Computing the nonabelian tensor squares of polycyclic groups

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Abstract

In this paper we develop the theory of computing the nonabelian tensor squares of polycyclic groups. The nonabelian tensor square $G \otimes G$ of any group G is isomorphic to a subgroup K of the derived subgroup of a cover group $\nu(G)$. We develop a general commutator calculus in K that models computations in $G \otimes G$. We show that if G is polycyclic, then the cover group $\nu(G)$ is also polycyclic, and we give a finite presentation for $\nu(G)$ based on a presentation for G. We are then able to describe a finite generating set for K, and hence for $G \otimes G$, without needing a polycyclic presentation for $\nu(G)$. We apply our results in two ways. We first develop an algorithm that can be implemented within a computer algebra system, such as GAP (Groups, Algorithms and Programming), to compute the nonabelian tensor square of any polycyclic group. Second, we use the commutator calculus and structural results for the cover group $\nu(G)$ to directly compute the nonabelian tensor squares for the free nilpotent groups of class 3 and finite rank. The computations for the free nilpotent groups of class 3 were guided by examining the structure of the nonabelian tensor squares of such groups of small rank that were found by computer calculation.

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1 Introduction

The nonabelian tensor square $H \otimes H$ of the group H is the group generated by the symbols $g \otimes h$, where $g, h \in H$, subject to the relations

$$gg' \otimes h = ({}^{g}g' \otimes {}^{g}h)(g \otimes h) \quad \text{and} \quad g \otimes hh' = (g \otimes h)({}^{h}g \otimes {}^{h}h')$$

for all $g, g, h, h' \in H$, where ${}^h g = hgh^{-1}$ is conjugation on the left. For consistency, we define the commutator $[h, g]$ of group elements h and g to be ${}^h g g^{-1}$.

By computing the nonabelian tensor square we mean finding a standard or simplified presentation for the nonabelian tensor square. One approach to computing the nonabelian tensor square for a finite group, used by Brown, Johnson and Robertson [7], is to start with the finite presentation given by the definition above and simplify the presentation using Tietze transformations. The simplified presentation is then examined to determine the isomorphism type of the nonabelian tensor square. This method was used in [7] to compute the nonabelian tensor square for each nonabelian group of order at most 30. Since the presentation given by the definition of $H \otimes H$ has $|H|^2$ generators and $2 \cdot |H|^3$ relations, this method is limited to groups of relatively small order.

A second approach to computing the nonabelian tensor square of a group H involves the group $\nu(H)$.

Definition 1 *Let be H be a group with presentation $\langle \mathcal{H} | \mathcal{R} \rangle$ and let H^φ be an isomorphic copy of H via the mapping $\varphi : h \mapsto h^\varphi$ for all $h \in H$. We define the group $\nu(H)$ to be*

$$\nu(H) = \langle \mathcal{H}, \mathcal{H}^\varphi \mid \mathcal{R}, \mathcal{R}^\varphi, {}^x[g, h^\varphi] = [{}^x g, ({}^x h)^\varphi] = {}^{x^\varphi}[g, h^\varphi],\ \forall x, g, h \in H \rangle. \quad (1)$$

The groups H and H^φ can be isomorphically embedded into $\nu(H)$ [14]. Hence we overload the labels H and H^φ to also denote the natural isomorphic copies of H and H^φ in $\nu(H)$.

The group $\nu(H)$ was independently investigated by Rocco [24] and Ellis and Leonard [14]. This group $\nu(H)$ can be found earlier (see [15]) in the language of crossed modules. The motivation for studying $\nu(H)$ is the following result found in both [24] and [14].

Theorem 2 *Let H be a group. The map $\phi : H \otimes H \to [H, H^\varphi] \lhd \nu(H)$ defined by $\phi(g \otimes h) = [g, h^\varphi]$ for all g and h in H is an isomorphism.*

Both papers [24] and [14] provide structural results for $\nu(H)$ that are summarized in the following theorems.

Theorem 3 ([24]) *Let H be a group.*

(i) *If H is finite then $\nu(H)$ is finite.*
(ii) *If H is a finite p-group then $\nu(H)$ is a finite p-group.*
(iii) *If H is nilpotent of class c then $\nu(H)$ is nilpotent of class at most $c + 1$.*
(iv) *If H is solvable of derived length d then $\nu(H)$ is solvable of derived length at most $d + 1$.*

Theorem 4 ([14]) *Let H be a group.*

(i) The group $\nu(H)$ is isomorphic to $((H \otimes H) \rtimes H) \rtimes H$.

(ii) Let $\iota : [H, H^{\varphi}] \to \nu(H)$ be the natural inclusion map and let $\xi : \nu(H) \to H \times H$ be the homomorphic extension of the map sending the generator $h \in H$ of $\nu(H)$ to $(h, 1)$ and the generator $h^{\varphi} \in H^{\varphi}$ of $\nu(H)$ to $(1, h)$. Then

$$1 \longrightarrow [H, H^{\varphi}] \xrightarrow{\ \iota\ } \nu(H) \xrightarrow{\ \xi\ } H \times H \longrightarrow 1 \tag{2}$$

is a short exact sequence.

If H is finite, then one can find a finite presentation for $\nu(H)$ using (1), and by Theorem 3(i) we know that $\nu(H)$ is finite. The problem then becomes one of finding a concrete representation of $\nu(H)$ such that the subgroup $[H, H^{\varphi}]$ can be computed. For example, if H is a finite p-group, then by Theorem 3(ii) a polycyclic representation of $\nu(H)$ can be found using a p-quotient algorithm. For an arbitrary finite group, coset enumeration can be employed to find a permutation representation of $\nu(H)$.

Computer implementations for computing the nonabelian tensor square for a finite group G by finding the subgroup $[H, H^{\varphi}]$ of $\nu(H)$ can be found in [14] using CAYLEY, in Ellis [13] using Magma [6], and by McDermott [21] and Rocco [25] using GAP [17].

The goal of this paper is to provide an analysis of the group $\nu(H)$ for H an arbitrary group (finite or infinite) and use this analysis as a vehicle for computing $H \otimes H$ and other homological functors. We then specialize this general analysis to polycyclic groups. Our results for polycyclic groups are summarized in the following theorem.

Theorem 5 *Let G be a polycyclic group with a finite presentation $\langle \mathcal{H} \mid \mathcal{R} \rangle$ and polycyclic generating set $\mathfrak{G}$. Then*

(i) The nonabelian tensor square $G \otimes G$ is polycyclic.

(ii) The group $\nu(G)$ is polycyclic.

(iii) The group $\nu(G)$ has a finite presentation that depends only on $\mathcal{H}$, $\mathcal{R}$ and $\mathfrak{G}$.

(iv) The nonabelian tensor square $G \otimes G$ is generated by the set

$$\{\mathfrak{g}^{\pm 1} \otimes \mathfrak{h}^{\pm 1} \mid \mathfrak{g}, \mathfrak{h} \text{ in } \mathfrak{G}\}.$$

Our general results define a presentation of $\nu(H)$ relative to a structure defined for any group H (infinite or finite) and provide a general commutator calculus for working in the normal subgroup $[H, H^{\varphi}]$ of $\nu(H)$. These general results provide a two-fold approach to computing the nonabelian tensor square of a polycyclic group G. The first approach allows for effective hand calculations within the subgroup $[G, G^\varphi]$ of $\nu(G)$, supported by the general commutator calculus we develop. Given a specific polycyclic group G, the second approach is to use computer methods to compute the nonabelian tensor square $G \otimes G$ by directly computing a polycyclic presentation of $\nu(G)$ and then computing the subgroup of $[G, G^\varphi]$ of $\nu(G)$. This is possible even for infinite polycyclic groups using the GAP package `Polycyclic` [10] and a polycyclic quotient algorithm, for example `nq` [23], when G is nilpotent.

In the outline below we indicate the steps we follow to obtain the results described above and briefly describe applications of those results.

In Section 2 we show that the nonabelian tensor square $G \otimes G$ of a polycyclic group G is polycyclic. In proving this result, we provide structure results for two abelian groups $J_2(G)$ and $\Gamma(G/G')$ related to the nonabelian tensor square.

In Section 3 we provide a list of commutator identities for working in the normal subgroup $[H, H^\varphi]$ of $\nu(H)$ for an arbitrary group H. We then show that for each finite normal series of H, including the trivial series $H \rhd 1$, there is a corresponding presentation (not necessarily finite) of $\nu(H)$. In the case when H is polycyclic, any polycyclic series of H leads to a finite presentation of $\nu(H)$. As noted above, one application of $\nu(H)$ is that it contains a subgroup isomorphic to $H \otimes H$. The group $\nu(H)$ also contains an isomorphic copy of the tensor center of H (Definition 22), which is a central subgroup of H.

We conclude Section 3 by developing a corresponding theory for computing the nonabelian exterior square of a group, which is a factor group of the nonabelian tensor square. We define the group $\tau(H)$ of $\nu(H)$ (Definition 19), which is a factor group of $\nu(H)$. The group $\tau(H)$ plays a similar role to $\nu(H)$, in that the normal subgroup $[H, H^\varphi]$ of $\tau(H)$ is isomorphic to the nonabelian exterior square $H \wedge H$ of H. The group $\tau(H)$ can also be used to compute the exterior center of H, which is also a central subgroup of H. Computing the exterior center is of interest because it determines exactly when a group H is capable, that is, whether H is a central factor of some group K.

Section 4 focuses on computing $G \otimes G$ when G is polycyclic. In this section we find a generating set for $[G, G^\varphi]$ relative to a given polycyclic generating set of G. We apply our results to compute the nonabelian tensor square of the infinite dihedral group by hand. We then develop an algorithm for computing $G \otimes G$ when G is polycyclic.

In Section 5, we conclude the paper with an application of our results to finding the nonabelian tensor square of the free nilpotent groups of class 3 and rank n. Using a computer implementation of the algorithm developed in Section 4, we compute $G \otimes G$ for G free nilpotent of class 3 and $n = 3, 4, 5, 6$. Using these direct computations to guide us and specializing the commutator identities of $[G, G^\varphi]$ found in Section 3 to nilpotent of class 3 groups, we fully describe the nonabelian tensor square of an arbitrary nilpotent of class 3 rank n group.

Theorem 6 *Let G be a free nilpotent group of class 3 and rank n. Then $G \otimes G$ is the direct product of a nilpotent of class 2 group minimally generated by $n(n-1)$ elements and a free abelian group of rank*

$$f(n) = \frac{n(3n^3 + 14n^2 - 3n + 10)}{24}.$$

Prior to the results presented in this paper, the only approach to computing the nonabelian tensor square of an infinite group H involved computing a crossed pairing $\Phi : H \times H \to L$ for some group L by hand and showing that its lift $\Phi^* : H \otimes H \to L$ is an isomorphism. See [5] for more details, definitions and a complete description of this method. The only computer assistance related to this method for infinite groups known to the authors involves symbolic checking to confirm whether a proposed mapping is indeed a crossed pairing. The crossed pairing method is not an algorithm since it first requires one to guess an appropriate group L and conjecture a crossed pairing. A computer algorithm that takes an infinite group and finds a group isomorphic to its nonabelian tensor square was thought not to exist; reducing the infinite presentation given in the Definition 1 to a finite presentation required often difficult hand calculations.

Checking that a proposed mapping is a crossed pairing by hand is manageable when the nonabelian tensor square is abelian (see for example [1] and [3]). This verification becomes significantly more difficult when the nonabelian tensor square is not abelian. For example, the nonabelian tensor square of the free 2-Engel group of rank n is nilpotent of class 2. The tensor square of the rank 3 case and then the general rank $n \geq 3$ case was computed using the crossed pairing method in two papers [2] and [5], which required 30 published pages. By contrast, the more complex case of computing the nonabelian tensor square of the free nilpotent groups of class 3 and rank n is completed in a few pages in Section 5 using the techniques we develop in the earlier sections. Moreover, the nonabelian tensor square of these groups can be computed directly for a given rank, using a computer implementation of Algorithm 28. A simple GAP implementation of this algorithm is given in the last section.

2 The nonabelian tensor square of a polycyclic group

In this section we show that if G is polycyclic then $G \otimes G$ is polycyclic, and hence $G \otimes G$ has a finite presentation. The question then becomes finding such a finite presentation, which will be the topic of Section 4.

Let H be any group. Then $H \otimes H$ is a central extension of H' and $J_2(H)$ (see [7]) and we have the exact sequence

$$0 \longrightarrow J_2(H) \longrightarrow H \otimes H \xrightarrow{\kappa} H' \longrightarrow 1, \tag{3}$$

where $\kappa(g \otimes h) = [g, h]$.

An immediate consequence of (3) is the following proposition.

Proposition 7 *Let $\mathfrak{X}$ be a class of groups that is closed under forming extensions and let H be any group. If $J_2(H)$ and H' are $\mathfrak{X}$-groups, then $H \otimes H$ is an $\mathfrak{X}$-group.*

Polycyclic groups are closed under forming extensions and taking subgroups. Suppose G is a polycyclic group. Then G' is polycyclic. If we can show that $J_2(G)$ is polycyclic, then $G \otimes G$ is polycyclic by Proposition 7. The following exposition shows that $J_2(G)$ is polycyclic.

Given an abelian group A, the Whitehead universal quadratic functor ΓA is the abelian group with generators γa, for all $a \in A$, and defining relations

$$\gamma(a^{-1}) = \gamma a \quad \text{and} \quad \gamma(abc)\gamma a\gamma b\gamma c = \gamma(ab)\gamma(bc)\gamma(ca)$$

for all $a, b, c \in A$. This group was introduced by Whitehead in Sections 5 and 6 of [28]. The following theorem captures the results we need.

Theorem 8 ([28]) *Let A be an abelian group.*

(i) If A is finitely generated then ΓA is finitely generated.
(ii) If A is free abelian of rank n then ΓA is free abelian of rank $\binom{n+1}{2}$.

We denote the nth dimensional integral homology group of a group H by $H_n(H)$. In [7] we find the following exact sequence:

$$H_3(H) \longrightarrow \Gamma(H^{ab}) \xrightarrow{\psi} J_2(H) \xrightarrow{\phi} H_2(H) \longrightarrow 0, \tag{4}$$

where $H^{ab} = H/H'$.

Lemma 9 *Let H be a finitely presented group. Then $J_2(H)$ is finitely generated.*

PROOF. Let H be a finitely presented group. Then $H_2(H)$ is a finitely generated abelian group [27], as is $\Gamma(H^{ab})$ by Theorem 8(i). Hence both $H_2(H)$ and $\Gamma(H^{ab})$ are polycyclic (see [26], Chapter 1, Lemma 4). Since the sequence (4) is exact, the image of ψ is equal to the kernel of ϕ and hence the kernel of ϕ is polycyclic. Therefore $J_2(H)$ is an extension of two polycyclic groups. It follows that $J_2(H)$ is polycyclic and therefore is finitely generated. □

We conclude with our desired result.

Proposition 10 *Let G be a polycyclic group. Then $G \otimes G$ is polycyclic.*

PROOF. Let G be a polycyclic group. Then G' is polycyclic and G is finitely presented. By Lemma 9 we have that $J_2(G)$ is polycyclic. Hence $G \otimes G$ is polycyclic by Proposition 7. $\square$

3 The groups $\nu(G)$ and $\tau(G)$

The theme of L.-C. Kappe's paper [20] titled *Nonabelian tensor products of groups: the commutator connection* is the similarity between the definitional relations of the nonabelian tensor square and the basic commutator expansion formulas. In this section we realize this connection explicitly by showing that all tensor computations for a group H can be translated into commutator computations within the subgroup $[H, H^{\varphi}]$ of $\nu(H)$. Moreover, if H is solvable or nilpotent we have structural results for $\nu(H)$ that also assist with these commutator computations. The commutator identities for $[H, H^{\varphi}]$ listed in Lemmas 11 – 14 below are found in Rocco [24]. Rocco uses right conjugation in his formalizations, which is nonstandard for most of the literature related to nonabelian tensor products. While both formulations are equivalent, for consistency we translate Rocco's identities to the corresponding identities that hold using conjugation from the left.

Lemma 11 *Let H be a group. The following relations hold in $\nu(H)$:*

(i) ${}^{[h_3,h_4^{\varphi}]}[h_1, h_2^{\varphi}] = {}^{[h_3,h_4]}[h_1, h_2^{\varphi}]$ *and* ${}^{[h_3^{\varphi},h_4]}[h_1, h_2^{\varphi}] = {}^{[h_3,h_4]}[h_1, h_2^{\varphi}]$ *for all* h_1, h_2, h_3, h_4 *in* H;
(ii) $[h_1^{\varphi}, h_2, h_3] = [h_1, h_2, h_3^{\varphi}] = [h_1^{\varphi}, h_2, h_3^{\varphi}]$ *and* $[h_1, h_2^{\varphi}, h_3] = [h_1^{\varphi}, h_2^{\varphi}, h_3] = [h_1, h_2^{\varphi}, h_3^{\varphi}]$ *for all* h_1, h_2, h_3 *in* H;
(iii) $[h, h^{\varphi}]$ *is central in* $\nu(H)$ *for all* h *in* H;
(iv) $[h_1, h_2^{\varphi}][h_2, h_1^{\varphi}]$ *is central in* $\nu(H)$ *for all* h_1, h_2 *in* H;
(v) $[h, h^{\varphi}] = 1$ *for all* h *in* H'.

Lemma 12 *Let H be a group and let x_i, y_i, for $i = 1, \ldots, s$, be elements of H. For $z = \prod_{i=1}^{s}[x_i, y_i]$, define $\tilde{z}$ to be $\prod_{i=1}^{s}[x_i^{\varphi}, y_i]$. Then the following identities hold in $\nu(H)$:*

(i) ${}^{[h_1^{\varphi},h_2]}\tilde{z} = {}^{[h_1,h_2]}\tilde{z}$ *for all* h_1, h_2 *in* H;
(ii) ${}^{\tilde{z}}[h_1, h_2^{\varphi}] = {}^{z}[h_1, h_2^{\varphi}]$ *for all* h_1, h_2 *in* H;
(iii) $[z, h^{\varphi}] = [\tilde{z}, h]$ *for all* h *in* H.

Lemma 13 *Let H be a group and let a, b and x be elements of H such that $[x, a] = 1 = [x, b]$. Then in $\nu(H)$,*

$$[a, b, x^{\varphi}] = 1 = [[a, b]^{\varphi}, x].$$

Lemma 14 *Let H be a group and let x and y be elements of H that commute. Then in $\nu(H)$,*

(i) $[x^n, y^{\varphi}] = [x, y^{\varphi}]^n = [x, (y^{\varphi})^n]$ *for all integers n;*
(ii) If x and y are torsion elements of H of orders $o(x)$ and $o(y)$ (respectively) in H, then the order of $[x, y^{\varphi}]$ in $\nu(H)$ divides the greatest common divisor of $o(x)$ and $o(y)$.

The following lemma records additional identities that will be used in the sequel.

Lemma 15 *Let h_1, h_2, h_3 and h_4 be elements of a group H. Then in $\nu(H)$*

(i) $[[h_1, h_2^{\varphi}], [h_2, h_1^{\varphi}]] = 1$;
(ii) $[[h_1, h_2], [h_3, h_4]^{\varphi}] = [[h_1, h_2^{\varphi}], [h_3, h_4^{\varphi}]]$;
(iii) $[h_1^n, h_2^{\varphi}] \cdot [h_2, (h_1^n)^{\varphi}] = [h_1, (h_2^n)^{\varphi}] \cdot [h_2^n, (h_1)^{\varphi}] = ([h_1, h_2^{\varphi}][h_2, (h_1)^{\varphi}])^n$;
(iv) $[h_1, (h_2^n h_3^m)^{\varphi}] \cdot [h_2^n h_3^m, h_1^{\varphi}] = ([h_1, h_2^{\varphi}][h_2, h_1^{\varphi}])^n \cdot ([h_1, h_3^{\varphi}][h_3, h_1^{\varphi}])^m$;
(v) $[h_1^n h_2^m, h_3^{\varphi}] \cdot [h_3, (h_1^n h_2^m)^{\varphi}] = ([h_1, h_3^{\varphi}][h_3, h_1^{\varphi}])^n \cdot ([h_2, h_3^{\varphi}][h_3, h_2^{\varphi}])^m$;
(vi) ${}^{h_1}[h_2, h_3^{\varphi}] = {}^{h_1^{\varphi}}[h_2, h_3^{\varphi}]$

PROOF. Lemma 11(iv) states that the product $[h_1, h_2^{\varphi}] \cdot [h_2, h_1^{\varphi}]$ is in the center of $\nu(H)$. Therefore

$$\begin{aligned} 1 &= [[h_1, h_2^{\varphi}] \cdot [h_2, h_1^{\varphi}], [h_2, h_1^{\varphi}]] \\ &= {}^{[h_1, h_2^{\varphi}]}[[h_1, h_2^{\varphi}], [h_2, h_1^{\varphi}]] \cdot [[h_2, h_1^{\varphi}], [h_2, h_1^{\varphi}]] \\ &= {}^{[h_1, h_2^{\varphi}]}[[h_1, h_2^{\varphi}], [h_2, h_1^{\varphi}]]. \end{aligned}$$

Hence $[[h_1, h_2^{\varphi}], [h_2, h_1^{\varphi}]] = 1$.

By Lemma 11(i) we have ${}^{[h_1, h_2^{\varphi}]}[h_3, h_4^{\varphi}] = {}^{[h_1, h_2]}[h_3, h_4^{\varphi}]$. Hence

$$[[h_1, h_2^{\varphi}], [h_3, h_4^{\varphi}]] \cdot [h_3, h_4^{\varphi}] = [[h_1, h_2], [h_3, h_4^{\varphi}]] \cdot [h_3, h_4^{\varphi}],$$

and therefore $[[h_1, h_2^{\varphi}], [h_3, h_4^{\varphi}]] = [[h_1, h_2], [h_3, h_4^{\varphi}]]$. Now by Lemma 11(ii)

$$\begin{aligned} [[h_1, h_2], [h_3, h_4^{\varphi}]] = [[h_3, h_4^{\varphi}], [h_1, h_2]]^{-1} &= [[h_3^{\varphi}, h_4^{\varphi}], [h_1, h_2]]^{-1} \\ &= [[h_1, h_2], [h_3, h_4]^{\varphi}]. \end{aligned}$$

We prove identity (iii) by induction on n. For $n = 1$ there is nothing to show. Suppose the result is true for $n = k$. Using the induction hypothesis and

Lemma 11(iv) for $n = k + 1$ we have

$$\begin{aligned}
[h_1^{k+1}, h_2^\varphi] \cdot [h_2, (h_1^{k+1})^\varphi] &= [h_1^k h_1, h_2^\varphi] \cdot [h_2, (h_1^k h_1)^\varphi] \\
&= {}^{h_1^k}[h_1, h_2^\varphi] \cdot [h_1^k, h_2^\varphi] \cdot [h_2, (h_1^k)^\varphi] \cdot {}^{(h_1^k)^\varphi}[h_2, h_1^\varphi] \\
&= {}^{h_1^k}[h_1, h_2^\varphi] \cdot {}^{h_1^k}[h_2, h_1^\varphi] \cdot [h_1^k, h_2^\varphi] \cdot [h_2, (h_1^k)^\varphi] \\
&= {}^{h_1^k}([h_1, h_2^\varphi][h_2, h_1^\varphi]) \cdot ([h_1, h_2^\varphi][h_2, h_1^\varphi])^k \\
&= [h_1, h_2^\varphi] \cdot [h_2, h_1^\varphi] \cdot ([h_1, h_2^\varphi][h_2, h_1^\varphi])^k \\
&= ([h_1, h_2^\varphi][h_2, h_1^\varphi])^{k+1}.
\end{aligned}$$

A similar argument gives $[h_1, (h_2^k)^\varphi] \cdot [h_2^k, (h_1)^\varphi] = ([h_1, h_2^\varphi][h_2, h_1^\varphi])^k$.

To prove identity (iv) we use (iii) and Lemma 11(iv) as follows:

$$\begin{aligned}
[h_1, (h_2^n h_3^m)^\varphi] \cdot [h_2^n h_3^m, h_1^\varphi] &= [h_1, (h_2^n)^\varphi] \cdot {}^{(h_2^n)^\varphi}[h_1, (h_3^m)^\varphi] \cdot {}^{(h_2^n)^\varphi}[h_3^m, h_1^\varphi] \cdot [h_2^n, h_1^\varphi] \\
&= [h_1, (h_2^n)^\varphi] \cdot [h_2^n, h_1^\varphi] \cdot [h_1, (h_3^m)^\varphi] \cdot [h_3^m, h_1^\varphi] \\
&= ([h_1, h_2^\varphi][h_2, h_1^\varphi])^n \cdot ([h_1, h_3^\varphi][h_3, h_1^\varphi])^m.
\end{aligned}$$

The proof of (v) is similar to (iv).

Finally, to prove (vi) we use Lemma 11(ii) as follows:

$$\begin{aligned}
{}^{h_1}[h_2, h_3^\varphi] &= [h_1, [h_2, h_3^\varphi]] \cdot [h_2, h_3^\varphi] \\
&= [h_2, h_3^\varphi, h_1]^{-1} \cdot [h_2, h_3^\varphi] \\
&= [h_2.h_3^\varphi, h_1^\varphi]^{-1} \cdot [h_2, h_3^\varphi] \\
&= [h_1^\varphi, [h_2.h_3^\varphi]] \cdot [h_2, h_3^\varphi] \\
&= {}^{h_1^\varphi}[h_2, h_3^\varphi].
\end{aligned}$$

□

We will apply the commutator calculus of these lemmas to computing the nonabelian tensor squares of infinite polycyclic groups in Sections 4 and 5. Let G be a polycyclic group. In Section 4 we determine a finite generating set for the subgroup $[G, G^\varphi]$ of $\nu(G)$ independent of any presentation of $\nu(G)$. Hence, in this case, we are able to use the commutator calculus above to compute the structure of $[G, G^\varphi] \cong G \otimes G$ (Theorem 2) directly using only general structural results about $\nu(G)$.

Since the presentation of $H \otimes H$ given in Definition 1 becomes computationally intractable for all but small finite groups, our next goal is to find conditions for when a presentation for $\nu(H)$ is small (finite) so that we can devise an algorithm to compute $[H, H^\varphi]$ using $\nu(H)$, even when H is a large finite or infinite group.

Definition 16 *Let H be a group and let $H = H_n \trianglerighteq \cdots \trianglerighteq H_1 \trianglerighteq H_0 = 1$ be a subnormal series for H. Let $\mathcal{T}_i$ denote a transversal for H_{i-1} in H_i and let $\mathcal{H}_i$ denote a lift of a generating set for H_i/H_{i-1} to $\mathcal{T}_i$. Set*

$$\mathcal{L}_i = \begin{cases} \mathcal{H}_i & \text{if } H_i/H_{i-1} \text{ is abelian} \\ \mathcal{T}_i & \text{otherwise.} \end{cases}$$

Then the set $\mathcal{L}_H$ relative to the subnormal series $H = H_n \trianglerighteq \cdots \trianglerighteq H_1 \trianglerighteq H_0 = 1$ is defined as

$$\mathcal{L}_H = \cup_{i=1}^n \mathcal{L}_i.$$

Note for any nonabelian group H that $H = \mathcal{L}_H$ relative to the subnormal series $H \trianglerighteq 1$. If H is polycyclic with some polycyclic series $H = H_n \trianglerighteq \cdots \trianglerighteq H_1 \trianglerighteq H_0 = 1$ and associated polycyclic generating sequence $\mathfrak{H}$, then $\mathcal{L}_H$ relative to this series is $\mathfrak{H}$.

The following theorem relates the set $\mathcal{L}_H$ to $\nu(H)$.

Theorem 17 *Let H be a group with presentation $\langle \mathcal{H} \mid \mathcal{R} \rangle$ and let $\mathcal{S}$ be any subnormal series of H. Then $\nu(H)$ is given by the following presentation:*

$$\begin{aligned} \nu(H) = \langle \mathcal{H}, \mathcal{H}^\varphi \mid \mathcal{R}, \mathcal{R}^\varphi, {}^x[a, b^\varphi] = [{}^x a, ({}^x b)^\varphi], {}^{x^\varphi}[a, b^\varphi] = [{}^x a, ({}^x b)^\varphi], \\ \forall a, b \in \mathcal{H}, \quad \forall x \in \mathcal{L}_H \text{ relative to } \mathcal{S} \rangle. \end{aligned} \tag{5}$$

PROOF. Let J be the set of words $\{{}^z[g, h^\varphi] \cdot [{}^z g, ({}^z h)^\varphi]^{-1}\}$ in the free product $H * H^\varphi$ such that $z \in H * H^\varphi$ and $g, h \in H$. The set J is a normal subset of $H * H^\varphi$ and we claim that

$$\nu(H) \cong H * H^\varphi / \langle J \rangle.$$

This claim reduces to demonstrating that the words of J can be written as conjugates of the relators of $\nu(H)$ given in the presentation (1). Set $W(z, g, h) = {}^z[g, h^\varphi] \cdot [{}^z g, ({}^z h)^\varphi]^{-1}$. For x and y in $H * H^\varphi$ and a and b in H we have

$$\begin{aligned} W(xy, a, b) &= {}^{xy}[a, b^\varphi] \cdot [{}^{xy}a, ({}^{xy}b)^\varphi]^{-1} \\ &= {}^x\left({}^y[a, b^\varphi][{}^y a, ({}^y b)^\varphi]^{-1}[{}^y a, ({}^y b)^\varphi]\right) \cdot [{}^{xy}a, ({}^{xy}b)^\varphi]^{-1} \\ &= {}^x W(y, a, b) \cdot {}^x[{}^y a, ({}^y b)^\varphi] \cdot [{}^{xy}a, ({}^{xy}b)^\varphi]^{-1} \\ &= {}^x W(y, a, b) \cdot W(x, {}^y a, {}^y b). \end{aligned}$$

Continuing the expansion we see that $W(xy, a, b)$ can eventually be written as the product of conjugates of words of the form $W(w, g, h)$ or $W(w^\varphi, g, h)$ for w in H. But these words are just the relators of $\nu(H)$.

Theorem 2.2.8 of [21] states that given $\mathcal{L}_H$ relative to some subnormal series for H, the set J is normally generated by the relators given in presentation (5). That is, the normal closure of these words generate all of J as required. $\square$

Let G be any polycyclic group with polycyclic generating sequence $\mathfrak{G}$. Taking $\mathcal{L}_G$ to be $\mathfrak{G}$, we see that $\nu(G)$ is finitely presented. In Section 4 we will use this fact to devise an algorithm for computing $[G, G^{\varphi}]$.

We conclude this section by using $\nu(H)$ to define a useful factor group of the nonabelian tensor square of a group H and two important central subgroups of H. These groups will be investigated in a future publication.

Note first by Lemma 11(iii) that the subgroup $\nabla(H) = \langle h \otimes h \mid h \in H\rangle$ is a central subgroup of $H \otimes H$.

Definition 18 *Let H be any group. Then the nonabelian exterior square $H \wedge H$ of H is defined to be*

$$H \wedge H = H \otimes H/\nabla(H).$$

For g and h in H we denote the coset $(g \otimes h)\nabla(H)$ by $g \wedge h$.

The subgroup $[H, H^{\varphi}]$ is a fully invariant subgroup of $\nu(H)$. Hence $\nu(H)$ contains a normal subgroup isomorphic to $\nabla(H)$.

Definition 19 *Let H be any group. Then we define $\tau(H)$ to be the quotient group $\nu(H)/\phi(\nabla(H))$, where $\phi : H \otimes H \to [H, H^{\varphi}]$ is defined in Theorem 2.*

Since ϕ isomorphically embeds $\nabla(H)$ in $\nu(H)$, it follows that

$$[H, H^{\varphi}]/\phi(\nabla(H)) \cong H \wedge H.$$

This isomorphism is realized by the mapping $\hat{\phi} : H \wedge H \to [H, H^{\varphi}]/\phi(\nabla(H))$ defined by $\hat{\phi}(g \wedge h) = [g, h^{\varphi}]\phi(\nabla(H))$ for all g, h in H. We will denote $[H, H^{\varphi}]/\phi(\nabla(H))$ by $[H, H^{\varphi}]_{\tau(H)}$. The following proposition is now evident.

Proposition 20 *Let H be any group. The map*

$$\hat{\phi} : H \wedge H \to [H, H^{\varphi}]_{\tau(H)} \lhd \tau(H)$$

defined by $\hat{\phi}(g \wedge h) = [g, h^{\varphi}]_{\tau(H)}$ is an isomorphism.

For any finitely generated group H, the subgroup $\nabla(H)$ is finitely generated, as the next lemma shows.

Lemma 21 *Let H be a group generated by $\mathcal{H}$. Then $\nabla(H)$ is generated by the set*

$$\{h \otimes h \mid h \in \mathcal{H}\} \cup \{(h \otimes h')(h' \otimes h) \mid h, h' \in \mathcal{H}\}$$

PROOF. Let x be an element of H. Then $x = h_1^{\alpha_1} \cdots h_n^{\alpha_n}$, where $h_i \in \mathcal{H}$ and $\alpha_i \in \mathbb{Z}$ for $i = 1, \ldots, n$. We will show that

$$[x, x^\varphi] = \prod_{1 \le i \le n} [h_i, h_i^\varphi]^{\alpha_i^2} \cdot \prod_{1 \le j < i \le n} ([h_i, h_j^\varphi][h_j, h_i^\varphi])^{\alpha_i \alpha_j} \tag{6}$$

by induction on n. For $n = 1$ we have $[x, x^\varphi] = [h_1^{\alpha_1}, (h_1^{\alpha_1})^\varphi] = [h_1, h_1^\varphi]^{\alpha_1^2}$ by Lemma 14(i). Suppose the result is true for $n = k$. Then for $n = k + 1$ we write $x = h_1^{\alpha_1} \cdots h_k^{\alpha_k} h_{k+1}^{\alpha_{k+1}} = y \cdot h_{k+1}^{\alpha_{k+1}}$. It follows from Lemma 11(iii) and (iv) and Lemma 15(vi) that

$$\begin{aligned}
[x, x^\varphi] &= [y \cdot h_{k+1}^{\alpha_{k+1}}, (y \cdot h_{k+1}^{\alpha_{k+1}})^\varphi] \\
&= {}^y\Big([h_{k+1}^{\alpha_{k+1}}, y^\varphi] \cdot {}^{y^\varphi}[h_{k+1}^{\alpha_{k+1}}, (h_{k+1}^{\alpha_{k+1}})^\varphi]\Big) \cdot [y, y^\varphi] \cdot {}^{y^\varphi}[y, (h_{k+1}^{\alpha_{k+1}})^\varphi] \\
&= [h_{k+1}^{\alpha_{k+1}}, y^\varphi] \cdot [y, (h_{k+1}^{\alpha_{k+1}})^\varphi] \cdot [y, y^\varphi] \cdot [h_{k+1}^{\alpha_{k+1}}, (h_{k+1}^{\alpha_{k+1}})^\varphi].
\end{aligned}$$

Now by repeated use of Lemma 15(iv) we have

$$\begin{aligned}
[h_{k+1}^{\alpha_{k+1}}, y^\varphi] \cdot [y, (h_{k+1}^{\alpha_{k+1}})^\varphi] &= [h_{k+1}^{\alpha_{k+1}}, (h_1^{\alpha_1} \cdots h_k^{\alpha_k})^\varphi] \cdot [h_1^{\alpha_1} \cdots h_k^{\alpha_k}, h_{k+1}^{\alpha_{k+1}}] \\
&= \prod_{1 \le j \le k} ([h_{k+1}, h_j^\varphi][h_j, h_{k+1})^\varphi])^{\alpha_{k+1}\alpha_j}.
\end{aligned}$$

Applying the induction hypothesis, we have

$$\begin{aligned}
[x, x^\varphi] = \prod_{1 \le j \le k} ([h_{k+1}, h_j^\varphi][h_j, h_{k+1})^\varphi])^{\alpha_{k+1}\alpha_j} \cdot \prod_{1 \le i \le k} [h_i, h_i^\varphi]^{\alpha_i^2} \cdot \\
\prod_{1 \le j < i \le k} ([h_i, h_j^\varphi][h_j, h_i^\varphi])^{\alpha_i \alpha_j} \cdot [h_{k+1}, (h_{k+1})^\varphi]^{\alpha_{k+1}^2}.
\end{aligned}$$

Since all factors are in the center of $\nu(H)$ we obtain the product (6).

It now follows that $\nabla(H)$ is generated by

$$\{h \otimes h \mid h \in \mathcal{H}\} \cup \{(h \otimes h')(h' \otimes h) \mid h, h' \in \mathcal{H}\}.$$

This completes the proof. $\square$

Let $H = \langle \mathcal{H} | \mathcal{R} \rangle$ be any group. Given $\mathcal{L}_H$ relative to some subnormal series of H, we obtain by Lemma 21, the presentation

$$\begin{aligned}
\tau(H) = \langle \mathcal{H}, \mathcal{H}^\varphi \mid \mathcal{R}, \mathcal{R}^\varphi, {}^x[a, b^\varphi] = [{}^x a, ({}^x b)^\varphi], {}^{x^\varphi}[a, b^\varphi] = [{}^x a, ({}^x b)^\varphi], \\
[a, a^\varphi], [a, b^\varphi][b, a^\varphi] \quad \forall a, b \in \mathcal{H}, \quad \forall x \in \mathcal{L}_H \rangle.
\end{aligned} \tag{7}$$

If H is finitely presented and $\mathcal{L}_H$ is a finite set, then (7) gives a finite presentation for $\tau(H)$. In particular, for any polycyclic group G the group $\tau(G)$ is finitely presented.

Definition 22 *Let H be a group. The nonabelian tensor center $Z^{\otimes}(H)$ and nonabelian exterior center $Z^{\wedge}(H)$ are subgroups of H defined respectively by*

$$Z^{\otimes}(H) = \langle h \in H \mid g \otimes h = 1_{H\otimes H} \text{ for all } g \in H \rangle \text{ and}$$
$$Z^{\wedge}(H) = \langle h \in H \mid g \wedge h = 1_{H\wedge H} \text{ for all } g \in H \rangle.$$

Ellis [16] shows that the tensor center $Z^{\otimes}(H)$ is a characteristic and central subgroup of H. Similarly, the exterior center $Z^{\wedge}(H)$ of H is a central and characteristic subgroup of H. The tensor center of H can be characterized as the largest subgroup A of H such that $(H/A) \otimes (H/A) \cong H \otimes H$ [16]. The exterior center is equal to $Z^*(H)$, the epicenter of H [12]. Groups with trivial epicenters are called capable [4].

The tensor center and exterior center of a group H can be alternatively described relative to $\nu(H)$ and $\tau(H)$, respectively.

Theorem 23 *Let H be any group. Then*

$$Z^{\otimes}(H) \cong H \cap Z(\nu(H)) \quad \text{and} \quad Z^{\wedge}(H) \cong H \cap Z(\tau(H)),$$

where H on the right hand side of each equation is interpreted as the natural isomorphic copy of H in $\nu(H)$ and $\tau(H)$ respectively.

PROOF. Let H be a group generated by $\mathcal{H}$. Then by (5) we have that $\nu(H)$ is generated by $\mathcal{H} \cup \mathcal{H}^{\varphi}$. Suppose $h \in Z^{\otimes}(H)$. Then $[h, x^{\varphi}] = 1_{\nu(H)}$ for all $x \in H$. In particular, all h in $Z^{\otimes}(H)$ commute with all g^{φ} in $\mathcal{H}^{\varphi}$. Since $Z^{\otimes}(H)$ is central in H , we have $[h, g] = 1_{\nu(H)}$ for all g in $\mathcal{H}$. Hence h is in the center of $\nu(H)$, since it commutes with all of its generators. Therefore $Z^{\otimes}(H)$ is contained in $H \cap Z(\nu(H))$. Suppose $h \in H \cap Z(\nu(H))$. Then we have $[h, x^{\varphi}] = 1_{\nu(H)}$ for all x in H. Therefore by the definition of the tensor center, $h \in Z^{\otimes}(H)$ and $H \cap Z(\nu(H))$ is contained in $Z^{\otimes}(H)$. The equality for the exterior center holds by a similar argument. □

4 Computing $G \otimes G$ when G is polycyclic

In Section 2 we showed that if G is polycyclic, then $G \otimes G$ is polycyclic. It follows that $G \otimes G$ has a polycyclic presentation. The goal of this section is finding such a polycyclic presentation so that the structure of $G \otimes G$ can be investigated. Finding a polycyclic presentation and determining the structure of $G \otimes G$ can be done by computing in $\nu(G)$. These computations can be completed using computer methods or by hand. Hand calculations are aided by Proposition 25, which gives a complete description of a set of generators of $[G, G^{\varphi}]$, so that the commutator calculus developed in Section 3 can be applied. Our computer methods rely on the structure of $\nu(G)$.

The following proposition is the starting point for our algorithm for computing the nonabelian tensor squares of polycyclic groups.

Proposition 24 *If G is polycyclic then $\nu(G)$ is polycyclic.*

PROOF. Let G be a polycyclic group. Then $G \times G$ is polycyclic. By Proposition 10, $G \otimes G \cong [G, G^{\varphi}]$ is polycyclic. Hence, by Theorem 4, $\nu(G)$ is an extension of two polycyclic groups and therefore is also polycyclic. $\square$

The following result allows us to explicitly write down a finite generating set for $[G, G^{\varphi}]$ in terms of a polycyclic generating set of G, independent of the polycyclic presentation of $\nu(G)$. This extends Rocco's result Theorem 2.1 [25] for finite solvable groups to all polycyclic groups.

Proposition 25 *Let G be a polycyclic group with a polycyclic generating sequence $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$. Then $[G, G^{\varphi}]$, a subgroup of $\nu(G)$, is generated by*

$$[G, G^{\varphi}] = \langle [\mathfrak{g}_i, \mathfrak{g}_i^{\varphi}], [\mathfrak{g}_i^{\epsilon}, (\mathfrak{g}_j^{\varphi})^{\delta}] \rangle$$

and $[G, G^{\varphi}]_{\tau(G)}$, a subgroup of $\tau(G)$, is generated by

$$[G, G^{\varphi}]_{\tau(G)} = \langle [\mathfrak{g}_i^{\epsilon}, (\mathfrak{g}_j^{\varphi})^{\delta}] \rangle$$

for $1 \leq i, j \leq k$, $i \neq j$, where

$$\epsilon = \begin{cases} 1 & \text{if } |\mathfrak{g}_i| < \infty \\ \pm 1 & \text{if } |\mathfrak{g}_i| = \infty \end{cases} \qquad \text{and} \qquad \delta = \begin{cases} 1 & \text{if } |\mathfrak{g}_j^{\varphi}| < \infty \\ \pm 1 & \text{if } |\mathfrak{g}_j^{\varphi}| = \infty. \end{cases}$$

Our proof of Proposition 25 requires the following lemma, which generalizes Lemma 8.39 of [19] from finite polycyclic groups to all polycyclic groups.

Lemma 26 *Let G be a polycyclic group with subgroups A and B having polycyclic generating sets $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ and $\mathfrak{b}_1, \ldots, \mathfrak{b}_m$ respectively. If $G = \langle A, B \rangle$ then $[A, B]$ is generated by $[\mathfrak{a}_i^{\epsilon}, \mathfrak{b}_j^{\delta}]$, where $1 \leq i \leq n$, $1 \leq j \leq m$,*

$$\epsilon = \begin{cases} 1 & \text{if } |\mathfrak{a}_i| < \infty \\ \pm 1 & \text{if } |\mathfrak{a}_i| = \infty \end{cases} \qquad \text{and} \qquad \delta = \begin{cases} 1 & \text{if } |\mathfrak{b}_j| < \infty \\ \pm 1 & \text{if } |\mathfrak{b}_j| = \infty. \end{cases}$$

PROOF. [Proof of Proposition 25] Let G be a polycyclic group with a polycyclic generating sequence $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$. From the generators and relations of

$\nu(G)$ we see that $\nu(G) = \langle G, G^\varphi \rangle$. By Lemma 14(i), we have $[\mathfrak{g}_i^n, (\mathfrak{g}_i^\varphi)^m] = [\mathfrak{g}_i, \mathfrak{g}_i^\varphi]^{nm}$ for all integers n, m. Hence by Lemma 26, $[G, G^\varphi]$ is generated as claimed. The generating set for $[G, G^\varphi]_{\tau(G)}$ follows immediately from Lemma 26. $\square$

Proposition 25, coupled with the commutator calculus from Section 3, provides enough information in principle to compute the nonabelian tensor square of any polycyclic group. As an example, we compute $D \otimes D$, the nonabelian tensor square of the infinite dihedral group

$$D = \langle a, b \mid a^2, {}^a b = b^{-1} \rangle. \tag{8}$$

Beuerle and Kappe [3] compute the nonabelian tensor squares for all the infinite metacyclic groups. For the infinite dihedral group they obtain the following result.

Theorem 27 ([3]) *Let D denote the infinite dihedral group. Then*

$$D \otimes D \cong C_2 \times C_2 \times C_2 \times C_0,$$

where C_0 denotes the infinite cyclic group.

We will prove Theorem 27 using our commutator calculus.

By Proposition 25 we have that $D \otimes D$ is isomorphic to the subgroup $[D, D^\varphi]$ of $\nu(D)$ generated by the set

$$\{[a, a^\varphi], [b, b^\varphi], [a, (b^\varphi)^{\pm 1}], [b^{\pm 1}, a^\varphi]\}.$$

We first show that the generators $[a, (b^\varphi)^{-1}]$ and $[b^{-1}, a^\varphi]$ can be written in terms of the other four generators. By the relations of D and Lemma 11(ii), we have that

$$\begin{aligned} [a, b^{-\varphi}] &= {}^{b^{-1}}[a, b^\varphi]^{-1} \\ &= [b^{-1}, [a, b^\varphi]^{-1}] \cdot [a, b^\varphi]^{-1} \\ &= [b^{-1}, [a, b]^{-\varphi}] \cdot [a, b^\varphi]^{-1} \\ &= [b^{-1}, (b^{-2})^{-\varphi}] \cdot [a, b^\varphi]^{-1} \\ &= [b, b^\varphi]^2 \cdot [a, b^\varphi]^{-1}. \end{aligned} \tag{9}$$

By a similar derivation we have

$$[b^{-1}, a^\varphi] = [b, b^\varphi]^2 \cdot [b, a^\varphi]^{-1}.$$

Hence $[D, D^\varphi]$ is generated by the set

$$\{[a, a^\varphi], [b, b^\varphi], [a, b^\varphi], [b, a^\varphi]\}.$$

By Lemma 11(iii), the commutators $[a, a^\varphi]$ and $[b, b^\varphi]$ are central in $\nu(D)$. The generators $[a, b^\varphi]$ and $[b, a^\varphi]$ commute by Lemma 15(i). Hence $D \otimes D \cong [D, D^\varphi]$ is abelian.

We will now show that $[a, a^\varphi]$, $[b, b^\varphi]$, and $[a, b^\varphi][b, a^\varphi]$ all have order 2 and that $[a, b^\varphi]$ has infinite order. Moreover, each of these generators is nontrivial. It follows that $D \otimes D \cong C_2 \times C_2 \times C_2 \times C_0$.

Using Lemma 14(i) and Lemma 15(iii) we have

$$\begin{aligned} 1 &= [a^2, a^\varphi] = [a, a^\varphi]^2; \\ 1 &= [a^2, b^\varphi][b, (a^2)^\varphi] = ([a, b^\varphi][b, a^\varphi])^2. \end{aligned}$$

Using (9) we see that

$$\begin{aligned} 1 &= [a^2, b^\varphi] = {}^a[a, b^\varphi] \cdot [a, b^\varphi] \\ &= [a, ({}^a b)^\varphi] \cdot [a, b^\varphi] = [a, b^{-\varphi}] \cdot [a, b^\varphi] \\ &= [b, b^\varphi]^2 \cdot [a, b^\varphi]^{-1} \cdot [a, b^\varphi] = [b, b^\varphi]^2. \end{aligned}$$

Suppose $[a, b^\varphi]^n = 1$ for some $n \geq 1$. Then under the mapping κ of (3) we have

$$\kappa([a, b^\varphi]^n) = \kappa([a, b^\varphi])^n = [a, b]^n = 1,$$

which is a contradiction since $[a, b]$ has infinite order in D. Hence $[a, b^\varphi]$ has infinite order.

Proposition 24 provides the basis for a computer implementation for computing the nonabelian tensor square and exterior tensor square of a polycyclic group G. The main problem is to find a polycyclic presentation for $\nu(G)$. If this can be done, then the subgroup $[G, G^\varphi] \cong G \otimes G$ can be computed and its structure examined. Putting these steps together we get the following algorithm.

Algorithm 28 *Given a finite presentation for the polycyclic group $G = \langle \mathcal{G} \mid \mathcal{R} \rangle$ with polycyclic generating sequence $\mathfrak{G}$, the nonabelian tensor square $G \otimes G$ is computed by the following procedure.*

(1) Construct a finite presentation of $\nu(G)$ from $\mathcal{G}$, $\mathcal{R}$ and $\mathfrak{G}$ using (5).
(2) Compute a polycyclic presentation for $\nu(G)$ (Proposition 24).
(3) Return the subgroup $[G, G^\varphi]$ of $\nu(G)$ as a polycyclic group (Theorem 2).

GAP has methods for constructing finitely presented groups as needed in step *(1)*. The GAP package `Polycyclic` [10] can be used to effectively compute with finite and infinite polycyclic groups. For example, this package can be used to compute the subgroup $[G, G^\varphi]$ in step *(3)*. Step *(2)* can be completed by employing some type of polycyclic quotient algorithm.

We attempted to use an experimental GAP package `Ipcq` [9] to find a polycyclic presentation of $\nu(D)$, where D is the infinite dihedral group. The program requested more than 12 gigabytes of memory before failing. We are uncertain whether the problem is inherent to the algorithm or in its implementation.

Since this work was initiated, another approach to computing the nonabelian tensor and exterior square has been developed. By exploiting the fact that both the nonabelian tensor and exterior square are central extensions, Eick and Nickel [11] can effectively compute these squares for many infinite polycyclic groups, including D. Their work may supersede our more direct computational approach, at least for the nonnilpotent polycyclic groups; however, as Eick and Nickel point out, their method is less efficient for nilpotent groups.

For finitely generated nilpotent groups, step *(2)* in the algorithm above reduces to using a nilpotent quotient algorithm, since by Theorem 3, if G is nilpotent, then $\nu(G)$ is nilpotent. The `nq` package [23] of GAP can find a polycyclic presentation of $\nu(G)$ as needed. Moreover, Theorem 3 gives a class bound for $\nu(G)$, which makes finding this quotient easier. The run times of our GAP implementation of Algorithm 28 using `nq` are tabulated in Section 5. This implementation is listed in Section 6. Timing results with our implementation when applied to nilpotent groups is at least an order of magnitude faster than those of [11].

In the next section we will use the theoretical results of Section 3 to compute the nonabelian tensor square of the free nilpotent groups of class 3 and rank n. Our GAP implementation of Algorithm 28 motivated this analysis in that we were able to compute examples of small rank from which relations for the general case could be determined.

5 Free nilpotent groups of class 3 with finite rank

In this section we apply our results to compute the nonabelian tensor square of the free nilpotent group of class 3 and rank n, which we will denote as G_n. Table 1 below records information about the structure of the nonabelian tensor square of G_n for $n = 3, 4, 5, 6$, as computed using the GAP implementation of our algorithm listed in Section 6. Table 1 shows that the nonabelian tensor square of G_n is a direct product $N \times A$, where N is nilpotent of class 2 and A is free abelian of the stated rank. These computations were performed using an ordinary laptop with 1GB of RAM running Linux. The GAP workspace used was 500MB; run times listed are `nq` run time/GAP run time in milliseconds with wall clock time approximately the sum of the two times. These computer calculations prompted conjectures that became theoretical results

n	Class of $G_n \otimes G_n$	Minimal Generators of N	Rank of A	Run time
3	2	6	26	240/388
4	2	12	69	3388/2644
5	2	20	150	36870/15841
6	2	30	286	553270/84733

Table 1
Run times to compute $G_n \otimes G_n$

in the analysis given in this section.

We begin with some observations about the commutator calculus in $\nu(G_n)$.

Note first of all from Theorem 3 we have that $\nu(G_n)$ is nilpotent of class at most 4. In fact, when $n > 2$, the group $\nu(G_n)$ must have class exactly 4. Otherwise $\nu(G_n)'$ would be abelian, which implies that $G_n \otimes G_n$ is abelian, a contradiction, since we know that the nonabelian tensor square of the free 2-Engel group of rank $n > 2$, a homomorphic image of G_n, is nilpotent of class 2 [2,5]. By direct computation we have that $\nu(G_2)$ is nilpotent of exactly class 4 even though $G_2 \otimes G_2$ is abelian.

We start our computation of $G_n \otimes G_n$ by specializing the commutator calculus presented in Section 3 to groups of class 3. First we extend Lemma 11(ii).

Lemma 29 *Suppose that the group G is nilpotent of class 3. Then in $\nu(G)$ the identity $[g, h^\varphi, k] = [g^\varphi, h, k]$ holds for all $g, h, k \in G$.*

PROOF. By Lemma 11(iv), $[[g, h^\varphi] \cdot [h, g^\varphi], k] = 1$ in $\nu(G)$ for all $g, h, k \in G$. Since $\nu(G)$ is nilpotent of class at most 4, the product expands linearly, that is, $[g, h^\varphi, k] \cdot [h, g^\varphi, k] = 1$. Hence $[g, h^\varphi, k] = [h, g^\varphi, k]^{-1} = {}^{[g^\varphi, h]^{-1}}[g^\varphi, h, k] = [g^\varphi, h, k]$. □

Corollary 30 *Suppose the group G is nilpotent of class 3. Then in $\nu(G)$ the six commutators of Lemma 11(ii) are all equal.*

As a consequence, when G is nilpotent of class at most 3, φ may be removed and introduced as is convenient within weight three and weight four commutators in $\nu(G)$ whenever, both in the initial commutator and in the resulting commutator, at least one of the terms is in G and at least one is in G^φ. We use this property in the sequel without further reference.

Lemma 31 *Suppose that G is a nilpotent group of class at most 3. Then for all $g, h, k \in G$ the following identities hold in $\nu(G)$:*

(i) $[[g,h]^{-1},k^{\varphi}]=[[g,h],k^{\varphi}]^{-1}$*;*

(ii) $[[g,h],k^{-\varphi}]=[[g,h,k],k^{\varphi}]^{-1}\cdot[[g,h],k^{\varphi}]^{-1}$*;*

(iii) $[[g,h]^{-1},k^{-\varphi}]=[[g,h,k],k^{\varphi}]\cdot[[g,h],k^{\varphi}]$*;*

(iv) $[g^{-1},h^{\varphi}]=[[g,h,g],g^{\varphi}]^{-1}\cdot[[g,h],g^{\varphi}]^{-1}\cdot[g,h^{\varphi}]^{-1}$*;*

(v) $[g,h^{-\varphi}]=[[g,h,h],h^{\varphi}]^{-1}\cdot[[g,h],h^{\varphi}]^{-1}\cdot[g,h^{\varphi}]^{-1}$*; and*

(vi) $[g^{-1},h^{-\varphi}]=[[g,h,g],g^{\varphi}]\cdot[[g,h,h],g^{\varphi}][[g,h,h],h^{\varphi}][[g,h],g^{\varphi}]\cdot[[g,h],h^{\varphi}]\cdot[g,h^{\varphi}]$.

PROOF. Although (i)-(iii) are special cases of (iv)-(vi), we use the former to prove the latter. From the fact that $\nu(G)$ is nilpotent of class at most 4, we have $[[g,h]^{-1},k^{\varphi}]={}^{[g,h]^{-1}}[[g,h],k^{\varphi}]^{-1}=[[g,h],k^{\varphi}]^{-1}$, that is, (i) holds. For the proof of (ii) we use Corollary 30:

$$\begin{aligned}
[[g,h],k^{-\varphi}] &= {}^{k^{-\varphi}}[[g,h],k^{\varphi}]^{-1}\\
&= [k^{-\varphi},[[g,h],k^{\varphi}]^{-1}]\cdot[[g,h],k^{\varphi}]^{-1}\\
&= [k^{\varphi},[[g,h],k^{\varphi}]]\cdot[[g,h],k^{\varphi}]^{-1}\\
&= [[[g,h],k^{\varphi}],k^{\varphi}]^{-1}\cdot[[g,h],k^{\varphi}]^{-1}\\
&= [[g,h,k],k^{\varphi}]^{-1}\cdot[[g,h],k^{\varphi}]^{-1},
\end{aligned}$$

which proves (ii). By (i) and (ii), $[[g,h]^{-1},k^{-\varphi}]=[[g,h],k^{-\varphi}]^{-1}=[[g,h,k],k^{\varphi}]\cdot[[g,h],k^{\varphi}]$, that is, (iii) holds. Next, using (ii),

$$\begin{aligned}
[g^{-1},h^{\varphi}] &= {}^{g^{-1}}[g,h^{\varphi}]^{-1}\\
&= [g^{-1},[g,h^{\varphi}]^{-1}]\cdot[g,h^{\varphi}]^{-1}\\
&= [[g,h^{\varphi}]^{-1},g^{-1}]^{-1}\cdot[g,h^{\varphi}]^{-1}\\
&= {}^{[g,h^{\varphi}]^{-1}}[[g,h^{\varphi}],g^{-1}]\cdot[g,h^{\varphi}]^{-1}\\
&= [[g,h^{\varphi}],g^{-1}]\cdot[g,h^{\varphi}]^{-1}\\
&= [[g,h],g^{-\varphi}]\cdot[g,h^{\varphi}]^{-1}\\
&= [[g,h,g],g^{\varphi}]^{-1}\cdot[[g,h],g^{\varphi}]^{-1}\cdot[g,h^{\varphi}]^{-1},
\end{aligned}$$

which proves (iv). We also use (ii) to prove (v):

$$\begin{aligned}
[g,h^{-\varphi}] &= {}^{h^{-\varphi}}[g,h^{\varphi}]^{-1}\\
&= [h^{-\varphi},[g,h^{\varphi}]^{-1}]\cdot[g,h^{\varphi}]^{-1}\\
&= [[g,h^{\varphi}]^{-1},h^{-\varphi}]^{-1}\cdot[g,h^{\varphi}]^{-1}\\
&= {}^{[g,h^{\varphi}]^{-1}}[[g,h^{\varphi}],h^{-\varphi}]\cdot[g,h^{\varphi}]^{-1}\\
&= [[g,h^{\varphi}],h^{-\varphi}]\cdot[g,h^{\varphi}]^{-1}\\
&= [[g,h],h^{-\varphi}]\cdot[g,h^{\varphi}]^{-1}\\
&= [[g,h,h],h^{\varphi}]^{-1}\cdot[[g,h],h^{\varphi}]^{-1}\cdot[g,h^{\varphi}]^{-1}.
\end{aligned}$$

Finally,

$$\begin{aligned}
[g^{-1},h^{-\varphi}] &= [[g,h^{-1},g],g^{\varphi}]^{-1}\cdot[[g,h^{-1}],g^{\varphi}]^{-1}\cdot[g,h^{-\varphi}]^{-1}\\
&= [[g,h,g],g^{\varphi}]\cdot[[g,h^{-\varphi}],g^{\varphi}]^{-1}\cdot[g,h^{\varphi}]\cdot[[g,h],h^{\varphi}]\cdot[[g,h,h],h^{\varphi}]\\
&= [[g,h,g],g^{\varphi}]\cdot[[[g,h,h],h^{\varphi}]^{-1}\cdot[[g,h],h^{\varphi}]^{-1}\cdot[g,h^{\varphi}]^{-1},g^{\varphi}]^{-1}\cdot\\
&\qquad [g,h^{\varphi}]\cdot[[g,h],h^{\varphi}]\cdot[[g,h,h],h^{\varphi}]\\
&= [[g,h,g],g^{\varphi}]\cdot[[[g,h],h^{\varphi}]^{-1}\cdot[g,h^{\varphi}]^{-1},g^{\varphi}]^{-1}\cdot\\
&\qquad [g,h^{\varphi}]\cdot[[g,h],h^{\varphi}]\cdot[[g,h,h],h^{\varphi}]\\
&= [[g,h,g],g^{\varphi}]\cdot(\,{}^{[[g,h],h^{\varphi}]^{-1}}[[g,h^{\varphi}]^{-1},g^{\varphi}]\cdot[[[g,h],h^{\varphi}]^{-1},g^{\varphi}])^{-1}\cdot\\
&\qquad [g,h^{\varphi}]\cdot[[g,h],h^{\varphi}]\cdot[[g,h,h],h^{\varphi}]\\
&= [[g,h,g],g^{\varphi}]\cdot(\,{}^{[g,h^{\varphi}]^{-1}}[[g,h^{\varphi}],g^{\varphi}]^{-1}\cdot\ {}^{[[g,h],h^{\varphi}]^{-1}}[[[g,h],h^{\varphi}],g^{\varphi}]^{-1})^{-1}\\
&\qquad [g,h^{\varphi}]\cdot[[g,h],h^{\varphi}]\cdot[[g,h,h],h^{\varphi}]\\
&= [[g,h,g],g^{\varphi}]\cdot[[[g,h],h^{\varphi}],g^{\varphi}][[g,h^{\varphi}],g^{\varphi}]\cdot\\
&\qquad [g,h^{\varphi}]\cdot[[g,h],h^{\varphi}]\cdot[[g,h,h],h^{\varphi}]\\
&= [[g,h,g],g^{\varphi}]\cdot[[g,h,h],g^{\varphi}][[g,h],g^{\varphi}]\cdot\\
&\qquad [g,h^{\varphi}]\cdot[[g,h],h^{\varphi}]\cdot[[g,h,h],h^{\varphi}]\\
&= [[g,h,g],g^{\varphi}]\cdot[[g,h,h],g^{\varphi}][[g,h,h],h^{\varphi}][[g,h],g^{\varphi}]\cdot[[g,h],h^{\varphi}]\cdot[g,h^{\varphi}],
\end{aligned}$$

which proves (vi). □

Our description of the structure of the nonabelian tensor squares of the free nilpotent groups of class 3 finite rank n makes use of basic commutators (see [18]). Fix F to be the free group of rank n with generating set $X = \{x_1, x_2, \ldots, x_n\}$. The particular collection of basic commutators on X depends on an arbitrary choice of ordering of those commutators. The orderings chosen for the basic commutators of weights 1 and 2 completely determine the set of basic commutators on X of weight no more than 4. For the weight 1 commutators on X we choose the natural ordering $x_1, x_2, \ldots, x_n$. For the basic commutators of weight 2 on X, namely those of form $[x_i, x_j]$ with $j < i$, we use right lexicographic ordering based on the subscripts: $[x_i, x_j] < [x_k, x_l]$ if $j < l$ or if $j = l$ and $i < k$. With this ordering the basic commutators of weight 3 on X have form

$$[x_i, x_j, x_k], \text{ where } j < i \text{ and } j \leq k, \tag{10}$$

and the basic commutators of weight 4 on X have the forms

$$[[x_i, x_j], [x_k, x_l]], \text{ where } j < i, l < k \text{ and } l \leq j, \text{ and if } j = l \text{ then } i > k \tag{11}$$

and

$$[x_i, x_j, x_k, x_l], \text{ where } j < i \text{ and } j \leq k \leq l. \tag{12}$$

Denote by $M(n, w)$ the number of basic commutators of rank n and weight w. The value of $M(n, w)$ is computed by Witt's formula (see [18]), which for

$w = 1, 2, 3,$ and 4 gives:

$$\begin{aligned}
M(n,1) &= n;\\
M(n,2) &= \binom{n}{2} = n(n-1)/2;\\
M(n,3) &= 2\binom{n}{3} + 2\binom{n}{2} = n(n^2-1)/3;\\
M(n,4) &= n^2(n^2-1)/4.
\end{aligned}$$

A fundamental property of basic commutators is that the basic commutators of weight w on X form a basis for the free abelian group $\gamma_w(F)/\gamma_{w+1}(F)$ ([18]).

Set $G_n \cong F/\gamma_4(F)$ to be the free nilpotent group of class 3 and rank n, generated by $g_1, \ldots, g_n$, where $g_i = x_i\gamma_4(F)$. The group G_n has a polycyclic generating sequence consisting of all the basic commutators of weights 1, 2 and 3 on the generating set $\{g_1, \ldots, g_n\}$, that is, consisting of the natural images in G of the basic commutators of weights 1, 2 and 3 on X.

Our goal is to prove that $G_n \otimes G_n$ is a direct product of the form $N \times A$, where N is a nilpotent group of class 2 and A is free abelian. We accomplish this by first computing the nonabelian exterior square $G_n \wedge G_n$, which is a homomorphic image of $G_n \otimes G_n$. The nonabelian exterior square $G_n \wedge G_n$ is isomorphic to the derived subgroup of the Schur cover of G_n (Corollary 2,[7]). Since the Schur cover of G_n is the free nilpotent group $F/\gamma_5(F)$ of class 4 and rank n, the following lemma describes the form of $G_n \wedge G_n$.

Lemma 32 *Let V_n be the free nilpotent group of class 4 and rank n generated by $v_1, \ldots, v_n$. Then the derived subgroup V_n' of V_n has the form*

$$V_n' \cong N \times A$$

where N is free nilpotent of class 2 and rank $M(n,2)$ and A is free abelian of rank $M(n,3) + M(n,4) - M(M(n,2),2)$.

PROOF. The derived subgroup of V_n is nilpotent of class 2 and is generated by a set of basic commutators of weights 2, 3, and 4 on the generators $v_1, \ldots, v_n$, that is, it is generated by the natural images in V_n of a set basic commutators of weights 2, 3 and 4 on X. Any abelian subgroup of a free nilpotent group is free abelian [22]. In particular, the center of V_n' is a free abelian group of rank $M(n,3) + M(n,4)$. The subgroup N of V_n generated by the basic commutators of weight 2 is a normal free nilpotent subgroup of V_n' of class 2 and rank $M(n,2)$. The derived subgroup of N is therefore free abelian of rank $M(M(n,2),2)$. Since N' is in the center of V_n', it follows that $Z(V_n')$ is isomorphic to $N' \times A$, where A is free abelian of rank

$M(n,3)+M(n,4)-M(M(n,2),2)$. Moreover, $N\cap A$ is trivial, because otherwise the center of N would have rank larger than $M(M(n,2),2)$, a contradiction. Therefore V_n' is the direct product of N and A. $\square$

We now classify the generators of $G_n\otimes G_n$ and relate them to $G_n\wedge G_n\cong V_n'$.

Definition 33 *In the group $[G_n,G_n^{\varphi}]$ the following commutators of weights $2,3$ and 4 are said to be of basic type:*

(i) $[g_i,g_j^{\varphi}]$, *if $[x_i,x_j]$ is a basic commutator on X.*
(ii) $[g_i,g_j,g_k^{\varphi}]$, *if $[x_i,x_j,x_k]$ is a basic commutator on X.*
(iii) $[g_i,g_j,g_k,g_l^{\varphi}]$, *if $[x_i,x_j,x_k,x_l]$ is a basic commutator on X.*
(iv) $[[g_i,g_j],[g_k,g_l]^{\varphi}]$, *if $[[x_i,x_j],[x_k,x_l]]$ is a basic commutator on X.*

Every element g of G_n can be written in the form

$$g=\prod_{1\le i\le n} g_i^{\alpha_i}\prod_{1\le j<i\le n}[g_i,g_j]^{\beta_{i,j}}\prod_{1\le j<i<k\le n}[g_i,g_j,g_k]^{\rho_{i,j,k}}[g_k,g_j,g_i]^{\sigma_{i,j,k}}$$
$$\prod_{1\le j<i\le n}[g_i,g_j,g_j]^{\gamma_{i,j}}[g_i,g_j,g_i]^{\delta i,j},$$

where the $\alpha_i,\beta_{i,j},\rho_{i,j,k},\sigma_{i,j,k},\gamma_{i,j}$ and $\delta_{i,j}$ are integers, and we therefore have a polycyclic generating set

$$\begin{array}{l}\{g_i\mid 1\le i\le n\}\quad\cup\\ \{[g_i,g_j]\mid 1\le j<i\le n\}\quad\cup\\ \{[g_i,g_j,g_k],[g_k,g_j,g_i]\mid 1\le j<i<k\le n\}\quad\cup\\ \{[g_i,g_j,g_j],[g_i,g_j,g_i]\mid 1\le j<i\le n\}\end{array}\tag{13}$$

for G_n.

Recall from Theorem 25 that the subgroup $[G_n,G_n^{\varphi}]\cong G_n\otimes G_n$ of $\nu(G_n)$ is generated by commutators $[\mathfrak{g}^{\epsilon},(\mathfrak{h}^{\varphi})^{\delta}]$, where $\mathfrak{g}$ and $\mathfrak{h}$ are elements of a polycyclic generating sequence for G_n and $\epsilon,\delta\in\{-1,1\}$. By Lemma 31 we need only retain commutators of the form $[\mathfrak{g},\mathfrak{h}^{\varphi}]$, where $\mathfrak{g}$ and $\mathfrak{h}$ are elements of a polycyclic generating sequence for G_n. Thus we consider which among the possible such commutators of elements (13) and their images under φ are needed to generate $[G_n,G_n^{\varphi}]$. Certainly only resulting commutators of weight at most 4 need be considered; also by Corollary 30 we need consider only commutators in which the weight of $\mathfrak{g}$ is at least the weight of $\mathfrak{h}^{\varphi}$. Among these generators lie all the commutators of basic type of weights 2, 3 and 4.

The commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
H_3(G_n) \arrow[r] & \Gamma(G_n^{ab}) \arrow[r, "\psi'"] \arrow[d, "1"] & J_2(G_n) \arrow[r, "\pi'"] \arrow[d] & H_2(G_n) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \nabla(G_n) \arrow[r, "\psi"] & {[G_n, G_n^{\varphi}]} \arrow[r, "\pi"] \arrow[d, "\kappa"] & V_n' \arrow[r] \arrow[d, "\kappa'"] & 1, \\
 & & G_n' \arrow[r, "1"] \arrow[d] & G_n' \arrow[d] & \\
 & & 1 & 1 &
\end{tikzcd} \tag{14}$$

follows from [8] and [7] where each sequence is exact and all extensions are central. We have substituted $[G_n, G_n^{\varphi}]$ for the nonabelian tensor product $G_n \otimes G_n$ and V_n' for the nonabelian exterior square $G_n \wedge G_n$. The natural mapping π, which maps the generators of $[G_n, G_n^{\varphi}]$ to V_n' via

$$\begin{aligned}
&[g_i, g_i^{\varphi}] \to 1_{V_n}, 1 \le i \le n,\\
&[g_i, g_j^{\varphi}] \to [v_i, v_j], 1 \le i, j \le n, i \ne j,\\
&[g_i, g_j, g_k^{\varphi}] \to [v_i, v_j, v_k], 1 \le i, j, k \le n,\\
&[[g_i, g_j], [g_k, g_l]^{\varphi}] \to [[v_i, v_j], [v_k, v_l]] \text{ and}\\
&[g_i, g_j, g_k, g_l^{\varphi}] \to [v_i, v_j, v_k, v_l], 1 \le i, j, k, l \le n,
\end{aligned}$$

is an epimorphism. Every basic commutator in V_n' on $v_1, \ldots, v_n$ has a preimage in $[G_n, G_n^{\varphi}]$ that is a commutator of basic type of the same weight. Hence the central subgroup of $[G_n, G_n^{\varphi}]$ generated by the commutators of basic type of weights 3 and 4 isomorphically embeds into the center of V_n', which is free abelian of rank $M(n,3) + M(n,4)$.

The abelianization G_n^{ab} of G_n is free abelian of rank n. Let η be the natural mapping from $G_n \to G_n^{ab}$. Then there is a epimorphism $\theta : G_n \otimes G_n \to G_n^{ab} \otimes G_n^{ab}$ defined by $x \otimes y \to \eta(x) \otimes \eta(y)$ (see [7]). Now $G_n^{ab} \otimes G_n^{ab}$ is free abelian of rank n^2, with standard basis $\{\eta(g_i) \otimes \eta(g_j) \mid 1 \le i, j \le n\}$. Choose alternately as basis for $G_n^{ab} \otimes G_n^{ab}$ the collection

$$\begin{aligned}
&\eta(g_i) \otimes \eta(g_i), \quad 1 \le i \le n,\\
&\eta(g_i) \otimes \eta(g_j), \quad 1 \le j < i \le n \text{ and}\\
&(\eta(g_i) \otimes \eta(g_j))(\eta(g_j) \otimes \eta(g_i)), \quad 1 \le j < i \le n.
\end{aligned}$$

Hence the generators of $\nabla(G_n)$ (see Lemma 21) map under θ to the distinct basis elements of a free abelian group and therefore they generate a free abelian group of rank $n + \binom{n}{2} = \binom{n+1}{2}$. Therefore, in $\nu(G_n)$ the generators $[g_i, g_i^\varphi]$ for $1 \leq i \leq n$ are all nontrivial central elements of infinite order, as are the products $[g_i, g_j^\varphi][g_j, g_i^\varphi]$ for $1 \leq j < i \leq n$. Since $\ker(\pi) = \psi(\nabla(G_n))$ and we have shown that the rank of $\ker(\pi)$ is at least equal to the rank of $\Gamma(G_n^{ab})$, we conclude that in fact ψ is injective and that $\ker(\pi)$ is free abelian of rank $\binom{n+1}{2}$. These observations have the following consequences:

Proposition 34 *Let G_n be the free nilpotent group of class 3 and rank n. Then*

(i) The map $H_3(G_n) \to \Gamma(G_n^{ab})$ in diagram (14) is the zero map;
(ii) $\Gamma(G_n^{ab}) \cong \nabla(G_n)$; and
(iii) $J_2(G_n) \cong \Gamma(G_n^{ab}) \oplus H_2(G_n)$.

Corollary 35 *Let G_n be the free nilpotent group of class 3 and rank n. Then $J_2(G_n)$ is a free abelian group of rank*

$$\frac{n(n+1)(n^2-n+2)}{4}.$$

PROOF. The result follows immediately from the facts that $\Gamma(G_n^{ab})$ is free abelian of rank $\binom{n+1}{2}$, the Schur multiplier $H_2(G_n)$ of G_n is free abelian of rank $\frac{n^2(n^2-1)}{4}$ and

$$\binom{n+1}{2} + \frac{n^2(n^2-1)}{4} = \frac{n(n+1)(n^2-n+2)}{4}.$$

□

Let $\mathcal{A}$ be the central subgroup of $[G_n, G_n^\varphi]$ generated by the elements $[g_i, g_i^\varphi]$ for $1 \leq i \leq n$, the commutators of basic type of weight 3 and those of weight 4 of type (iii) in Definition 33 on the generating set $\{g_1, \ldots, g_n\}$. Since these generators are independent, $\mathcal{A}$ is a free abelian group of rank

$$M(n,3) + M(n,4) - M(M(n,2),2) + M(n,1) = \frac{n(n+5)(3n^2-n+2)}{24},$$

where we make use of the fact that $\pi(\mathcal{A}) = A$ in H_n'.

Let $\mathcal{N}$ be the normal subgroup of $[G_n, G_n^\varphi]$ generated by $[g_i, g_j^\varphi]$ for $1 \leq i, j \leq n$ with $i \neq j$.

By Lemma 15(i) and Corollary 30 the relations

$$[[g_i, g_j^\varphi], [g_j, g_i^\varphi]] = 1 \tag{15}$$

for $1 \leq i, j \leq n, i \neq j$ and

$$\begin{aligned} [[g_i, g_j], [g_k, g_l]^\varphi] &= [[g_i, g_j], [g_l, g_k]^\varphi]^{-1} \\ &= [[g_j, g_i], [g_k, g_l]^\varphi]^{-1} \\ &= [[g_k, g_l], [g_i, g_j]^\varphi]^{-1} \end{aligned} \tag{16}$$

for $1 \leq i, j, k, l \leq n, i \neq j, k \neq l$ hold in $\mathcal{N}$. The relations (16) reflect the fact that the commutators of basic type of weight 2 in $\mathcal{N}$ suffice to generate $\mathcal{N}'$. Note that the images of the relations (15) and (16) hold trivially in $\pi(\mathcal{N})$. Since it is nilpotent of class 2, the structure of $\mathcal{N}$ is given by the group $\mathcal{W}_n$ described in Example 36.

Example 36 *Let U_n be a free nilpotent group of class 2 of rank $n(n-1)$ generated by the elements $u_{i,j}$, $1 \leq i, j \leq n$, $i \neq j$. Set R to be the set of the following words for all i, j, k, l for $1 \leq i, j, k, l \leq n$ with $i \neq j$ and $k \neq l$:*

$$[u_{i,j}, u_{j,i}], [u_{i,j}, u_{k,l}][u_{i,j}, u_{l,k}], [u_{i,j}, u_{k,l}][u_{j,i}, u_{k,l}]$$

for all generators of U_n. We set $\mathcal{W}_n = U_n/R$.

We have now established the main theorem of this section, which Theorem 6 from Section 1 summarizes.

Theorem 37 *Let G_n be a free nilpotent group of class 3 and rank n. Then $G_n \otimes G_n \cong \mathcal{W}_n \times \mathcal{A}$ where $\mathcal{W}_n$ is the group in Example 36 and $\mathcal{A}$ is the free abelian group of rank*

$$f(n) = \frac{n(n+5)(3n^2 - n + 2)}{24}.$$

6 GAP Implementation

The following is a simple GAP implementation of Algorithm 28 for nilpotent groups.

```
##
## GAP -- Groups, Algorithms and Programming Version 4.4.9 [17]
## Implementation to compute G\otimes G for G a nilpotent group
##
## Input:  G a nilpotent group represented as a PcpGroup
## Output: G\otimes G as a PcpGroup
##
## Dependencies: GAP Packages Polycyclic [11] and nq [23]
##
TensorSquare :=
    function(G)

    local FP,         # Free product G*G^\varphi
          lg,rg,      # Generators of the left and right base groups of
                      #   G*G^\varphi
```

```
      pairs,      # Generators of the left and right base group paired up
      lcj, lcomm, # Helper functions for left conjugation and commutation
      nurel,      # Helper function to create relation pairs for nu(G)
      mingp,      # Positions of the minimal generating set of G relative
                  #   to the polycyclic generating set (Igs)
      class,      # Nilpotency class of G
      nu,         # Finite presentation of nu(G)
      LG, RG,     # Images of the generators of the left and right base
                  #   groups in nu(G) as a finitely presented group
      epi,        # Epimorphism from nu(G)-> polycyclic presentation of
                  #   nu(G)
      LSG, RSG;   # left and right base groups as subgroups of the
                  #   polycyclic presentation of nu(G)

## Check to see if G is nilpotent and a Pcp group
##
if not (IsPcpGroup(G) and IsNilpotent(G)) then
    Error("Error: G must be a nilpotent pcp group \n");
fi;

## Define functions for left conjugation and commutation
##
lcj   := function (a,b) return a*b*a^-1; end;
lcomm := function(a,b) return lcj(a,b)*b^-1; end;

## Define a function to create the relation for \nu(G)
##    {^X}[a,b]=[{^x}a,({^x^\phi}b)^\varphi]
## where X=x (if i=1) or X=x^\phi (if i=2).
##
nurel := function(x,a,b,i)
    return lcj(x[i],lcomm(a,b)) / lcomm(lcj(x[1],a),lcj(x[2],b));
end;

## Create the free product G*G^\varphi and record the class of G
##
FP    := FreeProduct(G,G);
class := NilpotencyClassOfGroup(G);

## Record the positions in the polycyclic generating set (Igs) of
## the minimal generators of the group
##
mingp := List(MinimalGeneratingSet(G), i->Position(Igs(G),i));
if fail in mingp then
    Error("Error: Minimal generating set must be in the Igs");
fi;

## Image of the Igs (polycyclic generating set) of each base group
## in the underlying free group of the free product FP. And pair
## them up as needed to create the relations of nu(G).
##
lg    := List(Igs(G), g->UnderlyingElement(Image(Embedding(FP,1),g)));
rg    := List(Igs(G), g->UnderlyingElement(Image(Embedding(FP,2),g)));
pairs := List([1..Length(lg)],i->[lg[i],rg[i]]);

## Create a finite presentation of nu(G)
##
nu    := FreeGroupOfFpGroup(FP)/Concatenation(RelatorsOfFpGroup(FP),
             ListX(pairs,lg{mingp},rg{mingp},[1,2], nurel) );

## Record the images of the generators of the left and right base
## groups in nu(G) as a finitely presented group.
##
LG    := List(lg, g->ElementOfFpGroup(FamilyObj(nu.1),g));
RG    := List(rg, g->ElementOfFpGroup(FamilyObj(nu.1),g));

## Create an isomorphism from the finite presentation of nu(G) to
## the polycyclic presentation for nu(G) using the known
## nilpotency bound on nu(G).
```

```
##
epi   := NqEpimorphismNilpotentQuotient(nu, class+1);

## Obtain the images of the left and right base groups in the
## polycyclic presentation of nu(G)
##
LSG   := Subgroup(Image(epi),List(LG,g->Image(epi,g)));
RSG   := Subgroup(Image(epi),List(RG,g->Image(epi,g)));

## Return [G,G^\phi] which is isomorphic to the tensor square
##
return CommutatorSubgroup(LSG,RSG);

end;
```

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