alpha_2} = e^{\gamma_1} = e^{\gamma_2} = 1$. With the above established notations we proved (see [16]) the following result:
Let $f = (f_1, f_2)$ be a holomorphic map, without fixed points in the complex bidisc. If $f_1 \neq \pi_1$ and $f_2 \neq \pi_2$, then only the following five cases are possible:

i) $W(f) = \emptyset$ if and only if $f$ is of first type and $\lambda_i > 1$ for either $i = 1$ or $i = 2$;

ii) $W(f) = (1, 1)$ if and only if $f$ is of first type and $\lambda_i \leq 1$ for each $i = 1, 2$;

iii) $W(f) = \{(1) \times \Delta \} \cup \{(1, 1)\}$ if $f$ is of second type and $k_2 \leq 1$;

iv) $W(f) = \{(1) \times \Delta \}$ if $f$ is of second type and $k_2 > 1$;

v) $W(f) = \{(1) \times \Delta \} \cup \{(1, 1)\} \cup \{\Delta \times \{1\}\}$ if $f$ is of third type.

On the other hand, if $f_1(x, y) = x, \forall x \in \Delta$, i.e. if $f_1 = \pi_1$ (or respectively $f_2(x, y) = y, \forall x \in \Delta$, i.e $f_2 = \pi_2$) then:

vi) $W(f) = (1 \times \Delta) \cup (1, 1) \cup (\Delta \times 1) \cup (1, 1) \cup (1 \times \Delta)$ where 1 is the Wolff point of $f_2(x, \cdot)$

(or respectively $W(f) = (\Delta \times 1) \cup (1, 1) \cup (1 \times \Delta) \cup (1, 1) \cup (\Delta \times 1)$ where 1 is the Wolff point of $f_1(\cdot, y)$).

With the same techniques used in the proof of Theorem 39 ([15], [16]) we also get the following characterization of the generalized Wolff points of $f$:

**Theorem 40.** Let $f = (f_1, f_2)$ be a holomorphic map, without fixed points in the complex bidisc. If $f_1 \neq \pi_1$ and $f_2 \neq \pi_2$, then the following three cases are possible:

i) $W_G(f) = (1, 1)$ if and only if $f$ is of first type.

ii) $W_G(f) = \{(1) \times \Delta\} \cup \{(1, 1)\}$ if $f$ is of second type;

iii) $W_G(f) = \{(1) \times \Delta\} \cup \{(1, 1)\} \cup \{\Delta \times \{1\}\}$ if $f$ is of third type.

**Remark 41.** Let us notice that if $\tau$ is contained in a flat component of the boundary then $\tau$ is a Wolff point if and only if $\tau$ is a generalized Wolff point. On the other hand, let us consider a point $\tau$ of the Silov boundary of the bidisc. If $\tau$ is a Wolff point for $f$ then it is also a generalized Wolff point for $f$. The converse is, in general, false [16] (see also [15]). It is sufficient to look at the points $i$ of Theorems 39 and 40. Indeed those theorems show that if $f$ is of first type and $\lambda_i > 1$ for either $i = 1$ or $i = 2$, then the point $(1, 1)$ is not a Wolff point but it is a generalized Wolff point.

It is interesting to notice that, if $f$ is of first type, using the result of Hervé [19], about the target set of $f$, then we get that $W_G(f) \equiv T(f)$ otherwise $T(f) \subseteq W_G(f)$.

**References**