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On Commuting and Noncommuting Complexes

Jonathan Pakianathan

Department of Mathematics, University of Wisconsin–Madison, Madison, Wisconsin 53706
E-mail: pakianat@math.wisc.edu

and

Ergün Yalçın

Department of Mathematics & Statistics, McMaster University, Hamilton, Ontario, Canada L8S 4K1
E-mail: yalcine@math.mcmaster.ca

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In this paper we study various simplicial complexes associated to the commutative structure of a finite group G. We define NC(G) (resp. C(G)) as the complex associated to the poset of pairwise noncommuting (resp. commuting) sets of nontrivial elements in G.

We observe that NC(G) has only one positive dimensional connected component, which we call BNC(G), and we prove that BNC(G) is simply connected.

Our main result is a simplicial decomposition formula for BNC(G) which follows from a result of A. Björner, M. Wachs and V. Welker, on inflated simplicial complexes (2000, A poset fiber theorem, preprint). As a corollary we obtain that if G has a nontrivial center or if G has odd order, then the homology group $H_{n-1}(\mathrm{BNC}(G))$ is nontrivial for every n such that G has a maximal noncommuting set of order n.

We discuss the duality between NC(G) and C(G) and between their p-local versions $\mathrm{NC}_p(G)$ and $\mathrm{C}_p(G)$. We observe that $\mathrm{C}_p(G)$ is homotopy equivalent to the Quillen complexes $A_p(G)$ and obtain some interesting results for $\mathrm{NC}_p(G)$ using this duality.

Finally, we study the family of groups where the commutative relation is transitive, and show that in this case BNC(G) is shellable. As a consequence we derive some group theoretical formulas for the orders of maximal noncommuting sets.

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1. INTRODUCTION

Given a finite group G, one defines a noncommuting set to be a set of elements $\{g_1, \ldots, g_n\}$ such that g_i does not commute with g_j for $i \neq j$. The sizes of maximal noncommuting sets in a group are interesting invariants of the group. In particular, the largest integer n such that the group G has a noncommuting set of order n, which is denoted by $\mathrm{nc}(G)$, is known to be closely related to other invariants of G. For example, if $k(G)$ is the size of the largest conjugacy class in G then

$$k(G) \leq 4(\mathrm{nc}(G))^2.$$

(See [P].)
Also, if we define $\mathrm{cc}(G)$ to be the minimal number of abelian subgroups of G that covers G, then Isaacs (see [J]) has shown that

$$\mathrm{nc}(G) \leq \mathrm{cc}(G) \leq (\mathrm{nc}(G)!)^2.$$

Confirming a conjecture of Erdős, Pyber [P] has also shown that there is a positive constant c such that

$$\mathrm{cc}(G) \leq |G : Z(G)| \leq c^{\mathrm{nc}(G)}$$

for all groups G. Another interesting place where the invariant $\mathrm{nc}(G)$ appears is in the computation of the cohomology length of extra-special p-groups. (See [Y].)

In this paper we study the topology of certain complexes associated to the poset of noncommuting sets in a group G. Let $\mathrm{NC}(G)$ be the complex whose vertices are just the nontrivial elements of the group G and whose faces are the noncommuting sets in G. The central elements form point components in this complex and are not as interesting. So, we look at the subcomplex $\mathrm{BNC}(G)$ formed by noncentral elements of G. We show

Result 1 (3.2). If G is a nonabelian group, then $\mathrm{BNC}(G)$ is simply connected.

In general we also note that $\mathrm{BNC}(G)$ is equipped with a free $Z(G)$-action where $Z(G)$ is the center of G. It is also equipped in general with a $\mathbb{Z}/2\mathbb{Z}$-action whose fixed point set is exactly $\mathrm{BNC}_2(G)$, the corresponding complex where we use only the elements of order 2 (involutions). Thus if G is an odd order group or if G has nontrivial center, then the Euler characteristic of $\mathrm{BNC}(G)$ is not 1 and so it is not contractible.

On a more refined level, we use a recent simplicial decomposition result of Björner *et al.* [BWW] to show that there is a simplicial complex S, called the core of BNC(G), so that the following decomposition formula holds:

Result 2 (4.11). If G is a finite nonabelian group and S is the core of BNC(G), then

$$\mathrm{BNC}(G) \simeq S \vee \bigvee_{F \in S} [\mathrm{Susp}^{|F|}(\mathrm{Lk}\, F)]^{\gamma(F)},$$

where the F are the faces of S, Lk stands for the link of a face, Susp^k stands for a k-fold suspension, and $\vee$ stands for a wedge product. The number $\gamma(F) = \prod_{[x] \subseteq F}(m_x - 1)$ where m_x is the size of the centralizer class $[x]$.

It is clear from this decomposition formula that when the core of BNC(G) is contractible, then BNC(G) is a wedge of suspensions of spaces and hence has a trivial ring structure on its cohomology. This is true, for example, when $G = \Sigma_p$, the symmetric group on p letters, for some prime p.

The following is an important consequence of the above decomposition formula:

Result 3 (4.14). Let G be a finite nonabelian group, and let S_s denote the set of maximal noncommuting sets in G of size s. Then, for $s > 1$,

$$\mathrm{rk}(H_{s-1}(\mathrm{BNC}(G))) \geq \sum_{F \in S_s} \left[\prod_{x \in F} \left(1 - \frac{1}{m_x}\right) \right]$$

where m_x is the size of the centralizer class containing x.

In particular, if G is an odd order group or if G has a nontrivial center ($|G| \neq 2$), then

$$\widetilde{H}_{s-1}(NC(G)) \neq 0$$

whenever G has a maximal noncommuting set of size s.

There is also a p-local version of this theorem which, in particular, gives that $\widetilde{H}_{s-1}(\mathrm{NC}_p(G)) \neq 0$ whenever G has a maximal noncommuting p-set of size s and p is an odd prime. For $p = 2$, the same is true under the condition $2 \| Z(G)|$ and $|G| \neq 2$. Observe that this result has a striking formal similarity (in terms of their conclusions) to the following theorem proved by Quillen (Theorem 12.1 in [Q]):

THEOREM 1.1. (Quillen). *If G is a finite solvable group having no nontrivial normal p-subgroup, then*

$$\widetilde{H}_{s-1}(A_p(G)) \neq 0$$

whenever G has a maximal elementary abelian p-group of rank s.

A nice consequence of Result 3 can be stated as follows: If G is a group of odd order or a group with nontrivial center such that BNC(G) is spherical, i.e., homotopy equivalent to a wedge of equal-dimensional spheres, then all maximal noncommuting sets in G have the same size.

A natural question to ask is: For which groups is BNC(G) spherical? As a partial answer, we show that if G is a group where the commutation relation is transitive, then BNC(G) is spherical. We give examples of such groups and compute BNC(G) for these groups.

Observe that one could also define a commuting complex C(G) analogous to the way we defined NC(G) by making the faces consist of commuting sets of elements instead of non-commuting sets. However, this definition does not provide us with new complexes. For example, $\mathrm{C}_p(G)$, the commuting complex formed by the elements of prime order p, is easily shown to be G-homotopy equivalent to Quillen's complex $A_p(G)$. However, the definition of $\mathrm{C}_p(G)$ helps us to see a duality between $\mathrm{NC}_p(G)$ and $\mathrm{C}_p(G)$ where $\mathrm{NC}_p(G)$ is the subcomplex of NC(G) spanned by the vertices which correspond to elements of order p. Using a result of Quillen on $A_p(G)$, we obtain:

Result 4 (5.2). Let G be a group and p a prime with $p \| G|$. Pick a Sylow p-subgroup P of G and define N to be the subgroup generated by the normalizers $\mathrm{N}_G(H)$ where H runs over all the nontrivial subgroups of P.

Then $\mathrm{NC}_p(G)$ is $(|G:N|-2)$-connected. In fact, $\mathrm{NC}_p(G)$ is the $|G:N|$-fold join of a complex S with itself where S is "dual" to a path-component of $A_p(G)$.

Finally, under suitable conditions, BNC(G) is shellable and this yields some combinatorial identities. As an application we obtain

Result 5 (4.24). Let G be a nonabelian group with a transitive commuting relation, i.e., if $[g,h]=[h,k]=1$, then $[g,h]=1$ for every noncentral $g,h,k \in G$. Then,

$$\mathrm{nc}(G)(\mathrm{nc}(G)-1)+|G|(|G|-m)-2(\mathrm{nc}(G)-1)(|G|-|Z(G)|)\geq 0,$$

where m denotes the number of conjugacy classes in G.

2. BACKGROUND

We start the section with a discussion of complexes associated with posets of subgroups of a group G. For a complete account of these well-known results, see Chapter 6 in [B].

Given a finite poset $(P,\leq)$, one can construct a simplicial complex $|P|$ out of it by defining the n-simplices of $|P|$ to be chains in P of the form $p_0 < p_1 < \cdots < p_n$. This is called the simplicial realization of the poset P.

Furthermore, any map of posets $f\colon (P_1, \leq_1) \to (P_2, \leq_2)$ (map of posets means $x_1 \leq_1 x_2 \Rightarrow f(x_1) \leq_2 f(x_2)$) yields a simplicial map between $|P_1|$ and $|P_2|$ and hence one has in general a covariant functor from the category of finite posets to the category of finite simplicial complexes and simplicial maps. Thus if a (finite) group G acts on a poset P via poset maps (we say P is a G-poset) then G will act on $|P|$ simplicially.

Brown, Quillen, Webb, Bouc, Thévenaz, and many others constructed many finite G-simplicial complexes associated to a group G and used them to study the group G and its cohomology. In particular, the following posets of subgroups of G have been studied extensively:

(a) the poset $s_p(G)$ of nontrivial p-subgroups of G,

(b) the poset $a_p(G)$ of nontrivial elementary abelian p-subgroups of G, and

(c) the poset $b_p(G)$ of nontrivial p-radical subgroups of G. (Recall that a p-radical subgroup of G is a p-subgroup P of G such that $\mathrm{PN}_G(P)/P$ has no nontrivial normal p-subgroups.)

G acts on each of these posets by conjugation, and thus from each of these G-posets one gets a G-simplicial complex, $S_p(G)$, $A_p(G)$, and $B_p(G)$ respectively. $S_p(G)$ is usually called the Brown complex of G and $A_p(G)$ is usually called the Quillen complex of G where the dependence on the prime p is understood. Note again that the trivial subgroup is not included in any of these posets, since if it were the resulting complex would be a cone and hence trivially contractible.

It was shown via work of Quillen and Thévenaz that $S_p(G)$ and $A_p(G)$ are G-homotopy equivalent and via work of Bouc and Thévenaz that $B_p(G)$ and $S_p(G)$ are G-homotopy equivalent. Thus, in a sense, these three G-complexes capture the same information.

Recall the following elementary yet very important lemma (see [B]):

LEMMA 2.1. *If $f_0, f_1\colon P_1 \to P_2$ are two maps of posets such that $f_0(x) \leq f_1(x)$ for all $x \in P_1$ then the simplicial maps induced by f_0 and f_1 from $|P_1|$ to $|P_2|$ are homotopic.*

Using this, Quillen made the following observation: If P_0 is a nontrivial normal p-subgroup of G, then we may define a poset map $f\colon s_p(G) \to s_p(G)$ by $f(P) = P_0P$, and by the lemma above f would be homotopic to the identity map; but on the other hand, since $f(P)$ contains P_0 always, again by the lemma, f is also homotopic to a constant map. Thus we see that $\mathrm{S}_p(G)$ is contractible in this case. Quillen then conjectured

CONJECTURE 2.2 (Quillen). *If G is a finite group, $S_p(G)$ is contractible if and only if G has a nontrivial normal p-subgroup.*

He proved his conjecture in the case that G is solvable but the general conjecture remains open. Note though that if $\mathrm{S}_p(G)$ is G-homotopy equivalent to a point space then this does imply that G contains a nontrivial normal p-subgroup since in this case $\mathrm{S}_p(G)^G$ is homotopy equivalent to a point, which means in particular that $S_p(G)^G$ is not empty, yielding a nontrivial normal p-subgroup.

The purpose of this paper is to introduce some simplicial complexes associated to elements of a group rather than to subgroups of a group and use these to give a different perspective on some of the complexes above.

For this purpose we give the following definitions:

DEFINITION 2.3. Let G be a group. Define a simplicial complex $C(G)$ by declaring a n-simplex in this complex to be a collection $[g_0, g_1, \dots, g_n]$ of distinct nontrivial elements of G which pairwise commute.

Similarly, define a simplicial complex $\mathrm{NC}(G)$ by declaring an n-simplex to be a collection $[g_0, g_1, \dots, g_n]$ of nontrivial elements of G, which pairwise do not commute.

It is trivial to verify that the above definition does indeed define complexes on which G acts simplicially by conjugation.

Usually when one studies simplicial group actions, it is nice to have admissible actions, i.e., actions where if an element of G fixes a simplex, it actually fixes it pointwise. Although $\mathrm{C}(G)$ and $\mathrm{NC}(G)$ are not admissible in general, one can easily fix this by taking a barycentric subdivision. The resulting complex is of course G-homotopy equivalent to the original; however, it now is the realization of a poset.

Thus if we let $\mathrm{PC}(G)$ be the barycentric subdivision of $\mathrm{C}(G)$, it corresponds to the realization of the poset consisting of subsets of nontrivial, pairwise commuting elements of G, ordered by inclusion. Similarly, if we let $\mathrm{PNC}(G)$ be the barycentric subdivision of $\mathrm{NC}(G)$, it corresponds to the realization of the poset consisting of subsets of nontrivial, pairwise noncommuting elements of G, ordered by inclusion.

Depending on the situation, one uses either the barycentric subdivision or the original. For the purpose of understanding the topology, the original is easier but for studying the G-action the subdivision is easier.

Of course, we will want to work a prime at a time also, so we introduce the following p-local versions of $\mathrm{C}(G)$ and $\mathrm{NC}(G)$.

DEFINITION 2.4. Given a group G and a prime p, let $\mathrm{C}_p(G)$ be the subcomplex of $\mathrm{C}(G)$ where the simplices consist of sets of nontrivial, pairwise commuting elements of order p.

Similarly let $\mathrm{NC}_p(G)$ be the subcomplex of $\mathrm{NC}(G)$ where the simplices consist of sets of nontrivial, pairwise noncommuting elements of order p.

Of course, the same comments about the G-action and the barycentric subdivision apply to these p-local versions.

Our first order of business is to see that the commuting complexes $\mathrm{C}(G)$ and $\mathrm{C}_p(G)$ are nothing new. We will find the following standard lemma useful for this purpose (see [B]):

LEMMA 2.5. *If f is a G-map between admissible G-simplicial complexes X and Y with the property that for all subgroups $H \leq G$, f restricts to an ordinary homotopy equivalence between X^H and Y^H (recall that X^H is the subcomplex of X which consists of elements fixed pointwise by H), then f is a G-homotopy equivalence, i.e., there is a G-map $g: Y \to X$ such that $f \circ g$ and $g \circ f$ are G-homotopic to identity maps.*

THEOREM 2.6. *Let G be a finite group, then $C(G)$ is G-homotopy equivalent to the simplicial realization of the poset $A(G)$ of nontrivial abelian subgroups of G, ordered by inclusion and acted on by conjugation.*

Furthermore, if p is a prime, then $C_p(G)$ is G-homotopy equivalent to $A_p(G)$ (and thus to $S_p(G)$ and $B_p(G)$.)

Proof. First we will show homotopy equivalence and remark on G-homotopy equivalence later.

We work with $\mathrm{PC}(G)$, the barycentric subdivision. Note that the associated poset of $\mathrm{PC}(G)$ contains the poset $A(G)$ of nontrivial abelian subgroups of G as a subposet; they are merely the commuting sets whose elements actually form an abelian subgroup (minus identity). Let $i: A(G) \to \mathrm{PC}(G)$ denote this inclusion.

We now define a poset map $r: \mathrm{PC}(G) \to A(G)$ as follows: If S is a set of nontrivial, pairwise commuting elements of G, then $\langle S\rangle$, the subgroup generated by S, will be a nontrivial abelian subgroup of G; thus we can set $r(S) = \{\langle S\rangle - 1\}$. It is obvious that r is indeed a poset map, and that $S \subset r(S)$ and so $i \circ r$ is homotopic to the identity map of $\mathrm{PC}(G)$ by Lemma 2.1. Furthermore, it is clear that $r \circ i = \mathrm{Id}$ and so r is a deformation retraction of $\mathrm{PC}(G)$ onto $A(G)$.

Thus $\mathrm{PC}(G)$ is homotopy equivalent to $A(G)$. To see this is a G-homotopy equivalence, we just need to note that r is indeed a G-map and maps a commuting set invariant under conjugation by a subgroup H into a subgroup invariant under conjugation by H and thus induces a homotopy equivalence between $\mathrm{PC}(G)^H$ and $A(G)^H$ for any subgroup H. Thus r is indeed a G-homotopy equivalence by Lemma 2.5.

The p-local version follows exactly in the same manner, once one notes that the subgroup generated by a commuting set of elements of order p is an elementary abelian p-subgroup. ∎

Thus we see from Theorem 2.6 that the commuting complexes at a prime p are basically the $A_p(G)$ in disguise. However, for the rest of the paper we look at the noncommuting complexes and we will see that they are quite different from, and in some sense dual to, the commuting ones. However, before doing that we conclude this section by looking at a few more properties of the commuting complex.

Recall the following important proposition of Quillen [Q]:

PROPOSITION 2.7. *If $f: X \to Y$ is a map of posets and $y \in Y$, we define*

$$f/y = \{x \in X | f(x) \leq y\}$$
$$y\backslash f = \{x \in X | f(x) \geq y\}.$$

Then if f/y is contractible for all $y \in Y$ (respectively, $y\backslash f$ is contractible for all $y \in Y$) then f is a homotopy equivalence between $|X|$ and $|Y|$.

Using this we will prove:

PROPOSITION 2.8. *If G is a group with nontrivial center then $A(G)$ and hence $C(G)$ are contractible. Moreover, $A(G)$ is homotopy equivalent to $Nil(G)$ where $Nil(G)$ is the poset of nontrivial nilpotent subgroups of G.*

Proof. If G has a nontrivial center $Z(G)$, then $A(G)$ is conically contractible via $A \leq AZ(G) \geq Z(G)$ for any abelian subgroup A of G. Thus $C(G)$ is also contractible as it is homotopy equivalent to $A(G)$.

Let i: $A(G) \to \mathrm{Nil}(G)$ be the natural inclusion of posets. Take $N \in \mathrm{Nil}(G)$ and let us look at $i/N = \{B \in A(G) | B \subseteq N\} = A(N)$. However, since N is nilpotent, it has a nontrivial center Z and hence $A(N)$ is conically contractible. Thus by Proposition 2.7 the result follows. ∎

It is natural to ask:

CONJECTURE 2.9. *$C(G)$ is contractible if and only if G has a nontrivial center.*

3. NONCOMMUTING COMPLEXES

Fix a group G, and let us look at NC(G). The first thing we note is that any nontrivial central element in G gives us a point component in NC(G) and hence is not interesting. Thus we define:

DEFINITION 3.1. *$BNC(G)$ is the subcomplex of $NC(G)$ consisting of those simplices of $NC(G)$ which are made out of noncentral elements. Thus $BNC(G)$ is empty if G is an abelian group.*

The first thing we will show is that if G is a nonabelian group (so that BNC(G) is nonempty) then BNC(G) is not only path-connected but it is simply-connected. Note also that this means that the general picture of NC(G) is as a union of components, with at most one component of positive dimension and this is BNC(G) and it is simply connected. Also note that BNC(G) is invariant under the conjugation G-action, and the point components of NC(G) are just fixed by the G-action as they correspond to central elements.

THEOREM 3.2. *If G is a nonabelian group, then BNC(G) is a simply connected G-simplicial complex.*

Proof. First we show that BNC(G) is path-connected. Take any two vertices in BNC(G), call them g_0 and g_1, then these are two noncentral elements of G. Thus their centralizer groups $C(g_0)$ and $C(g_1)$ are proper subgroups of G.

It is easy to check that no group is the union of two proper subgroups for suppose that $G = H \cup K$ where H, K are proper subgroups of G. Then we can find $h \in G - K$ (it follows that $h \in H$) and $k \in G - H$ (hence $k \in K$). Then hk is not in H as $h \in H$ and $k \notin H$. Similarly, $hk \notin K$ so $hk \notin H \cup K = G$, which is an obvious contradiction. Thus no group is the union of two proper subgroups.

Thus we conclude that $C(g_0) \cup C(g_1) \neq G$, and so we can find an element w which does not commute with either g_0 or g_1 and so the vertices g_0 and g_1 are joined by an edge path $[g_0, w] + [w, g_1]$. (The $+$ stands for concatenation.) Thus we see that BNC(G) is path-connected. In fact, any two vertices of BNC(G) can be connected by an edge path involving at most two edges of BNC(G).

To show that it is simply connected, we argue by contradiction. If it was not simply connected, then there would be a simple edge loop which did not contract, i.e., did not bound a suitable union of 2-simplices. (A simple edge loop is formed by edges of the simplex and is of the form $L = [e_0, e_1] + [e_1, e_2] + \cdots + [e_{n-1}, e_n]$ where all the e_i are distinct except $e_0 = e_n$.)

Take such a loop L with minimal size n. (Note that n is just the number of edges involved in the loop.)

Since we are in a simplicial complex, certainly $n \geq 3$.

Suppose $n > 5$, then e_3 can be connected to e_0 by an edge path E involving at most two edges by our previous comments. This edge path E breaks our simple edge loop into two edge loops of smaller size which hence must contract since our loop was minimal. However, then it is clear that our loop contracts, which is a contradiction so $n \leq 5$.

So we see that $3 \leq n \leq 5$. Thus we have following three cases to consider:

(a) $n = 3$. Here $L = [e_0, e_1] + [e_1, e_2] + [e_2, e_3]$ with $e_3 = e_0$. But then it is easy to see that $\{e_0, e_1, e_2\}$ is a set of pairwise non-commuting elements and so this gives us a 2-simplex $[e_0, e_1, e_2]$ in BNC(G) which bounds the loop, which gives a contradiction.

(b) $n = 4$. Here $L = [e_0, e_1] + [e_1, e_2] + [e_2, e_3] + [e_3, e_4]$ with $e_4 = e_0$. Thus L forms a square. Note that by the simplicity of L, the diagonally opposite vertices in the square must not be joined by an edge in BNC(G), i.e., they must commute, thus e_0 commutes with e_2 and e_1 commutes with e_3.

Since e_0 and e_1 do not commute, $\{e_0, e_1, e_0e_1\}$ is a set of mutually noncommuting elements and so forms a 2-simplex of BNC(G). Since e_2 commutes with e_0 but not with e_1, it does not commute with e_0e_1 and thus $\{e_0e_1, e_1, e_2\}$ also is a 2-simplex in BNC(G). Similar arguments show that $\{e_0e_1, e_0, e_3\}$ and $\{e_0e_1, e_2, e_3\}$ form 2-simplices in BNC(G). The union of the four 2-simplices mentioned in this paragraph, bounds-our loop, giving us our contradiction.

Thus we are reduced to the final case:

(c) $n = 5$. Here $L = [e_0, e_1] + [e_1, e_2] + [e_2, e_3] + [e_3, e_4] + [e_4, e_5]$ with $e_5 = e_0$. Thus L forms a pentagon, and again by the simplicity of L nonadjacent vertices in the pentagon cannot be joined by an edge in BNC(G), thus they must commute. Arguments similar to those for the $n = 4$ case yield that $[e_0, e_1, e_0e_1]$, $[e_0e_1, e_1, e_2]$, and $[e_0e_1, e_0, e_4]$ are 2-simplices in BNC(G) and that the union of these three simplices contracts our loop L into one of length four, namely $[e_0e_1, e_2] + [e_2, e_3] + [e_3, e_4] + [e_4, e_0e_1]$, which by our previous cases must contract, thus yielding the final contradiction. ∎

From Theorem 3.2, we see that BNC(G) is simply connected for any nonabelian group G. One might ask if it is contractible? The answer is no, in general, although there are groups G where it is contractible. We look at these things next.

PROPOSITION 3.3. *For a general finite nonabelian group G, the center $Z(G)$ of G acts freely on $BNC(G)$ by left multiplication and hence $|Z(G)|$ divides the Euler characteristic of $BNC(G)$.*

For a group of odd order, the simplicial map A which maps a vertex g to g^{-1} is a fixed point free map on $BNC(G)$ and on $NC(G)$ of order 2. Thus the Euler characteristic of both $BNC(G)$ and $NC(G)$ is even in this case.

Thus if $BNC(G)$ is $\mathbb{F}$-acyclic for some field $\mathbb{F}$, then G must have trivial center and be of even order.

Proof. The remarks about Euler characteristics follow from the fact that if a finite group H acts freely on a space where the Euler characteristic is defined, then $|H|$ must divide the Euler characteristic. So we will concentrate mainly on finding such actions.

First for the action of $Z(G)$. $a \in Z(G)$ acts by taking a simplex $[g_0, \ldots, g_n]$ to a simplex $[ag_0, \ldots, ag_n]$. Note that this is well-defined since ag_i is noncentral if g_i is noncentral and since ag_i commutes with ag_j if and only if g_i commutes with g_j.

Furthermore, if a is not the identity element this does not fix any simplex $[g_0, \ldots, g_n]$ since if the set $\{g_0, \ldots, g_n\}$ equals the set $\{ag_0, \ldots, ag_n\}$ then ag_0 is one of the g_j's. However, ag_0 commutes with g_0 so it would have to be g_0, but $ag_0 = g_0$ gives $a = 1$, a contradiction.

Thus this action of nonidentity central a does not fix any simplex and so we get a free action of $Z(G)$ on BNC(G).

Now for the action of A. First note that A is well-defined since if $[g_0, \ldots, g_n]$ is a set of mutually noncommuting elements of G so is $[g_0^{-1}, \ldots, g_n^{-1}]$. Clearly $A \circ A = \mathrm{Id}$. Furthermore, if the two sets above are equal, then g_0^{-1} would have to be one of the g_j. But since g_0^{-1} commutes with g_0 it would have to be g_0, i.e., g_0 would have to have order 2. Similarly, all the g_i would have to have order 2. Thus in a group of odd order, A would not fix any simplex of BNC(G) or $NC(G)$, and hence would not have any fixed points.

The final comment is to recall that if BNC(G) were $\mathbb{F}$-acyclic, its Euler characteristic would be 1 and hence the center of G would be trivial and G would have to have even order by the facts we have shown above. ∎

Proposition 3.3 shows that BNC(G) is not contractible if G is of odd order or if G has a nontrivial center; for example, if G were nilpotent. There is a corresponding p-local version which we state next:

PROPOSITION 3.4. *If $2 \| Z(G)|$ or if G has odd order then $NC_2(G)$ is not contractible, in fact, the Euler characteristic is even. $NC_p(G)$ is never contractible for any odd prime p, in fact it always has even Euler characteristic.*

Proof. The proof follows from the proof of Proposition 3.3, once we note that left multiplication by a central element of order 2 takes the subcomplex $\mathrm{NC}_2(G)$ of NC(G) to itself and that the map A maps $\mathrm{NC}_p(G)$ into itself, as the inverse of an element has the same order as the element. For odd primes p, A is fixed point free on $\mathrm{NC}_p(G)$ always as no elements of order 2 are involved in $\mathrm{NC}_p(G)$. ∎

The fact that BNC(G) can be contractible sometimes is seen in the next proposition.

PROPOSITION 3.5. *If G is a nonabelian group with a self-centralizing involution, i.e., an element x of order 2 such that $C(x) = \{1, x\}$, then BNC(G) is contractible. In fact, BNC(G)=NC(G) in this case.*

Thus, for example, $\mathrm{NC}(\Sigma_3) = \mathrm{BNC}(\Sigma_3)$ *is contractible where Σ_3 is the symmetric group on three letters.*

Proof. Since x does not commute with any nontrivial element, the center of G is trivial and $\mathrm{NC}(G) = \mathrm{BNC}(G)$. Furthermore, it is clear that $\mathrm{BNC}(G)$ is a cone with x as its vertex and hence is contractible. ∎

To help show that the $\mathrm{NC}(G)$ complex of a group is not contractible, we note the following observation which uses Smith theory. (See [B].)

PROPOSITION 3.6. *If G is a group and $NC(G)$ is $\mathbb{F}_2$-acyclic where $\mathbb{F}_2$ is the field with two elements, then $NC_2(G)$ is also $\mathbb{F}_2$-acyclic. Furthermore, one always has $\chi(\mathrm{NC}(G)) = \chi(\mathrm{NC}_2(G))$ mod 2.*

Proof. We first recall that the map A from Proposition 3.3 has order 2 as a map of $\mathrm{NC}(G)$. However, it might have fixed points; in fact, from the proof of that proposition, we see that A fixes a simplex $[g_0, \ldots, g_n]$ of $\mathrm{NC}(G)$ if and only if each element g_i has order 2 and it fixes the simplex pointwise. Thus the fixed point set of A on $\mathrm{NC}(G)$ is nothing other than $\mathrm{NC}_2(G)$, the 2-local noncommuting complex for G. Since A has order 2, we can apply Smith Theory to finish the proof of the first statement of the proposition. The identity on the Euler characteristics follows once we note that under the action of A the cells of $\mathrm{NC}(G)$ break up into free orbits and cells which are fixed by A, and the fixed cells exactly form $\mathrm{NC}_2(G)$. ∎

We observed earlier that if $\mathrm{BNC}(G)$ is contractible then the center of G is trivial, i.e., $\mathrm{BNC}(G) = \mathrm{NC}(G)$, hence, by Proposition 3.6, $\mathrm{NC}_2(G)$ is $\mathbb{F}_2$-acyclic (and in particular nonempty).

DEFINITION 3.7. Let G be a finite group. We define $\mathrm{nc}(G)$ to be the maximum size of a set of pairwise noncommuting elements in G. Thus $\mathrm{nc}(G) - 1$ is the dimension of $\mathrm{NC}(G)$.

We now compute a general class of examples, the Frobenius groups. Recall that a group G is a Frobenius group if it has a proper nontrivial subgroup H with the property that $H \cap H^g = 1$ if $g \in G - H$. H is called the Frobenius complement of G. Frobenius showed the existence of a normal subgroup K such that $K = G - \cup_{g\in G}(H^g - \{1\})$. Thus G is a split extension of K by H, i.e., $G = K \times_\phi H$, for some homomorphism $\phi : H \to \mathrm{Aut}(K)$. This K is called the Frobenius kernel of G.

We have:

PROPOSITION 3.8. *If G is a Frobenius group with Frobenius kernel K and Frobenius complement H, then*

$$\mathrm{NC}(G) = \mathrm{NC}(K) * \mathrm{NC}(H)^{|K|},$$

where $*$ stands for a simplicial join and the superscript $|K|$ means that $NC(K)$ is joined repeatedly with $|K|$ many copies of $NC(H)$.

It also follows that $nc(G) = \mathrm{nc}(K) + |K|\mathrm{nc}(H)$.

Finally, if both H and K are abelian, then $NC(G)$ is homotopy equivalent to a wedge of $(|K|-2)(|H|-2)^{|K|}$*-many* $|K|$*-spheres.*

Proof. First, from the condition that $H \cap H^g = 1$ for $g \in G - H$ we see that no nonidentity element of H commutes with anything outside of H. Thus, conjugating the picture, no nonidentity element of any H^g commutes with anything outside H^g. Thus if $H^{g_1}, H^{g_2}, \ldots, H^{g_m}$ is a complete list of the conjugates of H in G, we see easily that $G - \{1\}$ is partitioned into the sets $K - \{1\}, H^{g_1} - \{1\}, \ldots, H^{g_m} - 1$, and two elements picked from different sets in this partition will not commute. Thus it follows that the noncommuting complex based on the elements of $G - \{1\}$ decomposes as a join of the noncommuting complexes based on each set in the partition. To complete the picture one notes that each H^g is isomorphic to H and so contributes the same noncommuting complex as H, and furthermore, since the conjugates of H make up $G - K$, a simple count gives that $m = (|G| - |K| / |H| - 1) = (|H||K| - |K| / |H| - 1) = |K|$.

The sentence about $\mathrm{nc}(G)$ follows from the fact that if we define $d(S) = \dim(S) + 1$ for a simplicial complex S, then $d(S_1 * S_2) = d(S_1) + d(S_2)$. Thus since $d(\mathrm{NC}(G)) = \mathrm{nc}(G)$. This proves the stated formula concerning $\mathrm{nc}(G)$.

Finally, when H and K are abelian, $\mathrm{NC}(H)$ and $\mathrm{NC}(K)$ are just sets of points, namely, the nonidentity elements in each group. The short exact sequence of the join together with the fact that $\mathrm{NC}(G) = \mathrm{BNC}(G)$ is simply connected finishes the proof. ∎

EXAMPLE 3.9. A_4 is a Frobenius group with kernel $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and complement $\mathbb{Z}/3\mathbb{Z}$. Thus Proposition 3.8 shows $\mathrm{NC}(A_4) \simeq S^4 \vee S^4$ and $\mathrm{nc}(A_4) = 5$.

Claim 3.10. $\mathrm{NC}_2(A_5)$ is a 4-spherical complex homotopy equivalent to a bouquet of 32 4-spheres. Thus it is 3-connected and has odd Euler characteristic. Hence $\mathrm{NC}(A_5)$ has odd Euler characteristic.

Proof. The order of A_5 is $60 = 4 \cdot 3 \cdot 5$, of course. It is easy to check that there are five Sylow 2-subgroups P which are elementary abelian of rank 2 and self-centralizing, i.e., $C_G(P) = P$, and are "disjoint," i.e., any two Sylow subgroups intersect only at the identity element.

Thus the picture for the vertices of $\mathrm{NC}_2(A_5)$ is as five sets $\{S_i\}_{i=1}^5$ of size 3. (Since each Sylow 2-group gives 3 involutions.) Now since the Sylow 2-groups are self-centralizing, this means that two involutions in two different Sylow 2-subgroups do not commute and thus are joined by an edge in $\mathrm{NC}_2(A_5)$. Thus we see easily that $\mathrm{NC}_2(A_5)$ is the join $S_1 * S_2 * S_3 * S_4 * S_5$.

Using the short exact sequence for the join (see p. 373, Exercise 3, in [M]), one calculates easily that $\mathrm{NC}_2(A_5)$ has the homology of a bouquet of 32 4-spheres. Since the join of the two path-connected spaces $S_1 * S_2$

and $S_3 * S_4 * S_5$ is simply connected, it follows that $\mathrm{NC}_2(A_5)$ is homotopy equivalent to a bouquet of 32 4-spheres, and since it is obviously four-dimensional, this completes all but the last sentence of the claim. The final sentence follows from Proposition 3.6 which says that $\mathrm{NC}(A_5)$ has the same Euler characteristic as $\mathrm{NC}_2(A_5)$ mod 2. ∎

At this stage, we would like to make a conjecture:

CONJECTURE 3.11. *If G is a nonabelian simple group, then* $\mathrm{NC}(G) = \mathrm{BNC}(G)$ *has odd Euler characteristic.*

Recall the following famous theorem of Feit and Thompson:

THEOREM 3.12 (Odd Order Theorem). *Every group of odd order is solvable.*

Note, that if the conjecture is true it would imply the odd order theorem. This is because it is easy to see that a minimal counterexample G to the odd-order theorem would have to be an odd-order nonabelian simple group. The conjecture would then say $\mathrm{BNC}(G)$ has an odd Euler characteristic and Proposition 3.3 would say that G was even order, which is a contradiction to the original assumption that G has odd order.

4. GENERAL COMMUTING STRUCTURES

Before we further analyze the $\mathrm{NC}(G)$ complexes introduced in the last section, we need to extend our considerations to general commuting structures.

DEFINITION 4.1. *A commuting structure is a set S together with a reflexive, symmetric relation $\sim$ on S. If $x, y \in S$ with $x \sim y$ we say that x and y commute.*

DEFINITION 4.2. Given a commuting structure $(S, \sim)$, the dual commuting structure $(S, \sim')$ is defined by

(a) $\sim'$ is reflexive and

(b) for $x \neq y$, $x \sim' y$ if and only if x does not commute with y in $(S, \sim)$.

When it is understood we write a commuting structure as S and its dual as S'. It is easy to see that $S'' = S$ in general.

DEFINITION 4.3. Given a commuting structure S, $\mathrm{C}(S)$ is the simplicial complex whose vertices are the elements of S and such that $[s_0, \dots, s_n]$ is a face of $\mathrm{C}(S)$ if and only if $\{s_0, \dots, s_n\}$ is a commuting set, i.e., $s_i \sim s_j$ for all i and j. Similarly, we define $\mathrm{NC}(S) = \mathrm{C}(S')$ and refer to the elements in a face of $\mathrm{NC}(S)$ as a noncommuting set in S.

Below are some examples of commuting structures which will be important in our considerations:

(a) The nontrivial elements of a group G form a commuting structure which we denote also by G, where $x \sim y$ if and only if x and y commute in the group G. In this case $\mathrm{C}(G)$ and $\mathrm{NC}(G)$ are the complexes considered in the previous sections.

(b) The noncentral elements of a group G form a commuting structure $G - Z(G)$ and $\mathrm{NC}(G - Z(G)) = \mathrm{BNC}(G)$. Similarly, the elements of order p in G form a commuting structure G_p and $\mathrm{C}(G_p) = \mathrm{C}_p(G)$, $\mathrm{NC}(G_p) = \mathrm{NC}_p(G)$.

(c) If V is a vector space equipped with a bilinear map $[\cdot, \cdot]\colon V \otimes V \to V$, then the nonzero elements of V form a commuting structure also denoted by V, where $v_1 \sim v_2$ if $[v_1, v_2] = 0$ or if $v_1 = v_2$.

(d) In the situation in (c), we can also look at the set $P(V)$ of lines in V. The commuting structure on V descends to give a well-defined commuting structure on $P(V)$, which we will call the projective commuting structure.

(e) If $1 \to C \to G \to Q \to 1$ is a central extension of groups with Q abelian then one can form $[\cdot, \cdot] : Q \to C$ by

$$[x, y] = \hat{x}\hat{y}\hat{x}^{-1}\hat{y}^{-1},$$

where $x, y \in Q$ and $\hat{x}$ is a lift of x in G, etc. It is easy to see this is well-defined, independent of the choice of lift, and furthermore that the bracket $[\cdot, \cdot]$ is bilinear.

Thus, by (c), we get a commuting structure on the nontrivial elements of Q via this bracket. We denote this commuting structure by $(G; C)$. Note in general it is not the same as the commuting structure of the group G/C, which is abelian in this case. More generally, even if Q is not abelian, one can define a commuting structure on the nontrivial elements of Q from the extension by declaring $x \sim y$ if and only if $\hat{x}$ and $\hat{y}$ commute in G. We will denote this commuting structure by $(G; C)$, in general.

EXAMPLE 4.4. If P is an extraspecial p-group of order p^3 then it has center Z of order p and P/Z is an elementary abelian p-group of rank 2. Let Symp denote the commuting structure obtained from a vector space of dimension two over $\mathbb{F}_p$ equipped with the symplectic alternating inner product $[x, y] = 1$ where $\{x, y\}$ is a suitable basis. Then it is easy to check that $(P; Z) = \mathrm{Symp}$.

One of the main results we will use in order to study the noncommuting complexes associated to commuting structures is a result of Björner et al. [BWW] on "blowup" complexes, which we describe next.

Let S be a finite simplicial complex with vertex set $[n]$. (This means the vertices have been labelled 1, ..., n.) To each vertex $1 \leq i \leq n$ we assign a positive integer m_i. Let $\bar{m} = (m_1, \ldots, m_n)$, then the "blowup" complex $S_{\bar{m}}$ is defined as follows.

The vertices of $S_{\bar{m}}$ are of the form (i, j) where $1 \leq i \leq n, 1 \leq j \leq m_i$. (One should picture m_i vertices in $S_{\bar{m}}$ over vertex i in S.)

The faces of $S_{\bar{m}}$ are exactly of the form $[(i_0, j_0), \ldots, (i_n, j_n)]$ where $[i_0, \ldots, i_n]$ is a face in S. (In particular $i_l \neq i_k$ for $l \neq k$.)

The result of Björner *et al.* describes $S_{\bar{m}}$ up to homotopy equivalence, in terms of S and its links. More precisely, we have

THEOREM 4.5 (Bjoӧrner *et al.* [BWW]). *For any connected simplicial complex S with vertex set $[n]$ and given n-tuple of positive integers $\bar{m} = (m_1, \ldots, m_n)$, we have*

$$S_{\bar{m}} \simeq S \vee \bigvee_{F \in S} [\mathrm{Susp}^{|F|}(\mathrm{Lk}(F))]^{\gamma(F)}.$$

Here the $\vee$ stands for the wedge of spaces, $Susp^k$ for k-fold suspension, $Lk(F)$ for the link of the face F in S, $|F|$ for the number of vertices in the face F (which is one more than the dimension of F), and $\gamma(F) = \prod_{i \in F}(m_i - 1)$. Thus in the decomposition above, $\gamma(F)$ copies of $\mathrm{Susp}^{|F|}(\mathrm{Lk}(F))$ *appear wedged together.*

Note that in [BWW] the empty face is considered a face in any complex. In the above formulation we are not considering the empty face as a face and thus have separated out the S term in the wedge decomposition.

For example, if we take a 1-simplex $[1, 2]$ as our complex S and use $\bar{m} = (2, 2)$, then it is easy to see that $S_{\bar{m}}$ is a circle. On the other hand, all links in S are contractible except the link of the maximal face $[1, 2]$, which is empty (a "(-1)-sphere"). Thus everything in the right-hand side of the formula is contractible except the two-fold suspension of this (-1)-sphere which gives a 1-sphere or a circle, as expected.

Also note that whenever some $m_i = 1$, the corresponding $\gamma(F) = 0$ and so that term drops out of the wedge decomposition. Thus if $m_1 = \cdots = m_n = 1$, the decomposition gives us nothing as $S_{\bar{m}} = S$.

Now for some examples of where this theorem applies:

COROLLARY 4.6. *If G is a finite group and $Z(G)$ is its center, then $BNC(G)$ is the blowup of $NC(G; Z(G))$ where each $m_i = |Z(G)|$. This is because everything in the same coset of the $Z(G)$ in G commutes with each other and whether or not two elements from different cosets commute is decided in $(G; Z(G))$. Thus we conclude*

$$BNC(G) \simeq NC(G; Z(G)) \vee \bigvee_{F \in NC(G; Z(G))} [Susp^{|F|}(Lk(F))]^{(|Z(G)|-1)^{|F|}}.$$

Before we say more, let us look at another example of Theorem 4.5 in our context. The proof is the same as that of Example 4.6 and is left to the reader.

COROLLARY 4.7. *Let $(V, [\cdot,\cdot])$ be a vector space over a finite field* $\mathbf{k}$ *equipped with a nondegenerate alternating bilinear form. Then $NC(V)$ is a blowup of $NC(P(V))$ and we have*

$$NC(V) \simeq NC(P(V)) \vee \bigvee_{F \in NC(P(V))} [Susp^{|F|}(Lk(F))]^{(|\mathbf{k}|-2)^{|F|}}.$$

For example, it is easy to see that in Symp, if $[x, y] = 0$ then x is a scalar multiple of y. Thus in P(Symp) two distinct elements do not commute and hence NC(P(Symp)) is a simplex. Thus all of its links are contractible except the link of the top face which is empty. This top face has $(p^2 - 1)/(p - 1) = p + 1$ vertices. Thus following Example 4.7, we see that

$$\text{NC(Symp)} \simeq \vee_{(p-2)^{(p+1)}} S^p.$$

REMARK 4.8. If $(V, [\cdot,\cdot])$ is a vector space over a finite field equipped with a nondegenerate symmetric inner product corresponding to a quadratic form Q, then one can restrict the commuting structure induced on $P(V)$ to the subset S consisting of singular points of Q, i.e., lines $< x >$ with $Q(x) = 0$. (S will be nonempty only if Q is of hyperbolic type.) Among the noncommuting sets in S are the subsets called ovoids defined by the property that every maximal singular subspace of V contains exactly one element of the ovoid. (See [G]) Thus these ovoids are a special subcollection of the facets of NC(S).

In general, we will find the following notion useful:

DEFINITION 4.9. If $(S, \sim)$ is a commuting set and $x \in S$, we define the centralizer of x to be

$$C(x) = \{y \in S | y \sim x\}.$$

This allows us to define

DEFINITION 4.10. If $(S, \sim)$ is a commuting set, we define an equivalence relation on the elements of S by $x \approx y$ if $C(x) = C(y)$. The equivalence classes are called the centralizer classes of $(S, \sim)$. We then define the core of S to be the commuting set $(\bar{S}, \sim)$ where the elements are the centralizer classes of S and the classes $[x]$ and $[y]$ commute in $\bar{S}$ if and only if the representatives x, y commute in S.

It is easy to see that NC(S) is the blowup of NC($\overline{S}$), where for the vertex $[x] \in$ NC($\bar{S}$) there are m_x vertices above it in NC(S), where m_x is the size of the centralizer class $[x]$. Thus once again Theorem 4.5 gives us:

THEOREM 4.11. *Let S be a commuting set and $\bar{S}$ be its core, and suppose that $NC(\bar{S})$ is connected. Then*

$$\mathrm{NC}(S) \simeq \mathrm{NC}(\bar{S}) \vee \bigvee_{F \in \mathrm{NC}(\bar{S})} [Susp^{|F|}(\mathrm{Lk}F)]^{\gamma(F)},$$

where $\gamma(F) = \prod_{[x]\subseteq F}(m_x - 1)$. Here again m_x is the size of the centralizer class $[x]$.

REMARK 4.12. *Note that if $NC(S)$ is connected, $NC(\bar{S})$ will automatically be connected as it is the image of $NC(S)$ under a continuous map.*

With some abuse of notation, we will call $\mathrm{NC}(\bar{S})$ the core of $\mathrm{NC}(S)$.

Following the notation above, in a finite group G the equivalence relation "has the same centralizer group" partitions G into centralizer classes. The central elements form one class and the noncentral elements thus inherit a partition.

Note that if $[x]$ is the centralizer class containing x, then any other generator of the cyclic group $< x >$ is in the same class. Thus $[x]$ has at least $\phi(n)$ elements where n is the order of x and ϕ is Euler's totient function. Thus if $n > 2$, then $[x]$ contains at least two elements. Also note that everything in the coset $xZ(G)$ is also in $[x]$ so if $Z(G) \neq 1$ we can also conclude that $[x]$ contains at least two elements.

DEFINITION 4.13. A noncommuting set S in G is a nonempty subset S such that the elements of S pairwise do not commute. A maximal noncommuting set S is a noncommuting set which is not properly contained in any other noncommuting set of G.

In general, not all maximal noncommuting sets of a group G have the same size.

One obtains the following immediate corollary of Theorem 4.11. (Assume $|G| > 2$ for the following results.)

COROLLARY 4.14. *Let G be a finite nonabelian group, and let S_s denote the set of maximal noncommuting sets in G of size s. Then, for $s > 1$,*

$$\mathrm{rk}(H_{s-1}(\mathrm{BNC}(G))) \geq \sum_{F \in S_s} \left[\prod_{x \in F} \left(1 - \frac{1}{m_x}\right)\right],$$

where m_x is the size of the centralizer class containing x.

In particular, if G is an odd order group or if G has a nontrivial center, then

$$\widetilde{H}_{s-1}(\mathrm{NC}(G)) \neq 0$$

whenever G has a maximal noncommuting set of size s.

Proof. Let X be the non-commuting complex associated to the core of BNC(G). Let us define a facet to be a face of a simplicial complex which is not contained in any bigger faces. Thus the facets of BNC(G) consist exactly of the maximal noncommuting sets of noncentral elements in G.

The first thing to note is that to every facet F of BNC(G) there corresponds a facet $\overline{F}$ of X, and furthermore this correspondence preserves the dimension of the facet (or equivalently, the number of vertices in the facet).

Since the link of a facet is always empty (a (-1)-sphere), in the wedge decomposition of Theorem 4.11 we get the suspension of this empty link as a contribution. If the facet F has n vertices in it, then we suspend n times to get a $(n-1)$-sphere. Thus to every maximal noncommuting set of size s we get a $s-1$ sphere contribution from the corresponding facet in X. In fact, we get $\gamma(F)$-many such spheres from the facet F. However, above each facet $\overline{F}$ in X, there correspond $\prod_{[x]\in\overline{F}}(m_x)$ many facets in BNC(G). Thus in the sum over the facets of BNC(G) stated in the theorem we divide $\gamma(F)$ by $\prod_{x\in F}(m_x)$ in order to count the contribution from the facet $\overline{F}$ in X the correct number of times.

Note that $\gamma(F)/\prod_{x\in F}(m_x) = \prod_{x\in F}(1-1/m_x) \geq 1/2^{|F|}$ if all the centralizer classes have size bigger than one. So if G has the property that the size of the centralizer classes of noncentral elements is always strictly bigger than 1, for example, if G is odd or if G has a nontrivial center, then

$$2^s \mathrm{rk}(H_{s-1}(\mathrm{BNC}(G))) \geq |S_s|$$

for all $s \in \mathbb{N}$. In particular, $H_{s-1}(\mathrm{BNC}(G)) \neq 0$ whenever G has a maximal noncommuting set of size s. Also observe that $\widetilde{H}_0(\mathrm{NC}(G)) \neq 0$ whenever G has a maximal noncommuting set of size 1, i.e., a singleton consisting of a nontrivial central element (except for the trivial case when G has order 2.) ∎

It is easy to see from this proof that a p-local version of Corollary 4.14 is also true. We state here only the last part of this result for odd primes.

COROLLARY 4.15. *Let p be an odd prime. Then,*

$$\widetilde{H}_{s-1}(\mathrm{NC}_p(G)) \neq 0$$

whenever G has a maximal noncommuting p-set of size s.

The same is true for $p = 2$ under the additional condition $2 \mid |Z(G)|$.

REMARK 4.16. Note that the conclusion of Corollary 4.14 is consistent with the simple connectedness of BNC(G), because there is no maximal noncommuting set of size 2. To see this, observe that whenever there is a non-commuting set $\{a, b\}$ with two elements, we can form a bigger non-commuting set $\{a, b, ab\}$.

Also note that this is no longer the case for the p-local case. For example, when $G = D_8 = \langle a, b | a^2 = b^2 = c^2 = 1, [a, b] = c,\ c\ central \rangle$, the complex $BNC_2(G)$ is a rectangle with vertices a, b, ac, bc which is the inflated complex corresponding to the maximal 2-set $\{a, b\}$. In particular, $\mathrm{BNC}_2(G)$ is not simply connected in general.

REMARK 4.17. One of the things that Corollary 4.14 says is that if one wants to calculate $\mathrm{nc}(G)$, the answer which is obviously the dimension of $\mathrm{BNC}(G)$ plus one can also be determined by finding the highest nonvanishing homology of $\mathrm{BNC}(G)$ in the case when $Z(G) \neq 1$ or G is of odd order. Thus the answer is determined already by the homotopy type of $\mathrm{BNC}(G)$ in this situation. If $Z(G) = 1$ and G has even order, this is no longer true, for example, $\mathrm{NC}(\Sigma_3)$ is contractible and so does not have any positive dimensional homology.

Sometimes one can show that the noncommuting complex for the core of $(S, \sim)$ is contractible, as the next lemma shows.

LEMMA 4.18. *If $(S, \sim)$ has a centralizer class $[x]$ where $[x] = C(x)$ then if the core is $\bar{S}$, $NC(\overline{S})$ is contractible.*

Proof. This is because it is easily seen that $\mathrm{NC}(\overline{S})$ is a cone on the vertex $[x]$ as everything outside $[x]$ does not commute with x as $[x] = C(x)$. ∎

Thus, for example, we have

EXAMPLE 4.19. Let p be a prime, then the core of $\mathrm{BNC}(\Sigma_p)$ is contractible.

Proof. The cycle $x = (1, 2, \ldots, p)$ in Σ_p has $C(x) =< x >$ by a simple calculation. Thus $C(x) = [x]$ and so the result follows from Lemma 4.18. ∎

We now study an important special case.

DEFINITION 4.20. A TC-group G is a group where the commuting relation on the noncentral elements is transitive. This is equivalent to the condition that all proper centralizer subgroups $C(x) \subset G$ are abelian. Examples of this are any minimal nonabelian group like S_3, A_4, or an extraspecial group of order p^3.

COROLLARY 4.21. *If $(S, \sim)$ is a commuting set where $\sim$ is also transitive (i.e., $\sim$ is an equivalence relation), then $NC(S)$ is homotopy equivalent to a wedge of spheres of the same dimension. (We allow the "empty" wedge, i.e., we allow the case $NC(S)$ to be homotopy equivalent to a point.) The dimension of the spheres is equal to $n - 1$ where n is the number of equivalence classes in $(S, \sim)$ and the number of spheres appearing is $\prod_{i=1}^{n}(m_i - 1)$ where m_i is the size of the equivalence class i.*

Thus if G is a TC-group, then $BNC(G)$ is homotopy equivalent to a wedge of spheres of dimension $nc(G)-1$ and the number of spheres is given by a product as above where the m_i are the orders of the distinct proper centralizer groups of G.

Proof. If $\sim$ is an equivalence relation, it is easy to see that the centralizer classes are exactly the $\sim$ equivalence classes. Thus one sees that the noncommuting complex for the core, $\mathrm{NC}(\overline{S})$, is a simplex. Thus in Theorem 4.11 all terms drop out except those corresponding to the maximum face in $\mathrm{NC}(\overline{S})$ where the link is empty. This link is suspended to give a sphere of dimension equal to the number of equivalence classes minus one. The number of these spheres appearing in the wedge decomposition is the product $\prod_{i=1}^{n}(m_i-1)$ where m_i is the size of the equivalence class i and the product is over all equivalence classes. (Thus this can be zero if one of the equivalence classes has size one, in which case the complex is contractible.)

In the case of $\mathrm{BNC}(G)$, for G a TC-group, one just has to note that $C(x)$ is the centralizer class $[x]$ for any noncentral element x. ∎

REMARK 4.22. In Corollary 4.21, one did not actually have to use the general result of Bjorner *et al.* [BWW] since it is easy to see that in this situation $\mathrm{NC}(S)$ is the join of each equivalence class as discrete sets, and a simple count gives the result.

Now note that in the case that $(S,\sim)$ has $\sim$ transitive, the above analysis shows that $\mathrm{NC}(S)$ is a join of discrete sets. Thus it is easy to see that $\mathrm{NC}(S)$ is shellable. (This is because the facets of $\mathrm{NC}(S)$ are just sets where we have chosen exactly one element from each of the $\sim$ equivalence classes. We can linearly order the equivalence classes and then lexicographically order the facets. It is easy to check that this is indeed a shelling.)

Given a shelling of a simplicial complex, there are many combinatorial equalities and inequalities which follow (see [S]). Since these are not as deep in the above general context, we will point out only the interpretation when applied to $\mathrm{BNC}(G)$. Recall that pure shellable just means shellable where all the facets have the same dimension.

PROPOSITION 4.23. *If G is a nonabelian group such that $BNC(G)$ is pure shellable, e.g., G a TC-group like Σ_3 or A_4, then if one sets nc_i to be the number of noncommutative sets of noncentral elements which have size i, one has*

$$(-1)^j C(\mathrm{nc}(G),j)+\sum_{k=1}^{j}(-1)^{j-k}C(nc(G)-k,j-k)\mathrm{nc}_k \geq 0$$

for all $1 \leq j \leq \mathrm{nc}(G)$ where $C(n,k)$ is the usual binomial coefficient.

Proof. The proof follows from a direct interpretation of the inequalities in [S, p. 4, Theorem 2.9]. One warning about the notation in that paper is that $|\sigma|$ means the number of vertices in σ and the empty face is considered a simplex in any complex. ∎

Using that $\mathrm{nc}_1 = |G| - |Z(G)|$, $\mathrm{nc}_2 = |G|/2(|G| - m)$, where m is the number of conjugacy classes in G, one gets, for example, from the inequality with $j = 2$ above,

COROLLARY 4.24. *Let G be a nonabelian group with a transitive commuting relation, i.e., if $[g, h] = [h, k] = 1$, then $[g, h] = 1$ for every noncentral $g, h, k \in G$. Then,*

$$\mathrm{nc}(G)(\mathrm{nc}(G) - 1) + |G|(|G| - m) - 2(\mathrm{nc}(G) - 1)(|G| - |Z(G)|) \geq 0,$$

where m denotes the number of conjugacy classes in G.

5. DUALITY

Let $(S, \sim)$ be a finite set with a commuting relation. Suppose the commuting complex $C(S)$ breaks up as a disjoint union of path components $C(S_1), \ldots, C(S_n)$ where of course we are using S_i to stand for the vertex set of component i.

Then note that in the corresponding noncommuting complex, $\mathrm{NC}(S)$, we have $\mathrm{NC}(S) = \mathrm{NC}(S_1) * \cdots * \mathrm{NC}(S_n)$ where $*$ stands for the join operation as usual.

We state this simple but useful observation as the next lemma.

LEMMA 5.1 (Duality). *Let $(S, \sim)$ be a commuting set, then if*

$$C(S) = \bigsqcup_{i=1}^{n} C(S_i),$$

where $\bigsqcup$ stands for disjoint union, then we have

$$\mathrm{NC}(S) = *_{i=1}^{n} \mathrm{NC}(S_i)$$

where $*$ stands for join.

Thus in some sense "the more disconnected $C(S)$ is, the more connected $\mathrm{NC}(S)$ is."

We can apply this simple observation to say something about the complexes $\mathrm{NC}_p(G)$ in general.

THEOREM 5.2. *Let G be a finite group and p a prime such that $p||G|$. Let P be a Sylow p-group of G and define N to be the subgroup generated by $N_G(H)$ as H runs over all the nontrivial subgroups of P.*

Then $NC_p(G)$ is $(|G:N|-2)$-connected and in fact it is the $|G:N|$-fold join of some complex with itself.

Proof. By Quillen [Q], if $S_1, \ldots, S_n$ are the components of $A_p(G)$, then under the G-action G acts transitively on the components with isotropy group N under a suitable choice of labelling. Thus the components are all simplicially equivalent and there are $|G:N|$ many of them.

However, we have seen that $A_p(G)$ is G-homotopy equivalent to $C_p(G)$ and so we have the same picture for that complex. Thus $C_p(G)$ is the disjoint union of $|G:N|$ copies of some simplicial complex S. Thus by lemma 1, $\mathrm{NC}_p(G)$ is the $|G:N|$-fold join of the dual of S with itself. To finish the proof one just has to note that a k-fold join of nonempty spaces is always $(k-2)$-connected. ∎

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