Simplicial Complexes in Finite Groups

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Introduzione

L'ambito generale di ricerca di questa tesi riguarda le interrelazioni tra la teoria dei gruppi finiti e alcuni aspetti della topologia algebrica. Il legame tra queste due aree risiede nel fatto che ad ogni gruppo finito sono associati in maniera naturale degli insiemi parzialmente ordinati (detti per brevità *posets*), i quali vengono considerati come complessi simpliciali, in modo standard. L'intento di fondo è quello di comprendere da un lato la struttura del gruppo finito dall'analisi, in chiave topologica (quali ad esempio proprietà di contraibilità e connessione) di tali posets associati ad esso, dall'altro come la topologia di questi possa dipendere da determinate caratteristiche algebriche del gruppo. Per chiarire meglio questi aspetti riporto qui sotto alcune considerazioni storiche sul complesso di Brown di un gruppo finito, che è proprio il complesso simpliciale che ha dato origine a questo tipo di studio, e su cui mi sono maggiormente concentrato.

Il complesso di Brown, così chiamato perchè appare per la prima volta in letteratura coi lavori di Kenneth Brown ([15] e [16]) del 1975, è il complesso simpliciale di tutti i sottogruppi non banali di un gruppo finito che hanno per ordine una potenza di p, dove p è un divisore primo dell'ordine del gruppo; viene denotato con $\mathcal{S}_p$. L'interesse di Brown è per lo più rivolto ad alcune applicazioni alla coomologia di gruppi discreti . Uno dei suoi risultati più interessanti, generalmente visto come una variante del teorema di Sylow, afferma che per ogni gruppo finito G la caratteristica di Eulero di $\mathcal{S}_p(G)$ è congruente ad 1 modulo la $p-$parte dell'ordine di G. È però nel 1978 coll'articolo di Daniel Quillen ([44]) che lo studio di $\mathcal{S}_p$ acquista una rilevanza intrinseca e indipendente da problemi di coomologia. Quillen osserva che $\mathcal{S}_p$ è omotopicamente equivalente al complesso dei $p-$sottogruppi abeliani elementari, nota che questo è contraibile se il gruppo ammette un $p-$sottogruppo normale non banale, e propone come congettura, che rimarrà famosa, la validità dell'implicazione opposta, ovvero: ogni gruppo il cui complesso di Brown per un certo primo p è contraibile deve avere necessariamente un $p-$sottogruppo normale non banale. Inoltre, Quillen si accorge che per i gruppi finiti di Chevalley in caratteristica p, $\mathcal{S}_p$ è omotopicamente equivalente al *building* di Tits. Ciò contribuisce ad arricchire l'interesse per questo complesso, che così viene visto come un analogo, per tutti i gruppi e per tutti i divisori primi dei loro ordini, di un concetto geometrico tipico dei gruppi di tipo Lie. Dall'articolo di Quillen in poi lo studio

del complesso di Brown si è sviluppato in diverse direzioni. Da un lato alcuni matematici (tra i quali ricordiamo T. Hawkes, M. Isaacs, J. Alperin, G. Robinson, P. Kleidman, M. Aschbacher e S. Schmidt) hanno dato significativi contributi alla dimostrazione della congettura di Quillen. Questa, sebbene rimanga tuttora un problema aperto, è stata provata sotto opportune restrizioni al gruppo, ad esempio se esso è $p-$risolubile, oppure semplice, quasisemplice, o più in generale se possiede un $p'-$sottogruppo $q-$iperelementare (cioè estensione di un sottogruppo ciclico per un $q-$gruppo, dove q è un primo diverso da p) che agisce per coniugio sui vertici di $\mathcal{S}_p$ senza avere punti fissi. La vastità dei casi esaminati porta a supporre che la congettura sia effettivamente vera. A tal riguardo è bene ricordare quello che, a rigore, è ritenuto il risultato più significativo, dovuto a M. Aschbacher e S. Schmidt ([5]), che dimostrano la validità della congettura per i primi $p \geq 7$ e per tutti i gruppi finiti privi di fattori di composizione isomorfi al gruppo unitario $U(n, q)$ con $q \equiv -1 \pmod{p}$.
Richiamo l'attenzione del lettore sul fatto che, almeno da un punto di vista teorico, non risulta difficile provare che un dato gruppo ammette un complesso di Brown, per qualche primo, non contraibile (basterebbe infatti mostrare che l'omologia del complesso non è banale, in genere questo si fa provando che il gruppo di omologia di dimensione massima non è nullo), d'altro canto uno studio di questo tipo è ben lontano dal determinare il tipo di omotopia effettivo del complesso simpliciale. Lo scopo del mio studio è stato quello di colmare parzialmente questa lacuna, e il Teorema 3.4.6 si colloca in quest'ottica.
In un'altra direzione, più combinatoria, si sono invece orientati altri studiosi, quali J. Thévenaz e P. Webb (in [53]) e T. Hawkes, I.M. Isaacs e M. Ozaydin (in [32]). Nei loro lavori si analizzano le caratteristiche di Eulero di sottocomplessi di $\mathcal{S}_p$ e di complessi ad essi associati. Questa analisi porta, tra l'altro, a trovare dimostrazioni alternative, apprezzabili per semplicità ed eleganza, di teoremi "classici" quali ad esempio il sopra citato risultato di K. Brown, nonché un teorema di Frobenius sul numero di elementi del gruppo G aventi ordine divisibile per un divisore naturale di $|G|$, oppure una generalizzazione di P. Hall a un teorema di Kulakoff (si veda 4.12 in [53]).
Altri aspetti topologici del complesso di Brown sono stati presi in considerazione, quali la connessione e la connessione semplice. Già Quillen nel suo lavoro dà condizioni necessarie e sufficienti, sulla struttura di un gruppo, affinché abbia un complesso di Brown connesso. Questo equivale infatti a richiedere che il gruppo in questione non possegga sottogruppi *strongly* $p-$*embedded*. La classificazione dei gruppi che ammettono sottogruppi strongly $p-$embedded fu completata da H. Bender (per $p = 2$, [7]) e da D. Gorenstein e R. Lyons (nel caso p dispari, [29]). I legami tra la connessione di $\mathcal{S}_p(G)$ e il teorema di fusione di J. Alperin sono stati messi in luce da L. Puig ([42]). Invece sembra essere più complicato determinare i gruppi con complesso di Brown semplicemente connesso. Ovviamente questo problema ha senso se il gruppo G non ha $p-$sottogruppi normali non banali, poiché in caso contrario $\mathcal{S}_p(G)$ è semplicemente connesso in quanto contraibile. Sotto questa assunzione, una condizione necessaria è che G abbia $p-$rango almeno 3, ovvero che

esistano dei sottogruppi abeliani elementari d'ordine p^4. M. Aschbacher congettura che, sotto questa condizione, $\mathcal{S}_p(G)$ è sempre semplicemente connesso, e la congettura è stata verificata in molti casi (si veda [4], [25], [26] e [48]).

Per ulteriori informazioni sul complesso di Brown e le problematiche ad esso inerenti, rimando il lettore, oltre ai lavori citati, all'articolo di Webb ([60]).

Tornando al mio lavoro, questa Tesi è articolata in cinque capitoli.

Il primo è un sunto di alcuni concetti e risultati di topologia algebrica. Si espone la teoria dei complessi simpliciali (finiti) mantenendo un approccio di fondo combinatorio, ovvero partendo dai posets e mostrando come ad essi è associato in maniera naturale un complesso simpliciale e di conseguenza una topologia. Si tratta di concetti ben noti che sembrano essere stati introdotti per la prima volta negli anni '40 da P.S. Aleksandrov, e che hanno indubbiamente avuto il loro fiorire coi lavori di Gian-Carlo Rota sui fondamenti della teoria combinatoria negli anni '60 ([45]).

Nel capitolo II vengono raccolti criteri per determinare il tipo di omotopia di uno spazio topologico. Nella prima sezione del capitolo espongo alcuni risultati classici (quali il "Lemma di incollamento" e il "Teorema del Nerve"), nonché il più recente Lemma della Fibra di Quillen, di cui farò largo uso nei capitoli successivi. La seconda metà del capitolo è concentrata su risultati di *teoria degli spazi e dei limiti diretti*. Questa è una teoria omologica a cui hanno collaborato intorno agli anni '60 vari matematici, quali D. Quillen, D. Anderson, G. Segal, A. Bousfield, D. Kan e più recentemente T. tom Dieck, R. Vogt, G. Ziegler e R. Živaljeviź. Il mio interesse in questo settore è rivolto principalmente alla problematica di determinare il tipo di omotopia di uno spazio topologico a partire da un arrangiamento di spazi che lo ricoprono. Le potenzialità di queste tecniche, proprio nello studio del complesso di Brown, sono state messe in luce da J. Pulkus e V. Welker (in [43]). L'uso che ne faccio sarà chiaro nei capitoli III e IV.

Il capitolo III tratta del già citato complesso di Brown. Le due sezioni introduttive sono una esposizione di risultati di base e del legame che intercorre coi buildings di Tits (per gruppi di tipo Lie). La terza sezione è invece un contributo originale allo studio del complesso di Bouc, $\mathcal{B}_p$; in particolare si prova una stima della $p-$lunghezza di un gruppo $p-$risolubile in rapporto alla dimensione di $\mathcal{B}_p$. Nella quarta sezione sta invece il cuore del capitolo e, direi, di tutto il lavoro. Affronto il problema di determinare il tipo di omotopia del complesso di Brown, almeno nel caso di gruppi "abbastanza trattabili" . Facendo uso delle tecniche topologiche descritte nel capitolo II, provo che, per gruppi risolubili di $p-$lunghezza 1, questo ha il tipo di omotopia di un bouquet di sfere di dimensione non necessariamente costante (Teorema 3.4.6); ciò fornisce parzialmente una risposta a una domanda di J. Thévenaz. Il capitolo si conclude poi con una sezione di carattere combinatorio, dove a partire dalla struttura di un gruppo risolubile G di $p-$lunghezza arbitraria costruisco un gruppo risolubile $\widetilde{G}$ di $p-$lunghezza 1. Provo che i due complessi $\mathcal{S}_p(G)$ e $\mathcal{S}_p(\widetilde{G})$ hanno la stessa caratteristica di Eulero. Si noti che, qualora risultasse che questi sono anche omotopicamente equivalenti, allora il teorema precedente sul tipo di omotopia si estenderebbe a tutta la classe dei gruppi risolubili.

Nel capitolo IV viene preso in considerazione il complesso di tutti i sottogruppi nilpotenti non banali di un gruppo finito. Dopo una sezione introduttiva, mi concentro sullo studio di questo complesso per gruppi metanilpotenti e provo, in questa situazione, una formula di decomposizione per il tipo di omotopia. Da tale formula si ricavano, per gruppi $p-$nilpotenti, risultati simili al caso del complesso di Brown.

Il capitolo V tratta di complessi simpliciali definiti da laterali di un gruppo. Nella prima sezione riporto alcuni risultati di K. Brown inerenti il complesso di tutti i laterali propri del gruppo. In particolare, mi soffermo sul legame tra la caratteristica di Eulero di questo e la funzione probabilistica generatrice dell'intero gruppo. Nello stesso lavoro Brown suggerisce di analizzare il sottocomplesso dei "$p-$laterali", ovvero di tutti i laterali che hanno come sottogruppo rappresentativo un $p-$sottogruppo, eventualmente banale. Nella seconda sezione del capitolo affronto questo studio e mostro con un'osservazione e un esempio che questo complesso è topologicamente ben lungi dall'essere unione di sfere, e può dare luogo a spazi topologici che sono unioni di iper-tori.

Introduction

The general research field of this Thesis concerns the interactions between finite group theory and some aspects of algebraic topology. The connection between these two areas lies in the fact that to each finite group there are associated, in natural ways, some partially ordered sets (called simply *posets*), which are thought to be simplicial complexes. The aim of this study is on one hand to understand the structure of the finite group from a topological analysis (for instance in terms of contractibility or connection) of these posets associated to it, and on the other hand how their topology can be influenced by some algebraic properties of the group. In order to be more explicit, I would like to report some historical notions on the Brown complex of a finite group. This is the simplicial complex that gave birth to this current of research and upon which I focused my work to a greater extent.

The Brown complex is given this name since its appearance for the first time in literature in the works of Kenneth Brown ([15], [16]) in 1975. It consists in the simplicial complex of all non trivial subgroups of a finite group whose orders are some powers of a prime number p, dividing the order of the group. It is denoted by $\mathcal{S}_p$. Brown's interest is above all devoted to the applications to cohomology of discrete groups. One of his more interesting results, generally seen as a variant of Sylow theorem, says that for every finite group G, the Euler characteristic of $\mathcal{S}_p(G)$ is congruent to 1 modulo the $p-$part of the order of G. Indeed it is in 1978, with the work of Daniel Quillen ([44]), that the study of $\mathcal{S}_p$ obtains a relevant and intrinsic interest, independent from cohomological problems. Quillen notes that $\mathcal{S}_p$ is homotopy equivalent to the complex of elementary abelian $p-$subgroups, and that this is contractible if the group has a non trivial normal $p-$subgroup. He conjectures also the opposite implication, which means: every finite group whose Brown complex at the prime p is contractible, necessarily contains a non trivial normal $p-$subgroup. Moreover, Quillen proves that for finite Chevalley groups in characteristic p, $\mathcal{S}_p$ is homotopy equivalent to the *Tits building*. This contributes to the increase in interest in this complex, as now it is seen as an analog, for every finite group and every prime divisor of its order, as a geometric concept typical of groups of Lie type. From the article of Quillen onwards, the study of the Brown complex developed into several directions. Some mathematicians (among whom

we remember T. Hawkes, M. Isaacs, J. Alperin, G. Robinson, P. Kleidman, M. Aschbacher and S. Schmidt) gave significant contribution to the proof of the Quillen conjecture. Even if this still remains an open question, it has been proven under some restrictions on the group, for instance if the group is $p-$soluble, or simple, semisimple, or, more in general, if it contains a $q-$hyperelementary $p'-$subgroup (this means a $p'-$subgroup which is an extension of a cyclic subgroup by some $q-$subgroup, where q is a prime number different from p) acting by conjugation on the vertices of $\mathcal{S}_p$ with no fixed points. The great amount of cases that have been checked, brings us to think that the conjecture is actually true. It is useful to remember what is considered the most significant result on this problem, due to M. Aschbacher and S. Schmidt ([5]). They prove the truth of the conjecture for prime numbers $p \geq 7$ and for all finite groups with no composition factors isomorphic to the unitary group $U(n, q)$, with $q \equiv -1 \pmod{p}$.
Let me draw the reader's attention to the fact that, at least from a theoretical point of view, it is not difficult to prove that a given finite group admits a non contractible Brown complex at some prime (in fact it would be enough to prove that the homology of the complex is not trivial, generally this is done by proving that the homology group at its greatest dimension is not zero). On the other hand a study of this type is far from determining the effective homotopy type of the simplicial complex. The aim of my research is partially to resolve this gap. Theorem 3.4.6 expresses this point of view.
Another direction, more combinatorial, was taken by some other mathematicians, such as J. Thévenaz and P. Webb (in [53]) and T. Hawkes, I.M. Isaacs and M. Ozaydin (in [32]). In their works, the Euler characteristics of some subcomplexes of $\mathcal{S}_p$ have been studied. This analysis brings, furthermore, to find alternative proofs, valuable as a result of their simplicity and elegance, of some classical facts, such as, the cited theorem by Brown, or a theorem by Frobenius on the number of elements of a group G whose order is a natural divisor of $|G|$, or a generalization of P. Hall to a theorem by Kulakoff (see 4.12 in [53]).
Other topological aspects of the Brown complex have been considered, such as its connection and simply connection. Quillen himself gives necessary and sufficient conditions, on the structure of a group, to have a connected Brown complex. This is equivalent to requiring that the group has no *strongly $p-$embedded* subgroups. The classification of the finite groups which admit strongly $p-$embedded subgroups was completed by H. Bender (for $p = 2$, [7]) and by D. Gorenstein e R. Lyons (for p odd, [29]). The links between the connection of $\mathcal{S}_p(G)$ and the theorem of J. Alperin have been pointed out by L. Puig ([42]). On the contrary, it seems more complicated determining the finite groups where the Brown complex is simply connected. Obviously this problem makes sense if the group G has no non trivial normal $p-$subgroups, otherwise the Brown complex is of course simply connected, being contractible. Under this assumption, a necessary condition is that the group G has $p-$rank at least 3, in other words that there exist elementary abelian $p-$subgroups of order p^4. M. Aschbacher conjectures that, under this condition, $\mathcal{S}_p(G)$ is always simply connected. The conjecture has been proven in

many cases (see [4], [25], [26] and [48]).

For further information on the Brown complex, and the problems involved, a keen reader can consult the cited references and the article by Webb ([60]).

Coming to my work, this Thesis is divided into five chapters.

The first is a summary of some concepts and results of algebraic topology. The theory of (finite) simplicial complexes is exposed under a combinatorial background approach, this means starting from posets and showing how a simplicial complex and, as a consequence, a topology are associated to them. These are well known concepts, introduced for the first time the 40s by P.S. Aleksandrov, and which flourished with the works of Gian-Carlo Rota on the foundation of combinatorial theory in the 60s (see [45]).

In Chapter II criteria have been brought together to determine the homotopy type of a topological space. In the first section of the chapter some classical results are exposed (such as the "Gluing Lemma" and the "Nerve Theorem"), as well as the more recent Fiber Lemma of Quillen, which will frequently be used in the following chapters. The second half of the chapter concentrates on some results of the *theory of spaces and direct limits*. This is an homological theory to which various mathematicians gave their contribution, such as D. Quillen, D. Anderson, G. Segal, A. Bousfield, D. Kan, in the 60s, and, more recently, T. tom Dieck, R. Vogt, G. Ziegler and R. Živaljeviź. My interest in this field is principally devoted to the problem of determining the homotopy type of a topological space, by knowing an arrangement of spaces which cover it. The potentialities of these techniques, in the study of simplicial complexes associated to finite groups, have been pointed out by J. Pulkus and V. Welker (in [43]). My use will be made clear in Chapters III and IV.

Chapter III deals with the aforementioned Brown complex. The two introductory sections are an exposition of some basic results and the link with the Tits buildings (for groups of Lie type). The third section is an original contribute to the study of the complex of Bouc, $\mathcal{B}_p$. In particular I prove a bound for the $p-$length of a $p-$soluble group in terms of the dimension of $\mathcal{B}_p$. The fourth section contains the "clou" of the whole Chapter, and, I could say, of all the Thesis. I study the homotopy type of the Brown complex, at least for groups which are "tractable enough". Making use of the topological techniques described in Chapter II, I prove that, the homotopy type of the Brown complex of a finite $p-$soluble group of $p-$length 1, is a wedge of spheres of possible different dimensions (Theorem 3.4.6). This partially answers a question by J. Thévenaz. The Chapter ends with a section of combinatorial nature. Starting from a soluble group G of arbitrary $p-$length, I build up a soluble group $\widetilde{G}$ of $p-$length 1. I prove the two simplicial complexes $\mathcal{S}_p(G)$ and $\mathcal{S}_p(\widetilde{G})$ have the same Euler characteristic. Note that, if these also proved to be homotopy equivalent, then the previous theorem on the homotopy type could be extended to the class of all soluble groups.

In Chapter IV the complex of all non trivial nilpotent subgroups of a finite group is taken in consideration. After an introductory section, I concentrate my study on this complex for metanilpotent groups. I prove, under this assumption, a decomposition formula for the homotopy type, which implies, in the case of $p-$nilpotent groups, results similar to the ones obtained for the Brown complex.

Chapter V treats the simplicial complexes which are defined by cosets in a group. In the first section I report some results of K. Brown on the complex of all proper cosets of a group. In particular, I stop to consider the connection between its Euler characteristic and the probabilistic function of generating a group. In the same article, K. Brown suggests studying the subcomplex of all "$p-$cosets", by these he means the cosets whose representative subgroup is either trivial or a $p-$group. In the second section, I deal with this study and I show with an observation and an example that this complex is far from being a set of spheres, as it can give birth to unions of hyper-tori.

Chapter 1

Algebraic Topology

This chapter is a review of basic facts concerning simplicial complexes. We suggest as good references the books of J.R. Munkres [40] and E.H. Spanier [51], as well as the valuable article of A. Björner in the Handbook of Combinatorics ([9]).

Throughout this chapter every map between topological spaces is silently assumed to be continuous.

1.1 Simplicial Complexes and Partially Ordered Sets

1.1.1 Simplicial Complexes

Definition 1.1.1. Let V be an arbitrary set. An *(abstract) simplicial complex* with vertex set V is a collection Δ of non empty finite subsets of V satisfying the following conditions:

1. for each $v \in V$, $\{v\} \in \Delta$;
2. if $\emptyset \neq \tau \subseteq \sigma \in \Delta$ then $\tau \in \Delta$.

Let Δ be a simplicial complex with vertex set V. We then write Δ^0 for the vertex set V of Δ, and call *simplices* (or *faces*) the elements of Δ. More specifically, if σ is a simplex of Δ, containing $n+1$ elements, we say that σ is a $n-$*simplex* (or also that σ is a simplex of dimension n). We define the *dimension* of Δ as the largest dimension of its simplices. Since we are above all interested in finite structures, from now on all simplicial complexes involved are assumed to be finite, thus the previous definition has always sense. For a $n-$simplex σ, we usually write $\sigma = \{v_0, v_1, \ldots, v_n\}$, where $v_i \in \Delta^0$. If σ is a simplex, any $\emptyset \neq \tau \subseteq \sigma$ is called a *face* of σ (by definition, $\tau \in \Delta$).

Note that this definition of simplicial complex allows the empty complex $\Delta = \emptyset$, to which we attribute dimension equal -1.
A subcomplex of a complex Δ is a subset of Δ which is closed by taking all faces of its elements (clearly a subcomplex is in turn a complex). If n is the dimension of Δ and $0 \le q \le n$, then the set of all simplices of Δ of dimension at most q is a subcomplex of Δ, which is called the *q-skeleton* of Δ and denoted by $\Delta^{\le q}$. We set $\Delta^r := \{r - \text{simplices of } \Delta\}$, so that $\Delta^{\le q} = \bigcup_{r \le q} \Delta^r$.
An $n-$dimensional complex Δ is said to be *pure* if every face is contained in an $n-$face, in this case the elements of Δ^n are called *facets* or *chambers*.

Definition 1.1.2. Let Δ_1 and Δ_2 be two simplicial complexes. A *simplicial map* $f : \Delta_1 \longrightarrow \Delta_2$ is an application between the vertex sets

$$f : \Delta_1^0 \longrightarrow \Delta_2^0$$

which maps all simplices of Δ_1 into simplices of Δ_2 (possibly of smaller dimension), i.e. such that

$$\text{if} \quad \sigma = \{x_0, x_1, \dots, x_q\} \in \Delta_1, \quad \text{then} \quad f(\sigma) = \{f(x_0), f(x_1), \dots, f(x_q)\} \in \Delta_2 \ .$$

f is an *isomorphism* between simplicial complexes if it is bijective and both f and f^{-1} are simplicial maps.

Definition 1.1.3. To any abstract simplicial complex Δ there is associated a topological (affine) space, called *geometrical realization* of Δ and denoted by $|\Delta|$. The elements of $|\Delta|$ are the formal "convex combinations"

$$\sum_{x \in \Delta^0} t_x x,$$

with t_x non-negative real numbers, $\sum_x t_x = 1$ and $\{x | t_x > 0\} \in \Delta$. The topology of $|\Delta|$ is so defined:

> $A \subseteq |\Delta|$ is a closed subset if and only if for every $\sigma \in \Delta$, the set
> $$\left\{ (t_x)_{x \in \sigma} \middle| \sum_{x \in \sigma} t_x x \in A \right\}$$
> is a closed subset in $\mathbb{R}^{|\sigma|}$.

Given a simplicial map $f : \Delta_1 \longrightarrow \Delta_2$, this induces, in a natural way, the affine map

$$\begin{array}{rcl} |f| : |\Delta_1| & \longrightarrow & |\Delta_2| \\ \sum_{x \in \Delta_1^0} t_x x & \longmapsto & \sum_{x \in \Delta_1^0} t_x f(x). \end{array}$$

Since affine maps are continuous on every closed simplex, $|f|$ is a continuous map between the two spaces $|\Delta_1|$ and $|\Delta_2|$. (Note that the finiteness of the complex Δ implies the topological space $|\Delta|$ to be compactly generated (see [59] on page 99), this means that a subset A of $|\Delta|$ is closed just when $A \cap C$ is closed in each compact subspace C of $|\Delta|$).

We set $\mathcal{S}$ the category whose objects are finite abstract simplicial complexes and whose morphisms are simplicial maps, and $\mathcal{T}$ the category whose objects are the (compactly generated) topological spaces and whose morphisms are continuous functions. Then the assignment $|\cdot|$ is a covariant functor from $\mathcal{S}$ to $\mathcal{T}$, and this allows us to introduce concepts and language from topology in the study of abstract simplicial complexes.

1.1.2 Partially Ordered Sets

Let $(P, \leq)$ be a partially ordered set (*poset* for short). A *chain* of *length* k in P is a totally ordered subset

$$x_0 < x_1 < \ldots < x_k$$

of $k+1$ elements of P. The *rank*, or *length*, of P is the maximal length of all chains in P. P is said *pure* if all maximal chains have the same finite length.

A poset P is a *lattice* if *every* pair of elements $x, y \in P$ has in P a least upper bound $(x \vee y)$, called *join*, and a greatest lower bound $(x \wedge y)$, called *meet*. If for every pair x, y in P is defined just the meet (join) we call P a *meet (join) semi-lattice*.

Let us fix some notation on posets.
Let P be a poset and x be one of its elements. We denote with $P_{\geq x}, P_{>x}, P_{\leq x}, P_{<x}$ respectively the subposets $\{y \in P | y \geq x\}$, $\{y \in P | y > x\}$, $\{y \in P | y \leq x\}$ and $\{y \in P | y < x\}$. We reserve the notation (x, y) for the *open interval* $P_{>x} \cap P_{<y}$, and $[x, y]$ for the *closed interval* $P_{\geq x} \cap P_{\leq y}$. A *bottom element* $\widehat{0}$ and a *top element* $\widehat{1}$ are elements of P satisfying $\widehat{0} \leq x$ and, respectively, $\widehat{1} \geq x$, for all $x \in P$. If both exist in P, then P is said *bounded* and with $\overline{P} := P \setminus \left\{\widehat{0}, \widehat{1}\right\}$ we denote the *proper part* of P. Moreover for arbitrary posets P, we set $\widehat{P} := P \cup \left\{\widehat{0}, \widehat{1}\right\}$ the extended poset by a new bottom, $\widehat{0}$, and a new top element, $\widehat{1}$.

Most of the simplicial complexes of this work derive from (finite) partially ordered sets. This construction goes back to Aleksandrov ([1]).

Definition 1.1.4. Let $(P, \leq)$ be any finite poset. To P there is associated a simplicial complex, written $\Delta(P)$ and called the *complex of chains* (or the *order complex*) of P, whose $q-$simplices are the chains of P of length q.

Note. If we indicate with P^{op} the *dual* poset of P, i.e. the poset which has the same underlying set of P and reversed ordering (namely, $x \leq_{\text{op}} y$ in P^{op} if and only if $x \geq y$ in P), then it is obvious that $\Delta(P) = \Delta(P^{\text{op}})$.

To any map between two posets $f : P \longrightarrow Q$, which is order-preserving (i.e. $x \leq y$ in P implies $f(x) \leq f(y)$), there corresponds a simplicial map

$$\Delta(f) : \Delta(P) \longrightarrow \Delta(Q)$$

so defined:

$$\text{if} \quad \sigma = \{x_0 < x_1 < \ldots < x_k\}\,, \quad \text{then} \quad \Delta(f)(\sigma) = \{f(x_0) \leq f(x_1) \leq \ldots \leq f(x_k)\}\,,$$

note that $\Delta(f)(\sigma)$ may have possible smaller dimension than σ.
In this way we have implicitly defined a covariant functor Δ from the category of finite posets and order-preserving maps, $\mathcal{P}$, to the one of finite simplicial complexes and simplicial maps, $\mathcal{S}$.
We remind the reader that this construction lies in a more general contest. In fact if the poset P is seen as a small category (i.e. a category for which the objects form a set), then $\Delta(P)$ is the so called *classifying space* of P (see [44] and [47]).

It is also possible to go in the opposite direction, as the following definition suggests

Definition 1.1.5. Let Δ be an abstract simplicial complex, and define $P(\Delta) := (\Delta, \subseteq)$ the *face poset* of Δ, consisting of all the faces of Δ partially ordered by inclusion.

We note that the poset $P(\Delta)$ is pure exactly when the complex Δ is pure. The rank of $P(\Delta)$ is just the dimension of Δ.

Definition 1.1.6. For a simplicial complex Δ, we define $\mathrm{sd}\Delta := \Delta(P(\Delta))$ the *(first) barycentric subdivision* of Δ.

A basic fact is that $|\Delta|$ and $|\mathrm{sd}\ \Delta|$ are homeomorphic topological spaces (see, for instance, [51], Ch. III, sec. 3). Therefore passing between $\mathcal{S}$ and $\mathcal{P}$, via the functors $P(\cdot)$ and $\Delta(\cdot)$, does not affect the topology. From a topological point of view, finite simplicial complexes and finite posets can so be considered equivalent.

By composing the functor $|\cdot|$ with $\Delta(\cdot)$ we obtain a covariant functor

$$|\Delta(\cdot)| : \mathcal{P} \longrightarrow \mathcal{T},$$

from (finite) posets to (compactly generated) topological spaces. In this way, finite posets get a topological structure, and, from now on, whenever we make a topological statement about a poset P, we will always have in mind its geometric realization $|\Delta(P)|$.

For the sake of simplicity we simply denote $|\Delta(P)|$ by $|P|$, when this does not cause confusion.

We inform the reader that this is not the only way to associate a useful topology to a poset P. At least another one, also due to Aleksandrov, consists in declaring open sets for P the order ideals of P (i.e. the subsets I of P such that if $x \in I$ and $y \leq x$, then $y \in I$). This gives rise to the so called *ideal topology* of P, in which, for instance, the continuous maps are exactly the order-preserving maps. For further readings on this topic we suggest [1], [6] and [63].

1.2 Operations on Spaces, Complexes and Posets

Here we briefly introduce some basic operations on topological spaces, simplicial complexes and posets we make wider use in the following chapters. We prove the functoriality of these operations under the geometrical realization functor

$$|\cdot| : \mathcal{P} \longrightarrow \mathcal{T}.$$

At the end of the section, some more terminology on complexes is introduced.

We begin with some remarks on topological spaces.
Given a topological space T and an equivalence relation on it, denoted by $\approx$, it is of course possible to associate a topological structure to the quotient set $T/\approx$, by declaring open sets the subsets $A \subseteq T/\approx$ such that $\pi^{-1}(A)$ is open in T (here $\pi : T \longrightarrow T/\approx$ is the projection map on the quotient). Thus, let $S_i, i \in I$, be pairwise disjoint subsets of a topological space T, we use the notation $T/(S_i)_{i\in I}$ to denote the quotient space obtained from T by identifying together all the points within each set S_i, $i \in I$. As examples

$$\mathbf{C}(T) := (T \times [0,1])\Big/(T \times \{1\})$$

is the *cone space* over T, and

$$\mathbf{S}(T) := (T \times [0,1])\Big/(T \times \{0\}, T \times \{1\})$$

is the *suspension* of T.
Making use of this notation, we define two operations between topological spaces.

Let $(T_i, x_i)_{i\in I}$ be a family of pointed spaces ($x_i \in T_i$) that are pairwise disjoint, then the *wedge space* of this family is by definition the quotient space

$$\bigvee_{i\in I} T_i := \bigcup_{i\in I} T_i\Big/(\bigcup_{i\in I} x_i).$$

Let T_1 and T_2 be two topological spaces, the *join space* of T_1 and T_2 is by definition the space

$$T_1 \star T_2 := (T_1 \times T_2 \times [0,1]) \Big/ (\{(\overline{x}, y, 0) | y \in T_2\}, \{(x, \overline{y}, 1) | x \in T_1\})_{\overline{x} \in T_1, \overline{y} \in T_2}.$$

According to the join operation, $\mathbf{C}(T) = T \star \{*\}$ and $\mathbf{S}(T) = T \star S^0$, where $\{*\}$ denotes the single point space and S^0 the $0-$dimensional sphere, i.e. the set constituted by two disjoint points, with trivial topology.

We define similar operations on simplicial complexes.

Let Δ_1 and Δ_2 be two simplicial complexes with disjoint vertex sets, the *product complex* $\Delta_1 \times \Delta_2$ is the complex whose underlying vertex set is the cartesian product $\Delta_1^0 \times \Delta_2^0$ of the vertex sets of Δ_1 and Δ_2, and whose $q-$simplices are the pairs (δ_1, δ_2) with $\delta_i \in \Delta_i$, $i = 1, 2$, and $\text{maxdim}\{\delta_1, \delta_2\} = q$.

The *join complex* is the complex

$$\Delta_1 \star \Delta_2 := \Delta_1 \cup \Delta_2 \cup \{\sigma \cup \tau | \sigma \in \Delta_1 \text{ and } \tau \in \Delta_2\},$$

whose $q-$simplices are either $q-$simplices of Δ_1, or of Δ_2, or of the form $\delta_1 \cup \delta_2$ with δ_i simplex of Δ_i for $i = 1, 2$, and $\dim(\delta_1) + \dim(\delta_2) = q$.

The *cone* and the *suspension* over the complex Δ can be respectively defined as $\mathbf{C}(\Delta) = \Delta \star \{*\}$ and $\mathbf{S}(\Delta) = \Delta \star S^0$.

Note that the geometrical realization functor $|\cdot|$ from $\mathcal{S}$ to $\mathcal{T}$ behaves well with respect the operations of direct product, " $\times$ ", and join, " $\star$ ". In fact, there are homeomorphisms

$$|\Delta_1 \times \Delta_2| \approx |\Delta_1| \times |\Delta_2|$$
$$|\Delta_1 \star \Delta_2| \approx |\Delta_1| \star |\Delta_2|.$$

In particular $|\mathbf{C}(\Delta)| \approx \mathbf{C}(|\Delta|)$ and $|\mathbf{S}(\Delta)| \approx \mathbf{S}(|\Delta|)$. (see [9], (9.5)).

We can go combinatorially deeper inside and define analogous operations on posets, too.

The *direct product*, $P \times Q$, of two posets P and Q is the poset whose underlying set is the cartesian product of P and Q and whose order is given by the following relation

$$(a, x) \leq (b, y) \iff a \leq b \text{ in } P \text{ and } x \leq y \text{ in } Q.$$

The *join poset* (or *ordinal sum*), $P \star Q$, is the poset whose underlying set is the disjoint union $P \sqcup Q$, and whose order is given by

$$x \leq y \iff \begin{cases} x \leq_P y & \text{if } x, y \in P \\ x \leq_Q y & \text{if } x, y \in Q \\ x \in P, y \in Q & \text{otherwise} \end{cases}$$

Practically, for finite posets, this means that we simply put Q over P in a way that *every* maximal elements of P is less then *every* minimal element of Q.

The cones and suspensions for finite posets are defined in the obvious way, i.e. $\mathbf{C}(P) := P \star \{*\}$ and $\mathbf{S}(P) := P \star \{0,1\}$, where $\{0,1\}$ is the poset consisting of two incomparable elements.

Clearly it holds that $\Delta(P \times Q) = \Delta(P) \times \Delta(Q)$ and $\Delta(P \star Q) = \Delta(P) \star \Delta(Q)$. Moreover, in the following Lemma, we collect the functoriality of the previous operations under the functor $|\cdot|$, from finite posets to topological spaces.

LEMMA 1.2.1. *Let P and Q be* finite *posets. Then there are canonical homeomorphisms:*

1. $|P \times Q| \approx |P| \times |Q|\,,$
2. $\left|(P \times Q)_{>(x,y)}\right| \approx |P_{>x}| \times |Q_{>y}|\,,$
3. $|P \star Q| \approx |P| \star |Q|\,,$
4. $|(\mathbf{C}(P) \times \mathbf{C}(Q)) \setminus \{(*_P, *_Q)\}| \approx |P| \star |Q|,$
   (where $*_R$ denotes the vertex of the cone $|\mathbf{C}(R)|$).

PROOF. The reader may find the proofs of these results in the articles of D. Quillen ([44], Section 1.2 and Proposition 1.9) and J. Walker ([59], Theorems 3.1, 3.2 and 5.1). Q.E.D.

In the following chapters we make also use of the following standard terminology

DEFINITION 1.2.1. Let Δ be a simplicial complex and δ be a simplex of Δ. We define:

the *star* of δ in Δ as the subcomplex of Δ of all the simplexes $\tau \in \Delta$ such that $\tau \cup \delta$ is still a simplex of Δ, i.e.

$$\mathrm{Star}_\Delta(\delta) := \{\tau \in \Delta | \tau \cup \delta \in \Delta\}\,;$$

the *link* of δ in Δ as the subcomplex of $\mathrm{Star}_\Delta(\delta)$ constituted by all the simplexes $\tau \in \mathrm{Star}_\Delta(\delta)$ disjoint from δ, i.e.

$$\mathrm{Link}_\Delta(\delta) := \{\tau \in \Delta | \tau \cup \delta \in \Delta \text{ and } \tau \cap \delta = \emptyset\}\,.$$

1.3 Homotopy & Homology.

Let T_1 and T_2 be two topological spaces, and set $I := [0,1]$ be the closed unit interval of $\mathbb{R}$.

DEFINITION 1.3.1. Two continuous maps $f_0, f_1 : T_1 \longrightarrow T_2$ are *homotopic* (written $f_0 \sim f_1$) if there exists a continuous map (called *homotopy*)

$$F : T_1 \times I \longrightarrow T_2$$

such that $F(t,0) = f_0(t)$ and $F(t,1) = f_1(t)$ for all $t \in T_1$.

DEFINITION 1.3.2. The spaces T_1 and T_2 are *homotopy equivalent* (or *of the same homotopy type*) if there exist two continuous maps $f_1 : T_1 \longrightarrow T_2$ and $f_2 : T_2 \longrightarrow T_1$ such that the compositions $f_2 \circ f_1 \sim \mathrm{id}_{T_1}$ and $f_1 \circ f_2 \sim \mathrm{id}_{T_2}$. In this case we write $T_1 \simeq T_2$.

Let $S^{n-1} := \{x \in \mathbb{R}^n | \, \|x\| = 1\}$ and $B^n := \{x \in \mathbb{R}^n | \, \|x\| \leq 1\}$ denote respectively the standard $(n-1)-$sphere and the $n-$ball of $\mathbb{R}^n$. Note that S^{-1} is the empty set, S^0 the set constituted by two disjoint points and B^0 is the single point set, also denoted by $\{*\}$.

DEFINITION 1.3.3. A topological space T is said *contractible* if it is homotopy equivalent to a point. T is said *$n-$spherical* if its homotopy type is the same as a wedge of some $n-$dimensional spheres. We reserve the terminology *set of spheres* to denote topological spaces any of whose connected component is a wedge of spheres of possible different dimensions.

We note that the class of spheres is closed (up to homeomorphism) under the join operation, this is the content of the next

PROPOSITION 1.3.1. $S^n \star S^m \approx S^{n+m+1}$.

PROOF. For a direct proof we refer to [8], Ch.II, sec.4. Otherwise it can be proved using formula (1.3.1) of the next section (see Corollary 1.3.4, *(ii)*). Q.E.D.

A similar statement of course holds for the classes of spherical spaces and of sets of spheres.

1.3.1 $k-$CONNECTEDNESS AND HOMOTOPY GROUPS.

DEFINITION 1.3.4. A space T is *$k-$connected* if for all $0 \leq i \leq k$ each continuous map $f : S^i \longrightarrow T$ can be extended to a continuous map $\widehat{f} : B^{i+1} \longrightarrow T$ such that for all $x \in S^i \subset B^{i+1}$, $\widehat{f}(x) = f(x)$.

Note that $0-$connected means just arcwise connected, and $(-1)-$connected simply means nonempty. Every space is always $k-$connected for $k \leq -2$.

Remark 1.3.1. The sphere S^n, and therefore any $n-$spherical space, is $(n-1)-$connected but not $n-$connected. Moreover, for simplicial complexes of finite dimension $n \geq 0$, being $(n-1)-$connected is properly equivalent to being $n-$spherical ([9], (9.19)).

$k-$connectedness is a homotopical property, in the sense that it is preserved passing between spaces of the same homotopy type. In fact it can be completely characterized via the homotopy groups of the space. For the rigorous definition of the *homotopy groups,* $\pi_i(T, x)$, $x \in T$, the reader can consult the cited references ([51], Ch. VI, sec. 2, or [9], 9.10 for instance), here, we just state in a separate remark the link between homotopy groups and $k-$connectedness.

Remark 1.3.2. A topological space T is $k-$connected if and only if for all $0 \leq i \leq k$ and all $x \in T$ the ith homotopy group $\pi_i(T, x)$ is trivial.

1.3.2 Homology Groups

For the definitions of the *homology groups* $H_i(\Delta, G)$ and the *reduced homology groups* $\widetilde{H}_i(\Delta, G)$, of a complex Δ with coefficients in an abelian group G, we refer to the books of J.R. Munkres ([40]) and E.H. Spanier ([51]).

We simplify the notation and set $\widetilde{H}_i(\Delta) := \widetilde{H}_i(\Delta, \mathbb{Z})$ for reduced the homology groups with integer coefficients. We set the degenerate cases: if Δ is the empty space, then $\widetilde{H}_{-1}(\emptyset) \simeq \mathbb{Z}$ and $\widetilde{H}_i(\emptyset) = 0$ for all $i \neq -1$, for $\Delta \neq \emptyset$, $\widetilde{H}_i(\Delta) = 0$ for all $i < 0$ and all $i > \dim \Delta$. Moreover $\widetilde{H}_0(\Delta) \approx \mathbb{Z}^{c-1}$ where c is the number of connected components of Δ.
The relationship between the homology groups and the reduced ones is expressed by the following equivalence

$$H_i(\Delta) \approx \begin{cases} \widetilde{H}_0(\Delta) \oplus \mathbb{Z} & \text{if } i = 0 \\ \widetilde{H}_i(\Delta) & \text{otherwise} \end{cases}$$

A basic fact is that the homology is preserved by homotopy equivalence in the sense of the following

Theorem 1.3.2. *Two homotopy equivalent spaces have isomorphic homology groups in all dimensions.*

Proof. We refer to [40], Theorem 19.2. Q.E.D.

Definition 1.3.5. A simplicial complex Δ is said $k-$*acyclic over* G if for all $i \leq k$, $\widetilde{H}_i(\Delta, G)$ is trivial. Δ is said *acyclic over* G if it is $k-$acyclic for all k.

Note that $(-1)-$acyclic just means nonempty, and $0-$acyclic means nonempty and connected.

We now list some results between homotopy and homology properties of a complex Δ. These are consequences of the Theorems of Hurewicz and Whitehead (see [51] Ch. VII, secs. 4,5, or [61], Ch. IV, sec. 7).

- Δ is $k-$connected if and only if Δ is $k-$acyclic (over $\mathbb{Z}$) and simply connected (i.e. connected and with trivial fundamental group $\pi_1(|\Delta|)$).
- Δ is contractible if and only if Δ is $\mathbb{Z}-$acyclic and simply connected.
- If Δ is simply connected and $\widetilde{H}_i(\Delta) = 0$ for all $i \neq d > 1$, while $\widetilde{H}_d(\Delta) \approx \mathbb{Z}^k$, for some k, then Δ is $d-$spherical (it is the wedge of exactly k $d-$spheres).
- The classes of acyclic and connected complexes are closed under the join operation. In fact if, for $i = 1, 2$, Δ_i are k_i-acyclic (k_i-connected) complexes, then $\Delta_1 \star \Delta_2$ is $(k_1 + k_2 + 2)-$acyclic ($-$connected). See Milnor [39].

The next result expresses the behavior of the homology groups with respect to the join operation

<u>Proposition</u> 1.3.3. *Let Δ_1 and Δ_2 be two finite simplicial complexes and suppose that at least one of $\widetilde{H}_p(\Delta_1)$ or $\widetilde{H}_q(\Delta_2)$ is torsion-free, whenever $p+q=i-1$. Then*

$$\widetilde{H}_{i+1}(\Delta_1 \star \Delta_2) \approx \bigoplus_{p+q=i} (\widetilde{H}_p(\Delta_1) \otimes \widetilde{H}_q(\Delta_2)) \ . \tag{1.3.1}$$

Note. The same decomposition formula holds, without any restrictions for reduced homology with coefficients in a field. For the proof of Proposition 1.3.3 and for further details we refer to the work of J. Milnor ([39]).

As immediate consequences of Proposition 1.3.3, we have the following

<u>Corollary</u> 1.3.4. *(i) The join of two complexes, one of which is contractible, is still contractible.*

(ii) The join of two spherical complexes is still a spherical complex.

Note that *(ii)* is an alternative proof of Lemma 1.3.1.

Associated to any (compactly generated) topological space there is an important homotopy invariant called *Euler characteristic* of the space. For finite simplicial complexes this can be defined in different ways. Let Δ be a complex and, for $i \geq 0$, let $\beta_i := \mathrm{rk}(\widetilde{H}_i(\Delta))$ be the torsion free rank of $i-$th homology group of Δ. The β_i are called *Betti numbers* and they satisfy the *Euler-Poincaré formula*:

$$\sum_{i\geq 0}(-1)^i \left|\Delta^i\right| = \sum_{i\geq 0}(-1)^i \beta_i$$

(a proof of this can be found in [51], Ch. 4, Theorem 14). Either side of the previous equation can be taken as the definition of the *Euler characteristic* of Δ, denoted by $\chi(\Delta)$. The *reduced Euler characteristic* is, by definition, $\widetilde{\chi}(\Delta) := \chi(\Delta) - 1$. In the following section we will see the combinatorial meaning of this.

1.4 Möbius Function

Definition 1.4.1. Let P be a finite poset. The *Möbius Function* on P is the application

$$\mu_P : P \times P \longrightarrow \mathbb{Z}$$

defined by, for all $a, b \in P$:

(i) $\mu_P(a,a) = 1$;

(ii) $\mu_P(a,b) = 0$ if $a \not\leq b$;

(iii) if $a < b$ then $\mu_P(a,b)$ is the number of chains of even length in (a,b) minus the number of chains of odd length (including the empty chain, which we view as having length -1).

From the definition we get immediately the following observation, which seems to have been first noticed by P. Hall, and which expresses the link with the Euler characteristic of $\Delta(P)$.

Proposition 1.4.1. *Let P be a finite poset. Then, for all $a, b \in P$ with $a < b$:*

$$\mu_P(a,b) = \chi(\Delta(a,b)) - 1 = \widetilde{\chi}(\Delta(a,b))) \ .$$

In particular $\mu(P) := \mu_P(\widehat{0}, \widehat{1}) = \widetilde{\chi}(\Delta(P))$.

If P is a poset, and $a, b \in P$ with $a < b$. Let us denote by $\mathcal{C}_b$ the set of all chains σ such that $a < \inf \sigma$ and $b = \sup \sigma$. Then the chains of even (odd) length in (a,b) are in 1-1 correspondence to the chains of odd (even) length of $\mathcal{C}_b$. Thus

$$\mu_P(a,b) = \sum_{\sigma \in \mathcal{C}_b} (-1)^{|\sigma|} \ .$$

Having observed that, we now prove a useful property of Möbius function.

Proposition 1.4.2. *Let P be a poset, and $a, b \in P$ with $a < b$. Then*

$$(iv) \qquad \sum_{a \leq x \leq b} \mu_P(a,x) = 0 = \sum_{a \leq x \leq b} \mu_P(x,b) \ .$$

Proof. Let $a, b \in P$ with $a < b$. We denote by $\mathcal{C}$ the set of all non-empty chains of (a,b), and for $\sigma \in \mathcal{C}$ we write $m(\sigma) = \sup \sigma$. Then, by definition of Möbius function:

$$\mu(a,b) = -1 - \sum_{\sigma \in \mathcal{C}} (-1)^{|\sigma|} = -\mu(a,a) - \sum_{a<x<b} \sum_{m(\sigma)=x} (-1)^{|\sigma|}$$

whence, by the observation made above,

$$\mu(a,b) = -\mu(a,a) - \sum_{a<x<b} \mu(a,x) = - \sum_{a \leq x < b} \mu(a,x)$$

proving the first equality. The second one is proved in a similar way. Q.E.D.

Indeed, it is important and not difficult to observe (see Rota [45]) that properties *(i), (ii)* and *(iv)* completely determine the function μ_P.

The fundamental property of Möbius function is the so-called *Möbius Inversion Formula.*

<u>Theorem</u> 1.4.3. *Let P be a poset, and G an abelian (additive) group.*

(i) Given an application $f : P \longrightarrow G$, define a map $F : P \longrightarrow G$ by setting, for all $x \in P$

$$F(x) = \sum_{x \leq y} f(y) \ .$$

Then, for all $x \in P$

$$f(x) = \sum_{x \leq y} \mu_P(x, y) F(y).$$

(ii) Conversely, given a map $H : P \longrightarrow G$, if $h : P \longrightarrow G$ is defined by $h(x) = \sum_{x \leq y} \mu_P(x, y) H(y)$, then

$$H(x) = \sum_{x \leq y} h(y) \ .$$

Proof. *(i)* For all $x \in P$, by applying property *(iv)* we get

$$\begin{aligned}\sum_{x \leq y} \mu_P(x, y) F(y) &= \sum_{x \leq y} \Big(\mu_P(x, y) \sum_{y \leq z} f(z) \Big) = \\ &= \sum_{x \leq z} \Big(f(z) \sum_{x \leq y \leq z} \mu_P(x, y) \Big) = f(x) \mu_P(x, x) = f(x).\end{aligned}$$

(ii) For all $x \in P$, by applying again property *(iv)* we get

$$\begin{aligned}\sum_{x \leq y} h(y) &= \sum_{x \leq y} \Big(\sum_{y \leq z} \mu_P(x, z) H(z) \Big) = \\ &= \sum_{x \leq z} \Big(H(z) \sum_{x \leq y \leq z} \mu_P(y, z) \Big) = H(x) \mu_P(x, x) = H(x).\end{aligned}$$

Q.E.D.

Note. It is clear that Möbius inversion formula holds, exchanging $\leq$ with $\geq$, also for the function $F^{\mathrm{op}}(x) = \sum_{y \leq x} f(y)$. More generally, if P^{op} is the opposite poset of P, then for all $a, b \in P$, $\mu_{P^{\mathrm{op}}}(b, a) = \mu_P(a, b)$.

We end up this part with an observation useful in computing the reduced Euler characteristic of disjoint unions, products, joins and wedges of simplicial complexes.

<u>Proposition</u> 1.4.4. *Let Δ_1, Δ_2 be two finite simplicial complexes. Then*

(i) $\widetilde{\chi}(\Delta_1 \sqcup \Delta_2) = \widetilde{\chi}(\Delta_1) + \widetilde{\chi}(\Delta_2) + 1.$

(ii) $\widetilde{\chi}(\Delta_1 \times \Delta_2) = \widetilde{\chi}(\Delta_1) \cdot \widetilde{\chi}(\Delta_2).$

(iii) $\widetilde{\chi}(\Delta_1 \star \Delta_2) = -\widetilde{\chi}(\Delta_1) \cdot \widetilde{\chi}(\Delta_2).$

(iv) $\widetilde{\chi}(\Delta_1 \vee \Delta_2) = \widetilde{\chi}(\Delta_1) + \widetilde{\chi}(\Delta_2).$

Proof. We propose combinatorial proofs for *(i)*, *(ii)* and *(iii)* when the simplicial complexes are order complexes of finite posets (even if these hold in the general statement); for *(iv)* we remind the reader to [51], IV.
Let P and Q be two finite posets.
(i) It is enough to notice that for any x in $P \sqcup Q$, $\mu_{P\sqcup Q}(\widehat{0}, x)$ is either $\mu_P(\widehat{0}, x)$ or $\mu_Q(\widehat{0}, x)$ if x lies respectively in P or in Q. In fact then

$$\begin{aligned}\widetilde{\chi}(\Delta(P \sqcup Q)) = \mu_{P\sqcup Q}(\widehat{0}, \widehat{1}) &= -\sum_{x \in \widehat{P \sqcup Q}} \mu_{P\sqcup Q}(\widehat{0}, x) = -\sum_{x \in P \sqcup Q} \mu_{P\sqcup Q}(\widehat{0}, x) - 1 = \\ &= -\sum_{x \in P} \mu_P(\widehat{0}, x) - \sum_{x \in Q} \mu_Q(\widehat{0}, x) - 1 = \mu_P(\widehat{0}, \widehat{1}) + \mu_Q(\widehat{0}, \widehat{1}) + 1 + 1 - 1 \\ &= \widetilde{\chi}(\Delta(P)) + \widetilde{\chi}(\Delta(Q)) + 1.\end{aligned}$$

(ii) We make use of the fact that the Möbius function is completely determined by the *(i)*, *(ii)* of definition 1.4.1 and condition *(iv)* of Proposition 1.4.2 and show that if $(x, y) \leq (x', y')$ in $P \times Q$ then

$$\mu_{P\times Q}((x, y), (x', y')) = \mu_P(x, x') \cdot \mu_Q(y, y').$$

In fact we have

$$\begin{aligned}\sum_{(x,y)\leq(u,v)\leq(x',y')} \mu_P(x, u)\mu_Q(y, v) &= \Big(\sum_{x\leq u\leq x'} \mu_P(x, u)\Big) \cdot \Big(\sum_{y\leq v\leq y'} \mu_Q(y, v)\Big) = \\ &= \delta_{x,x'} \cdot \delta_{y,y'} = \delta_{(x,y),(x',y')}.\end{aligned}$$

(iii) We prove that if $x \leq y$ in $P \star Q$, the function so defined

$$\lambda(x, y) := \begin{cases} \mu_P(x, y) & \text{if } y \in P \\ -\mu_P(x, \widehat{1})\mu_Q(\widehat{0}, y) & \text{if } x \in P, y \in Q \\ \mu_Q(x, y) & \text{if } x \in Q \end{cases}$$

satisfies condition *(iv)* of Proposition 1.4.2 and so, by uniqueness, it is indeed

$\mu_{P\star Q}(x,y)$. Just consider the case $x \in P$ and $y \in Q$, as the others are trivial, then

$$\begin{aligned}
\sum_{a:\, x\le a\le y} \lambda(x,a) &= \sum_{a:\, x\le a\in P} \lambda(x,a) + \sum_{a:\, Q\ni a\le y} \lambda(x,a) \\
&= \sum_{a:\, x\le a\in P} \mu_P(x,a) + \sum_{a:\, Q\ni a\le y} \mu_{P\star Q}(x,a) \\
&= -\mu_P(x,\widehat{1}) - \sum_{a:\, Q\ni a\le y} \mu_P(x,\widehat{1})\mu_Q(\widehat{0},a) \\
&= -\mu_P(x,\widehat{1}) - \mu_P(x,\widehat{1}) \cdot \sum_{a:\, Q\ni a\le y} \mu_Q(\widehat{0},a) \\
&= -\mu_P(x,\widehat{1}) - \mu_P(x,\widehat{1})(-\mu_Q(\widehat{0},\widehat{0})) \;=\; 0.
\end{aligned}$$

Finally for $x = \widehat{0}$ and $y = \widehat{1}$ we obtain the claimed result. Q.E.D.

1.5 Cohen-Macaulay Complexes

The concept of Cohen-Macaulay simplicial complex is not standardly defined. We make use of the same definition of A. Björner [9] and D. Quillen [44].

Definition 1.5.1. Let Δ be a finite dimensional simplicial complex of dimension n. Δ is *Cohen-Macaulay* if it is $n-$spherical and if the link of each $l-$simplex of Δ is $(n-l-1)-$spherical.

A poset P is by definition Cohen-Macaulay when the associated simplicial complex $\Delta(P)$ has this property.

Cohen-Macaulay property is preserved by the join operation. In fact, just using the same arguments of the proof of Proposition 1.3.1, one can prove the following

Lemma 1.5.1. *Let Δ_1 and Δ_2 be two Cohen-Macaulay complexes of dimensions respectively d_1 and d_2. Then $\Delta_1 \star \Delta_2$ is Cohen-Macaulay of dimension d_1+d_2+1.*

Chapter 2

Combinatorial Homotopy Theorems

In this chapter we present some techniques for manipulating homotopies and homotopy types of simplicial complexes. These topological and combinatorial tools in general permit to find out the homotopy type of a given complex in terms of the homotopy types of some of its subcomplexes.

The chapter is divided in two sections. In the former we collect basic results dealing specially with the homotopy type of chain complexes. The second section is instead a summing up of some lemma on re-arrangement of homotopies, and it ends with an exposition on how, in this contest, the homotopy theory of *diagram of spaces and direct limits* can be applied.

2.1 Homotopy Theorems

Many of the fundamental homotopy equivalence criteria have their basis on the so called Carrier Lemma.

Definition 2.1.1. Let Δ_1 and Δ_2 be two simplicial complexes. A *carrier* of Δ_1 in Δ_2 is a mapping

$$\mathcal{C} : \Delta_1 \longrightarrow 2^{|\Delta_2|}$$

from the complex Δ_1 to the power set of the geometrical realization of Δ_2, that assigns to each simplex σ of Δ_1 a subspace $\mathcal{C}(\sigma)$ of $|\Delta_2|$ such that if τ is a face of σ then $\mathcal{C}(\tau) \subseteq \mathcal{C}(\sigma)$. In other words, $\mathcal{C}$ is an order preserving map between the face poset, $P(\Delta_1)$, and the power set $2^{|\Delta_2|}$ ordered by inclusion.

Let $\mathcal{C}$ be a carrier of Δ_1 in Δ_2, we say that a mapping $f : |\Delta_1| \longrightarrow |\Delta_2|$ is *carried* by $\mathcal{C}$ if, for all $\sigma \in \Delta_1$, $f(|\sigma|) \subseteq \mathcal{C}(\sigma)$. Moreover, for $k \geq 0$, we say that the carrier

$\mathcal{C}$ is contractible, $k-$connected or $k-$acyclic, exactly whenever for all σ in Δ_1, $\mathcal{C}(\sigma)$ is contractible, $k-$connected or $k-$acyclic.
Suppose k is either a nonnegative integer or ∞.

<u>LEMMA</u> 2.1.1 (CARRIER LEMMA). *Let $\mathcal{C}$ be a carrier from Δ_1 to Δ_2, and assume $\mathcal{C}$ is min$(k, \dim(\sigma))-$connected for all $\sigma \in \Delta_1$. Then*

(i) there exists a mapping from the $k+1$ skeleton of Δ_1, $|\Delta_1^{\leq k+1}|$, to $|\Delta_2|$, carried by $\mathcal{C}$, and

(ii) if $f, g : |\Delta_1^{\leq k}| \longrightarrow |\Delta_2|$ are both carried by $\mathcal{C}$, then $f \sim g$.

PROOF. We present the proof given in [58](Lemma 2.1), with small changes.
(i) We construct the mapping $g : |\Delta_1^{\leq k+1}| \longrightarrow |\Delta_2|$ by induction on k.
For $k = -1$, just set, for each vertex v, $g(v)$ any point of $\mathcal{C}(v)$.
Suppose we have defined $g : |\Delta_1^{\leq k}| \longrightarrow |\Delta_2|$ in a continuous way and such that for all $\sigma \in \Delta_1^{\leq k}$, $g(|\sigma|)) \subseteq \mathcal{C}(\sigma)$, and let $\tau \in \Delta_1^{k+1}$ be a $(k+1)-$simplex. In particular for each proper face σ of τ, $g(|\sigma|) \subseteq \mathcal{C}(\sigma) \subset \mathcal{C}(\tau)$, so $g(|\partial\tau|) \subset \mathcal{C}(\tau)$, where $\partial\tau$ denotes the boundary of τ. Since $|\tau|$ is a $(k+1)-$cell and $\mathcal{C}(\tau)$ is $\min(k, \dim(\sigma))-$connected, by the definition of $k-$connectedness, the mapping g can be continuously extended over $|\tau|$. Moreover, since $g : |\Delta_1^{\leq k+1}| \longrightarrow |\Delta_2|$ is continuous if and only if it is continuous on each simplex, the induction is complete.
(ii) We suppose $f, g : |\Delta_1^{\leq k}| \longrightarrow |\Delta_2|$ are both carried by $\mathcal{C}$ and we will inductively construct a homotopy $H : |\Delta_1^{\leq k}| \times I \longrightarrow |\Delta_2|$ from f to g. Observe that $|\Delta_1^{\leq k}| \times I$ is the union of cells of the form $|\sigma| \times I$, and a map on $|\Delta_1^{\leq k}| \times I$ is continuous if and only if it is continuous on each such cell.
We define $H : |\Delta_1^{\leq k}| \times \{0, 1\} \longrightarrow |\Delta_2|$ by $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Note that if v is a vertex of Δ_1 then $f(v)$ and $g(v)$ lie in $\mathcal{C}(v)$, so $H : \{v\} \times \{0, 1\} \longrightarrow |\Delta_2|$ can be continuously extended to $\{v\} \times I$. By the inductive hypothesis H is defined continuously on $|\Delta_1| \times \{0, 1\} \cup |\Delta_1^{\leq k}| \times I$ and $H(|\sigma| \times I) \subset \mathcal{C}(\sigma)$ for each $\sigma \in |\Delta_1^{\leq k}|$. Therefore we can repeat the argument of point *(i)* to the function H, and conclude the proof. Q.E.D.

Note that if in the previous Lemma $\mathcal{C}$ is a contractible carrier, then $k = \infty$ and $|\Delta_1|$ can replace the skeletons in *(i)* and *(ii)*, thus we have

<u>COROLLARY</u> 2.1.2. *If $\mathcal{C}$ is a contractible carrier between Δ_1 and Δ_2, then, up to equivalence, there exists one and only one morphism f from $|\Delta_1|$ to $|\Delta_2|$ carried by $\mathcal{C}$.*

In order to state one of the most important applications of the Carrier Lemma, we have to enlarge the definition of simplicial map. The reader is now prayed to permit us this little abuse of language. We call a map

$$f : \Delta \longrightarrow P$$

from a complex Δ to a poset P, *simplicial* if it sends vertices of Δ into elements of P in such a way that each simplex $\sigma \in \Delta$ is mapped by f into a chain of P. In practice, f is an order preserving map from the face poset $P(\Delta)$ to the poset P. Note that, in particular, any order preserving map between two posets $Q \longrightarrow P$ is indeed a simplicial map between $\Delta(Q) \longrightarrow P$, in this sense.
The following known Lemma is due to D. Quillen

<u>THEOREM</u> 2.1.3 (FIBER LEMMA). *Let $f : \Delta \longrightarrow P$ be a simplicial map from a simplicial complex Δ to a poset P.*

(i) Suppose all upper (lower) fibers $\left|f^{-1}(P_{\geq x})\right|$, $(\left|f^{-1}(P_{\leq x})\right|)$, at the vary of x in P, are contractible. Then f induces a homotopy equivalence between $|\Delta|$ and $|P|$.

(ii) Suppose all upper (lower) fibers $\left|f^{-1}(P_{\geq x})\right|$, $(\left|f^{-1}(P_{\leq x})\right|)$, at the vary of x in P, are $k-$connected. Then $|\Delta|$ is $k-$connected if and only if $|P|$ is $k-$connected.

PROOF. Suppose that all upper fibers are contractible. We define the carrier

$$\mathcal{C} : \Delta(P) \longrightarrow 2^{|\Delta|}$$

$$\mathcal{C}(\sigma) := \left|f^{-1}(P_{\geq \min(\sigma)})\right|$$

where $\min(\sigma)$ denotes the least element of the chain σ of P. The contractibility of the fibers implies that $\mathcal{C}$ is a contractible carrier too. By Corollary 2.1.2 there exists a continuous map $g : |\Delta(P)| \longrightarrow |\Delta|$ carried by $\mathcal{C}$. This means that for any chain $\sigma \in \Delta(P)$, $g(|\sigma|) \subseteq \left|f^{-1}(P_{\geq \min\sigma})\right|$. The map g is the homotopy inverse to f. In fact, making use again of Corollary 2.1.2, we define other two contractible carries, namely

$$\mathcal{C}' : \Delta(P) \longrightarrow 2^{|\Delta(P)|}$$

so defined

$$\mathcal{C}'(\sigma) := \left|P_{\geq \min(\sigma)}\right|$$

and

$$\mathcal{C}'' : \Delta \longrightarrow 2^{|\Delta|}$$

so defined

$$\mathcal{C}''(\tau) := \left|f^{-1}(P_{\geq \min(f(\tau))})\right| .$$

$\mathcal{C}'$ carries $f \circ g$ and $\mathrm{id}_{|P|}$ and $\mathcal{C}''$ carries $g \circ f$ and $\mathrm{id}_{|\Delta|}$. Hence, by uniqueness of Corollary 2.1.2, $f \circ g \sim \mathrm{id}_{|P|}$ and $g \circ f \sim \mathrm{id}_{|\Delta|}$, Δ and P being therefore homotopy equivalent.

Part *(ii)* is proved analogously by passing to $(k+1)-$skeletons and using $k-$connected carries as in Lemma 2.1.1 *(ii)*. Q.E.D.

In particular this lemma applies in the situation of an order-preserving map between two posets, when the simplicial complexes involved are chain complexes of posets. For reader's convenience, we state the most useful part of this version in a separated corollary

<u>Corollary</u> 2.1.4. *Let $f : P \longrightarrow Q$ be an order preserving map among the two posets P and Q. Suppose all (the geometric realizations of the) upper (lower) fibers $f^{-1}(Q_{\geq x})$, ($f^{-1}(Q_{\leq x})$), at the vary of $x \in Q$, are contractible as topological spaces. Then f induces a homotopy equivalence between $|P|$ and $|Q|$.*

There are many important consequences of Fiber Lemma. The following little variation will be applied in Chapter 3 (Lemma 3.1.1 on page 32).

<u>Proposition</u> 2.1.5. *Let P be a poset of finite length and Q a subposet of P, and assume that for every $a \in P \setminus Q$, $P_{<a}$ (resp. $P_{>a}$) is contractible. Then the inclusion map $Q \longrightarrow P$ is a homotopy equivalence.*

Proof. The key point of the proof is constructing a chain of posets

$$P = P_0 \supset P_1 \supset \cdots \supset P_n = Q$$

by removing at each step from P_i those elements which are maximal subject not to be in Q and calling the resulting subposet P_{i+1}. Since P is of finite length we surely reach Q after a finite number of steps. Denote with $f_i : P_{i+1} \longrightarrow P_i$ the inclusion map, and consider the lower fiber $f_i^{-1}((P_i)_{\leq a})$ for all $a \in P_i$. Note that if a is chosen in P_{i+1}, then $f_i^{-1}((P_i)_{\leq a}) = (P_{i+1})_{\leq a}$ and so it is contractible having a as a maximal element. On the other hand if $a \notin P_{i+1}$, then $f_i^{-1}((P_i)_{\leq a}) = P_{<a}$, since a is removed in such a way that no element below it has been removed yet. By assumption $P_{<a}$ is contractible. Therefore, applying Corollary 2.1.4 to the inclusion map f_i, we deduce the homotopy equivalence between P_{i+1} and P_i, therefore between Q and P. Q.E.D.

Another consequence of the Fiber Theorem 2.1.4 is the following, also due to D. Quillen

<u>Theorem</u> 2.1.6 (Ideal Relation Theorem). *Let P and Q be two finite posets and suppose $R \subseteq P \times Q$ is an ideal relation (this means that if $(x, y) \leq (x', y') \in R$, then $(x, y) \in R$.) Suppose furthermore that the subposets $R_x := \{y \in Q | (x, y) \in R\}$ and $R_y := \{x \in P | (x, y) \in R\}$ are contractible for all $x \in P$ and all $y \in Q$. Then P and Q are homotopy equivalent.*

Proof. We just prove that P and R are homotopy equivalent and use the symmetry and the transitivity of equivalence relation to deduce the statement.
Let $\pi_P : R \longrightarrow P$ be the projection map onto P ($\pi_P(x, y) = x$, $\forall (x, y) \in R$), and show that the upper fibers $F_x := \pi^{-1}(P_{\geq x}) = \{(z, y) \in R | z \geq x\}$ are contractible; in virtue of Corollary 2.1.4 *(ii)*, we obtain the equivalence between P and R. Consider the application

$$\rho : F_x \longrightarrow R_x$$

$$(z, y) \longmapsto y.$$

Note that the upper fiber of ρ,

$$\rho^{-1}((R_x)_{\geq y}) = \{(z, w) \in F_x | w \geq y\} = \{(z, w) \in R | (z, w) \geq (x, y)\}$$

is the cone over (x, y) and hence it is contractible. This holds obviously for all $y \in R_x$, therefore, by Fiber Lemma 2.1.4, F_x is homotopy equivalent to R_x, which is by assumption contractible. Q.E.D.

<u>THEOREM</u> 2.1.7 (ORDER HOMOTOPY THEOREM). *Let $f, g : \Delta \longrightarrow P$ be two simplicial maps from Δ to P. If for every vertex x of Δ, $f(x)$ and $g(x)$ are comparable, then f and g are homotopy equivalent.*

PROOF. Define the following carrier $\mathcal{C}$ of Δ in $\Delta(P)$,

$$\mathcal{C}(\sigma) := f(\sigma) \cup g(\sigma).$$

Note that either the minimal element of the chain $f(\sigma)$ or the minimal element of the chain $g(\sigma)$ is, by assumption, below every other element of $\mathcal{C}(\sigma)$. Thus, $\mathcal{C}(\sigma)$ is a cone over either $\min(f(\sigma))$ or $\min(g(\sigma))$, in any case it is contractible. Since this holds for all $\sigma \in \Delta$, $\mathcal{C}$ is a contractible carrier. Finally by Corollary 2.1.2, we conclude $f \sim g$. Q.E.D.

As an immediate consequence we have the following

<u>COROLLARY</u> 2.1.8. *Let $f : P \longrightarrow P$ be an order preserving map such that for all $x \in P$, $f(x)$ and x are comparable elements of P. Then f induces an homotopy equivalence between P and $f(P)$.*

DEFINITION 2.1.2. A poset P is said *join-contractible* (resp. *meet-contractible*) if it has some element $x_0 \in P$ such that for all $x \in P$, the join $x_0 \vee x$ exists in P (resp. the meet $x_0 \wedge x$ exists P). In this case the element x_0 is said *conjunctive* (resp. *subjunctive*).

<u>COROLLARY</u> 2.1.9. *Join-contractible and meet-contractible complexes are contractible as topological spaces.*

PROOF. We just prove the version for join-contractible complexes, being the other dual.
Suppose x_0 is a conjunctive element of P, then the order-preserving map

$$f_{x_0} : P \longrightarrow P$$

$$x \longrightarrow x_0 \vee x$$

is such that for all $x \in P$, $x_0 \leq f_{x_0}(x) > x$, and so in virtue of Theorem 2.1.7, the identity map on P, the constant map onto x_0 and f_{x_0} are all homotopy equivalent. This of course means the space $|P|$ is contractible. Q.E.D.

We end up this section with an application of these results in lattice theory. Given a lattice L and one of its elements, say $z \in L$, by a *complement* of z in L we mean an element $x \in L$ such that $z \wedge x = 0$ and $z \vee x = 1$ (where, as usual, 0 and 1 are respectively the bottom and the top elements of L). We denote by $\mathcal{C}_0(z)$ the set of all complements of z in L.

<u>Theorem</u> 2.1.10 (Homotopy Complementation Theorem). *Let L be a bounded lattice, $\overline{L} := L \setminus \{0,1\}$ be its proper part, z be an element of $\overline{L}$. Then the poset $\overline{L} \setminus \mathcal{C}_0(z)$ is contractible. In particular, if L is non-complemented then $\overline{L}$ is contractible.*

Proof. For each chain σ in $P = \overline{L} \setminus \mathcal{C}_0(z)$, let $C(\sigma) = \{x \in P | x \geq z\} \cup \{x \in P | x \leq \max(\sigma)\}$. Note that either $z \vee \max(\sigma)$ exists in P, in which case $C(\sigma)$ is meet-contractible, or else $z \wedge \max(\sigma)$ exists, and $C(\sigma)$ is join-contractible. In any case C is a contractible carrier. Therefore by Lemma 2.1.1 the constant map z is homotopy equivalent to the identity map on P. Q.E.D.

2.2 Manipulation of Homotopies

2.2.1 The "Nerve" Theorem

We begin this section with a basic Lemma of gluing spaces. This can be deduced simply from Mayer-Vietoris long exact sequence and Seifert-van Kampen theorem, using the properties of $k-$connectedness (see for instance [51], IV.6, and [35], 23).

<u>Lemma</u> 2.2.1 (Gluing Lemma). *Let Δ_1 and Δ_2 be two simplicial subcomplexes of a complex. Then the following holds:*

(i) *if Δ_1 and $\Delta_1 \cap \Delta_2$ are contractible, then $\Delta_1 \cup \Delta_2 \simeq \Delta_2$,*

(ii) *if Δ_1 and Δ_2 are $k-$connected and $\Delta_1 \cap \Delta_2$ is $(k-1)-$connected, then $\Delta_1 \cup \Delta_2$ is $k-$connected,*

(iii) *if $\Delta_1 \cup \Delta_2$ and $\Delta_1 \cap \Delta_2$ are $k-$connected, then so are Δ_1 and Δ_2.*

The Gluing Lemma can be also seen as a consequence of a more general setting.

Let X be an arbitrary set, and let $\mathcal{F} := (X_i)_{i \in I}$ be a family of subsets of X. We define *the intersection poset* of $\mathcal{F}$ (written $P^{\mathcal{F}}$) as the partially ordered set of all the non trivial intersections

$$X_J := \bigcap_{i \in J} X_i \quad \text{for } J \subseteq I \ ,$$

with reversed inclusion as order relation, i.e. $X_J \leq X_K$ in $P^{\mathcal{F}}$ iff $X_J \supseteq X_K$.

The *nerve* of the family $\mathcal{F}$ is the simplicial chain complex of $P^{\mathcal{F}}$, $\Delta(P^{\mathcal{F}})$, and it is generally denoted by $\mathrm{N} := \mathrm{N}(\mathcal{F})$.

The following is famous theorem in combinatorial topology, and it has many applications and many formulations. The earliest of these is purely topological and seems to be attributed to J. Leray. We propose a more combinatorial version due to K. Borsuk, A. Björner and others.

<u>THEOREM</u> 2.2.2 (NERVE THEOREM). *Let Δ be a simplicial complex and let $\mathcal{F} := (\Delta_i)_{i\in I}$ be a family of subcomplex which covers Δ, i.e. $\Delta = \bigcup_{i\in I} \Delta_i$.*

(i) Suppose that every nonempty finite intersection $\Delta_{i_1} \cap \Delta_{i_2} \cap \ldots \cap \Delta_{i_l}$ is contractible. Then Δ and the nerve $\mathrm{N}(\mathcal{F})$ are homotopy equivalent.

(ii) Suppose every nonempty finite intersection $\Delta_{i_1} \cap \Delta_{i_2} \cap \ldots \cap \Delta_{i_l}$ is $(k-l+1)-$connected. Then Δ is $k-$connected if and only if $\mathrm{N}(\mathcal{F})$ is $k-$connected.

PROOF. We prove the Theorem under the assumption that Δ is finite (in particular the covering $\mathcal{U}$ of Δ consists of finite elements), even if this results holds for arbitrary complexes and coverings.

(i). For simplicity, set $Q = P(\Delta)$ the face poset of Δ and $P = P^{\mathcal{F}}$ the intersection poset of $\mathcal{F}$. Define a mapping $f : Q \longrightarrow P$ by

$$\sigma \mapsto \bigcap_{\sigma\in\Delta_i} \Delta_i \ .$$

f is a well defined order preserving map, therefore the induced map $f : \Delta(Q) = \mathrm{sd}(\Delta) \longrightarrow P$ is simplicial. The upper fiber of f over the element Γ of P is

$$f^{-1}(P_{\geq\Gamma}) = \{\sigma \in Q | f(\sigma) \geq \Gamma\} = \left\{\sigma \in Q \,\middle|\, \bigcap_{\sigma\in\Delta_i} \Delta_i \subseteq \Gamma\right\} = \Gamma$$

and therefore it is contractible by assumption. Finally Fiber Lemma 2.1.3 *(i)* completes the proof of the first part.

For statement *(ii)* we refer to Björner et al. ([10]). Q.E.D.

In general it happens that the hypotheses of the Nerve Lemma are too strong to be satisfied, therefore it becomes necessary to have a method which permits to reconstruct the homotopy type of a given complex, by some of its coverings, even when the non-empty intersections are neither contractible nor $k-$connected. One of these methods is explained in the next section.

2.2.2 DIAGRAMS OF SPACES AND DIRECT HOMOTOPY LIMITS.

The first application of the homological methods described in this section to the study of simplicial complexes associated to finite groups is due to J. Pulkus and

V. Welker ([43]). I will refer to their article, and to the work of G. M. Ziegler and R. T. Živaljević ([64]), as, there, the setting is the one of partially ordered sets. Diagrams of spaces can in fact be introduced for arbitrary small categories. The reader interested in the general theory may wish to consult the works of M. Bousfield, D. M. Kan ([14]), R. M. Vogt ([57]), and T. tom Dieck ([56]). Since we work essentially only on posets the version given in [43] and [64] will be enough for our purposes.

Any finite partially ordered set $(P, \leq)$ can be seen as a small category, where the objects are the vertices of P and the morphisms are given by the order relation (namely $p \to q$ means $p \geq q$). We denote this category with $(P, \leq)$ and with CW-Top we denote the category whose objects are CW-complexes and whose arrows are continuous functions (see [40], [51]).

Definition 2.2.1. A $P-$diagram of spaces is a functor

$$\mathcal{D} : (P, \leq) \longrightarrow \text{CW-Top}$$

Explicitly, a $P-$diagram of spaces is a poset P together with a space D_p for every $p \in P$ and maps $d_{qp} : D_q \longrightarrow D_p$ for $p \leq q$ preserving functoriality, i.e. $d_{qp} \circ d_{rq} = d_{rp}$ and $d_{pp} = \mathrm{id}_{D_p}$ for any $p \leq q \leq r$ in P.

Example 2.2.1. The basic example of a diagram of spaces is the following. Let X be a topological space (a CW-complex) which admits a covering $\mathcal{U}$ by a finite number of subspaces X_i, $i \in I$. Assume the empty set is allowed as an element of the intersection poset $P^{\mathcal{U}}$ of the covering $\mathcal{U}$, so that $X_I := \bigcap_{i \in I} X_i$ results the unique maximal element of $P^{\mathcal{U}}$ (remember that the order is reversed in $P^{\mathcal{U}}$). In the case that there exist two different subsets $J, J' \subseteq I$ such that $X_J = X_{J'}$, then, of course, X_J and $X_{J'}$ represent the same element in $P^{\mathcal{U}}$. Since any vertex p in the poset intersection $P^{\mathcal{U}}$ is indeed a topological space, there is a natural diagram of spaces associated to $P^{\mathcal{U}}$ so defined: for any $p = X_J \in P^{\mathcal{U}}$ we set $D_p = X_J$ and for $q > p$ let D_{qp} be the natural inclusion.
We call this diagram *the natural diagram on the space X associated to the covering $\mathcal{U} = \{X_i\}_{i \in I}$* and we denote it by $\mathcal{D}^{\mathcal{U}}$.

To any $P-$diagram of spaces $\mathcal{D}$ it is possible to associate (in a unique way up to equivalence) a topological space, called *the homotopy direct limit of* $\mathcal{D}$. This is so defined.

Definition 2.2.2. Let $\mathcal{D}$ be a diagram of spaces over the finite poset P. The *homotopy direct limit* of $\mathcal{D}$ is, by definition, the topological space obtained by the disjoint union $T := \bigsqcup_{p \in P} \big(\Delta(P_{\leq p}) \times D_p\big)$ by making the following identification. Let $Y := \bigsqcup_{p > q} \big(\Delta(P_{\leq q}) \times D_p\big)$ and consider the two maps α, β from Y to T so defined: α consists of the maps from $\Delta(P_{\leq q}) \times D_p$ to $\Delta(P_{\leq p}) \times D_p$ induced by the

inclusions $\Delta(P_{\leq q}) \hookrightarrow \Delta(P_{\leq p})$ for $p > q$, β consists of the maps from $\Delta(P_{\leq q}) \times D_p$ to $\Delta(P_{\leq q}) \times D_q$ induced by the maps $d_{pq} : D_p \longrightarrow D_q$.
The space obtained from T by identifying $\alpha(x, u)$ with $\beta(x, u)$, $\forall (x, u) \in Y$ is the *homotopy direct limit* of $\mathcal{D}$. We denote it by $\underrightarrow{\mathrm{holim}}\, \mathcal{D}$.

Note. In the case of the natural diagram $\mathcal{D}$ on a space X associated to a covering $\mathcal{U} = \{X_i\}_{i \in I}$, there is a natural "collapsing" map $\xi : \underrightarrow{\mathrm{holim}}\, \mathcal{D} \longrightarrow X$, namely the one which associates to any equivalence class $[X_J, x_0]$ of $\mathcal{D}$ the representative point x_0.

The key point for us of this machinery consists in the fact that if a covering $\mathcal{U}$ of a space X is "sufficiently nice", then the space X has the same homotopy type of the homotopy direct limit of the natural diagram $\mathcal{D}^{\mathcal{U}}$, i.e. $X \simeq \underrightarrow{\mathrm{holim}}\, \mathcal{D}^{\mathcal{U}}$. This is in fact the content of the next

Lemma 2.2.3 (Projection Lemma 1.6 in [64]). *Let $\mathcal{U} : X = \bigcup_{i \in I} X_i$ be a covering of the CW−complex X by a finite number of closed subcomplexes X_i, then $\underrightarrow{\mathrm{holim}}\, \mathcal{D}^{\mathcal{U}} \simeq X$.*

Sometimes we need to modify the maps d_{rq} and the spaces D_r in a way that the homotopy type of the homotopy direct limit remains unchanged. This can be done with accuracy, using the following

Lemma 2.2.4 (Homotopy Lemma 1.7 in [64]). *Let $\mathcal{D} = (D_r, d_{q,r})$ and $\mathcal{E} = (E_r, e_{q,r})$ be two diagrams of spaces on the poset P. Assume that for each $r \in P$ there is a map $f_r : D_r \longrightarrow E_r$ such that f_r induces a homotopy equivalence between D_r and E_r and that for every $q \leq r$ in P the following diagrams commute*

$$\begin{tikzcd} D_r \arrow[r, "d_{rq}"] \arrow[d, "f_r"'] & D_q \arrow[d, "f_q"] \\ E_r \arrow[r, "e_{rq}"'] & E_q \end{tikzcd}$$

Then $\underrightarrow{\mathrm{holim}}\, \mathcal{D} \simeq \underrightarrow{\mathrm{holim}}\, \mathcal{E}$.

Finally it is possible to compute explicitly the homotopy type of $\underrightarrow{\mathrm{holim}}\, \mathcal{D}$ when some (strong) assumptions are guaranteed, in the sense of the following

Lemma 2.2.5 (Wedge Lemma 1.8 in [64]). *Let $\mathcal{D} = (D_r, d_{q,r})$ be a diagram of spaces over the poset P with a unique maximal element $\widehat{1}$ such that for every $q \in P$, $q \neq \widehat{1}$, there is a point $c_q \in D_q$ such that for all $r > q$, $d_{rq} : D_r \longrightarrow D_q$ is the constant map on c_q. Then $\underrightarrow{\mathrm{holim}}\, \mathcal{D}$ is homotopy equivalent to*

$$\big(D_{\widehat{1}} \star \big|\Delta(P_{<\widehat{1}})\big|\big) \vee \bigvee_{r \in P} \big(D_r \star |\Delta(P_{<r})|\big)$$

where the wedge is formed by identifying, for every $r < \widehat{1}$ in P, the point $c_r \in D_r \star |\Delta(P_{<r})|$ with the element $r \in D_{\widehat{1}} \star \left|\Delta(P_{<\widehat{1}})\right|$.

We postpone the outlines of the proofs of these three Lemma in the appendix 2.2.2 on page 27. We prefer to give now the statement and the proof of a Corollary, due to J. Pülkus and V. Welker (Corollary 2.4 in [43]), which consists in the starting point for all our applications of these techniques.

In order to facilitate the reading we identify any complex with its geometrical realization, so we simply write Δ in place of $|\Delta|$.

COROLLARY 2.2.6. *Let $f : P \longrightarrow Q$ be an order preserving map between the two finite posets P and Q. Assume*

(i) Q is a meet semi-lattice. In particular Q contains a unique minimal element $\widehat{0}$.

(ii) The image of f contains $Q \setminus \{\hat{0}\}$.

(iii) For any $q \in Q_{>\widehat{0}}$ there exists an element $c_q \in f^{-1}(Q_{\leq q})$ such that the inclusion map $i_q : \Delta(f^{-1}(Q_{<q})) \hookrightarrow \Delta(f^{-1}(Q_{\leq q}))$ is homotopy equivalent to the constant map sending $\Delta(f^{-1}(Q_{<q}))$ to c_q.

Then the chain complex of P is homotopy equivalent to the wedge

$$\left(\Delta(f^{-1}\{\hat{0}\}) \star \Delta(Q_{>\widehat{0}})\right) \vee \bigvee_{q \in Q_{>\widehat{0}}} \left(\Delta(f^{-1}(Q_{\leq q})) \star \Delta(Q_{>q})\right) \tag{2.2.1}$$

where for $q \in Q_{>\widehat{0}}$ the point $c_q \in \Delta(f^{-1}(Q_{\leq q}) \subseteq \Delta(f^{-1}(Q_{\leq q})) \star \Delta(Q_{>q})$ is identified with $q \in \Delta(f^{-1}\{\hat{0}\}) \star \Delta(Q_{>\widehat{0}})$.

PROOF. Set $\mathcal{U} := \left\{\Delta(f^{-1}(Q_{\leq q}))\right\}_{q \in Q}$. $\mathcal{U}$ is a finite covering of $\Delta(P)$ by a finite number of closed subcomplexes, therefore by Projection Lemma 2.2.3, $\underrightarrow{\mathrm{holim}}\, \mathcal{D}^{\mathcal{U}} \simeq \Delta(P)$. Note that the intersection poset $\mathcal{P}^{\mathcal{U}}$ is isomorphic to Q^{op} via the map sending $\Delta(f^{-1}(Q_{\leq q}))$ into q. Now we define a second diagram $\mathcal{E} := (E_q, e_{p,q})$ of spaces over Q^{op} by setting $E_q = D_q = \Delta(f^{-1}(Q_{\leq q}))$ for $q \in Q^{\mathrm{op}}$, $e_{pq} := c_q$ for $q < p$ in Q^{op} and $e_{pp} = id_{E_p}$. By hypotheses *(iii)* there exists a homotopy $f_q : D_q \longrightarrow E_q$ such that $f_q(d_{pq}(D_p)) = c_q$ for all $p < q$ in Q^{op}, i.e. for all $p < q$ in Q^{op} there exist maps commuting the diagrams

$$\begin{array}{ccc} D_p & \xrightarrow{d_{pq}} & D_q \\ {\scriptstyle f_p}\downarrow & & \downarrow{\scriptstyle f_q} \\ E_p & \xrightarrow[c_q]{} & E_q \end{array}$$

Therefore all the conditions of the Homotopy Lemma 2.2.4 are satisfied. We can so deduce that $\underrightarrow{\operatorname{holim}}\,\mathcal{E} \simeq \underrightarrow{\operatorname{holim}}\,\mathcal{D} \simeq \Delta(P)$. Finally note that the diagram $\mathcal{E}$ has just been constructed in order to apply to it the Wedge Lemma 2.2.5; this gives the claimed homotopy equivalence in the formula (2.2.1). Q.E.D.

Remark 2.2.1. In general the most difficulties in applying Corollary 2.2.6 arise in proving condition *(iii)*. This is of course satisfied if all the maps involved are homotopically trivial, or if, for any $q \in Q$ at least one of the complexes $\Delta(f^{-1}(Q_{<q}))$, $\Delta(f^{-1}(Q_{\leq q}))$ is contractible.
More generally, for spherical spaces, if the homotopy types are determined, in some way, by the dimensions of the spaces, then condition *(iii)* is again satisfied (see what happens for instance in Theorem 3.4.5 of Chapter 3 on page 42).

Remark 2.2.2. It is worth while to notice that if we replace condition *(iii)* by the assumption that $\Delta(f^{-1}(Q_{\leq q}))$ is contractible for every $q > \widehat{0}$ in Q, then, using the fact that the join of two spaces is always contractible when at least one of the two spaces is (see Corollary 1.3.4 on page 10), Corollary 2.2.6 yields $\Delta(P) \simeq \Delta(f^{-1}\left\{\hat{0}\right\}) \star \Delta(Q_{>\widehat{0}})$. Moreover, if additionally f is a map onto $Q_{>\widehat{0}}$ (i.e. $f^{-1}(\left\{\hat{0}\right\})$ is the empty set), then we obtain $\Delta(P) \simeq \Delta(Q_{>\widehat{0}})$, which is the well-known version of the Fiber Lemma (2.1.3) of D. Quillen.

Remark 2.2.3. The Nerve Theorem 2.2.2 can be easily deduced from the Wedge Lemma 2.2.5. Indeed, this is the simplest case in which all inclusion maps are between contractible spaces.

The following result will be applied in the proof of our main Theorem of Chapter 3.

<u>Lemma</u> 2.2.7. *Let $\{\Delta_i\}_{i\in I}$ be a covering of the complex Δ by a finite number of subcomplexes Δ_i such that each proper intersection $\bigcap_{j\in J}\Delta_j$ (with $|J| \geq 2$) is non empty and contractible. Then Δ is homotopy equivalent to the wedge of all Δ_i, i.e.*

$$\Delta \simeq \bigvee_{i\in I} \Delta_i$$

Proof. Set $\mathcal{U}$ the collection of all possible intersections of Δ_i, $i \in I$ and let $\mathcal{D} := \mathcal{D}^{\mathcal{U}}$ be the natural diagram of spaces associated to the covering $\mathcal{U}$. Since, by hypothesis, every proper intersection is contractible, the inclusion maps in the intersection poset $P^{\mathcal{D}}$ are homotopy equivalent to constant maps, therefore we can immediately apply to $\mathcal{D}$ the Wedge Lemma 2.2.5 and deduce that

$$\underrightarrow{\operatorname{holim}}\,\mathcal{D} \simeq \bigvee D_r \star \Delta(P_{<r}) \simeq \bigvee_{i\in I} \Delta_i$$

where in the last equivalence we used the fact that $D_r \star \Delta(P_{<r})$ is non contractible if and only if D_r is non contractible, which happens just when $r \in \{\Delta_i\}$, and $P_{<r} = \emptyset$. Finally, by Projection Lemma 2.2.3, $\underrightarrow{\operatorname{holim}}\,\mathcal{D} \simeq \Delta$. Q.E.D.

As another application of these techniques we propose an alternative proof of a result due to Y. Segev and P. Webb (Theorem 2.4 in [49]). We have to introduce some notation before.

Given a finite poset P and a subposet $Q \leq P$ we define a new poset P_Q in this way: the vertices of P_Q are the vertices of P plus a new element, say 0_Q, the order relation in P_Q is given by $x \leq y$ in P_Q if and only if either $x, y \in P$ and $x \leq y$ in P, or $x = 0_Q$ and $y \in Q$. Practically we pass from P to P_Q just by joining a least element 0_Q only under the minimal elements of Q. We mention the reader to the fact that the version of the Theorem in [49] lies in a more general contest, i.e. for general finite posets and not only for posets derived from lattices, but for our aim this version suffices.

Theorem 2.2.8. *Let L be a finite lattice and let $P := L \setminus \{0, 1\}$ be its proper part. If Q is a subposet of P such that for any element $p \in P$ either $p \in Q$ or p is a minimal element of P, then the poset P_Q is homotopically equivalent to the wedge*

$$\bigvee_{p \in P \setminus Q} \mathbf{S}\big(\Delta(P_{>p})\big)$$

of the suspensions over intervals of P.

Proof. Define a new poset $\widehat{P}$ simply by adding a least element $\widehat{0}$ under *all* the minimal elements of P. We consider the injection map between posets

$$f : P_Q \longrightarrow \widehat{P}$$

sending any element $p \in P$ into itself, seen as an element of $\widehat{P}$, and 0_Q into $\widehat{0}$. The map f is a surjective order preserving map between the two posets. Moreover $\widehat{P}$ is a meet-semilattice since, by hypothesis P is the proper part of a lattice and $\widehat{0}$ is the minimum of $\widehat{P}$. In order to apply Corollary 2.2.6 we just have to verify hypothesis *(iii)*, i.e. we have to evaluate the injections

$$i_q : \Delta(f^{-1}(P_{<p})) \hookrightarrow \Delta(f^{-1}(P_{\leq p}))$$

between the lower fibers of f. Take p any element of $\widehat{P}_{>\widehat{0}} = P$ and consider its preimage under f. Note that as a set

$$f^{-1}(\widehat{P}_{<p}) = (P_Q)_{<p} \cup \{0_Q\}$$

and similarly

$$f^{-1}(\widehat{P}_{\leq p}) = (P_Q)_{\leq p} \cup \{0_Q\}\,.$$

We claim that, for every p in P, either the former or the second space is contractible. In fact, if p lies in $P \setminus Q$, since by assumption p is a minimal element of P, $(P_Q)_{<p}$ is empty and so $f^{-1}(\widehat{P}_{<p})$, being equal to $\{0_Q\}$, is a contractible complex. If otherwise p lies in Q, $f^{-1}(\widehat{P}_{<p})$ is not necessary a contractible complex,

since there can be isolated points in $P \setminus Q$ below p and not comparable with any other elements of $(P_Q)_{<p} \cup \{0_Q\}$, but, in this situation is the complex $f^{-1}(\widehat{P}_{\leq p})$ to be contractible. In fact it is equal to $(P_Q)_{\leq p}$, as it contains $\{0_Q\}$; so it has p as a maximum element. In any case, with the use of Remark 2.2.1, we proved that the inclusion maps satisfy condition *(iii)* of Corollary 2.2.6, and we can then apply it and obtain the following sequence of homotopy equivalences

$$\begin{aligned}
\Delta(P_Q) &\simeq \big(\Delta(f^{-1}\{\hat{0}\}) \star \Delta(\widehat{P}_{>\hat{0}})\big) \vee \bigvee_{p\in P} \big(\Delta(f^{-1}(\widehat{P}_{\leq p})) \star \Delta(\widehat{P}_{>p})\big) \simeq \\
&\simeq \big(\{0_Q\} \star \Delta(P)\big) \vee \bigvee_{p\in P} \big(\Delta(f^{-1}(\widehat{P}_{\leq p})) \star \Delta(P_{>p})\big) \simeq \\
&\simeq \bigvee_{p\in P\setminus Q} \big(\Delta(\{p, 0_Q\}) \star \Delta(P_{>p})\big) \simeq \\
&\simeq \bigvee_{p\in P\setminus Q} \mathbf{S}(\Delta(P_{>p}))
\end{aligned}$$

where we used repeatedly the fact that the join of a contractible space with any space is always contractible. Q.E.D.

Appendix

In this appendix we outline the proofs of the basic lemmas (Lemma 2.2.3, 2.2.4 and 2.2.5) concerning the homotopy types of diagrams. This is a part of the article [64] of G. M. Ziegler and R. T. Živaljević.

Throughout this section we silently assume that all the inclusion maps between spaces $i : A \hookrightarrow B$ are closed cofibrations, this means that they possess the so-called homotopy extension property (HEP), i.e. roughly speaking that any local homotopy defined on A can be extended to a global homotopy in all B. Equivalent conditions is saying that the space $(B \times \{0\}) \cup (A \times I)$ is a retract of $B \times I$ (where I denotes the closed interval of all real numbers from 0 to 1), or, also saying that the pair (B, A) is an NDR-pair in the sense of [61] I.5.
Under these assumptions the following lemma, we make frequently use, can be proved as in [61], I.5.4.

Lemma 2.2.9. *Let $X = A \cup B$ and $Y = C \cup D$ be spaces. Suppose there exists a continuous map $f : X \longrightarrow Y$ such that $f(A) \subset C$ and $f(B) \subset D$. If the restrictions $f_{|A} : A \longrightarrow C$, $f_{|B} : B \longrightarrow D$ and $f_{|A\cap B} : A \cap B \longrightarrow C \cap D$ are all homotopy equivalences, then the map f is a homotopy equivalence too.*

Proof of the Projection Lemma 2.2.3.
Here and in the following, we set, for simplicity, $P := P^{\mathcal{U}}$ and $\|\mathcal{D}\|$ the direct homotopy limit $\underrightarrow{\mathrm{holim}}\,\mathcal{D}$.

We proceed by induction on the size of $(P, \leq)$.
If $|P| = 1$ there is nothing to prove, therefore we suppose $|P| > 1$ and we chose a minimal element p_0 in P. Let $\mathcal{A} := \{\text{intersections of elements of } \mathcal{U}\}$ be the arrangement obtained from $\mathcal{U}$, $\mathcal{A}^1 := \mathcal{A} \setminus \{D_{p_0}\}$ be the arrangement obtained by deleting the space associated to p_0, and $\mathcal{A}^2 := \{D_p \cap D_{p_0} | D_p \in \mathcal{A}\}$ be the arrangement of all intersections with D_{p_0}. Moreover we set $\mathcal{A}^{12} := \mathcal{A}^1 \cap \mathcal{A}^2$ the arrangement of all *proper* intersections with D_{p_0}. The unions of the arrangements are $X = \bigcup \mathcal{A}$, $X^1 = \bigcup \mathcal{A}^1$, $X^2 = \bigcup \mathcal{A}^2 = D_{p_0}$ and $X^{12} = \bigcup \mathcal{A}^{12} = X^1 \cap D_{p_0}$. The corresponding posets are respectively P, $P^1 := P \setminus \{p_0\}$, $P^2 := P_{\geq p_0}$ and $P^{12} := P_{>p_0} = P^1 \cap P^2$. Each of these inherits from $\mathcal{D}$ a natural diagram structure. The corresponding homotopy direct limits are so related

$$\begin{tikzcd} \left\|\mathcal{D}^{12}\right\| \arrow[r] \arrow[d] & \left\|\mathcal{D}^{1}\right\| \arrow[d] \\ \left\|\mathcal{D}^{2}\right\| \arrow[r] & \|\mathcal{D}\| \end{tikzcd}$$

The collapsing map $\xi : \|\mathcal{D}\| \longrightarrow X$, Remark 2.2.2, is an extension of the collapsing maps $\xi^1 : \left\|\mathcal{D}^1\right\| \longrightarrow X^1$, $\xi^2 : \left\|\mathcal{D}^2\right\| \longrightarrow X^2$ and $\xi^{12} : \left\|\mathcal{D}^{12}\right\| \longrightarrow X^{12}$.
Note that the map ξ^2 is a homotopy equivalence because it is the collapsing map of a mapping cylinder to its base. Moreover the maps ξ^1 and ξ^{12} are homotopy equivalences by the inductive hypothesis. Our aim is to prove that the maps ξ is also a homotopy equivalence. In order to show this we would apply Lemma 2.2.9. Therefore we need to show that all the pairs of spaces involved are NDR-pairs, namely we need a theorem which guarantee us that any pair of the type $(\|\mathcal{D}\|, \|\mathcal{E}\|)$ is an NDR-pair for any subdiagram $\mathcal{E}$ of $\mathcal{D}$ obtained by restricting $\mathcal{D}$ to a filter P' of P. This statement can be proved by a parallel inductive argument, in a similar way than the argument used above. This completes the proof of the Projection Lemma. Q.E.D.

Proof of the Homotopy Lemma 2.2.4.
Even here we set for simplicity $P := P^{\mathcal{U}}$, and proceed by induction on the size of $(P, \leq)$.
We denote with $\alpha = (\alpha_p)_{p \in P}$ a morphism between the two $P-$diagrams $\mathcal{D}$ and $\mathcal{E}$ such that $\alpha_p : D_p \longrightarrow E_p$ is a homotopy equivalence for any $p \in P$. Chose p_0 a minimal element in P and, as in the proof of the Projection Lemma, we set $P^1 := P \setminus \{p_0\}$, $P^2 := P_{\geq p_0}$ and $P^{12} := P_{>p_0} = P^1 \cap P^2$. Moreover we denote the restrictions of diagrams $\mathcal{D}$ and $\mathcal{E}$ respectively by $\mathcal{D}^1$, $\mathcal{E}^1$, etc. The restrictions of the morphism α to $\mathcal{D}^1$, $\mathcal{D}^2$ and $\mathcal{D}^{12}$ will be denoted respectively by α^1, α^2 and α^{12}, each of these induce a collapsing map ($\overline{\alpha^1}$, $\overline{\alpha^2}$ and $\overline{\alpha^{12}}$) of the corresponding homotopy limits. If we denote respectively by $\xi : \left\|\mathcal{D}^2\right\| \longrightarrow D_{p_0}$ and $\eta : \left\|\mathcal{E}^2\right\| \longrightarrow E_{p_0}$ the collapsing maps associated to $\left\|\mathcal{D}^2\right\|$ and $\left\|\mathcal{E}^2\right\|$, we have the following commutative

diagram

$$\begin{tikzcd} \|\mathcal{D}^2\| \arrow[r, "\overline{\alpha^2}"] \arrow[d, "\xi"'] & \|\mathcal{E}\|^2 \arrow[d, "\eta"] \\ D_{p_0} \arrow[r, "\alpha_{p_0}"'] & E_{p_0} \end{tikzcd}$$

and, since α_{p_0} is by assumption a homotopy equivalence, then is the map $\overline{\alpha^2}$. Moreover by the inductive assumption $\overline{\alpha^1}$ and $\overline{\alpha^{12}}$ are homotopy equivalences. By the remark at the end of Projection Lemma all spaces involved are NDR-pair, so, by Corollary 2.2.9, the map $\overline{\alpha}$ is a homotopy equivalence too. Q.E.D.

Proof of the Wedge Lemma 2.2.5.
For every $p < \widehat{1}$ in P, let $\mathcal{D}[p]$ be the diagram over P whose spaces are so defined: for $q \in P$ set

$$\mathcal{D}[p]_q := \begin{cases} D_p & \text{if } q = p \\ \{c_q\} & \text{if } q < p \\ \emptyset & \text{otherwise} \end{cases}$$

and whose connecting functions are the same as $\mathcal{D}$. Then we have an obvious map $\gamma[p] : \mathcal{D}[p] \longrightarrow \mathcal{D}$ which induces an embedding $\overline{\gamma[p]} : \|\mathcal{D}[p]\| \longrightarrow \|\mathcal{D}\|$ between the homotopy limits.

Note that $\|\mathcal{D}[p]\|$ is homeomorphic to the join space $\Delta(P_{<p}) \star D_p$ and that $\|\mathcal{D}[p]\| \supseteq \Delta(P_{<P}) \star \{c_p\} = \Delta(P_{\leq p})$. Now the structure of $\mathcal{D}$ can be described in terms of the "building blocks" $\|\mathcal{D}[p]\| \simeq \Delta(P_{<p}) \star D_p$. In fact we observe that $\|\mathcal{D}[p]\| \cap \Delta(P \setminus \{\hat{1}\}) = \Delta(P_{\leq p})$, which is a contractible set, and that

$$\|\mathcal{D}[p]\| \cap \|\mathcal{D}[q]\| = \Delta(P_{\leq p}) \cap \Delta(P_{\leq p}) \subseteq \Delta(P \setminus \widehat{1})$$

for $q \neq p$ and $p, q < \widehat{1}$. Hence the space $\|\mathcal{D}\|$ is obtained from $\Delta(P \setminus \widehat{1}) \star D_{\widehat{1}}$ by attaching for every $p < \widehat{1}$ the space $\Delta(P_{<p}) \star D_p$ to $\Delta(P \setminus \widehat{1}) \star D_{\widehat{1}}$ along the common copy of $\Delta(P_{\leq p})$. By replacing the attaching maps of $\|\mathcal{D}[p]\|$ to $\Delta(P_{\widehat{1}}) \star D_{\widehat{1}}$ with the wedge operator, we get the claimed formula

$$\|\mathcal{D}\| \simeq \big(\Delta(P_{<\widehat{1}}) \star D_{\widehat{1}}\big) \vee \bigvee_{p<\widehat{1}} \big(\Delta(P_{<p}) \star D_p\big).$$

Q.E.D.

Chapter 3

The complex of $p-$subgroups of a finite group

In this chapter we start our analysis of posets and complexes that arise naturally from the structure of a given finite group.

We begin with the simplicial complex that has had more fortune in literature: the complex of $p-$subgroups of a group. This complex coincides with a natural generalization, for each finite group and for each prime number dividing the order, of the notion of *Tits building* typical of groups of Lie type. Together with the conjugation action of the group, it carries the information inherent of the $p-$local structure of the entire group. Its importance is so manifestly justified.
For a well-done introduction to the $p-$subgroups complex we suggest the article D. Quillen ([44]) which is considered the starting point of its study.

The chapter is structured in this way. In section one we present some different ways, up to homotopy, of looking at this complex and we expose the main topics of its study. Section two analyzes the connections with buildings for groups of Lie type. Section three focuses on the Bouc complex. In section four we prove our main result (Theorem 3.4.6) concerning the homotopy type of the $p-$subgroups complex of a $p-$nilpotent finite soluble group. A combinatorial analysis of the Euler characteristic of the $p-$subgroups complex is explained in section five.

Throughout this chapter we fix G a finite group and p a prime number dividing its order.

3.1 Basic Results

Since the article by D. Quillen [44], much is known about the $p-$subgroups complex. Here we try to expose the topics more connected with our research.
Of course, an analysis of all the principal aspects will lead us off the track. Readers may wish to consult also the following papers [5], [15], [32], [43], [53], [60].

The following notation is standard in literature.

$$\mathcal{S}_p(G) := \{\text{non-trivial } p\text{−subgroups of } G\}$$
$$\mathcal{A}_p(G) := \{\text{non-trivial elementary abelian } p - \text{subgroups of } G\}$$
$$\mathcal{B}_p(G) := \{H \leq G | H = O_p(N_G(H)),\ H \neq 1\}$$
$$\mathcal{I}_p(G) := \{\text{all non-trivial intersections of } p - \text{Sylow subgroups of } G\}$$

All these classes of subgroups of G are to be thought ordered by natural inclusion. Each of these is therefore a poset and we will generally denote with the same symbols both the posets and the simplicial complexes of chains associated respectively to each of these. We will soon see that these posets represent the same space up to homotopy equivalence.

The complex $\mathcal{S}_p(G)$ is known as *the Brown complex* (of G at the prime p), in honor of Kenneth Brown who introduced it for describing group cohomology over a finite field ([15] and [16]). The study of the $p-$subgroups complex, as an independent mathematical object, starts effectively with D. Quillen in the mentioned paper ([44]). He proves the equivalence between $\mathcal{S}_p(G)$ and $\mathcal{A}_p(G)$ and points out the basic questions inherent to it (above all the so-called *Quillen's conjecture*), giving solutions in most cases. This is why $\mathcal{A}_p(G)$ is known in literature as *the Quillen complex*. $\mathcal{B}_p(G)$ seems to be first introduced by S. Bouc in [13], thus it is known as *the Bouc complex*. Its importance consists in the fact that when G is a finite group of Lie type in characteristic p, the elements of $\mathcal{B}_p(G)$ are precisely the unipotent radicals of the parabolic subgroups of G and the Bouc complex is exactly the (first barycentric subdivision of the) Tits building of G (see section 3.2).

<u>Proposition</u> 3.1.1 (Quillen, Bouc et al.). *The following homotopy equivalences hold*

$$\mathcal{S}_p(G) \simeq \mathcal{A}_p(G) \simeq \mathcal{B}_p(G) \simeq \mathcal{I}_p(G).$$

Proof. First we prove that $\mathcal{S}_p(G) \simeq \mathcal{A}_p(G)$.
Let $i : \mathcal{A}_p(G) \longrightarrow \mathcal{S}_p(G)$ be the inclusion map. Then i is of course an order preserving map between the two posets, let $i^{-1}_{\leq P} := \{A \in \mathcal{A}_p(G) | A \leq P\}$ be the lower fiber under an element P of $\mathcal{S}_p(G)$. Then if $\Omega_1(Z(P))$ denotes the subgroup of all the elements of the center of P of order less or equal to p, $\Omega_1(Z(P))$ is a conjunctive element of $i^{-1}_{\leq P}$, in fact for all A in the fiber $i^{-1}_{\leq P}$ we can form the

subgroup $A \cdot \Omega_1(Z(P))$ which is again an elementary abelian $p-$subgroup of G contained in P. In virtue of Corollary 2.1.9, the fiber is contractible and the map i realizes an homotopy equivalence between $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$, by Fiber Lemma 2.1.3.
In order to show the equivalence between $\mathcal{S}_p(G)$ and $\mathcal{B}_p(G)$ we apply Proposition 2.1.5, of page 18, to the inclusion map of $\mathcal{B}_p(G)$ into $\mathcal{S}_p(G)$. The assumptions of Proposition 2.1.5 are satisfied once we show that for any $P \in \mathcal{S}_p(G) \setminus \mathcal{B}_p(G)$ the upper set $\mathcal{S}_p(G)_{>P}$ is contractible. In order to do this, first note that $\mathcal{S}_p(G)_{>P}$ and $\mathcal{S}_p(N_G(P))_{>P}$ are homotopy equivalent complexes, in fact if Q is a $p-$subgroup of G strictly containing P, then, by a known property of finite $p-$groups, $N_Q(P)$ strictly contains P and so the map $Q \mapsto N_Q(P)$ is a well defined order preserving map between the posets $\mathcal{S}_p(G)_{>P}$ and $\mathcal{S}_p(N_G(P))_{>P}$. It is an easy verification to show that the upper fibers are contractible. Thus we limit our analysis to showing that $\mathcal{S}_p(N_G(P))_{>P}$ is contractible. Since P is chosen not to be an element of the Bouc complex, then $\overline{P} := O_p(N_G(P))$ strictly contains P, i.e. $\overline{P} \in \mathcal{S}_p(N_G(P))_{>P}$. Moreover $\overline{P}$ is normal in $N_G(P)$, therefore chosen any other $Q \in \mathcal{S}_p(N_G(P))_{>P}$ we have that $\overline{P}Q$ is again a $p-$subgroup of $N_G(P)$ strictly containing P, i.e. $\overline{P}Q \in \mathcal{S}_p(N_G(P))_{>P}$, and $\overline{P}$ is a thus conjunctive element of $\mathcal{S}_p(N_G(P))_{>P}$ forcing it to be contractible, in virtue of Corollary 2.1.9. Finally Fiber Lemma 2.1.3 completes the proof of the equivalence $\mathcal{S}_p(G) \simeq \mathcal{B}_p(G)$.
The last equivalence, $\mathcal{S}_p(G) \simeq \mathcal{I}_p(G)$ is not specific of the $p-$subgroups complex but it holds in general contest, as an easy consequence of the Nerve Theorem 2.2.2 of page 21 applied to the complex $\Delta(\mathcal{S}_p(G))$, covered by the family of subposets $\mathcal{S}_p(G)_{\leq P}$ at the vary of P among the maximal $p-$subgroups of G. Otherwise, it can be easily proved by considering the order preserving map f from $\mathcal{S}_p(G)$ to $\mathcal{I}_p(G)$ sending any $p-$subgroup X into the intersection of all $p-$Sylow subgroups containing it, and showing the contractibility of the lower fibers of f. Q.E.D.

Note. J. Thévenaz and P. Webb showed in [53] that the homotopies of Proposition 3.1.1 can be chosen preserving the conjugation actions of G on the complexes. In doing this one has first to prove the analogous of the Fiber Lemma for $G-$homotopies, i.e. for homotopies between $G-$maps on $G-$posets.

Note. There are some other homotopy equivalent ways of looking at the complex of $p-$subgroups of G. Here we have just mentioned the ones we make use repeatedly in this chapter, in the next the reader can find another simplicial complex equivalent to $\mathcal{S}_p(G)$, called the "commuting complex at p" (Chapter 4, Remark 4.1.1).

Speaking about the $p-$subgroups complex we can not avoid speaking about Quillen's conjecture.

If a group G admits a nontrivial normal $p-$subgroup, then this would result a conjunctive element for the poset $\mathcal{S}_p(G)$, imposing it to be a contractible space. In other terms:

if G is such that $O_p(G)$ is non trivial, then $\mathcal{S}_p(G)$ is contractible.

The converse implication –*if $\mathcal{S}_p(G)$ is contractible then G has a nontrivial normal $p-$subgroup*– is known as the Quillen's conjecture and it is still in general an open question. Apart from this, in many cases it has been proved true. Quillen himself proved it for soluble groups and general linear groups in characteristic p ([44], section 12), later, it has been confirmed for all groups of Lie type and for all almost simple groups by J. Alperin, M Aschbacher, R. Lyons, P. Kleidman, S. Smith and many others. Moreover, using the Classification Theorem of non-abelian finite simple groups, the proof of Quillen of the soluble case can be extended to the class of $p-$soluble groups. Nowadays the most general result is due to M. Aschbacher and S. Smith [5] and is the following one.

THEOREM 3.1.2. *Let G be a finite group and p a prime divisor of $|G|$ with $p \geq 7$. Suppose $O_p(G) = 1$ and G has no composition factor isomorphic to a unitary group $U(n, q)$ with $q \equiv -1 \ (mod\ p)$. Then $\widetilde{H}_*(\mathcal{S}_p(G)) \neq 0$, which means that $\mathcal{S}_p(G)$ is not acyclic and, a fortiori, not contractible.*

The reader may note that in order to prove that the Quillen complex for a certain group, and a certain prime, is not contractible, it suffices to show that at least one reduced homology group (in general the one in the biggest dimension) does not vanish. If this is enough to prove the non-contractibility of a complex, on the other hand it is far away from describing its homotopy type. Therefore a natural question arises.

> *Which is the homotopy type of the $p-$subgroups complex of a given finite group?*

As mentioned in the introduction of [43], Thévenaz raised the question whether $\mathcal{S}_p(G)$ is always homotopy equivalent to a set of spheres in the sense of our Definition 1.3.3 on page 8. This problem has been first faced by J. Pulkus and V. Welker in the case of a $p-$nilpotent group. In section 3.4 of this chapter we end up their arguments and give a positive answer in this particular situation.

3.2 TITS BUILDINGS

In this section, our attempt is to better understand the philosophy applied for studying the $p-$subgroups complex of a finite group G and the posets $\mathcal{S}_p(G), \mathcal{A}_p(G)$, $\mathcal{B}_p(G)$. Apart from the fact that a vast quantity of literature concerning the aforementioned complexes already exists, as we have partly seen in the previous section, and the fact that there are also many applications in cohomology and representation theory, their study can be considered as an important generalization for all finite groups of the notion of *Tits building* typical of classical groups and groups of Lie type. In this sense the words of J.L. Alperin ([2]) can be significant:

> "[...] there are many aspects of Lie type groups that appear in the study of general finite groups. Our thesis here is that these are not

unrelated set of interesting and provocative accidents, but evidence of an important unifying principle which should be taken very seriously and which is easy to enunciate:

If G is an arbitrary finite group and p is any prime divisor of its order, then there exist interesting and important analogs of all the aspects of the structure of Lie type groups whose natural characteristic is p."

The main aspect of this analogy consists in the fact that for groups of this type, the Bouc poset $\mathcal{B}_p(G)$ is isomorphic to the first barycentric subdivision of the building, and therefore, from a homotopy point of view, the Tits building and the $p-$subgroups complex are the same object. Before giving the right references for this identification, we state just few ingredients about groups of Lie type (in this we consulted the book of R. Carter [22], to which we refer the interested reader). Finite groups of Lie type in some characteristic p satisfy the axioms of $BN-$pairs introduced by J. Tits ([55] and [54]). To any group G with a $BN-$pair it is possible to associate a geometrical structure, called the *Tits building* of G and denoted by $\Delta := \Delta(G, B)$. A subgroup Q of such a group G is said *parabolic* if it contains the normalizer of some $p-$Sylow subgroup of G (in the case of classical groups, it turns out that each parabolic subgroup is a split extension of a $p-$group, called the *unipotent radical*, by a direct product of linear groups of smaller dimensions). An important fact is that the Tits building of G can be defined just in terms of the parabolic subgroups, since these represents the stabilizers of the simplices. Thus the first barycentric subdivision $\mathrm{sd}(\Delta)$ is the complex of chains of proper parabolic subgroups. We can now state the following

<u>Theorem</u> 3.2.1. *Let G be a finite group of Lie type in characteristic p. Then a $p-$subgroup U lies in $\mathcal{B}_p(G)$ if and only if U is the unipotent radical of a parabolic subgroup of G. Moreover the inclusion relationship between parabolic subgroups are reversed on taking the corresponding unipotent radicals.*

Sketch of the proof. The implication that unipotent radicals lie in $\mathcal{B}_p(G)$ is in [24], (69.10) and makes use of the fact that groups of Lie type possess split $BN-$pair. The converse application can be deduced from a theorem of Borel and Tits [12]. An elementary argument may also be found in [19]. Q.E.D.

By this, it is now clear that $\mathcal{B}_p(G)$ represents just the opposite poset which normally defines the building. But a poset and its dual give rise to the same simplicial complex of chains, therefore $\mathcal{B}_p(G)$ is isomorphic to (the first barycentric subdivision of) the building of G.

The homotopy type of a Tits building is well known as it is described by the following famous Theorem of L. Solomon and J. Tits

<u>Theorem</u> 3.2.2. *Every building is a Cohen-Macaulay complex. In particular every building is a $(n-1)-$spherical complex, where n is the Lie rank of the group.*

PROOF. A proof of this result may be found in Chapter IV.6 of [17]. Q.E.D.

In particular we state the result concerning symplectic groups, which will be used in section 3.4.

THEOREM 3.2.3. *The building of a symplectic group $G = Sp(2n, q)$ of rank n over the field $\mathbb{F}_q$ of q elements is a spherical complex of dimension $n-1$.*

There are many other analogies between the structure of Lie type groups and the "Lie structure" of finite groups. For instance there are analogs of Hecke algebras in representation theory and also there exists a concept of weight for arbitrary groups. The reader who want to know more about this, is referred to the article of J. Alperin ([2]) and the references there mentioned.

3.3 $\mathcal{B}_p(G)$ AND $\mathcal{I}_p(G)$

As posets $\mathcal{B}_p(G)$ and the one arising from the intersections of $p-$Sylow subgroups, $\mathcal{I}_p(G)$, are very close to each other, so that, under certain circumstances, they are "more than homotopy equivalent."

LEMMA 3.3.1. *For any finite group G and any prime p the following hold*

(i) $\mathcal{B}_p(G) \subseteq \mathcal{I}_p(G)$.

(ii) If G has abelian $p-$Sylows, then $\mathcal{B}_p(G) = \mathcal{I}_p(G)$.

(iii) If G is a $p-$soluble group with $p-$length $l_p(G) = 1$, then $\mathcal{B}_p(G) = \mathcal{I}_p(G)$.

PROOF. *(i).* Let B be an element of the Bouc complex of G and let I_B the intersection of all $p-$Sylow subgroups of G containing B. Assume by contradiction that $B < I_B$. Then, since I_B is a $p-$group, $B < N_{I_B}(B) = N_G(B) \cap I_B$. Choose Q a Sylow $p-$subgroup of $N_G(B)$ and suppose $Q \leq S \in Syl_p(G)$. In particular $B \leq Q \leq S$ and $N_{I_B}(B) \leq I_B \leq S$. Thus, $Q \leq \langle Q, N_{I_B}(B)\rangle \leq N_G(B) \cap S \in Syl_p(N_G(B))$ and so, by maximality of Q, $Q = \langle Q, N_{I_B}(B)\rangle$, i.e. $N_{I_B}(B) \leq Q$. Therefore, since $B \in \mathcal{B}_p(G)$,

$$N_{I_B}(B) \leq \bigcap_{Q \in Syl_p(N_G(B))} Q = O_p(N_G(B)) = B$$

which is the wished contradiction.
(ii). Assume now that G has abelian Sylow $p-$subgroups. Take an element of $\mathcal{I}_p(G)$, say $A = \bigcap_{i \in I} S_i$ with S_i some Sylow $p-$subgroups of G. Since the S_i are abelian they normalize A and so

$$A \leq O_p(N_G(A)) = \bigcap_{T \in Syl_p(N_G(A))} T = \bigcap_{\substack{A \leq S_i \\ S_i \in Syl_p(G)}} S_i = A$$

i.e. $A = O_p(N_G(A))$.
Indeed any element of $\mathcal{I}_p(G)$ which is normal in every $p-$Sylow containing it, is indeed an element of $\mathcal{B}_p(G)$.
(iii). We suppose now that G is a finite $p-$soluble group of $p-$length one. We can assume that $G = N \rtimes S$ with S a $p-$Sylow subgroup of G acting faithfully on N, which is a $p'-$group. Let X be an element of $\mathcal{I}_p(G)$ and prove that $X = O_p(N_G(X))$. Without loss of generality, we can suppose $X \leq S$. Observe that for any $g \in N$ we have $X \leq S^g$ if and only if $g \in C_N(X)$, in fact if $\langle X, X^{g^{-1}} \rangle \leq S$ then we have $X = NX \cap S = NX^{g^{-1}} \cap S = X^{g^{-1}}$ and so $g \in N_N(X) = C_N(X)$. Thus the normalizer of X in G is $N_G(X) = C_N(X) \rtimes N_S(X)$, and then

$$X \leq O_p(N_G(X)) = \bigcap_{g \in C_N(X)} (N_S(X))^g = \bigcap_{g \in C_N(X)} N_{S^g}(X) \leq$$
$$\leq \bigcap_{g \in C_N(X)} S^g = X$$

where the last equivalence is given by the fact that X lies in $\mathcal{I}_p(G)$. Q.E.D.

Remark 3.3.1. Note that if G is a $p-$soluble group with $p-$length greater than 1, $\mathcal{B}_p(G)$ can be different from $\mathcal{I}_p(G)$ as the following example shows.
Consider the symmetric group on four objects, S_4, with its natural action on a $4-$dimensional module V over the prime field $\mathbb{F}_q$, $q \geq 3$, and let G be the semidirect product $G = V \rtimes S_4$ (i.e. G is the permutational wreath product $C_q wr S_4$). G is a $2-$soluble group of $2-$length equal two and $2-2'-$series

$$1 < V < V \rtimes K_4 < V \rtimes A_4 < G$$

where K_4 and A_4 denote respectively the normal Klein group and the alternating group in S_4. In a fixed complement S_4 of V, take the subgroup $P := \langle K_4, (34) \rangle$. P is a $2-$Sylow subgroup of S_4, and so of G, and its center is the group $X := \langle x \rangle := \langle (12)(34) \rangle$. We claim that $X \in \mathcal{I}_2(G) \setminus \mathcal{B}_2(G)$. First note that X is an intersection of $2-$Sylow subgroups of G. Of course X is contained in the intersection of all $2-$Sylow subgroups of S_4, which is K_4. Moreover if $\{u_1, u_2, u_3, u_4\}$ is a basis for V, the element $c = u_1 + u_2$ of V is centralized by x and so $X \leq P^c \leq S_4^c$, thus $X \leq K_4 \cap P^c = X$, since c does not centralize the entire K_4. This shows $X \in \mathcal{I}_2(G)$.
Now

$$N_G(X) = C_V(X)P = \langle u_1 + u_2, u_3 + u_4 \rangle \, P$$

and so $O_2(N_G(X)) = C_P(C_V(X)) = \langle x, (34) \rangle \neq X$, proving our claim.
(Note by the way that $Y = \langle x, (34) \rangle$ is an element of $\mathcal{B}_2(G)$).

In order to find a bound for the $p-$length, $l_p(G)$, of a $p-$soluble group G in terms of the dimension of the Bouc complex, we make use of the following

<u>Lemma</u> 3.3.2. *Let N be a normal subgroup of G and B/N an element of $\mathcal{B}_p(G/N)$. Then any $p-$Sylow subgroup of B lies in $\mathcal{B}_p(G)$.*

<u>Proof.</u> We use the "bar" notation to denote the quotients groups modulo N, and let $\overline{B}$ be an element of $\mathcal{B}_p(\overline{G})$. Chose A a Sylow $p-$subgroup of B, then, since $\overline{B}$ is a $p-$subgroup, $B = NA$. Moreover, as $A \in Syl_p(B)$, we have that $N_{\overline{G}}(\overline{A}) = \overline{N_G(A)}$. Call $C = O_p(N_G(A))$, then $\overline{C}$ is a normal $p-$subgroup of $N_{\overline{G}}(\overline{A})$, and so $\overline{C} \leq O_p(N_{\overline{G}}(\overline{A})) = \overline{A}$, which implies $\overline{C} = \overline{A}$, since C contains A. Thus $NC = NA = B$ and, as A is a Sylow $p-$subgroup of B, $A = C \in \mathcal{B}_p(G)$. Q.E.D.

We can now prove the following

<u>Theorem</u> 3.3.3. *Let G be a finite $p-$soluble group with $O_p(G) = 1$. Then $l_p(G) \leq \dim \mathcal{B}_p(G) + 1$.*

Proof. We work by induction on $l_p(G)$.
If $l_p(G) = 1$ the result is obvious, so suppose $l_p(G) > 1$, i.e. p divides $|G/O_{p',p}(G)|$. We claim that

$$\dim \mathcal{B}_p(G) > \dim \mathcal{B}_p\Big(\frac{G}{O_{p',p}(G)}\Big) \tag{3.3.1}$$

In fact using the previous Lemma with $O_{p',p}(G)$ in place of N, we immediately have that $\dim \mathcal{B}_p(G) \geq \dim \mathcal{B}_p(\overline{G})$.
To show the strictly inclusion we let $X \in Syl_p(O_{p',p}(G))$ and prove that it is an element of $\mathcal{B}_p(G)$. Using Frattini argument, we have that

$$G = O_{p',p}(G)N_G(X) = O_{p'}(G)N_G(X),$$

therefore if it were $O_p(N_G(X)) =: Y > X$ we would have that $O_{p'}(G)Y > O_{p',p}(G)$ and so $O_p(G/O_{p',p}(G)) \neq 1$ which contradicts our hypotheses. Therefore we have that any maximal chain in $\mathcal{B}_p(G/O_{p',p}(G))$ can be extended to a maximal chain in $\mathcal{B}_p(G)$ just by taking the preimages in G of its elements and adding a suitable Sylow $p-$subgroup of $O_{p',p}(G)$ as a bottom element. This proves the claim (3.3.1), from which, using the inductive step, we obtain the following

$$\dim \mathcal{B}_p(G) + 1 \geq \dim \mathcal{B}_p(G/O_{p',p}(G)) + 2 \geq l_p(G/O_{p',p}(G)) + 1 = l_p(G)$$

Q.E.D.

Remark 3.3.2. There exist $p-$soluble groups G for which $l_p(G) = \dim \mathcal{B}_p(G)$. One of these is, for instance, the group of the Remark 3.3.1, where $l_2(G) = 2$ and $\mathcal{B}_2(G)$ contains as maximal chain

$$\langle (34) \rangle < \langle (12), (34) \rangle < \langle K_4, (34) \rangle$$

Question. Are there groups G for which $l_p(G) = \dim \mathcal{B}_p(G) + 1$?

3.4 The Quillen complex of a soluble $p-$nilpotent group.

Making use of the topological techniques described in Chapter 2, we find out the homotopy type of the $p-$subgroups complex of a soluble $p-$nilpotent finite group.

Throughout this section we set G the semidirect product

$$G := N \rtimes P$$

where N is a soluble normal $p'-$subgroup, P a Sylow $p-$subgroup of G acting on N and p an odd prime number. Our main result of this section (Theorem 3.4.6) can so be enunciated

Theorem. *The Quillen complex at p for G is homotopy equivalent to a wedge of spheres.*

Indeed we prove that $\mathcal{A}_p(G)$ is a *"set of spheres"*, where this terminology is to be intended in the sense of Definition 1.3.3 on page 8. Our Theorem can be stated in this form by the fact that, for soluble groups, $\mathcal{A}_p(G)$ is not connected if and only if $O_p(G) = 1$ and G has $p-$rank 1 (Proposition 5.4 in [44]), but this precisely means that $\mathcal{A}_p(G)$ is a set of disjoint points, alias a wedge of $0-$spheres.

We divided the proof of the Theorem in different Lemmas, in order to get it more digestible to the reader. The first of these Lemmas is due to D. Quillen ([44]) who treated this situation when P is an elementary abelian $p-$group. Another elegant proof of this can be found in [43] (Lemma 3.3).

Lemma 3.4.1. *Let $G = N \rtimes A$ be a finite semidirect product of a normal soluble $p'-$subgroup N and an elementary abelian $p-$subgroup A. Then $\mathcal{A}_p(G)$ is Cohen-Macaulay of dimension $rk(A) - 1$.*

In proving our Theorem we will need to understand the homotopy type of the upper intervals $\Delta(\mathcal{A}_p(H)_{>X})$, at the vary of X among the elementary abelian $p-$subgroups of H. This is a difficult problem when H is arbitrary, but if we limit our attention to the case when H is a $p-$group and p is odd, we are able to prove that these upper intervals are always homotopy equivalent to sets of spheres (Theorem 3.4.5). In order to show this claim we need two more Lemmas. In the following three statements we then work inside a $p-$group, say P, of *odd* prime power order.

Lemma 3.4.2. *Let A be an elementary abelian subgroup of P such that A is central in P and set $\mathcal{M} := \mathcal{M}_A := \{X \leq P | A < X = \Omega_1(X), X/A$ is elementary abelian$\}$. Suppose $X_1, X_2, \ldots, X_m$ are the maximal elements of $\mathcal{M}$. Then the upper interval $\Delta(\mathcal{A}_p(P)_{>A})$ is homotopy equivalent to the wedge*

$$\bigvee_{i=1,\ldots,m} \Delta(\mathcal{A}_p(X_i)_{>A}).$$

PROOF. The proof consists in applying Lemma 2.2.7 of page 25 to the covering $\{\Delta(\mathcal{A}_p(X_i)_{>A})\}_{i=1,\dots,m}$. We just have to show that if J is a subset of indices of cardinality more than one, then $\bigcap_{i\in J} \Delta(\mathcal{A}_p(X_i)_{>A})$ is a contractible space. This follows once we prove the following claim

($\star$) Chosen $n > 1$ and $X_1, X_2, \dots, X_n$ maximal elements of $\mathcal{M}$ and set $Y := \bigcap_{i=1,2,\dots,n} X_i$, then $\Omega_1(Z(Y))$ strictly contains A.

Note in fact that if this is true, then $\Omega_1(Z(Y))$ will be a conjunctive element of $\mathcal{A}_p(Y)_{>A}$, therefore, in virtue of Corollary 2.1.9, $\Delta(\mathcal{A}_p(Y)_{>A}) = \bigcap_{i\in J} \Delta(\mathcal{A}_p(X_i)_{>A})$ will be conically contractible.
So we limit to prove the claim ($\star$). Since p is odd and, for $i = 1, 2 \dots, n$, $X_i' \le A \le Z(X_i)$, we have that each X_i has exponent p, so

$$\exp(Y) = p. \tag{3.4.1}$$

We set $W = \langle X_1, \dots, X_n\rangle$, making induction on the order of W, we can suppose $W = P$. By contradiction we assume $A = \Omega_1(Z(Y))$.
Let $Z_2/A = Z(W/A)$, then we claim

$$\Omega_1(Z_2) = Y. \tag{3.4.2}$$

In fact, as X_i/A is abelian for every i, $Y/A \le Z(W/A)$ and so, by (3.4.1), $Y \le \Omega_1(Z_2)$. Conversely, for every i, $\Omega_1(Z_2)X_i$ lies in $\mathcal{M}$ and so, by the maximality of the X_i, $\Omega_1(Z_2) \le X_i$ for all i, i.e. $\Omega_1(Z_2) \le Y$.
Now $[W, Y, W] \le [A, W] = 1$, so, using the 3$-$subgroups lemma, we have that $W' \le C_W(Y)$. Since $W = \Omega_1(W)$, we obtain $W' = \Phi(W)$ and so

$$\Phi(W) \le C_W(Y). \tag{3.4.3}$$

Using (3.4.1) and (3.4.3) we obtain that

$$\Phi(W) \cap Y \le A. \tag{3.4.4}$$

Set $S := (X_1)_W$ the normal core of X_1 in W. Then we claim that

$$S \cap \Phi(W) \le A. \tag{3.4.5}$$

In fact let $R = (S \cap \Phi(W))A$ and suppose, by contradiction that $R > A$. Then, since R is normal in W we have that

$$Z(W/A) \cap R/A = B/A$$

is not trivial. Moreover $\exp(B) = p$ as $B \le S \le X_1$. Then for every $i = 1, \dots, n$, $BX_i \in \mathcal{M}$ and so $B \le X_i$ for every $i = 1, \dots, n$, i.e. $B \le Y$ and so the contradiction

$B \leq Y \cap \Phi(W)A = A$.
Using 3.4.5 it follows that

$$[W, S] \leq W' \cap S = \Phi(W) \cap S \leq A$$

and so $S/A \leq Z(W/A)$. As $\Omega_1(S) = S$ we have $S \leq \Omega_1(Z_2) = Y$ and so $S = Y$. Now, as $n \geq 2$, $Y \neq X_1$. So $Y = (X_1)_W = X_1 \cap X_1^{w_1} \cap \ldots X_s^{w_s}$ with $w_1, \ldots w_s \in W$ and we have at least two different conjugates. Obviously $X_1^{w_i}$ are maximal elements in $\mathcal{M}$, but

$$\langle X_1, X_1^{w_1}, \ldots, X_1^{w_s} \rangle \leq X_1^W < W$$

and by the inductive hypothesis $\Omega_1(Z(Y)) > A$, contradiction. Q.E.D.

Remark 3.4.1. We note that for $p = 2$ the claim $(\star)$ fails, as the example of the dihedral group of order 16, D_{16}, shows. In this case, with the notation of the previous Lemma, A is the center of the group, and it has order 2. $\mathcal{M}_A$ has two maximal elements which are the two non cyclic maximal subgroups of the entire group, call them X_1, X_2. Their meet is the Frattini subgroup, which is a cyclic group of order 4 strictly containing A, therefore $\Omega_1(X_1 \cap X_2) = A$ and $(\star)$ fails. Note that $\mathcal{A}_2(D_{16})_{>A} = \bigsqcup_{i=1,2} \Delta(\mathcal{A}_2(X_i)_{>A})$ is the set of 4 disjoint points.

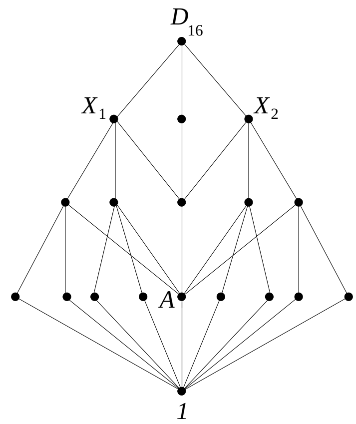


Figure 3.1: the lattice subgroup of D_{16}.

Our study brings us therefore to analyze the case of special $p-$groups, i.e. groups of $p-$power order such that modulo the center are elementary abelian. The simplest situation is when P is an extraspecial $p-$group.

<u>Lemma</u> 3.4.3. *Let $p \neq 2$ and P be an extraspecial $p-$group of exponent p. Then $\Delta(\mathcal{A}_p(P)_{>Z(P)})$ is $(rk(P) - 2)-$spherical.*

Proof. Remember that in an extraspecial $p-$group of exponent p the Frattini subgroup, the commutator subgroup and the center of the group all coincide in a subgroup Z of order p. We set $Z = \langle z \rangle$ and $V = P/Z$. In V there is a naturally defined symplectic form, via $f(xZ, yZ) = \eta$ where $[x, y] = z^\eta$, by which V gets the structure of a symplectic space over the field of order p of dimension, say $2n$. The upper interval $\mathcal{A}_p(P)_{>Z}$ is then a poset isomorphic to the poset of nonzero isotropic subspaces of V. Thus the associated simplicial complex is the building for the symplectic group of V, in particular, applying Theorem 3.2.3, we deduce that it is Cohen-Macaulay (and so spherical) of dimension $n-1$. Finally note that $\mathrm{rk}(P) = n + 1$. Q.E.D.

The following is a natural generalization for slightly different $p-$groups.

Lemma 3.4.4. *Let $p \neq 2$ and P be a finite special $p-$group of exponent p such that the Frattini subgroup, $\Phi(P)$, lies in the center of P and the commutator subgroup, P', has order p. Then $\Delta(\mathcal{A}_p(P)_{>Z(P)})$ is $(rk(P) - rk(Z(P)) - 1)-$spherical.*

Proof. This is an immediate consequence of the previous result, once noting that a group of this form can be seen as a central product $P = Z \circ E$ by its center Z and an extraspecial group E of order p^{2n+1} (where $2n$ is the dimension of the vector space P/Z), and observing that the poset $\mathcal{A}_p(P)_{>Z}$ is isomorphic to $\mathcal{A}_p(E)_{>Z(E)}$ via the map sending any $X = Z \times A$ to A. Q.E.D.

We can know describe the homotopy type of the upper intervals for arbitrary $p-$groups.

Theorem 3.4.5. *For every elementary abelian $p-$subgroup A of P the complex $\Delta(\mathcal{A}_p(P)_{>A})$, if not contractible, is homotopy equivalent to a set of spheres (of possibly different dimensions).*

Proof. We make some reductions on A. First we can restrict to the case A lies in the center of P. In fact $\mathcal{A}_p(P)_{>A} = \mathcal{A}_p(C_P(A))_{>A}$ and if A is not central in P we can always substitute P with the centralizer in P of A. Moreover, we can suppose $A = \Omega_1(Z(P))$. If not then $\Omega_1(Z(P))$ would be an conjunctive element of $\mathcal{A}_p(P)_{>A}$, which therefore would result conically contractible. Therefore we set $A = \Omega_1(Z(P)) =: Z$. Proposition 3.4.2 permits us to do another reduction assuming that P has exponent p and P/Z is elementary abelian. Consider first the case Z has order p. Then P is an extraspecial $p-$group and $\Delta(\mathcal{A}_p(P)_{>Z})$ is homotopy equivalent to a wedge of spheres of dimension $(\mathrm{rk}(P) - 2)$, in virtue of Lemma 3.4.3. Therefore it remains to treat the case Z has order greater than p. In this situation we will see that the complex $\Delta(\mathcal{A}_p(P)_{>Z})$ (which is no more spherical, see Remark 3.4.2 on page 46), is still a set of spheres. In order to prove this claim, we set $Z = X \times Y$ with X of order p and Y non trivial, and we factor out modulo X. Using the "bar" notation to denote the quotient groups, we write $\mathcal{M}$ for the set of all abelian subgroups $\overline{R}$ of $\overline{P}$ strictly containing $\overline{Z}$. $\mathcal{M}$ induces the covering $\mathcal{U} := \{\Delta(\mathcal{A}_p(R)_{>Z})|\overline{R} \in \mathcal{M}\}$ of $\mathcal{A}_p(P)_{>Z}$ made by closed subcomplexes. As usual, let $\mathcal{D} := \mathcal{D}^{\mathcal{U}}$ be the natural diagram of spaces associated to $\mathcal{U}$. $\mathcal{D}$ has intersection poset $\widehat{\mathcal{M}} := \mathcal{M} \cup \{\widehat{1}\}$ with reversed order relation, and the homotopy limit of $\mathcal{D}$ is equivalent to $\Delta(\mathcal{A}_p(P)_{>Z})$, in virtue of Projection Lemma 2.2.3. We claim that the spaces $\Delta(\mathcal{A}_p(R)_{>Z})$, at the vary of $\overline{R}$ inside $\mathcal{M}$, are either contractible or spherical. In fact consider $T/Y := Z(R/Y)$, if it strictly contains Z/Y, then, as $\overline{R} \in \mathcal{M}$, T is elementary abelian and therefore it is a conjunctive element of $\mathcal{A}_p(R)_{>Z}$ which would therefore result conically contractible. On the other hand, if $Z(R/Y) = Z/Y$, then R/Y is an extraspecial group of exponent p, therefore, by Lemma 3.4.3, $\mathcal{A}_p(R)_{>Z}$ is a wedge of, say, $(m-1)-$spheres where $2m$ is the dimension of the vector space R/Z. In both cases we fix a point $c_{\overline{R}}$ of the space $\Delta(\mathcal{A}_p(R)_{>Z})$, and we define another diagram of spaces $\mathcal{D}'$ on the same poset $\widehat{\mathcal{M}}$ such that $\mathcal{D}'$ has the same homotopy type of $\mathcal{D}$ and it is easier to calculate via the Wedge Lemma. We define $\mathcal{D}'$ in this way: $\mathcal{D}'$ has the same set of spaces as $\mathcal{D}$ (i.e. for every $\overline{R} \in \mathcal{M}$, $D'_{\overline{R}} = \mathcal{A}_p(R)_{>Z}$ and $D'_{\widehat{1}} = \emptyset$) and if $\overline{R} > \overline{S}$ in $\widehat{\mathcal{M}}$, then the map $d'_{\overline{RS}}$ is the contraction on $c_{\overline{S}}$, while $d'_{\overline{RR}}$ the identity map on $\Delta(\mathcal{A}_p(R)_{>Z})$. Using the Homotopy Lemma, we prove that $\underrightarrow{\operatorname{holim}}\,\mathcal{D}' = \underrightarrow{\operatorname{holim}}\,\mathcal{D}$. We define the map $f_{\overline{R}}$ of Proposition 2.2.4 in this way: if $\Delta(\mathcal{A}_p(R)_{>Z})$ is a contractible space, then $f_{\overline{R}}$ is the contraction on $c_{\overline{R}}$. Otherwise, by the observation made above, $\Delta(\mathcal{A}_p(R)_{>Z})$ is a wedge of, say, r spheres of some dimension, say k. We fix r points $e_1, e_2, \dots e_r$ in the space $\Delta(\mathcal{A}_p(R)_{>Z}) \setminus \bigcup_{S \subseteq R} \Delta(\mathcal{A}_p(S)_{>Z})$ each point in one of the r $k-$dimensional spheres. Thus we can define $f_{\overline{R}}$ as the map that contracts the complemented spaces of the balls with centers in e_i and sufficient small rays into $c_{\overline{R}}$. In this way all the conditions of the Homotopy Lemma are satisfied, and the diagrams $\mathcal{D}$ and $\mathcal{D}'$ have homotopy equivalent direct limits; we can express this direct limit using the Wedge Lemma 2.2.5 applied to $\mathcal{D}'$. We obtain the following sequence

$$\begin{aligned}
\Delta(\mathcal{A}_p(P)_{>Z}) &\simeq \underrightarrow{\operatorname{holim}}\,\mathcal{D}' \simeq \\
&\simeq \bigvee_{\overline{R} \in \widehat{\mathcal{M}}} D'_{\overline{R}} \star \Delta(\widehat{\mathcal{M}}_{<\overline{R}}) \simeq \\
&\simeq \Delta(\mathcal{M}) \vee \bigvee_{\overline{R} \in \mathcal{M}} \Delta(\mathcal{A}_p(R)_{>Z}) \star \Delta(\mathcal{M}_{<\overline{R}})
\end{aligned}$$

where every $\overline{R} \in \mathcal{M}$ is identified to the point $c_{\overline{R}} \in \Delta(\mathcal{A}_p(R)_{>Z}) \star \Delta(\mathcal{M}_{<\overline{R}})$. The poset $\mathcal{M}^{\mathrm{op}}$ is isomorphic to $\mathcal{A}_p(\overline{P})_{>\overline{Z}}$ via the identity map (consequently $\mathcal{M}_{<\overline{R}} \approx \mathcal{A}_p(\overline{P})_{>\overline{R}}$) and so we obtain the formula for the homotopy type of $\Delta(\mathcal{A}_p(P)_{>Z})$

$$\Delta(\mathcal{A}_p(\overline{P})_{>\overline{Z}}) \vee \bigvee_{\overline{R} \in \mathcal{A}_p(\overline{P})_{>\overline{Z}}} \Delta(\mathcal{A}_p(R)_{>Z}) \star \Delta(\mathcal{A}_p(\overline{P})_{>\overline{R}}) \tag{3.4.6}$$

where the wedge is made such that for every $\overline{R}$ in $\mathcal{A}_p(\overline{P})_{>\overline{Z}}$ a point $c_{\overline{R}}$ in $\Delta(\mathcal{A}_p(\overline{P})_{>\overline{R}})$ is identified with $\overline{R} \in \Delta(\mathcal{A}_p(\overline{P})_{>\overline{Z}})$. Finally by the inductive hypothesis on the order of P, we can suppose that $\Delta(\mathcal{A}_p(\overline{P})_{>\overline{R}})$ is a set of spheres for every $\overline{R}$ elementary abelian $p-$subgroup of $\overline{P}$. As the class sets of spheres is closed under the operations of joins and wedge (Lemma 1.3.1 on page 8), $\Delta(\mathcal{A}_p(R)_{>Z}) \star \Delta(\mathcal{A}_p(\overline{P})_{>\overline{R}})$ is still a wedge of spheres. So formula (3.4.6) implies that $\Delta(\mathcal{A}_p(P)_{>Z})$ is itself a set of spheres. Q.E.D.

We can now state and prove our main Theorem

<u>Theorem</u> 3.4.6. *Let $G = N \rtimes P$ with N a soluble normal $p'-$subgroup of G and P a Sylow $p-$subgroup. Then the Quillen complex of G is homotopy equivalent to a wedge of spheres of possible different dimensions.*

Proof. We use the apparatus of the diagrams of spaces described in Chapter 2 section 2.2.2. Factor out modulo N, and denote with $\overline{G}$ the factor group G/N Our aim is to apply Corollary 2.2.6 to the natural projection map f from the poset $P := \mathcal{A}_p(G)$ to $Q := \mathcal{A}_p(\overline{G}) \cup \left\{\widehat{0}\right\}$. Assumptions (i) and (ii) are of course satisfied. For (iii) note that the lower fibers of f under a subgroup $\overline{A} \in \mathcal{A}_p(\overline{G})$ is

$$f^{-1}(Q_{\leq \overline{A}}) = \mathcal{A}_p(N \rtimes A)$$

and so $f^{-1}(Q_{<\overline{A}})$ is $\mathcal{A}_p(N \rtimes A)$ deprived by its $p-$Sylow subgroups, in particular it is a $(\mathrm{rk}A - 2)-$space. Now by Lemma 3.4.1, the Quillen complex of $N \rtimes A$ is Cohen-Macaulay of dimension $\mathrm{rk}A - 1$, and so it is $(\mathrm{rk}A - 2)-$connected. Then, by standard topology, the injections $f^{-1}(Q_{<\overline{A}})$ into $f^{-1}(Q_{\leq \overline{A}})$ are all homotopy to constant maps, and condition (iii) is satisfied. So we can express the homotopy type of $\Delta(\mathcal{A}_p(G))$ by the formula

$$\Delta(\mathcal{A}_p(\overline{G})) \vee \bigvee_{\overline{A} \in \mathcal{A}_p(\overline{G})} \Delta(\mathcal{A}_p(NA)) \star \Delta(\mathcal{A}_p(\overline{G})_{>\overline{A}}) \tag{3.4.7}$$

Moreover, since $\Delta(\mathcal{A}_p(\overline{G}))$ is contractible, being $\overline{G}$ a $p-$group, we obtain

$$\Delta(\mathcal{A}_p(G)) \simeq \bigvee_{\overline{A} \in \mathcal{A}_p(\overline{G})} \Delta(\mathcal{A}_p(NA)) \star \Delta(\mathcal{A}_p(P)_{>A}).$$

By Lemma 3.4.1, $\Delta(\mathcal{A}_p(NA))$ is Cohen-Macaulay and in particular spherical. By Theorem 3.4.5, $\Delta(\mathcal{A}_p(P)_{>A})$ is a set of spheres. Thus, since the class of set of spheres is preserved by the operations of join and wedge, every $\Delta(\mathcal{A}_p(NA)) \star \Delta(\mathcal{A}_p(P)_{>A})$ is wedge of spheres of possible different dimensions, which forces $\Delta(\mathcal{A}_p(G))$ to be itself homotopy equivalent to a set of spheres. Q.E.D.

We conclude this section with a couple of Remarks.

There are examples of $p-$nilpotent groups whose Quillen complexes are wedges of spheres of different dimensions. We propose two different examples. The former is due to Alperin, the latter to J. Pulkus and V. Welker. Both examples can be found in [43].

Example 3.4.1. Let G be the standard wreath product of a Frobenius group of order pq (where p and q are different primes and p divides $q-1$) with the cyclic group of order p. Then $\mathcal{A}_p(G)$ is homotopy equivalent to a wedge of $(q-1)^p$ spheres of dimension $p-1$ with $p^{p-1}(q^p - q^{p-1})$ spheres of dimension 1.
In fact, factoring out modulo $N := O_q(G) \cong C_q^p$ and denoting with $\overline{G}$ the quotient group, $\overline{G}$ is isomorphic to $C_p wr C_p$), and so $\overline{G}$ has two conjugacy classes of maximal elementary abelian $p-$subgroups, one consists of one single subgroup, the base group of $\overline{G}$ of order p^p, and the second class of p^{p-2} subgroups of order p^2, namely the centralizers in $\overline{G}$ of elements of order p lying outside the base group. By applying formula (3.4.7) we obtain

$$\mathcal{A}_p(G) \simeq \mathcal{A}_p(N \rtimes C_p^p) \vee p^{p-2} \cdot \mathcal{A}_p(N \rtimes C_p^2) \vee \mathcal{A}_p(\overline{G})_{>Z(\overline{G})} \star \mathcal{A}_p(N \rtimes C_p).$$

One can check that spheres of dimensions $p-1$ come out from the first summand, while $1-$dimensional spheres arise from the second and the third summand.

Example 3.4.2. This second example is in a certain sense even stranger, since here all maximal elementary abelian $p-$subgroups have the same dimension.
Consider P the group of all upper unitriangular matrices 3×3 over the finite field of order p^n, $\mathbb{F}_{p^n}$ (where p is an odd prime), equivalently consider P a Sylow $p-$subgroup of the general linear group $\mathrm{GL}(3, p^n)$. Choose a faithful module N of co-prime order with p, and let $G := N \rtimes P$ be the semidirect product of N and P. Then $\mathcal{A}_p(G)$ is homotopy equivalent to a wedge of spheres of dimensions n and $2n-1$.
In fact we can argue as follows. Since p is odd, the group $\overline{G} \cong P$ has exponent p. Its center Z is constituted by matrices (a_{ij}) with $a_{12} = a_{21} = 0$, and it is contained in every maximal elementary abelian subgroup. P/Z is isomorphic to two copies of the additive group of $\mathbb{F}_{p^n}$, and the maximal elementary abelian subgroups of P correspond to the one-dimensional $\mathbb{F}_{p^n}-$subspaces of P/Z. Moreover any two of them intersect in Z. Therefore using formula (3.4.7) of Theorem 3.4.6, we have

$$\mathcal{A}_p(G) \simeq \mathcal{A}_p(\overline{G})_{>Z(\overline{G})} \star \mathcal{A}_p(N \rtimes Z) \vee \bigvee_{A \in \mathcal{P}} \mathcal{A}_p(NA) \ ,$$

where $\mathcal{P}$ denotes the set of the one-dimensional $\mathbb{F}_{p^n}-$subspaces of P/Z. Now $\mathcal{A}_p(\overline{G})_{>Z(\overline{G})}$ is a wedge of $0-$spheres, and, by Lemma 3.4.1, $\mathcal{A}_p(N \rtimes Z)$ is homotopy equivalent to a wedge of $(n-1)-$spheres, thus the first summand gives a contribution of a wedge of $n-$dimensional spheres. Again Lemma 3.4.1 yields that any $\mathcal{A}_p(NA)$, $A \in \mathcal{P}$, has the homotopy type of a wedge of $(2n-1)-$spheres, and so the claim is proved.

Remark 3.4.2. In the course of the proof of Theorem 3.4.5, we mentioned the fact that the "upper Quillen interval" for special groups may be not spherical. Here we give an example of a finite $p-$group P such that $\mathcal{A}(P)_{>\Omega(Z(P))}$ is a *non connected* set of spheres of *different* dimensions.
Fix p an odd prime and construct a $p-$group P by "fusing" together an extraspecial group F of order p^9 and exponent $p > 2$, and the group, E, of unitriangular $3\times 3-$matrices over the field $\mathbb{F}_{p^4}$, in such a way that $\mathcal{A}(P)_{>\Omega(Z(P))}$ will consist of some points (i.e. $0-$spheres) and some wedges of spheres of dimensions one. We show the details of this construction.
The group E consists of all unitriangular matrices

$$\begin{pmatrix} 1 & a & c \\ \cdot & 1 & b \\ \cdot & \cdot & 1 \end{pmatrix}$$

where a, b, c lie in the field $\mathbb{F}_{p^4}$ of order p^4. E is a special group satisfying $E' = Z(E) = \Phi(E)$, and it has p^4+1 maximal elementary abelian subgroups, call them H_i, which have all order p^8 and intersect pairwise in the center $Z(E)$. This of course implies that $\mathcal{A}_p(E)_{>Z(E)}$ is homotopy equivalent to p^4+1 points.
The group F is an extraspecial $p-$group of order p^9. Its maximal elementary abelian subgroups have order p^5, they all contain the center $Z(F)$ and $\mathcal{A}_p(F)_{>Z(F)}$, being equivalent to the building of a symplectic group of rank 4, is spherical of dimension 3.
Both $E/Z(E)$ and $F/Z(F)$ are isomorphic vector spaces over $\mathbb{F}_p$ of dimension 8. Choose an isomorphism ϕ between $E/Z(E)$ and $F/Z(F)$ in such a way that, at least one, but not all, $H_i/Z(E)$ is mapped by ϕ into a maximal isotropic subspace of $F/Z(F)$. The mapping ϕ will permit us to define the group G as the preimage in the direct product $E \times F$ via the projection map

$$\pi : E \times F \longrightarrow \frac{E \times F}{Z(E \times F)}$$

of the subgroup

$$V = \left\{ \big(eZ(E), \phi(eZ(E))\big) | eZ(E) \in E/Z(E) \right\}.$$

G is a group of order p^{13}, its center is $Z := Z(G) = Z(E)Z(F)$ and $G/Z(E) \cong F$, $G/Z(F) \cong E$. The structure of $\mathcal{A}_p(E)_{>Z(E)}$ has the effect of separating into components the elementary abelian subgroups of G over Z. In fact, if A is such a subgroup then $A/Z(F)$, being an element of $\mathcal{A}_p(E)_{>Z(E)}$, is contained in one single H_i, and if we denote with K_i the subgroups of G containing Z such that for all $i = 1, 2, \ldots, p^4+1$, $K_i/Z(F)$ corresponds to H_i, A will be contained in one single K_i. This proves that $\mathcal{A}_p(G)_{>Z}$ is the disjoint union

$$\bigsqcup_{i=1,2,\ldots,p^4+1} \mathcal{A}_p(K_i)_{>Z}. \tag{3.4.8}$$

The structure of $\mathcal{A}_p(K_i)_{>Z}$ is controlled by $\mathcal{A}_p(F)_{>Z(F)}$ and by the way we choose ϕ. In fact if ϕ sends $H_i/Z(E)$ into an isotropic subspace of $F/Z(F)$ then, being K_i abelian, $\mathcal{A}_p(K_i)_{>Z}$ is contractible and therefore its contribution in the formula (3.4.8) is one point. Otherwise if $\phi(H_i/Z(E))$ is not an isotropic subspace, then K_i is a group satisfying the assumptions of Lemma 3.4.4, so $\mathcal{A}_p(K_i)_{>Z}$ is spherical of dimension $\mathrm{rk}(K_i) - \mathrm{rk}(Z) - 1 = 1$.
Our choice of ϕ permits to have both $0-$dimensional and $1-$dimensional spheres into disjoint subcomplexes.

3.5 On the Euler Characteristic

In this part we show how we can "simplify" the structure of a given $p-$soluble group without changing the Euler characteristic of its $p-$subgroups complex.

Let G be a finite p-soluble group, with p a prime dividing the order of G. We fix P a Sylow $p-$subgroup of G and a p,p'-chief series of G

$$1 = N_0 < N_1 < \ldots < N_k = G$$

For every $i = 1, \ldots, k$ we set W_i the chief factor group N_i/N_{i-1}. G acts, by conjugation on every W_i. Define $\mathcal{A} := \{W_i | W_i \text{ is a } p'-\text{group}\}$ and set A the direct product of the W_i in $\mathcal{A}$,

$$A := \prod_{W_i \in \mathcal{A}} W_i$$

Then A is a p'-group on which P acts by conjugation componentwise (the action of P on every component is the restriction of the G-conjugation action). Finally, we define the semidirect product

$$\widetilde{G} := A \rtimes P.$$

We claim the following

<u>Theorem</u> 3.5.1. *The $p-$subgroups complexes of G and $\widetilde{G}$ have the same Euler characteristic, i.e.* $\chi(\mathcal{S}_p(G)) = \chi(\mathcal{S}_p(\widetilde{G}))$.

For every subgroup X of P, set $\alpha_G(X)$ (respectively $\alpha_{\widetilde{G}}(X)$) to be the number of conjugates of X in G (respectively in $\widetilde{G}$) that are contained in P. Then

<u>Lemma</u> 3.5.2. *For every X subgroup of P the following holds:*

(i) $N_{\widetilde{G}}(X) = N_P(X)C_A(X)$.

(ii) $|P : N_P(X)| = \alpha_{\widetilde{G}}(X)$.

PROOF. *(i)* Let $g = ay \in N_{\widetilde{G}}(X)$ with $a \in A$ and $y \in P$, then $X^a = X^{y^{-1}} \leq P$ and so $[X, a] \leq P \cap A = 1$, i.e. $a \in C_A(X)$, which implies $y \in N_P(X)$ and so $N_{\widetilde{G}}(X) = N_P(X)C_A(X)$.
(ii) It is an immediate consequence of *(i)*. Q.E.D.

<u>LEMMA</u> 3.5.3. *For every $X \leq P$,*

$$\alpha_G(X) \cdot |N_G(X)| = \alpha_{\widetilde{G}}(X) \cdot |N_{\widetilde{G}}(X)|$$

PROOF. For the proof we make use of Theorems 2.6 and 2.8 in [23], which imply

$$\alpha_G(X) \cdot |N_G(X)| = |P| \cdot \prod_{W_i \in \mathcal{A}} |C_{W_i}(X)| \ .$$

Thus, by Lemma 3.5.2

$$\alpha_G(X) \cdot |N_G(X)| = |P| \cdot |C_A(X)| = \alpha_{\widetilde{G}}(X) \cdot |N_{\widetilde{G}}(X)|.$$

Q.E.D.

<u>COROLLARY</u> 3.5.4. *If X and Y are two subgroups of P that are $G-$conjugate, then $C_A(X) = C_A(Y)$.*

<u>LEMMA</u> 3.5.5. *There exists a bijective correspondence ψ between the sets $\mathcal{S}_p(G)$ and $\mathcal{S}_p(\widetilde{G})$, preserving group isomorphism types.*

PROOF. We can look at P in two ways: as a subgroup of the group G and as a subgroup of the group $\widetilde{G}$. Fix X a subgroup of P, and denote with $\mathrm{Cl}_P^G(X)$ the set of subgroups of P that are conjugated to X by the action of G (similarly use $\mathrm{Cl}_P^{\widetilde{G}}(X)$ when the action is the conjugation by elements of $\widetilde{G}$). When we pass from G to $\widetilde{G}$, two subgroups of P that are conjugate in G may be no more conjugate in the group $\widetilde{G}$. Thus the set $\mathrm{Cl}_P^G(X)$ splits into a partition

$$\mathrm{Cl}_P^G(X) = \mathrm{Cl}_P^{\widetilde{G}}(X_1) \sqcup \mathrm{Cl}_P^{\widetilde{G}}(X_2) \sqcup \ldots \sqcup \mathrm{Cl}_P^{\widetilde{G}}(X_k),$$

where $X_1, X_2, \ldots, X_k$ are the representative subgroups in P of the $\widetilde{G}-$conjugacy classes in which the $G-$conjugacy class of X has been decomposed. Therefore passing to cardinalities we have

$$\alpha_G(X) = \sum_{j=1}^{k} \alpha_{\widetilde{G}}(X_j)$$

so, by Lemma 3.5.2 *(ii)*,

$$\alpha_G(X) = \sum_{j=1}^{k} |P : N_P(X_j)|$$

Moreover,

$$\begin{aligned}
\sum_{j=1}^{k} |\widetilde{G} : N_{\widetilde{G}}(X_j)| &= \sum_{j=1}^{k} |A : C_A(X_j)| \cdot |P : N_P(X_j)| \\
&= |A : C_A(X_1)| \cdot \sum_{j=1}^{k} |P : N_P(X_j)| \\
&= |A : C_A(X_1)| \cdot \alpha_G(X_1) \\
&= \frac{|G|}{|P|} \cdot \frac{\alpha_G(X_1)}{\prod_{W_i \in \mathcal{A}} |C_{W_i}(X_1)|} \\
&= |G : N_G(X_1)| \qquad (3.5.1)
\end{aligned}$$

where we made use of Corollary 3.5.4 and Lemma 3.5.3 for the last equality. The structure of $\widetilde{G}$ implies that every $p-$subgroup of $\widetilde{G}$ isomorphic to X has a representative of its $\widetilde{G}-$conjugacy class in P. Thus we let T be a set of representative subgroups of P for these $\widetilde{G}-$conjugacy classes. The number of $p-$subgroups of G isomorphic to X is therefore

$$\sum_{X_i \in T} |G : N_G(X_i)|$$

which, by equation (3.5.1), is equal to

$$\sum_{X_i \in T} \sum_{j=1}^{k} |\widetilde{G} : N_{\widetilde{G}}(X_j)|$$

but this number is exactly the number of $p-$subgroups of $\widetilde{G}$ isomorphic to X. This proves the existence of such a bijection. Q.E.D.

We can now give the proof of Theorem 3.5.1

Proof of the Theorem. For every $X \in \mathcal{S}_p(G)$ we indicate with $n_d(X)$ the number of chains $\mathcal{C}$ of length d in G such that $\sup(\mathcal{C}) = X$ (and use a similar notation for $\widetilde{G}$). Then, in virtue of Lemma 3.5.5,

$$\sum_{X \in \mathcal{S}_p(G)} n_d(X) = \sum_{X \in \mathcal{S}_p(G)} n_d(\psi(X)).$$

Therefore

$$
\begin{aligned}
\chi(\mathcal{S}_p(G)) &= \mu(\mathcal{S}_p(G)) + 1 \\
&= \sum_d \sum_{X \in \mathcal{S}_p(G)} (-1)^d n_d(X) + 1 \\
&= \sum_d \sum_{X \in \mathcal{S}_p(G)} (-1)^d n_d(\psi(X)) + 1 \\
&= \sum_d \sum_{\psi(X) \in \mathcal{S}_p(\widetilde{G})} (-1)^d n_d(\psi(X)) + 1 \\
&= \mu(\mathcal{S}_p(\widetilde{G})) + 1 \\
&= \chi(\mathcal{S}_p(\widetilde{G})).
\end{aligned}
$$

Q.E.D.

Note. The structures of the posets $\mathcal{S}_p(\widetilde{G})$ and $\mathcal{S}_p(G)$ are very close. In fact they both have the same Euler characteristic, as Theorem 3.5.1 says; moreover chosen arbitrary any element X in $\mathcal{S}_p(G)$, there corresponds a unique $\psi(X)$ in $\mathcal{S}_p(\widetilde{G})$ such that the lower intervals $\mathcal{S}_p(G)_{<X}$ and $\mathcal{S}_p(\widetilde{G})_{<\psi(X)}$ are isomorphic as posets. One may wonder if this is enough to ensure an isomorphism between the two entire posets.
This is not the case as the following example shows.

Example 3.5.1. Consider the case of the symmetric group on four objects, $G = S_4$. In this situation $\widetilde{G} = C_3 \rtimes D_8$ and the structures of the two $2-$subgroups posets are expressed by Figures 3.2 and 3.3.

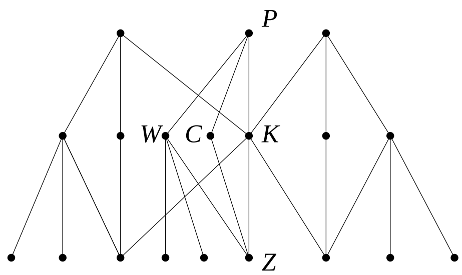


Figure 3.2: $\mathcal{S}_2(G)$

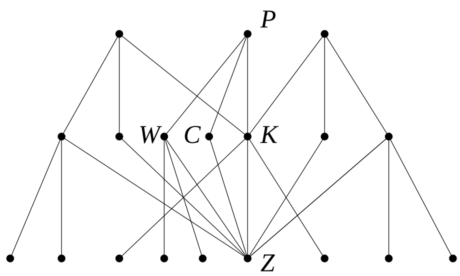


Figure 3.3: $\mathcal{S}_2(\widetilde{G})$

We note that for the group $\widetilde{G}$ there is a $2-$subgroup, namely the center of the entire group, such that it is below all non-minimal elements of $\mathcal{S}_2(\widetilde{G})$. This situation is of course impossible for $\mathcal{S}_2(G)$.

In both situations of the previous example, the Quillen complexes are contractible, thus one may natural ask the following

Question. Are always $\mathcal{S}_p(G)$ and $\mathcal{S}_p(\widetilde{G})$ homotopy equivalent?

We make use of the previous result to give an elegant proof of the Quillen's conjecture in the case of a soluble group whose Sylow $p-$subgroups are elementary abelian. A different proof of this result has been given by T. Hawkes and I.M. Isaacs in [31].
Any group of this type has $p-$length one, thus we set $G = N \rtimes P$ with N normal $p'-$subgroup and P an elementary abelian $p-$group of order p^n.

<u>Lemma</u> 3.5.6. $\overline{\chi}(\mathcal{S}_p(G)) = (-1)^n|\overline{\chi}(\mathcal{S}_p(G)|$.

Proof. By induction on $n + |N|$.
If $n = 1$, then P is cyclic of order p, and there are exactly $|N : C_N(P)|$ Sylow $p-$subgroups. Thus $\overline{\chi}(\mathcal{S}_p(G)) = 1 - |N : C_N(P)| = -\left|\overline{\chi}(\mathcal{S}_p(\widehat{G}))\right|$.
Let $n > 1$ and let N/M be a chief factor of G. N/M is an elementary abelian $q-$group, for q a prime number different from p, thus, if we denote with C the centralizer in P of this section, we have, by a known property of coprime actions, that P/C is a cyclic group. Thus $|P/C| \leq p$ and so $|C| \geq p^{n-1}$. We set $G_1 := MP$ and $G_2 := MC$. The reduced Euler characteristic of $\mathcal{S}_p(G)$ can be written as

$$\overline{\chi}(\mathcal{S}_p(G)) = \sum_{A \leq P} \mu(A)|N : C_N(A)|$$

and so

$$\overline{\chi}(\mathcal{S}_p(G)) = \sum_{A \leq C} \mu(A)|N : C_N(A)| + \sum_{A \nleq C} \mu(A)|N : C_N(A)|. \tag{3.5.2}$$

If $A \leq C$ then $MC_N(A) = N$ and therefore

$$|N : C_N(A)| = \frac{|M|}{|C_N(A) \cap M|} = |M : C_M(A)|.$$

If, on the contrary, $A \nleq C$ then $C_N(A) \leq M$ and so

$$|N : C_N(A)| = |N/M||M : C_M(A)|.$$

Therefore, (3.5.2) becomes

$$\overline{\chi}(\mathcal{S}_p(G)) = \sum_{A \leq C} \mu(A)|M : C_M(A)| + |N/M| \sum_{A \nleq C} \mu(A)|M : C_M(A)| \ .$$

Putting, for simplicity, $t = |N/M|$, we can write

$$\begin{aligned} \overline{\chi}(\mathcal{S}_p(G)) &= t \sum_{A \leq P} \mu(A)|M : C_M(A)| - (t-1) \sum_{A \leq C} \mu(A)|M : C_M(A)| \\ &= t\overline{\chi}(\mathcal{S}_p(G_1)) - (t-1)\overline{\chi}(\mathcal{S}_p(G_2)) \ . \end{aligned}$$

Using inductive hypothesis

$$\begin{aligned}\overline{\chi}(\mathcal{S}_p(G)) &= t(-1)^n|\overline{\chi}(\mathcal{S}_p(G_1))| - (t-1)(-1)^{n-1}|\overline{\chi}(\mathcal{S}_p(G_2))| \\ &= (-1)^n(t|\overline{\chi}(\mathcal{S}_p(G_1))| + (t-1)|\overline{\chi}(\mathcal{S}_p(G_2))|)\ ,\end{aligned}$$

from which we deduce

$$\overline{\chi}(\mathcal{S}_p(G)) = (-1)^n|\overline{\chi}(\mathcal{S}_p(G))|.$$

Q.E.D.

<u>Corollary</u> 3.5.7. *If* $\overline{\chi}(\mathcal{S}_p(G)) = 0$ *then* $C_P(N) \neq 1$*, i.e.* $O_p(G)) \neq 1$*.*

Proof. We proceed by induction on n.
If $n = 1$ the p−Sylow of G are cyclic and $\overline{\chi}(\mathcal{S}_p(G)) = 0$ implies there is only one p−Sylow, therefore $N = C_N(P)$ and $O_p(G) = P \neq 1$.
Let $n > 1$, using formula (3.5.3) we have that $0 = t|\overline{\chi}(\mathcal{S}_p(G_1))| + (t-1)|\overline{\chi}(\mathcal{S}_p(G_2))|$ and, since $t > 1$, it follows that $\overline{\chi}(\mathcal{S}_p(G_2)) = 0$ and, by induction $C_C(M) \neq 1$. Then, by the coprime action,

$$[N, C_C(M)] = [N, C_C(M), C_C(M)] \leq [N, C, C_C(M)] \leq [M, C_C(M)] = 1$$

and so $1 \neq C_C(M) = C_C(N) = C_P(N)$. Q.E.D.

<u>Corollary</u> 3.5.8. *If* G *is a soluble group with elementary abelian Sylow* p−*subgroups, then the following are equivalent*

(i) $\overline{\chi}(\mathcal{S}_p(G)) = 0$

(ii) $O_p(G)) \neq 1$.

Chapter 4

The Complex of Nilpotent Subgroups of A Finite Group

In this chapter we analyze the structure of the simplicial complex whose elements are chains of non-trivial nilpotent subgroups of a given finite group.

This complex can be seen half way between the complex of non-trivial p-subgroups, presented in the previous Chapter, and the complex of all non-trivial proper subgroups of the group (for the latter one we suggest consulting [13], [52] and in particular the recent work of Shareshian, [50], in which the author completely characterizes the class of finite soluble groups in terms of a combinatorial property of their subgroup lattice).

In section one, we describe homotopy equivalent formulations of the complex of nilpotent subgroups. We present some known results on it, in particular we see which group theoretical properties depend on its connectivity. In section two we start our analysis on the homotopy type, concentrating on the case of metanilpotent groups. We find a decomposition formula for the homotopy type of this complex for such groups, and look for group conditions that force the complex to be contractible.

Throughout this Chapter we make use of the following notation. For every finite group G, we set

$$\begin{aligned} \mathcal{N}(G) &:= \{\text{non-trivial nilpotent subgroups of } G\}\ , \\ \mathcal{AB}(G) &:= \{\text{non-trivial abelian subgroups of } G\}\ , \\ \mathcal{A}(G) &:= \{\text{non-trivial direct products of elementary abelian subgroups of } G\}\,. \end{aligned}$$

All these posets are ordered by the natural inclusion between subgroups of G. We reserve the same terminology for the simplicial complexes associated respectively to each of these. Moreover, we denote by $\mathcal{D}(G)$ the poset (and the associated simplicial complex) of all non-trivial subgroups of G ordered by subnormality; namely, taken two non-trivial subgroups of G, H and K, we say that $H \leq_{\mathcal{D}(G)} K$ if and only if H is a subnormal subgroup of K (written $H \lhd\lhd K$), this means that there exists a chain $H = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = K$ of subgroups from H to K each normal in the following.

4.1 Introductory Results

The complex $\mathcal{N}(G)$ admits different formulations, all equivalent up to homotopy. Reporting from the work of S. Lucido [36], we formulate the following

Lemma 4.1.1. *The simplicial complexes $\mathcal{N}(G)$, $\mathcal{AB}(G)$ and $\mathcal{A}(G)$ are all homotopy equivalent.*

Proof. The inclusion map

$$\begin{aligned} i : \mathcal{AB}(G) &\longrightarrow \mathcal{N}(G) \\ A &\longmapsto A \end{aligned}$$

is order preserving and it has contractible fibers. In fact, for every $X \in \mathcal{N}(G)$ the lower fiber $i^{-1}_{\leq X} := \{A \in \mathcal{AB}(G) | A \leq X\}$ contains the center of X as a conjunctive element. By Fiber Lemma 2.1.4 on page 18, $\mathcal{AB}(G)$ and $\mathcal{N}(G)$ are homotopy equivalent.
Similarly, the inclusion map

$$\begin{aligned} j : \mathcal{A}(G) &\longrightarrow \mathcal{AB}(G) \\ A &\longmapsto A \end{aligned}$$

induces a homotopy equivalence. In this case the lower fiber of every $X \in \mathcal{AB}(G)$ contains

$$\Omega_1(X) := \langle x \in X | x \text{ has prime order}\rangle$$

as the unique maximal element. Thus, by 2.1.9 on page 19, the fiber is contractible, and $\mathcal{AB}(G) \equiv \mathcal{A}(G)$.
By the transitivity of homotopy equivalence relation, the three complexes are homotopy equivalent. Q.E.D.

In the case of soluble groups the following holds

Lemma 4.1.2. *If G is a soluble group, then $\mathcal{D}(G)$ is homotopy equivalent to $\mathcal{N}(G)$.*

Proof. We consider the inclusion map

$$\begin{aligned} i : \mathcal{N}(G) &\longrightarrow \mathcal{D}(G) \\ X &\longmapsto X \end{aligned}$$

Since all subgroups of a nilpotent group are subnormal in the group, the map i is well defined and order preserving. The lower fiber of any $H \in \mathcal{D}(G)$ is

$$i^{-1}_{\leq H} := \{K \in \mathcal{N}(G) | K \lhd\!\lhd H\}\,.$$

By assumption H is soluble, therefore its Fitting subgroup, $\mathrm{Fit}(H)$, is non trivial and so it is an element of $i^{-1}_{\leq H}$. Moreover, it is well known that in a finite group every nilpotent subnormal subgroup is contained in the Fitting subgroup. Thus $\mathrm{Fit}(H)$ is the maximum element of $i^{-1}_{\leq H}$, which, thus, results contractible. Again Quillen Fiber Lemma 2.1.4 completes the proof. Q.E.D.

Note. In both previous Lemmas the homotopy equivalences can be chosen in such a way that the $G-$action by conjugation is preserved (see remark after the proof of Lemma 3.1.1 on page 32).

Note. For non-soluble groups Theorem 4.1.2 fails, as the example of the alternating group on five objects, A_5, shows. $\mathcal{N}(A_5)$ is equivalent to a set of 21 isolated points, while $\mathcal{D}(A_5)$ is equivalent to 22 points.

It would be interesting to analyze the connections of these posets in general, for instance whether or not the contractibility of one implies that of the other.
According to this point of view, we observe that the complex $\mathcal{D}(G)$ can be also characterized in another way

Lemma 4.1.3. *Set $\mathcal{PS}(G) := \{H \leq G | H$ is product of non-trivial simple groups$\}$ ordered by subnormality. Then $\mathcal{D}(G)$ is homotopy equivalent to $\mathcal{PS}(G)$.*

Proof. Let

$$\begin{aligned} i : \mathcal{PS}(G) &\longrightarrow \mathcal{D}(G) \\ X &\longmapsto X \end{aligned}$$

be the inclusion map.
i is an order preserving map, as both posets are ordered by subnormality. Chosen any non-trivial subgroup H of G we denote, as usual with

$$i^{-1}_{\leq H} := \{K \in \mathcal{PS}(G) | K \lhd\!\lhd H\}$$

the lower fiber of H and prove this is a contractible space. Write the socle of H as a direct product $\mathrm{Soc}(H) = S_1 \times S_2$, where S_1 is the direct product of the abelian minimal normal subgroups of H and S_2 is the direct product of the non-abelian simple subnormal subgroups of H.

Consider first the case in which S_2 is non trivial. We claim that S_2 is a conjunctive element of $i^{-1}_{\leq H}$. Let K be in $i^{-1}_{\leq H}$ and write $K = K_1 \times K_2$, with K_1 the product of the abelian direct factors of K and K_2 the product of the non abelian ones. Of course both K_1 and K_2 are subnormal subgroups of H. Moreover any indecomposable direct factor R of S_2, being non-abelian simple and subnormal in H, is a component of H, thus, in virtue of a known result (see for instance [3], Lemma 31.4), for every subnormal subgroup T of H, either R is a component of T, or R and T do commute. In particular choosing K_1 in place of T, K_1 commutes with every direct factor of S_2, and so $[K_1, S_2] = 1$. Arguing similarly with K_2, we have $K_2 \leq S_2$, and so the subgroup $\langle K, S_2\rangle = K_1 \times S_2 \in i^{-1}_{\leq H}$, proving our claim.
We suppose now that S_2 is trivial. Of course this implies that H has no non-abelian simple subnormal subgroups, and then the fiber reduces to

$$\{K \in \mathcal{A}(G) | K \lhd\lhd H\}\,.$$

In this situation it is not difficult to show that the subgroup generated by the elements of prime order of the center of the Fitting subgroup of H, namely $B := \Omega_1(Z(\mathrm{Fit}(H)))$, is a conjunctive element of the fiber. In fact any $K \in i^{-1}_{\leq H}$, being subnormal in H, is contained in $\mathrm{Fit}(H)$ and so HB still lies in $\mathcal{A}(G)$ and it is still a subnormal subgroup of H, i.e. $HB \in i^{-1}_{\leq H}$.
We proved that in any case the lower fibers of the map i are contractible spaces. By Fiber Lemma 2.1.4, the homotopy equivalence of the two complexes is guaranteed. Q.E.D.

Another way of looking at the complex of nilpotent subgroups is via the so-called *"commuting complexes"*. These are defined as follows.
To every non-empty subset U of a group G, there is associated a *commuting complex* $\Gamma(U)$, whose vertices are the elements of U and whose simplices are all finite non-empty subsets of U with pairwise commuting elements; equivalently, in other terms, the finite subsets of U that generate a non-trivial abelian subgroup.

<u>Lemma</u> 4.1.4. *If $G^\sharp := G \setminus \{1\}$, then $\Gamma(G^\sharp)$ is homotopy equivalent to $\mathcal{N}(G)$.*

Proof. Let $\mathrm{sd}(\Gamma(G^\sharp))$ be the first barycentric subdivision of $\Gamma(G^\sharp)$, and note that $\mathrm{sd}(\Gamma(G^\sharp))$ contains $\mathcal{AB}(G)$ as a subposet. Let $i : \mathcal{AB}(G) \longrightarrow \mathrm{sd}(\Gamma(G^\sharp))$ be the map sending a subgroup A to the set of its non-trivial elements $A^\sharp$. Define an "opposite" map $r : \mathrm{sd}(\Gamma(G^\sharp)) \longrightarrow \mathcal{AB}(G)$ by sending any set S of non-trivial pairwise commuting elements of G into $\langle S\rangle$, the subgroup generated by S. It is obvious that both i and r are poset maps, moreover for any subset T of $G^\sharp$, T is always contained in $r(T)$ and so the map $i \circ r$ is homotopy to the identity map on $\mathrm{sd}(\Gamma(G^\sharp))$, in virtue of Theorem 2.1.7 on page 19. Furthermore, $r \circ i = \mathrm{id}_{\mathcal{AB}(G)}$, so r is a deformation retract of $\mathrm{sd}(\Gamma(G^\sharp))$ into $\mathcal{AB}(G)$, and the two complexes are homotopy equivalent. Q.E.D.

Remark 4.1.1. In a similar way one can prove that if U is the set consisting of all non-trivial elements of G of order a power of p (where p is a fixed prime number dividing the order of G), then $\Gamma(U)$ is homotopy equivalent to the $p-$subgroups complex of G of Chapter 3.

The duality between the commuting complex of G, $\Gamma(G^\sharp)$, and the non-commuting one, i.e. the complex generated by pairwise non-commuting elements of $G^\sharp$, has recently been analyzed by J. Pakianathan and E. Yalin in [41].

The problem of establishing when $\mathcal{N}(G)$ is connected has been studied accurately by S. Lucido in [37]. To state her result we recall the notion of the *prime graph* $\Pi(G)$ of a group G. The vertices of $\Pi(G)$ are all prime divisors of the order of G, and two vertices p and q are joined by an edge if and only if G admits an element of order pq.

<u>Proposition</u> 4.1.5. ([37]). *The complex $\mathcal{N}(G)$ is connected if and only if the prime graph $\Pi(G)$ is connected.*

In an unpublished work, K. Grüenberg and O. Kegel give the general structure of a finite group whose prime graph is not connected. In particular, they describe all finite soluble groups with non-connected prime graph. These, from the point of view of the complex $\mathcal{N}(G)$ can be summarized up in the following result

<u>Theorem</u> 4.1.6. ([37]). *Let G be a finite group, $F_1(G) := \mathrm{Fit}(G)$ be the Fitting subgroup of G and $F_2(G)$ be the inverse image in G of $\mathrm{Fit}(G/F_1(G))$. Assume G is such that $F_2(G) \neq F_1(G)$, and suppose that $\mathcal{N}(G)$ is not connected. Then the following hold*

(i) Either G is a Frobenius group, or both $F_2(G)$ and $G/F_1(G)$ are Frobenius groups and $G/F_2(G)$ is cyclic.

(ii) $\mathcal{N}(G)$ is equivalent to a set of $|F_1| + 1$ isolated points and G acts on it by fixing one point and transitively permuting the others.

Simple groups with non-connected prime graph have been classified by Williams ([62]) and by A.S. Kondratev ([34]), and the quasi simple ones by S. Lucido ([38]).

The study of the homotopy type of $\mathcal{N}(G)$ and, in particular, the analysis of the structure of groups whose complex is contractible, will be the content of the next section, but before proceeding we need to introduce some more observations.

<u>Lemma</u> 4.1.7. *Let G_1 and G_2 be two finite groups. Then*

$$\mathcal{N}(G_1 \times G_2) \simeq \mathcal{N}(G_1) \star \mathcal{N}(G_2).$$

PROOF. In virtue of Lemma 1.2.1, *(iii)* of page 7 we know that the complex $\mathcal{N}(G_1) \star \mathcal{N}(G_2)$ is homotopy equivalent to the complex

$$\Delta := \mathbf{C}(\mathcal{N}(G_1)) \times \mathbf{C}(\mathcal{N}(G_2)) \setminus \{(1,1)\}$$

which consists in the collection of all non-trivial nilpotent subgroups of the form $K_1 \times K_2$, $K_i \in \mathcal{N}(G_i) \cup \{1\}$, for $i = 1, 2$. We prove that the inclusion map

$$j : \Delta \longrightarrow \mathcal{N}(G_1 \times G_2)$$

is an homotopy equivalence. In order to show this, let, for $i = 1, 2$,

$$\pi_i : G_1 \times G_2 \longrightarrow G_i$$

be the projection map onto G_i, and note that if K is a non-trivial nilpotent subgroup of $G_1 \times G_2$, $\pi_1(K) \times \pi_2(K)$ is still a nilpotent subgroup of $G_1 \times G_2$ containing K. Moreover, the upper fiber $j^{-1}_{\geq K}$ contains $\pi_1(K) \times \pi_2(K)$ as the unique minimal element, and so it is conically contractible. By Fiber Lemma 2.1.5, the map j is an equivalence. Q.E.D.

Note. A similar statement holds for the complex $\mathcal{D}(G)$.

LEMMA 4.1.8. *If G has non-trivial center, then $\mathcal{N}(G)$ is a contractible space.*

PROOF. If $Z(G) \neq 1$ it is immediate to see that it is a conjunctive element of the poset $\mathcal{N}(G)$, which, by Lemma 2.1.9 of page 19, is contractible. Q.E.D.

We will see in the next section that there are groups with trivial center and contractible complex of nilpotent subgroups, so having a non-trivial center is not the right condition corresponding to the contractibility of this complex. This, in particular, answers in a negative way to conjecture 2.9 of [41].

4.2 $\mathcal{N}(G)$ FOR METANILPOTENT GROUPS

Throughout this section we analyze the homotopy type of the simplicial complex of nilpotent subgroups of a finite metanilpotent group.

Before starting we collect well known facts on soluble groups we make frequently use in this section.

A *Carter subgroup* C of a group G is by definition a nilpotent self-normalizing subgroup of G (the latter condition means that C coincides with its normalizer $N_G(C)$).
Among the class of finite soluble groups, these subgroups satisfy nice properties, as R. Carter pointed out in [21]. Above all the following

<u>Proposition</u> 4.2.1. ([21]) *Every finite soluble group G has Carter subgroups. Moreover these form a complete conjugacy class of subgroups of G.*

If, in particular, G is a finite metanilpotent group, then it can be written in the form

$$G = NM$$

where N is a normal nilpotent subgroup of G and M is a Carter subgroup of G (see for instance Theorem 12.4 in [33]).

Beginning our study, we examine first the situation in which N and M do not intersect trivially.

<u>Proposition</u> 4.2.2. *Let G be a group of the form $G = NM$, with N normal nilpotent subgroup of G, and M a Carter subgroup of G. If $N \cap M \neq 1$, then $\mathcal{N}(G)$ is contractible.*

Proof. We prove that the complexes $\mathcal{N}(G)$ and $\mathcal{N}(N)$ are homotopy equivalent, by showing that the map

$$\begin{aligned} f : \mathcal{N}(G) &\longrightarrow \mathcal{N}(N)^{\mathrm{op}} \\ A &\longmapsto C_N(A) \end{aligned}$$

which sends any nilpotent subgroup of G into its centralizer in N, is a well-defined order preserving map between posets with contractible fibers. The nilpotency of the group N forces $\mathcal{N}(N)$, and so $\mathcal{N}(G)$, to be a contractible space.
The first step is to prove that f is well-defined; this means that for every nilpotent subgroup A of G, $C_N(A)$ is not trivial. By contradiction, suppose there exists a nilpotent subgroup A of G such that $C_N(A) = 1$. Then A is a Carter subgroup of the group NA and $NA = NM \cap NA = N(M \cap NA)$. Let N_1 be the normalizer in NA of $M \cap NA$. Then $N_1 = (N_1 \cap N) \times (M \cap NA)$ is nilpotent. If N_1 is not a Carter subgroup of NA, we take N_2 its normalizer in NA and note that N_2 is again nilpotent. We keep on proceeding in this way, defining $N_i = N_{NA}(N_{i-1})$, as soon as we find a subgroup K self-normalizing and so a Carter subgroup of NA containing $NA \cap M$. Then, by Theorem 4.2.1, K and A are conjugate by some element n of N. Therefore $N \cap K = N \cap A^n = (N \cap A)^n = 1$, but that is a contradiction since $N \cap K \geq N \cap M \neq 1$ by hypothesis. Thus the map f is well-defined. Of course, as we reversed the ordering in $\mathcal{N}(N)$, f is an order preserving map between $\mathcal{N}(G)$ and $\mathcal{N}(N)^{\mathrm{op}}$. It remains to prove that the upper fibers of f are contractible. On this purpose, we choose any subgroup L of N and consider the upper fiber

$$f^{-1}_{\geq L} := \{X \in \mathcal{N}(G) | C_N(X) \geq L\} .$$

Since L is nilpotent, its center $Z(L)$ is non trivial, thus it is an element of $f^{-1}_{\geq L}$. We note it is a conjunctive element. In fact, for any $X \in f^{-1}_{\geq L}$, the subgroup $XZ(L)$ is still nilpotent, as L centralizes X, and such that

$$C_N(XZ(L)) \geq C_N(X) \cap C_N(Z(L)) \geq L \ ,$$

thus $XZ(L) \in f^{-1}_{\geq L}$, which proves the contractibility of the fibers in virtue of Corollary 2.1.9 on page 19. Fiber Lemma 2.1.4 implies the homotopy equivalence, and this completes the proof. Q.E.D.

Note. In order to have a (metanilpotent) group G with contractible complex $\mathcal{N}(G)$ and trivial center it is enough to take, with the notation of the previous Proposition, $G = NM$ with M acting faithfully on N and $N \cap M \neq 1$. In this situation in fact $\mathcal{N}(G)$ is contractible by Proposition 4.2.2, and $Z(G) \leq C_M(N) = 1$. For instance, a group which is an extension of the extraspecial group of order 27 and exponent 3, by an involution fixing one of the two generators and inverting the other, is of this type. In this case N is the 3−Sylow subgroup of G and M a cyclic group of order 6.

In virtue of the just proved Proposition 4.2.2, from now on we limit our analysis to split metanilpotent groups; this means, in our notation, that we suppose $N \cap M = 1$ and G the semidirect product

$$G = N \rtimes M.$$

We can immediately make another reduction, assuming the action of M on N is faithful. Otherwise, having G a non trivial center, $\mathcal{N}(G)$ is contractible by Lemma 4.1.8. Thus from now on we suppose also that $C_M(N) = 1$. Moreover, since M is a Carter subgroup of G, and in particular it is self-normalizing, $C_N(M)$ must be trivial too.
The nilpotent subgroups of G acting faithfully on N, are located in G in a nice way, as the following focal observation shows.

<u>Lemma</u> 4.2.3. *If X is a nilpotent subgroup of G such that $C_N(X)$ is trivial, then X is contained in just one single complement of N in G.*

Proof. First prove that X is contained in a Carter subgroup of G.
Note that $X \cap N = 1$, in fact if not $X \cap N$ will be a proper normal subgroup of the nilpotent group X, and so in particular $Z(X) \cap N \neq 1$, but $Z(X) \cap N \leq C_N(X) = 1$ which is a contradiction. Therefore X and N intersect trivially. If X is self-normalizing, then X is a Carter subgroup of G and so it is a conjugate of M, therefore a complement of N. Suppose $X < N_G(X)$. Being $C_N(X) = 1$, $N_G(X)$ intersects trivially N, in particular $N_G(X)$ is nilpotent. So we can argue as before with $N_G(X)$ instead of X. If it is self-normalizing, then it is a Carter subgroup of G containing X, otherwise its normalizer is bigger and still disjoint from N, and so we can repeat the previous argument for it. Since the index of X in G is finite, after a finite number of steps we reach a Carter subgroup of G containing X, and this is a complement of N in G containing X.
Now we prove the uniqueness of the complement containing X.
Suppose by contradiction X is contained in at least two different complements, say M and M_1. As in soluble groups Carter subgroups are conjugated, we assume

$M_1 = M^n$ for some n in N. Therefore X and $X^{n^{-1}}$ are subgroups of $NX \cap M$. By repeated use of modular law,

$$X^{n^{-1}} = X^{n^{-1}}(N \cap M) = NX^{n^{-1}} \cap M = NX \cap M = X(N \cap M) = X$$

and so $X^{n^{-1}} = X$, i.e. n is an element of N normalizing X, but $C_N(X) = 1$, so $n = 1$ and $M_1 = M$. Q.E.D.

Define the following subposets of $\mathcal{N}(G)$

$\mathcal{B} := \{X \in \mathcal{N}(G) | C_N(X) \neq 1\}$,

$\mathcal{C} := \{\text{complements of } N \text{ in } G\}$,

$\mathcal{F} := \mathcal{F}(G) = \mathcal{B} \cup \mathcal{C}$.

As usual, we denote with the same symbols the simplicial complexes associated.

<u>Lemma</u> 4.2.4. *The following hold.*

(i) The subposet $\mathcal{B}$ is an ideal of $\mathcal{F}(G)$, and, as a complex, it has the same homotopy type as $\mathcal{N}(N)$, and so it is contractible.

(ii) The complexes $\mathcal{N}(G)$ and $\mathcal{F}(G)$ are homotopy equivalent.

Proof. *(i).* As when passing to centralizers of subgroups the order relation is reversed, it is clear that $\mathcal{B}$ is an ideal of $\mathcal{F}(G)$. Moreover, the map sending each $X \in \mathcal{B}$ into its centralizer in N, is a well-defined order reversing map with contractible fibers from $\mathcal{B}$ to $\mathcal{N}(N)$ (the contractibility of the fibers can be proved as in Proposition 4.2.2). Being N nilpotent, N is the unique maximal element of $\mathcal{N}(N)$, which is therefore contractible as $\mathcal{B}$ is so.
(ii) Consider i the inclusion map of $\mathcal{F}(G)$ into $\mathcal{N}(G)$ and prove that it induces a homotopy equivalence. Of course i is order preserving. In order to use Quillen criterion 2.1.4 of homotopy, we just have to prove the contractibility of the upper fibers

$$i^{-1}_{\geq Y} := \{X \in \mathcal{F}(G) | X \geq Y\},$$

at the vary of Y among the nilpotent subgroups of G. Choose therefore $Y \in \mathcal{N}(G)$. If $C_N(Y) \neq 1$, then Y is of course an element of $i^{-1}_{\geq Y}$ and therefore it is the unique minimal element of it. Suppose otherwise that $C_N(Y) = 1$, then, by Lemma 4.2.3, Y is contained in just one complement of N in G. It follows that the upper fiber $i^{-1}_{\geq Y}$ is just the set constituted by only this complement. In both cases therefore the fiber is contractible, and by 2.1.4, $\mathcal{N}(G)$ is homotopy equivalent to $\mathcal{F}(G)$. Q.E.D.

Using the previous Lemma we are now able to give a wedge decomposition of the homotopy type of the complex of nilpotent subgroups of G.
For any complement M of N in G we denote with $M_{(N)}$ the poset (and the

simplicial complex associated) of all non-trivial subgroups of M whose centralizers in N is not trivial, i.e.

$$M_{(N)} := \{X \leq M | X \neq 1 \text{ and } C_N(X) \neq 1\}.$$

Then

<u>THEOREM</u> 4.2.5. *The complex $\mathcal{N}(G)$ has the same homotopy type of the bouquet*

$$\bigvee_{x \in N} \mathbf{S}(M^x_{(N)})$$

where the wedge is taken on all complements of N in G, and the different suspensions $\mathbf{S}(M^x_{(N)})$, $x \in N$, are glued all together in one single point.

PROOF. In virtue of Lemma 4.2.4*(ii)*, we work with the complex $\mathcal{F} := \mathcal{F}(G)$. Note that any element of $\mathcal{M} := \mathcal{F} \backslash \mathcal{B}$ is a complement of N; therefore, it is a maximal element in the poset $\mathcal{F}$. Moreover, setting $\mathcal{P} := \mathcal{F}^{\text{op}}$ and $\mathcal{Q} := \mathcal{B}^{\text{op}}$, by 4.2.4*(i)*, all the hypotheses of Theorem 2.2.8 on page 26 are satisfied. This yields the wished formula

$$\begin{aligned} \mathcal{F}(G) &\simeq \bigvee_{x \in N} \mathbf{S}(\mathcal{F}(G)_{<M^x}) \\ &\simeq \bigvee_{x \in N} \mathbf{S}(M^x_{(N)}). \end{aligned} \tag{4.2.1}$$

Q.E.D.

The posets we have denoted by $M_{(N)}$ can be seen inside N, in this sense: we define $\mathcal{C}_N$ to be the poset of all non-trivial centralizers in N of non-trivial subgroups of M, i.e.

$$\mathcal{C}_N := \{C_N(X) \neq 1 | 1 \neq X \leq M\}.$$

Then

<u>LEMMA</u> 4.2.6. *$M_{(N)}$ and $\mathcal{C}_N$ are homotopy equivalent complexes.*

PROOF. To any X element of $M_{(N)}$ we associate its centralizer in N. This establishes an order reversing map, call it f, from $M_{(N)}$ to $\mathcal{C}_N$. An "opposite" map g from $\mathcal{C}_N$ to $M_{(N)}$ is the one which associates to any centralizer C in N of some non-trivial subgroup of M, the subgroup $C_M(C)$. It is straightforward to see that g is well defined and order reversing. Moreover $f \circ g = \text{id}_{\mathcal{C}_N}$ and for every $X \in M_{(N)}$, $g(f(X)) \leq X$, so f and g induce an homotopy equivalence between the two complexes. Q.E.D.

As an application of Theorem 4.2.5, we consider now what happens in the case that the complements M of N are elementary abelian $p-$subgroups. In this situation a bit more can be said.

<u>THEOREM</u> 4.2.7. *Let $G = N \rtimes P$ with N a finite nilpotent normal subgroup of G and P a Carter subgroup which is a finite elementary abelian $p-$group acting faithfully on N. Then the complex $\mathcal{N}(G)$ is spherical of dimension $(rk(P) - 1)$.*

PROOF. Note that the order of N is coprime with p. In fact otherwise there exists some element of N which centralizes P, and this is in contradiction to the fact that P is self-normalizing and $P \cap N = 1$. Thus P is a Sylow $p-$subgroup of G.
We prove by induction on the rank of P, that, with the notation of Theorem 4.2.5, $P_{(N)}$ is $(\mathrm{rk}(P) - 2)-$spherical. Formula (4.2.1) will give the wished result.
If $\mathrm{rk}(P) = 1$ then, as $C_N(P) = 1$, $P_{(N)}$ is empty, i.e. it is $(-1)-$spherical. So suppose P has rank at least two. By the so called "generation Lemma" (see for instance Lemma 1.9 in [11]) the group N can be generated by the centralizers in N of the maximal subgroups of P. In particular, there exists a maximal subgroup, say K, of P such that $C_N(K) \neq 1$. Set $\mathcal{P} := P_{(N)}$ and $\mathcal{Q} := \{X \in \mathcal{P} | X \cap K \neq 1\}$. Note that $\mathcal{Q}^{\mathrm{op}}$ is an ideal of $\mathcal{P}^{\mathrm{op}}$, and any element X of $\mathcal{M} = \mathcal{P} \setminus \mathcal{Q}$ is a complement of K in P, and so it is a minimal element of $\mathcal{P}$. Moreover, the map which associates to any $X \in \mathcal{Q}$ its intersection with K establishes a homotopy equivalence between the complex $\mathcal{Q}$ and the complex of all non-trivial subgroups of K; this latter, having a unique maximal element is contractible, and so $\mathcal{Q}$ is. This means that $\mathcal{P}$ is a minimal extension by the contractible space $\mathcal{Q}$, and we can apply Theorem 2.2.8 of page 26, obtaining the following formula for $P_{(N)}$

$$P_{(N)} = \mathcal{P} \simeq \bigvee_{X \in \mathcal{M}} \mathbf{S}(\mathcal{P}_{>X}).$$

Finally note that $\mathcal{P}_{>X} = \{Y \in P_{(N)} | X < Y\} = \{Y | X < Y < P\}$, which is spherical of dimension $rk(P) - rk(X) - 2 = rk(P) - 3$, since it is an interval of a Cohen-Macaulay poset of dimension $(\mathrm{rk}(P) - 2)$ ([44], Remark after Proposition 8.6). As making the suspensions the dimension increases by one, $P_{(N)}$ is a spherical complex of dimension $(\mathrm{rk}(P) - 2)$ as we claimed. Q.E.D.

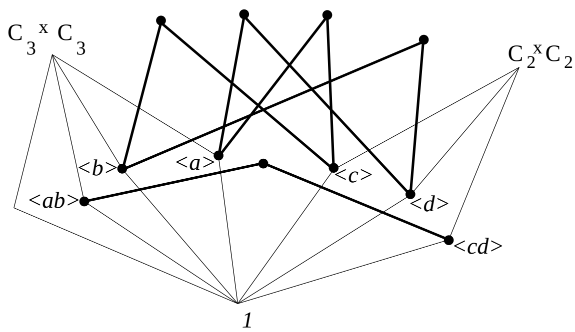
The following example shows that the situation may be different in the case that M not a $p-$group. The complex $M_{(N)}$ may be no more spherical. This of course, forces $\mathcal{N}(G)$ to be not spherical too.

Example 4.2.1. Let α be a third root of unity in $\mathbb{Z}_7$ and take M the abelian group $\langle a, b, c, d \rangle \simeq C_3 \times C_3 \times C_2 \times C_2$ acting on five copies of C_7 in this way:

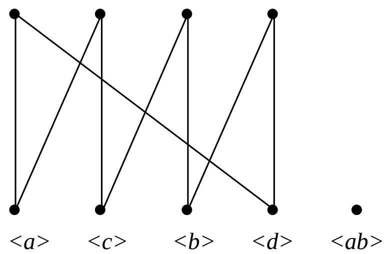
$$\begin{aligned} a &= \mathrm{diag}(\alpha, \alpha, 1, 1, \alpha), & b &= \mathrm{diag}(1, 1, \alpha, \alpha, \alpha^2), \\ c &= \mathrm{diag}(1, -1, 1, -1, -1), & d &= \mathrm{diag}(-1, 1, -1, 1, -1) \end{aligned}$$

(here we used a notation from linear algebra and $\mathrm{diag}(a, b, c, d, e)$ means the diagonal matrix with entries a, b, c, d, e on the principal diagonal and zeros otherwise).

The poset $M_{(N)}$ is constituted by the bolded lines in the following diagram



thus $M_{(N)}$ is equivalent to



So it has the homotopy type of the wedge $S^1 \vee S^0$. By Theorem 4.2.5

$$\mathcal{N}(G) \simeq \bigvee_{|N|} \mathbf{S}(S^1 \vee S^0)$$

which is a set of spheres of two different dimensions.

One may conjecture that, even if these complexes are not always spherical, their homotopy type is still the same as a set of spheres in the sense of Definition 1.3.3 of page 8. This is true at least when p is an odd prime and the Sylow $p-$subgroup is a Carter subgroup. In fact we can prove the following

<u>Theorem</u> 4.2.8. *Let $G = N \rtimes P$, with P an arbitrary $p-$group, for an odd prime number p, acting faithfully on a nilpotent $p'-$subgroup N and suppose also $C_N(P) = 1$. Then $\mathcal{N}(G)$ is a set of spheres (of possible different dimensions).*

Proof. We can suppose P is not elementary abelian, otherwise we are in the situation of Theorem 4.2.7 and $\mathcal{N}(G)$ will be spherical. Moreover, we work with the complex $\mathcal{A}(G)$, which is equivalent to $\mathcal{N}(G)$, by Lemma 4.1.1.

Consider the map

$$f : \mathcal{A}(G) \longrightarrow \widehat{\mathcal{A}}(\overline{G}) := \mathcal{A}(G/N) \cup \{\widehat{1}\}$$
$$A \longmapsto \overline{A}$$

induced by the projection on the quotient group.
f is well defined and order preserving. Moreover, the lower fiber of some $\overline{Y} \in \widehat{\mathcal{A}}(\overline{G})$ is

$$f^{-1}_{\leq \overline{Y}} = \mathcal{A}(NY).$$

This gives rise to a diagram of spaces on the poset $\widehat{\mathcal{A}}(\overline{G})$, to which we want to apply Corollary 2.2.6 on page 24. We have to analyze the inclusion maps between $\mathcal{A}(NY_1)$ and $\mathcal{A}(NY_2)$ whenever $NY_1 < NY_2$, for Y_1, Y_2 elementary abelian $p-$subgroups. We claim that, in this situation, i.e. when the complement of N is a $p-$group, these inclusions are always homotopy equivalent to contractions. This will suffice us to apply Corollary 2.2.6. Observe that, in virtue of Theorem 4.2.7, the subcomplexes $\Delta_Y := \mathcal{A}(NY)$ can be of three different types:

1. Δ_Y is contractible, and this happens if and only if $C_N(Y) \neq 1$,
2. Δ_Y is $0-$spherical not contractible, exactly when NY is a Frobenius group with kernel N and complement Y, and
3. Δ_Y is $(rk(Y) - 1)-$spherical, if and only if $C_N(Y) = 1$ and NY is not a Frobenius group.

Moreover, in our situation, Frobenius groups can appear only when $|Y| = p$ (this because Y is elementary abelian). Our claim is then proved if for each Δ_Y we consider the maps f_Y so defined:

1. f_Y is the contraction to a point if Δ_Y is contractible,
2. f_Y is the identity map if Δ_Y is a set of at least two points, and
3. if Δ_Y is a wedge of, say r, spheres of dimension $s > 0$, we fix r points $e_1, e_2, \ldots e_r$ in the space $\Delta_Y \setminus \bigcup_{X \in \mathcal{A}p(NY)} \Delta(\mathcal{A}(NY)_{>NX})$ each point in one of the r $k-$dimensional spheres, and we define f_Y as the map that contracts the complemented spaces of the balls with centers in e_i $(i = 1, 2, \ldots, r)$ and sufficient small rays into a common point.

These maps f_Y preserve the homotopy type of the spaces and make the diagram $\{\mathcal{A}(NY)\}_{\overline{Y} \in \mathcal{A}(\overline{G})}$ with inclusion maps to be equivalent to the natural one. All the hypotheses of Corollary 2.2.6 are then satisfied. So we deduce the formula

$$\begin{aligned}\Delta(\mathcal{A}(G)) &\simeq \big(\Delta(\mathcal{A}(N)) \star \Delta(\mathcal{A}_p(P))\big) \vee \bigvee_{\overline{X} \in \mathcal{A}(\overline{G})} \Delta(\mathcal{A}(NX)) \star \Delta(\mathcal{A}_p(\overline{G})_{>\overline{X}}) \simeq \\ &\simeq \bigvee_{X \in \mathcal{A}_p(P)} \Delta(\mathcal{A}(NX)) \star \Delta(\mathcal{A}_p(P)_{>X}).\end{aligned}$$

Since any $\Delta(\mathcal{A}(NX))$ is spherical and any $\Delta(\mathcal{A}_p(P)_{>X})$ is either empty or a set of spheres (respectively by Theorems 4.2.7 and 3.4.5), and since the class of set of spheres is closed under the operations of join and wedge, the previous formula proves that the complex $\mathcal{A}(G)$ is a set of spheres. Q.E.D.

Chapter 5

Complexes of Cosets of Finite Groups

In this chapter we analyze order complexes of posets whose elements are, instead of subgroups, right cosets of a given finite group.

In the first section, we examine the structure of $\mathcal{C}(G)$ the poset of *all* right cosets of G, different from G itself. The complex associated has a significant importance, in particular for what concerns the connections of its Euler characteristic and probability functions of generating the group G, as K. Brown pointed out recently in [18]. We will discuss these facts and present an analysis of its homotopy type, for soluble groups.
In section two, we consider the $p-$local structure of $\mathcal{C}(G)$, i.e. the subposet $\mathcal{C}_p(G)$ of $\mathcal{C}(G)$ constituted by cosets Hx whose subgroups, H, vary among the class of $p-$subgroups and the trivial subgroup of G. This complex is indeed topologically much different from $\mathcal{C}(G)$, for example it gives rise to spaces that are not sets of spheres.

5.1 The Complex of Proper Right Cosets of a Finite Group

As just said in the introduction of this Chapter, we denote by $\mathcal{C}(G)$ the poset

$$\{Hx \subset G | H < G \text{ and } x \in G\}$$

of all proper right cosets of G ordered by natural inclusion, and we call $(\mathcal{C}(G), \subseteq)$ the *coset poset* of G.
Note that if Hx and Ky are elements of $\mathcal{C}(G)$, then

$$Hx \subseteq Ky \iff H \leq K \text{ and } xy^{-1} \in K.$$

The content of this section is partly taken from the article [18] of K. Brown.

5.1.1 Definitions and Homotopy Type of $\mathcal{C}(G)$.

The coset poset $\mathcal{C}(G)$ can be seen as a part of a lattice; if we add to it a least and a greatest element (respectively the empty set and the entire group G), then the poset $\widehat{\mathcal{C}}(G) := \mathcal{C}(G) \cup \{\emptyset, G\}$ becomes a lattice with the usual operations of meet and join of subsets which assume these expressions

$$\begin{aligned} Hx \cap Ky &:= \begin{cases} (H \cap K)z & \text{where } z \text{ is any element of } Hx \cap Ky \\ \emptyset & \text{otherwise.} \end{cases} \\ Hx \cup Ky &:= \langle H, K, yx^{-1} \rangle x. \end{aligned}$$

To be more precise, $(\widehat{\mathcal{C}}(G), \cap, \cup)$ is a complete meet-sublattice of G.

An account of the properties of the lattice $\widehat{\mathcal{C}}(G)$ can be found in the book of R. Schmidt ([46], 9.4). In particular, for finite groups, $\widehat{\mathcal{C}}(G)$ is always a complemented lattice.
It is easy to see that $\mathcal{C}(G)$ is always connected unless G is a cyclic $p-$group for some prime p, in which case $\mathcal{C}(G)$ is an antichain of p points ([18], Proposition 14).
For what concerns the homotopy type of $\mathcal{C}(G)$, this has the structure of a nerve of spaces.

<u>Theorem</u> 5.1.1. *Suppose that $x_1, x_2, \ldots, x_n$ are the elements of G, then the complex associated to $\mathcal{C} := \mathcal{C}(G)$ is equivalent to $\mathrm{N}(\mathcal{F})$, the nerve of the family $\mathcal{F} = \{\mathcal{C}_{\geq x_1}, \mathcal{C}_{\geq x_2}, \ldots, \mathcal{C}_{\geq x_n}\}$ (here for $x \in G$, $\mathcal{C}_{\geq x}$ denotes the set $\{Hx | 1 \leq H < G\}$).*

Proof. According the Nerve Theorem 2.2.2 *(i)* on page 21, we just have to verify that any finite intersection of the family $\mathcal{F}$ is either empty or contractible. On this purpose let J be a non-empty subset of $\{1, 2, \ldots, n\}$ and consider the subcomplex

$$\bigcap_{j \in J} \mathcal{C}_{\geq x_j}.$$

An iterated use of the formula for the intersections yields

$$\bigcap_{j \in J} \mathcal{C}_{\geq x_j} = \mathcal{C}_{\geq \left\langle x_j x_{j_0}^{-1} \middle| j \in J \right\rangle x_{j_0}}$$

where j_0 is an arbitrary index in J. Therefore this intersection is empty if the subgroup $\left\langle x_j x_{j_0}^{-1} | j \in J \right\rangle$ is the entire group G, otherwise it is a contractible space, as the coset $\left\langle x_j x_{j_0}^{-1} | j \in J \right\rangle x_{j_0}$ is the unique minimal element of it. Q.E.D.

A more sophisticated argument permits, for non-simple groups, to express the homotopy type of $\mathcal{C}(G)$ as an iterated join of spaces.

Suppose G admits a proper non-trivial normal subgroup N, and denote with the "bar" notation the quotient groups modulo N. Moreover, set

$$\mathcal{C}(G,N) := \{Hx \in \mathcal{C}(G) | G = NH\}$$

the poset of all cosets whose representing subgroup is a supplement of N in G. The connection between $\mathcal{C}(G)$ and $\mathcal{C}(\overline{G})$ is expressed by the next Lemma, which emphasizes the topological meaning of a combinatorial result due to Gaschütz ([28]).

<u>Lemma</u> 5.1.2. *$\mathcal{C}(G)$ is homotopy equivalent to the join $\mathcal{C}(\overline{G}) \star \mathcal{C}(G,N)$.*

Proof. Consider the map

$$f : \mathcal{C}(G) \longrightarrow \mathcal{C}(\overline{G}) \star \mathcal{C}(G,N)$$

which sends Hx into $\overline{Hx}$ if $G \neq NH$, otherwise into itself seen as an element of $\mathcal{C}(G,N)$.
By the way we defined the join of two posets, it is immediate to notice that f is a well-defined order-preserving map.
Let choose $D \in \mathcal{C}(\overline{G}) \star \mathcal{C}(G,N)$, and show the contractibility of the lower fiber

$$f^{-1}_{\leq D} = \{C \in \mathcal{C}(G) | f(C) \leq D\}\,.$$

If $D \in \mathcal{C}(\overline{G})$, say $D = \overline{Hx}$, then it is trivial to notice that the coset HNx is the unique maximal element of $f^{-1}_{\leq D}$, which therefore would result contractible. So we may assume $D \in \mathcal{C}(G,N)$, moreover we may restrict to consider D a subgroup K of G such that $KN = G$. Now

$$f^{-1}_{\leq D} = \mathcal{A} \cup \mathcal{B}$$

where $\mathcal{A} := \{Hx \in \mathcal{C}(G) | KN \neq G\}$ is the poset of all the cosets whose subgroups associated do not supplement N, and $\mathcal{B} := \{Hx \in \mathcal{C}(G) | Hx \subseteq K\}$. Note that the projection map $q : \mathcal{A} \longrightarrow \mathcal{C}(\overline{G})$, sending any coset Hx of $\mathcal{A}$ into $\overline{Hx}$, induces a homotopy equivalence, so $\mathcal{A} \simeq \mathcal{C}(\overline{G})$. On the contrary $\mathcal{B}$ possesses K as greatest element, and so it is contractible. The intersection

$$\mathcal{A} \cap \mathcal{B} = \{Hx \in \mathcal{C}(K) | HN \neq G\}$$

and so the restriction of the quotient map q to the coset poset of the subgroup K induces a homotopy equivalence between $\mathcal{A} \cap \mathcal{B}$ and $\mathcal{C}(\overline{G})$. Thus the inclusion $\mathcal{A} \cap \mathcal{B} \hookrightarrow \mathcal{A}$ is a homotopy equivalence and so the subcomplex $\mathcal{A} \cap \mathcal{B}$ is a strong deformation retract of $\mathcal{A}$. In conclusion we have the that the complex $f^{-1}_{\leq D}$ is a strong deformation retraction onto $\mathcal{B}$, and in particular it is contractible, proving our claim. Fiber Lemma 2.1.4 implies the homotopy equivalence between the two complexes. Q.E.D.

Remark 5.1.1. According to the previous Lemma, in order to establish the homotopy type of $\mathcal{C}(G)$, one needs to know the homotopy type of the subcomplex $\mathcal{C}(G,N)$. This has in general a complicated structure. But if N is a minimal normal subgroup of G, and it is abelian, then every supplement of N must necessary be a complement, which means that $\mathcal{C}(G,N)$ is an antichain consisting of exactly $c\,|N|$ points, where c is the number of complements of N in G.

Using this fact, the following Theorem of K. Brown is immediate to prove.

<u>Theorem</u> 5.1.3. *Let G be a finite soluble group and let*

$$1 = N_0 < N_1 < \ldots < N_k = G$$

be a chief series of G. Then $\mathcal{C}(G)$ has the homotopy type of a bouquet of spheres of dimension $d-1$, where d is the number of indices $i = 1, 2, \ldots, k$ such that N_i/N_{i-1} has a complement in G/N_{i-1}.

Sketch of the proof. The key ingredients of the proof are the previous Lemma and Remark, the fact that in a finite soluble group the minimal normal subgroups are elementary abelian, and Lemma 1.3.1 on page 8, which expresses the join of spheres. Q.E.D.

Note. We will see that this result is no more true when passing from $\mathcal{C}(G)$ to its $p-$local version $\mathcal{C}_p(G)$ (see section 5.2, in particular example 5.2.1 on page 77).

In general the complex $\mathcal{C}(G)$ seems to be far away from being contractible and, for non-soluble groups it seems to have a very complicated structure. For further details we refer to [18].

5.1.2 The Euler Characteristic of $\mathcal{C}(G)$ and Probability Function

In this part we point out the connections between the coset complex $\mathcal{C}(G)$ and the probabilistic function $P(G,s)$ of generating the group G with $s-$tuples of its elements.

In 1936 P. Hall, in his work on Eulerian functions of a group ([30]), introduced the valuation $\phi(G,s)$ defined, for positive integers s, to be the number of ordered $s-$tuples of (not necessary distinct) elements of the group G that generate it. By the way, this seems to be the first time that Möbius function and Möbius inversion formula make their appearances in group theory.
Hall's observation for calculating $\phi(G,s)$ is easy and elegant.
For any subgroup K of G, we have

$$|K|^s = \sum_{H \leq K} \phi(H,s) \tag{5.1.1}$$

since every s−tuple of K generates some subgroup H of K. Then if we denote with μ the Möbius function on the lattice of all subgroups of G, and apply Möbius inversion formula (Theorem 1.4.3 on page 12) to equation (5.1.1) we obtain, for $K = G$, the formula

$$\phi(G, s) = \sum_{H \leq G} \mu(H, G)\, |H|^s \tag{5.1.2}$$

The function $P(G, s)$ expresses the probability that an ordered s−tuple of elements of G generates it, thus

$$P(G, s) = \frac{\phi(G, s)}{|G|^s}$$

and from (5.1.2) we have the following expression as a finite Dirichlet series

$$P(G, s) = \sum_{H \leq G} \frac{\mu(H, G)}{|G : H|^s} \tag{5.1.3}$$

We can attribute significance to this formula for arbitrary s in the complex field $\mathbb{C}$, as a formal identity in the ring of Dirichlet series. (It is worth noting that recently E. Detomi and A. Lucchini, in [27], answered a question of Bouc, proving a complete characterization of finite soluble groups in terms of the Euler expansion of $P(G, s)$).
We can attribute sense to formula (5.1.3) for $s = -1$,

$$P(G, -1) = \sum_{H \leq G} \mu(H, G)|G : H| \tag{5.1.4}$$

This value has a topological meaning as S. Bouc pointed out.

<u>Proposition</u> 5.1.4 (Bouc). *For any finite group G, the reduced Euler characteristic of $\mathcal{C}(G)$ coincides exactly with the opposite of $P(G, -1)$.*

Proof. We recall that if H is a subgroup of G, $\mu(H, G)$ can be computed by counting with sign the chains of subgroups between H and G. Any chain of the type

$$H = H_0 < H_1 < \ldots < H_l = G$$

gives a contribution of $(-1)^l$. Thus the term $\mu(H, G)\, |G : H|$ in (5.1.4) is the signed count of chains of right cosets

$$Hx = C_0 < C_1 < \ldots < C_l = G.$$

The sum in (5.1.4) expresses therefore the signed count of chains cosets

$$C_0 < C_1 < \ldots < C_l = G$$

for arbitrary C_0 and, since these are in bijective correspondence with chains of proper cosets and we are counting them with the opposite of the usual sign, this sum is exactly $-\widetilde{\chi}(\mathcal{C}(G))$. Q.E.D.

Kenneth Brown computed the function $P(G,-1)$ for many groups; he proved that in general this number is divisible by a big quantity of $|G|$. To state his main Theorem we need to introduce the following notation. Let p be a prime, and Q a p-subgroup of the group G; we denote by $N^*(Q)$ the subgroup of G generated by all $x \in G$ such that $\langle Q, Q^x \rangle$ is a p-group.

<u>Theorem</u> 5.1.5. ([18]) *Let G be a group, and p a prime. Denote by P a Sylow p-subgroup of G, and by Q a subgroup of P of maximal order such that $N^*(Q) = G$. Then $|P : Q|$ divides $P(G,-1)$.*

We conclude this section introducing an alternative formula for the computation of $P(G,-1)$, and so of $\widetilde{\chi}(\mathcal{C}(G))$, different from (5.1.4).
Consider the action of G on the poset $\mathcal{C}(G)$ by right multiplication. This is a faithful action, and by this G is a finite group of automorphisms of the poset $\mathcal{C}(G)$. Therefore G induces a group of automorphisms on the barycentric subdivision of the complex $\Delta(\mathcal{C}(G))$. Moreover this action is *regular* in the sense that every subgroup X of G satisfies the following condition:

($\star$) for every chain $C_0 < C_1 < \ldots < C_l$ in $\Delta(\mathcal{C}(G))$ and for every $x_0, x_1, \ldots, x_l$ of elements of X such that $C_0x_0 < C_1x_1 < \ldots < C_lx_l$ still lies in $\Delta(\mathcal{C}(G))$, there exists an element $x \in X$ such that

$$C_0x_0 < C_1x_1 < \ldots < C_lx_l = C_0x < C_1x < \ldots < C_lx$$

(any element of C_0x_0 is for instance such an x).
This condition of regularity enables us to construct the complex of orbits $\Delta(\mathcal{C}(G)/G)$, whose simplexes are chains of representative elements for the equivalence classes of the action.
Making the usual abuse of notation and calling with the same name both a poset and the simplicial complex associated, we can express the link between the Euler characteristic of $\mathcal{C}(G)/G$ and the one of $\mathcal{C}(G)$ by the next formula, which is an application of the so-called "Orbit-Counting" Lemma,

$$\sum_{g \in G} \chi(\mathcal{C}^g(G)) = |G|\chi(\mathcal{C}/G)$$

where for every $g \in G$, $\mathcal{C}^g(G)$ denotes the subcomplex of $\langle g \rangle$ −fixed simplexes of $\mathcal{C}(G)$, i.e. the subcomplex given by the subposet

$$\{Hx \in \mathcal{C} | Hxg = Hx\} = \{Hx \in \mathcal{C} | g = H^x\}$$

Since the poset $\mathcal{C}(G)/G$ is clearly isomorphic to $\mathcal{L}(G)\backslash\{G\}$ (via the map sending any classes $\{Hx\}$ to its identifying subgroup H), the orbit complex $\mathcal{C}(G)/G$, having a least element, is contractible, in particular $\chi(\mathcal{C}/G) = 1$ and so

$$\sum_{g \in G} \chi(\mathcal{C}^g(G)) = |G|. \qquad (5.1.5)$$

Moreover the following holds

<u>LEMMA</u> 5.1.6. *If g_1 and g_2 are two conjugated elements of G, then $\mathcal{C}^{g_1}(G)$ and $\mathcal{C}^{g_2}(G)$ are isomorphic posets.*

PROOF. Let $g_1 = {g_2}^t$, for some $t \in G$, then the following map between posets

$$\begin{aligned} f : \mathcal{C}^{g_1}(G) &\longrightarrow \mathcal{C}^{g_2}(G) \\ Hx &\longmapsto Hxt^{-1} \end{aligned}$$

is an isomorphism. Q.E.D.

In particular we can rewrite formula (5.1.5) in this way

$$\sum_{[g]\in Cl(G)} [G : C_G(g)]\chi(\mathcal{C}^g(G)) = |G|$$

where g is any representative element of the conjugacy class $[g]$ in G, and $Cl(G)$ is the set of all conjugacy classes of G. Using equation class for G, and the definition of the reduced Euler characteristic, we obtain

$$\widetilde{\chi}(\mathcal{C}(G)) = -\sum_{1\neq[g]\in Cl(G)} [G : C_G(g)]\widetilde{\chi}(\mathcal{C}^g(G)). \tag{5.1.6}$$

5.1.3 A Connection with the Bouc Complex for Groups of Lie Type

We end this section with an observation for groups of Lie type in characteristic p. If G is such a group the Bouc complex of G at p can be seen as a filter of the coset poset of G.

Fix B a Borel subgroup of G, i.e. B is the normalizer of some Sylow $p-$subgroup U of G, and call $\mathcal{P}_B(G)$ the poset (and the simplicial complex associated) of the parabolic subgroups of G over B

$$\mathcal{P}_B(G) := \{P < G | B \leq P\}.$$

Moreover set $\mathcal{D} := \{Hg | H \in \mathcal{P}_B(G)\}$ the coset poset made on the subgroups in $\mathcal{P}_B(G)$. We prove the following

<u>THEOREM</u> 5.1.7. *The poset $\mathcal{D}$ is isomorphic to the poset of Bouc $\mathcal{B}_p(G)$.*

PROOF. Define the following application between posets $f : \mathcal{D} \to \mathcal{B}_p(G)^{\mathrm{op}}$ which sends Hg to $f(Hg) := O_p(H)^g$. This map is well-defined by Theorem 3.2.1 of page 35; moreover f preserves the ordering, in fact if Hg and Kx are elements of $\mathcal{D}$ such that $Hg \leq Kx$, then we have that $O_p(H) \geq O_p(K)$ and $xg^{-1} \in K$ and

so $O_p(H)^g \geq O_p(K)^x$.
Conversely let $g : \mathcal{B}_p(G)^{\mathrm{op}} \to \mathcal{D}$ be the map defined as follows: if $L \in \mathcal{B}_p(G)$, then $N_G(L)$ is a parabolic subgroup of G, and therefore there exists an element y of G such that $N_G(L)^y \in \mathcal{P}_B$, set $g(L) := N_G(L)^y y^{-1}$.
Prove that $g(L)$ is well-defined, i.e. that $N_G(L)^y y^{-1}$ does not depend on the "conjugacy element" y. In fact, let y_1 and y_2 be two different elements of G such that $N_G(L)^{y_1}$, $N_G(L)^{y_2} \in \mathcal{P}_B$. Then B^{y_1} and B^{y_2} are two Borel subgroups of $N_G(L) \in \mathcal{P}_{B^{y_1}}$, and so, by Proposition 43.7(5) of [3], $y_1 y_2^{-1} \in N_G(L)$, i.e. $N_G(L)^{y_1} = N_G(L)^{y_2}$, and $N_G(L)^{y_1} y_1^{-1} = N_G(L)^{y_2} y_2^{-1}$, therefore $g(L)$ is well-defined. Now g preserves the ordering: if $L \leq M$ in $\mathcal{B}_p(G)^{\mathrm{op}}$, then $L \geq M$ in G, so $N_G(L) \leq N_G(M)$ and therefore $g(L) \leq g(M)$.
It is an easy verification to show that the maps f and g are one the inverse of the other. Q.E.D.

5.2 The Complex of $p-$cosets, $\mathcal{C}_p(G)$

In this section, following a suggestion of K. Brown, we study the "$p-$local version" of the coset poset.

For any prime number p dividing the order of G, we denote with $\mathcal{C}_p(G)$ the poset

$$\{Hx \subset G | H \in \mathcal{S}_p(G) \cup \{1\} \text{ and } x \in G\}$$

with the natural inclusion as order relation.

Our first observation deals with the connectivity of $\mathcal{C}_p(G)$.

<u>Proposition</u> 5.2.1. *For any finite group G, $\mathcal{C}_p(G)$ is the disjoint union of $[G : O^{p'}(G)]$ copies of $\mathcal{C}_p(O^{p'}(G))$. In particular the number of connected components of $\mathcal{C}_p(G)$ is $[G : O^{p'}(G)]$.*

Proof. It is suffice to work on the elements of the group.
We claim that for any $x \in G$ the elements that are joined to x by a path in $\mathcal{C}_p(G)$ are just the elements of the coset $O^{p'}(G)x$.
On this purpose, note that the minimal elements of $\mathcal{C}_p(G)$ directly connected to x (i.e. such that there exists a coset Hx in $\mathcal{C}_p(G)$ containing them) are the elements a of G, such that ax^{-1} lies in a $p-$subgroup, in particular $ax^{-1} \in O^{p'}(G)$. Similarly any minimal element b of G directly connected to any of these a, is such that $ba^{-1} \in O^{p'}(G)$, and so on. Thus if y is an element of G lying in the same component of x, there exists a sequence of elements of G

$$x = x_1, x_2, \ldots, x_{k-1}, x_k = y$$

such that for $i = 1, 2, \dots, k-1$,

$$x_{i+1}x_i^{-1} \in O^{p'}(G).$$

But then

$$yx^{-1} = (yx_{k-1}^{-1})(x_{k-1}x_{k-2}^{-1}) \cdot \ldots \cdot (x_2x^{-1}) \in O^{p'}(G)$$

and so $y \in O^{p'}(G)x$. Conversely if $y \in O^{p'}(G)x$, then we can write

$$yx^{-1} = z_1z_2 \dots z_r$$

with z_i $p-$elements. We prove by induction on r that y and x lie in the same connected component of $\mathcal{C}_p(G)$. If $r = 1$ then $yx^{-1} \in \langle z_1 \rangle$ which is a $p-$subgroup, thus $y \in \langle z_1 \rangle\, x \ni x$ and we have done. Suppose $r > 1$, then $yx^{-1}z_r^{-1} = y(z_rx)^{-1} = z_1 \dots z_{r-1}$. By the inductive hypothesis y and z_rx lie in the same component. Moreover $x \in \langle z_r \rangle\, x \ni z_rx$ and so x lies in the same component of z_rx, but then x and y belong to the same component and the claim is proved.
We can then fix a transversal $T := \{t_i\}$ for $O^{p'}(G)$ in G, and set $\mathcal{C}_{t_i}$ for the connected component of $\mathcal{C}(G)$ containing t_i (choose $t_1 = 1$). Of course $\mathcal{C}_p(G)$ is the disjoint union

$$\mathcal{C}_p(G) = \bigsqcup_{t_i \in T} \mathcal{C}_{t_i},$$

so it remains to prove that the $\mathcal{C}_{t_i}$ are all isomorphic posets. But this is trivial since, for any $t \in T$

$$\mathcal{C}_t = \left\{ Hxt \in \left| Hx \in \mathcal{C}_p(O^{p'}(G)) \right. \right\}$$

and the right multiplication by t induces an isomorphism of posets between $\mathcal{C}_1 = \mathcal{C}_p(O^{p'}(G))$ and $\mathcal{C}_t$. Q.E.D.

By the previous Proposition, in the study of the homotopy type of $\mathcal{C}_p(G)$ one can limit to the analysis of groups G which have no $p'-$epimorphic images, and for which $\mathcal{C}_p(G)$ is connected.
Moreover the following Proposition enables us to suppose also $O_p(G) = 1$.

<u>Proposition</u> 5.2.2. *Let G be a finite group. Then $\mathcal{C}_p(G) \simeq \mathcal{C}_p(G/O_p(G))$.*

Proof. We denote with $\overline{G}$ the quotient group $G/O_p(G)$, and consider the map

$$\begin{aligned} f : \mathcal{C}_p(G) &\longrightarrow \mathcal{C}_p(\overline{G}) \\ Hx &\longmapsto \overline{H}\overline{x} \end{aligned}$$

induced by the projection map on the quotient. This is clearly an order-preserving map. Let choose an element $\overline{H}\overline{x}$ in $\mathcal{C}_p(\overline{G})$ and denote with $f^{-1}_{\leq \overline{H}\overline{x}}$ the lower fiber of $\overline{H}\overline{x}$. Then $f^{-1}_{\leq \overline{H}\overline{x}}$ contains the coset $HO_p(G)x$ as a maximum element, therefore it is contractible and by Fiber Lemma 2.1.4, $\mathcal{C}_p(G) \simeq \mathcal{C}_p(\overline{G})$. Q.E.D.

As an immediate consequence of the previous result, or of Proposition 5.2.1, we have

COROLLARY 5.2.3. *Suppose G contains a normal $p-$Sylow subgroup, then $\mathcal{C}_p(G)$ is homotopic equivalent to an antichain of $|G|_{p'}$ points.*

The homotopy types of $\mathcal{C}(G)$ and $\mathcal{C}_p(G)$ are, even among the class of soluble groups, completely different. The following Lemma tells us in fact, that products of spaces are now involved.

LEMMA 5.2.4. *Let $G = A \times B$ and suppose that p is a prime number dividing the order of G. Then $\mathcal{C}_p(G)$ is homotopically equivalent to the product space $\mathcal{C}_p(A) \times \mathcal{C}_p(B)$.*

PROOF. Set $\mathcal{P} := \{Sx \in \mathcal{C}_p(G) | S = (S \cap A) \times (S \cap B)\}$. The inclusion map $i : \mathcal{P} \hookrightarrow \mathcal{C}_p(G)$ realizes an homotopy equivalence between the complexes $\mathcal{P}$ and $\mathcal{C}_p(G)$. In fact, it is surely a poset homomorphism and, chosen an arbitrary element, say Tx, in $\mathcal{C}_p(G)$, the upper fiber over it

$$i^{-1}_{\geq Tx} := \{Sy \in \mathcal{P} | Sy \geq Tx\}$$

has a unique minimal element $(\pi_A(T) \times \pi_B(T))x$, and thus it is conically contractible (here π_A and π_B denote the projection map of G respectively onto the first and second component). Fiber Lemma 2.1.4 proves that i is an homotopy equivalence between $\mathcal{P}$ and $\mathcal{C}_p(G)$.
Moreover, $\mathcal{P}$ is homotopically equivalent to the product space $\mathcal{C}_p(A) \times \mathcal{C}_p(B)$ via the map (between posets) so defined

$$\begin{array}{rcl} g : \mathcal{P} & \longrightarrow & \mathcal{C}_p(A) \times \mathcal{C}_p(B) \\ Sab & \longmapsto & ((S \cap A)a, (S \cap B)b) \end{array}$$

In fact, if $Sa_1b_1 \leq Ta_2b_2$ in $\mathcal{P}$, then $a_1a_2^{-1} \in T \cap A$ and $b_1b_2^{-1} \in T \cap B$, therefore $(T \cap A)a_1 = (T \cap A)a_2$ and $(T \cap B)b_1 = (T \cap B)b_2$, from which it results $g(Sa_1b_1) \leq g(Ta_2b_2)$. Taken an arbitrary element (Sa, Tb) in $\mathcal{C}_p(A) \times \mathcal{C}_p(B)$, the lower fiber

$$g^{-1}_{\leq (Sa,Tb)} := \{R\widetilde{a}\widetilde{b} \in \mathcal{P} | ((R \cap A)\widetilde{a}, (R \cap B)\widetilde{b}) \leq (Sa, Tb)\}$$

has a greatest element $(S \times T)ab$, so it is conically contractible and the complex $\mathcal{P}$ is equivalent to $\mathcal{C}_p(A) \times \mathcal{C}_p(B)$ again by Fiber Lemma.
The transitivity relation of homotopy equivalence completes the proof. Q.E.D.

Note. We note that the previous result is not true for the entire coset poset $\mathcal{C}(G)$. In fact the equivalence $\mathcal{C}(A \times B) \simeq \mathcal{C}(A) \times \mathcal{C}(B)$ fails (if $|A|$ and $|B|$ are coprime, then $\mathcal{C}(A \times B) \simeq \mathcal{C}(A) \star \mathcal{C}(B)$ holds, but in general even this is not satisfied, see [18], section 8.3 for more details).

Example 5.2.1. Consider the group $G = S_3 \times S_3$ the direct product of two copies of the symmetric group on three objects, and the prime $p = 2$. Then $\mathcal{C}_2(G)$ has the same homotopy type of the product space of two wedges each consisting of four spheres of dimension 1, which is of course far away from being a set of spheres.

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Asato Ma Sadgamaya
Tamaso Ma Jyotirgamaya
Mritior Ma Amritam Gamaya

Sarvesham Swasti Bhavatu
Sarvesham Shantir Bahavatu
Sarvesham Purnam Bhavatu
Sarvesham Mangalam Bhavatu

Lokah Samastah Sukhino Bhavantu

Om Shantih Shantih Shantih

Hari Om

From illusion lead me to truth.
From darkness lead me to light.
From death lead me to immortality.
Let all beings live in joy.
Let all beings live in fullness.
Let all beings live in happiness.
Let the whole world live in bliss.
Let the joy be in the whole world.