SOME REMARKS ON GROUPS WITH NILPOTENT MINIMAL COVERS

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Abstract

A *cover* of a group is a finite collection of proper subgroups whose union is the whole group. A cover is *minimal* if no cover of the group has fewer members. It is conjectured that a group with a minimal cover of nilpotent subgroups is soluble. It is shown that a minimal counterexample to this conjecture is almost simple and that none of a range of almost simple groups are counterexamples to the conjecture.
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1. Introduction

A finite collection of proper subgroups of a group is a *cover* if its union is the whole group, *irredundant* if no proper sub-collection is also a cover. A *minimal* cover is irredundant and no collection of subgroups with fewer members is a cover. The earliest results on minimal covers appear in Cohn [6] and Tomkinson [17]. In [3] minimal covers of $\mathrm{GL}_2(q)$ and related groups are described. The articles of Maróti [11] and Holmes-Maróti [9] give deep information about the size of minimal covers of the alternating and symmetric groups, and for a wide class of linear groups. In [8], [9] the sizes of minimal covers for a selection of sporadic simple groups are determined. Every finite group has, of course, an irredundant cover of abelian, even cyclic, subgroups. However the present authors showed in [4] that a group with a *minimal* cover of abelian subgroups is soluble of very restricted structure. In this note we collect some results on groups that admit a minimal cover with all members nilpotent, a *nilpotent minimal cover* in short, and we conjecture that such groups are soluble. As a first step towards a possible proof of this we show, in the next section, that if there is an insoluble group admitting a nilpotent minimal cover there is a finite, almost simple one.

In Section 3 we derive a number of conditions necessary in order that a group admit a nilpotent minimal cover. In Section 4 we limit the range of possible counter-examples to our conjecture by showing that the groups of certain classes of almost simple groups violate these conditions.

For ease of reference we list here two easy lemmas concerning a minimal cover $\mathcal{A} = \{A_1, \ldots, A_n\}$ of a group G: proofs are left to the reader.

LEMMA 1.1. *If $N \trianglelefteq G$, then either $G = A_iN$ for some i or $\{A_iN/N : 1 \leq i \leq n\}$ is a minimal cover for G/N.*

LEMMA 1.2. *For $1 \leq i < j \leq n$, $\langle A_i, A_j \rangle = G$.*

2. Minimal counterexample

Let G be an insoluble group with a nilpotent minimal cover, $\mathcal{A} = \{A_1, \ldots, A_n\}$ say. Write D for the intersection of the cover so that $|G : D|$ is finite by a result of Neumann [13]. Since $C := \mathrm{core}_G(D)$ is nilpotent, G/C is insoluble. Moreover G/C is finite, as it embeds in the symmetric group of degree $|G : D|$ and, by Lemma 1.1, $\{A_iC/C : 1 \leq i \leq n\}$ is a nilpotent minimal cover of G/C. We prove more.

Proposition 2.1. *If there is an insoluble group admitting a nilpotent minimal cover there is a finite, almost simple one.*

We suppose G to be a finite, insoluble group of smallest order admitting a nilpotent minimal cover $\{A_1, A_2, \ldots, A_n\}$. Our proof that G is almost simple begins with two lemmas.

Lemma 2.2. *G is monolithic with non-abelian monolith.*

Proof. If W is an arbitrary minimal normal subgroup of G then G/W is soluble either because $WA_i = G$ for some i, and then $G/W \cong A_i/A_i \cap W$ is nilpotent; or because there is no such i and then $\{A_iW/W : 1 \leq i \leq n\}$ is a nilpotent minimal cover for G/W by Lemma 1.1 so, by the minimality of G, G/W would be soluble. This shows at once that W is not abelian, or G would be soluble. If X is a minimal normal subgroup other than W, then $W \cap X = 1$ and G embeds in $G/W \times G/X$ which would be soluble, a contradiction. We have therefore proved what was claimed. □

Now let U be the monolith of G so that $U = S_1 \times S_2 \times \cdots \times S_m$ where $S_1 \cong S_i$ $(1 \leq i \leq m)$ and S_1 is non-abelian and simple. Moreover $\{S_1, S_2, \ldots, S_m\}$ is a conjugacy class of G. If $m = 1$ then G is almost simple, so suppose that $m > 1$. We will denote by N the normaliser in G of S_1. Notice that $U \leq N \neq G$.

Lemma 2.3. *If $A_i \not\leq N$ and $A_i \cap S_1 \neq 1$ then $|A_i : A_i \cap N|$ and $|A_i \cap S_1|$ are powers of the same prime, p say; and p'-elements of A_i are in N.*

Proof. Since A_i is nilpotent there is a composition chain from $A_i \cap N$ to A_i: let V_1/V_2 be one of its factors whose order, a prime, is p, say. A p-element in $V_1 \setminus V_2$ centralises each p'-element of $S_1 \cap A_i$ since A_i is nilpotent but, on the other hand, conjugates it into an S_j whose intersection with S_1 is trivial. That is $A_i \cap S_1$ is a p-group. The same argument shows that all factors in the chain above $A_i \cap N$ are of this same order p. Consequently $|A_i : A_i \cap N|$ is a power of p. Since $A_i \cap N$ is subnormal in A_i all p'-elements of A_i are in N. □

We resume the proof of Proposition 2.1. At most one of the A_is is in N by Lemma 1.2; and not all of the A_is not in N intersect S_1 trivially as S_1 is not contained in an A_j. We re-number the A_is, if necessary, so that $A_i \not\leq N$ $(1 \leq i \leq n-1)$ and $A_i \cap S_1 \neq 1$ $(1 \leq i \leq r)$ where $r \leq n-1$.

Let us suppose, for now, that G/U is nilpotent. We write $\mathcal{P}$ for the set of those primes that divide the indices $|A_i : A_i \cap N|$ $(1 \leq i \leq r)$; and for $p \in \mathcal{P}$ let T_p/U be the Sylow p-subgroup of G/U. If $p \mid |A_i : A_i \cap N|$ then $A_i \leq T_pN$ by Lemma 2.3; note that $T_p \trianglelefteq G$ so that T_pN is a subgroup. By Lemma 1.2 no two A_is are in the same T_pN unless $T_pN = G$. That is either $|\mathcal{P}| = r$ or $|\mathcal{P}| = 1$.

Now

$$S_1 = (S_1 \cap A_1) \cup (S_1 \cap A_2) \cup \cdots \cup (S_1 \cap A_r) \cup (S_1 \cap A_n)$$

where the first r terms in the union are subgroups of prime-power order either all for the same prime or for r different primes. In the first case choose another prime, q say, dividing $|S_1|$, and then all Sylow q-subgroups of S_1 are in $S_1 \cap A_n$ yielding $S_1 \leq A_n$, a contradiction to the nilpotence of A_n. In the second case for each $i \in \{1, 2, \ldots, r\}$ there is a Sylow subgroup X_i of S_1, not containing $S_1 \cap A_i$, but involving the same prime. Therefore $X_i \leq S_1 \cap A_n$; and of course Sylow subgroups of S_1 for all primes not in $\mathcal{P}$ are all in $S_1 \cap A_n$ so $S_1 \leq A_n$, again a contradiction.

Hence G/U is not nilpotent and so $UA_i \neq G$ $(1 \leq i \leq n)$. The subgroups UA_i/U together form a nilpotent minimal cover of G/U so, by the minimality of G, G/U is soluble. Theorem 11 of [4] gives the following information about such a group. Let Z/U be the hypercentre of G/U. Then G/Z is monolithic: let K/Z be its monolith, an elementary abelian t-group where

t is prime. G/K is cyclic of order co-prime to t. (The group G/Z is Frobenius.) Also $Z = UA_i \cap UA_j$ $(1 \leq i < j \leq n)$; and one of the members of this cover is K/U; we will suppose it to be UA_n/U. The others are, modulo Z/U, the complements for K/U in G/U. Note too that $U \leq Z \leq UA_i$ gives

$$Z = U(Z \cap A_i)\ (1 \leq i \leq n) \text{ and } Z = (Z \cap N)(Z \cap A_i)\ 1 \leq i \leq n. \qquad (1)$$

Case 1: $Z \not\leq N$.

Then $Z \cap A_i \not\leq N$ $(1 \leq i \leq n)$. By Lemma 2.3 $|Z \cap A_i : A_i \cap Z \cap N|$ is a prime-power for some prime p and $S_1 \cap A_i$ is a p-subgroup whenever $S_1 \cap A_i \neq 1$. But (1) shows that

$$|Z : Z \cap N| = |Z \cap A_i : A_i \cap Z \cap N|$$

so p is the same for all i for which $S_1 \cap A_i \neq 1$, meaning that S_1 is a union of p-subgroups, a contradiction.

Case 2: $Z \leq N$.

First we prove a useful lemma.

Lemma 2.4. *Let $H = VL$ be a Frobenius group where the kernel V is elementary abelian and L is a complement for V. Suppose L_1, L_2 are proper subgroups of L whose indices in L are co-prime and where $L_2 \trianglelefteq H$. Let $1 \neq v \in V$. Then $\langle L_1, L_2^v \rangle = H$.*

Proof. Write $T := \langle L_1, L_2^v \rangle$. Modulo V, $H = T$ so T acts, by conjugation, irreducibly on V. Also $T \cap V \trianglelefteq H$ so, if $T \cap V \neq 1$, $T = H$, as required. Suppose that $T \cap V = 1$. Then $T \cap L \geq L_1 \neq 1$ so $T = L$ as H is Frobenius; and then $[L_2, v] \leq \langle L_2, L_2^v \rangle \leq L \cap V = 1$. However L_2 contains every Sylow subgroup of H for primes dividing $|L : L_1|$ so there is a non-trivial normal subgroup of L with non-trivial centraliser in V contradicting that L acts faithfully and irreducibly on V. □

Now $N \neq G$ as $m > 1$ so no two of $\{A_i : 1 \leq i \leq n-1\}$ are in N; let us say $A_i \not\leq N$ $(1 \leq i \leq n-2)$. We suppose that $S_1 \cap A_i \neq 1$ $(1 \leq i \leq s \leq n-2)$;

note that $s \geq 2$ as S_1 is not coverable by three or fewer of the A_is. By Lemma 2.3 each $|A_i : A_i \cap N|$ $(1 \leq i \leq s)$ is a prime-power. If two of these indices, say for $i = 1, 2$, were co-prime then, by Lemma 2.4 and working modulo Z,

$$N \geq \langle N \cap A_1, N \cap A_2 \rangle = G$$

a contradiction. It follows that

$$S_1 = X_1 \cup X_2 \cup \cdots \cup X_s \cup (S_1 \cap A_{n-1}) \cup (S_1 \cap A_n)$$

where X_j $(1 \leq j \leq s)$ are p-groups for the same prime p. Note that $p \neq t$. If $UA_n = K \not\leq N$, so that $A_n \not\leq N$, it follows from Lemma 2.3 that $S_1 \cap A_n$ is a t-group (possibly trivial) S_1 is insoluble so, by Burnside's Theorem, there is a prime q, different from both p, t, dividing $|S_1|$. All Sylow q-subgroups of S_1 are therefore in $S_1 \cap A_{n-1}$ yielding $S_1 \leq A_{n-1}$ contradicting the nilpotence of A_{n-1}. This leaves us to consider $UA_n = K \leq N$. But then $A_i \not\leq N$ $(1 \leq i \leq n-1)$ by Lemma 1.2 so, using the argument at the beginning of this paragraph, we find that also $S_1 \cap A_{n-1}$ is a p-group. Consequently all Sylow q-subgroups of S_1 are in $S_1 \cap A_n$ giving $S_1 \leq A_n$, another contradiction.

This completes the proof of Proposition 2.1. □

3. Further reduction

Next we derive a number of necessary conditions on finite groups admitting a nilpotent minimal cover. These will allow us to qualify further the almost simplicity of a minimum counter-example. Throughout $\mathcal{A} := \{A_1, A_2, \ldots, A_n\}$ is a nilpotent minimal cover of a group G.

Lemma 3.1. *The intersection of a nilpotent cover for an almost simple group is* 1.

Proof. Let G be almost simple with socle U. Now $C_G(U) = 1$. If S were a Sylow p-subgroup of the non-trivial intersection of a nilpotent cover for G it would centralise every p'-element of U and therefore centralise U, a contradiction. □

The following lemma is well known; it was proved in [12], but we give a proof for the convenience of the reader.

Lemma 3.2. *A finite abelian group with a partition is elementary.*

Proof. Suppose that G is an abelian group with a partition $\{B_i : 1 \leq i \leq n\}$. Let $b \in B_1$ have prime order p. We prove that all elements of $G \setminus B_1$ have order p. If $a \in G \setminus B_1$, say $a \in B_j$ where $j > 1$, then $ab \notin B_1 \cup B_j$ so $ab \in B_k$ where $1 \neq k \neq j$. Now $a^p = (ab)^p \in B_j \cap B_k = 1$ so all elements outside B_1 have order p as, similarly, do all elements outside B_2. Therefore G has exponent p and is elementary. □

Proposition 3.3. *Let G be a finite group with trivial centre and a nilpotent minimal cover $\mathcal{A}$. Let p be a prime dividing $|G|$ and suppose that $P \in \mathrm{Syl}_p(G)$ is abelian and not normal in G. Then*

(a) either *P is elementary abelian but not of order p;* or

(b) *P is a Sylow p-subgroup of some A_i and*

  (i) *$N_G(P)$ is a maximal subgroup of G;*

  (ii) *$N_G(P)$ is strongly p-embedded in G (i.e. $|N_G(P) \cap N_G(P)^g|_p = 1$ for all $g \in G \setminus N_G(P)$);*

  (iii) *and $C_G(P) = A_i$.*

Proof. Suppose that P is in no A_i; in particular P is not of order p. Using Lemma 1.2

$$A_i \cap A_j \cap P \leq Z(G) = 1.$$

That is, P admits a partition so, by Lemma 3.2, it is elementary.

Next suppose that P is not elementary or is of order p. Then P is a Sylow p-subgroup of A_i for some i. If $g \in G$ and $P^g \neq P$ then P^g is a Sylow p-subgroup of A_j for some $j \neq i$. Suppose that $N_G(P)$ is not a maximal subgroup of G so that $N_G(P) < M < G$ for some proper subgroup M of G. Choose $g \in M \setminus N_G(P)$. Then $M \geq \langle N_G(P), N_G(P)^g \rangle \geq \langle A_i, A_j \rangle = G$, a contradiction so $N_G(P)$ is maximal in G. Note that $N_G(P) \geq A_i$. Next note that, for $g \in G \setminus N_G(P)$, $N_G(P)^g = N_G(P^g) \geq A_j$ for some $j \neq i$. We have $|N_G(P) \cap N_G(P^g)|_p = 1$ because, if $1 \neq x \in N_G(P) \cap N_G(P)^g$, with $|\langle x \rangle| = p$, then $C_G(x) \geq \langle A_i, A_j \rangle = G$, a contradiction to $Z(G) = 1$. So $N_G(P)$ is strongly p-embedded in G.

Now we prove that $C_G(P) = A_i$. Suppose, in order to obtain a contradiction, that A_i is contained properly in $C_G(P)$. There is a p'-element $x \in C_G(P) \setminus A_i$. With $1 \neq a \in P$, $ax \notin A_i$ so $ax \in A_j$ for some $j \neq i$ yielding $a \in A_j$ and so $C_G(a) \geq \langle A_i, A_j \rangle = G$, another contradiction to $Z(G) = 1$. □

Corollary 3.4. *Let G be a finite group with $Z(G) = 1$, $\mathcal{A}$ a nilpotent minimal cover of G and let $P \in \mathrm{Syl}_p(G)$ be cyclic, or abelian but not elementary, and not normal in G. Then $C_G(P)$ is nilpotent and $N_G(P)$ is the unique maximal subgroup of G containing $C_G(P)$.*

Proof. First $P \leq A_i$ for some i. Let $g \in G \setminus N_G(P)$. Then $P^g \leq A_j$ for some $j \neq i$ and therefore $A_j = C_G(P^g) = C_G(P)^g = A_i^g$. From this we see that

$$\langle A_i, g \rangle \geq \langle A_i, A_i^g \rangle = \langle A_i, A_j \rangle = G.$$

Consequently every proper subgroup of G containing A_i is contained in $N_G(P)$. In other words $N_G(P)$ is, as claimed, the unique maximal subgroup of G containing $C_G(P)$. □

It is this result that allows us to see that various insoluble groups do not admit nilpotent minimal covers. In particular an almost simple group G with an abelian Sylow subgroup P which is cyclic or not elementary does not admit a nilpotent minimal cover if either $C_G(P)$ is not nilpotent or if $N_G(P)$ is not maximal.

COROLLARY 3.5. *With the same hypotheses as in the last corollary each member of $\mathcal{A}$ either contains a Sylow p-subgroup or is a p'-group; those containing Sylow p-subgroups form a conjugacy class.*

Since no group is the union of a conjugacy class of subgroups there are, under these hypotheses, p'-subgroups in every nilpotent, minimal cover of the group.

4. Applications

Here we demonstrate the use of Corollary 3.4 in showing that several classes of potential minimal counter-examples to the solubility of a group with a nilpotent minimal cover are not, in fact, minimally coverable by nilpotent subgroups.

4.1. Symmetric Groups

LEMMA 4.1. *The alternating groups of degree $n \geq 5$ and the symmetric groups of degree $n \geq 4$ do not have nilpotent minimal covers.*

PROOF. If $P = \langle (123) \rangle \in \mathrm{Syl}_3(\mathrm{Alt}_5)$ then $C_G(P) = P \leq \mathrm{Sym}_3 \cap \mathrm{Alt}_4$ and consequently there are two maximal subgroups of Alt_5 containing $C_G(P)$. This contradicts Corollary 3.4, so Alt_5 does not admit a nilpotent minimal cover. (In any case nilpotent subgroups of Alt_5 are abelian so a nilpotent minimal cover would be an abelian minimal cover making Alt_5 soluble by [4].)

Suppose now that $n \geq 6$ and denote Alt_n by G. Seeking a contradiction, we suppose that G does have a nilpotent minimal cover. Bertrand's Postulate ensures that there is a prime p satisfying $\frac{1}{2}n < p < n$. Note that $p \geq 5$ and that $2p > n$ so $p^2 \nmid |G|$. Let $P = \langle (12\ldots p) \rangle \in \mathrm{Syl}_p(\mathrm{Alt}_n)$. Then, if H is the subgroup of permutations in Alt_n fixing each of $1, 2, \ldots, p$, $C_G(P) \leq \mathrm{Alt}_p \times H$. Consequently, since $\mathrm{Alt}_p \times H < G$, $\mathrm{Alt}_p \leq \mathrm{Alt}_p \times H \leq N_G(P)$, using Corollary 3.4, contradicting the simplicity of Alt_p.

The proof for the symmetric groups with $n \geq 4$ is similar. $\square$

4.2. Suzuki Groups

LEMMA 4.2. *None of the Suzuki groups $Sz(q)$ has a nilpotent minimal cover.*

PROOF. Let $G = Sz(q)$ be a finite simple Suzuki group with $q = 2^{2m+1}$ and let $\mathcal{A} = \{A_1, \ldots, A_n\}$ a nilpotent minimal cover of G. Let $S \in \mathrm{Syl}_2(G)$ and $N = N_G(S)$. Then $|N| = q^2(q-1)$ and N is a Hall subgroup of G (see [14]). The subgroups of order $q-1$ are cyclic Hall subgroups of G. Moreover if $|H| = q-1$, then $C_G(y) = H$ for all non-identity y in H. It follows by Proposition 3.3 that $\mathcal{A}$ must contain all subgroups of order $q-1$ among its members. On the other hand N contains distinct subgroups H_1, H_2 of order $q-1$. If $H_1 \leq A_1, H_2 \leq A_2$, say, then

$$G = \langle A_1, A_2 \rangle = \langle H_1, H_2 \rangle \leq N,$$

a contradiction. Therefore none of the simple Suzuki groups admits a nilpotent minimal cover $\square$

4.3. Linear groups

THEOREM 4.3. *Let $\mathrm{PSL}_n(q) \leq G \leq \mathrm{PGL}_n(q)$ where $n \geq 3$, or $n = 2$ and $q \geq 4$. Then G admits no nilpotent minimal cover.*

We will suppose, seeking a contradiction, that some such G does admit a nilpotent minimal cover and produce a contradiction. To this end let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ be a nilpotent minimal cover of G. We write Z for the centre of $\mathrm{GL}_n(q)$ and define $\overline{G}$ and $\overline{A}_i$, subgroups of $\mathrm{GL}_n(q)$, by $\overline{G}/Z = G$ and $\overline{A}_i/Z = A_i$ $(1 \leq i \leq n)$. Note that $\overline{\mathcal{A}} := \{\overline{A}_i \ : 1 \leq i \leq n\}$ is a nilpotent, irredundant cover of $\overline{G}$. Denote by V the natural vector space on which $\mathrm{GL}_n(q)$ acts.

We will need in the proof of Theorem 4.3 the seemingly well known fact that whenever $PSL_n(q)$ is simple then G satisfying $PSL_n(q) \leq G \leq PGL_n(q)$ is almost simple so, in particular, its centre is trivial; a proof follows easily from Theorem 9.9 of Suzuki [15]. This will be needed in the proof of Theorem 4.3, which we divide into cases according as $n \geq 4$ or $n < 4$.

Case 1: $n \geq 4$. Suppose first of all that $q^{n-1} - 1$ has a primitive prime divisor, p. Since $q - 1 = (q^n - 1) - q(q^{n-1} - 1)$ it follows that $p \nmid q^n - 1$. Let V_1 be a subspace of V with dimension one. Write $V = V_1 \oplus V_2$ and let L be the Levi component of this decomposition of V in the stabiliser of the flag (V_1, V): abstractly $L \cong \mathrm{GF}(q)^\times \times \mathrm{GL}(V_2)$.

Now let $\overline{P}$ be the Sylow p-subgroup of a Singer cycle of $\mathrm{GL}(V_2)$; on order considerations it is in $\mathrm{SL}(V_2)$. Extend its action to the whole of V via trivial action on V_1. Order considerations also show that $\overline{P}$ is a Sylow p-subgroup of $\mathrm{GL}_n(q)$; it is in $\mathrm{SL}_n(q)$ and so it is a (cyclic) Sylow subgroup of $\overline{G}$.

If $h \in N_{\overline{G}}(\overline{P})$ then it is easy to see that V_1h, V_2h both admit the action of $\overline{P}$. However V_1, V_2 are non-isomorphic as $\overline{P}$-modules so are the unique proper, non-trivial submodules of $V|_{\overline{P}}$. Therefore $V_1h = V_1$ and $V_2h = V_2$, so V_1, V_2 are $N_{\overline{G}}(\overline{P})$-submodules of V. This shows that $N_{\overline{G}}(\overline{P}) \leq L \cap \overline{G}$.

Now $\overline{G} \not\leq L$ as $\mathrm{SL}_n(q) \not\leq L$. But Proposition 3.3 requires that $N_G(\overline{P}Z/Z)$ be maximal in G and so $N_{\overline{G}}(\overline{P})$ is maximal in $\overline{G}$. Therefore $N_{\overline{G}}(\overline{P}) = L \cap \overline{G}$.

However

$$\mathrm{SL}(V_2) \leq \mathrm{SL}_n(q) \cap L \leq L \cap \overline{G} = N_{\overline{G}}(\overline{P})$$

a contradiction since $\overline{P}Z/Z$ is not normal in G.

If $q^{n-1}-1$ has no primitive prime divisor then, by Zsygmondy's Theorem, $n-1=6$ and $q=2$. That is $G=\mathrm{GL}_7(2)$. G has a Sylow subgroup P of order 31 whose action splits V as $U \oplus W$ where $\dim U = 5$ and on W the action of P is trivial. Therefore $C_G(P)$ contains a copy of $\mathrm{GL}_2(2)$ which is not nilpotent, contradicting Corollary 3.4; so Case 1 does not arise.

Case 2: $n \leq 3$. A Singer cycle of $\overline{G}$ is the intersection of $\overline{G}$ with a Singer cycle of $\mathrm{GL}_n(q)$. Every Singer cycle of $\overline{G}$ is, of course, in some member of $\overline{\mathcal{A}}$. Denote by $\overline{\mathcal{A}}_S$ the subset of $\overline{\mathcal{A}}$ of those members containing a Singer cycle. Also let $\mathcal{T}$ be the set of stabilisers in $\overline{G}$ of one-dimensional subspaces of V.

LEMMA 4.4. *1. Each $\overline{A} \in \overline{\mathcal{A}}_S$ contains exactly one Singer cycle. This Singer cycle has index at most 2 in $\overline{A}$ and if its index is exactly 2 then $n=2$ and $q=2^\beta-1$ with $\beta \geq 3$.*

2. $\overline{G} = (\cup\overline{\mathcal{A}}_S) \cup (\cup\mathcal{T})$ and no member of $\overline{\mathcal{A}}_S$ is omissible from this union.

(Here $\cup\overline{\mathcal{A}}_S$ denotes the union of the members of $\overline{\mathcal{A}}_S$ and $\cup\mathcal{T}$ the union of the members of $\mathcal{T}$.)

PROOF. Let S be a Singer cycle of $\overline{G}$ with $S \leq \overline{A} \in \overline{\mathcal{A}}_S$. Suppose firstly that q^n-1 has a primitive prime divisor, p say. Then, on order considerations, the Sylow p-subgroup $\overline{P}$ of S is a Sylow subgroup of $\overline{G}$, even of $\mathrm{SL}_n(q)$. $\overline{A} \leq C_{\overline{G}}(\overline{P})$ since $\overline{A}$ is nilpotent. As $\overline{P}$ acts irreducibly on V, $S = C_{\overline{G}}(\overline{P})$ which is in $\overline{A}$ so $S=\overline{A}$ confirming (1) in this case.

If, on the other hand, q^n-1 has no primitive prime divisor then, by Zsygmondy's Theorem, $n=2$ and $q=2^\beta-1$ for some $\beta \geq 3$. The Sylow 2-subgroups of $\mathrm{SL}_2(q)$ and $\mathrm{GL}_2(q)$ are generalised quaternion and semidihedral respectively (see pp. 142-3 of Carter and Fong [5]) of orders $2^{\beta+1}, 2^{\beta+2}$ so a Sylow 2-subgroup of $\overline{G}$ is one or other of these. S has Sylow 2-subgroup C, cyclic of index two in a Sylow subgroup D of $\overline{G}$ and is the unique cyclic subgroup of index 2 in D. C acts irreducibly on V and so $S \leq C_{\overline{G}}(C) \leq S$, and that is $S = C_{\overline{G}}(C)$. Since $\overline{A}$ is nilpotent S is of index at most 2 in $\overline{A}$. Since a Sylow 2-subgroup of $\overline{G}$ has a unique cyclic subgroup of index 2, S is the only Singer cycle in $\overline{A}$. This completes the proof of (1).

Now the centraliser of an element acting irreducibly on V is a Singer cycle so lies in some $\overline{A} \in \overline{\mathcal{A}}_S$. On the other hand an element whose action on V is reducible is in some member of $\mathcal{T}$. This is obvious when $n = 2$ so we may suppose that $g \in \mathrm{GL}_3(q)$ acts reducibly on V with U a two-dimensional, irreducible submodule. Write $g = g_0 g_1$ where g_0 is an s-element, supposing $q = s^\delta$, and g_1 an s'-element. Then $U(g_0 - 1)$ is a proper $\langle g\rangle$-submodule of U so it is zero; hence U is irreducible for $\langle g_1\rangle$. Then, by Maschke's Theorem, $V = U \oplus W$ where W admits the action of g_1. However, Wg_0 admits g_1 so, as the decomposition $V|_{\langle g_1\rangle} = U \oplus W$ is unique, $Wg_0 = W$ meaning that W is a one-dimensional subspace of V stabilised by g. The non-omissibility of members of $\overline{\mathcal{A}}_S$ follows since no Singer cycle stabilises a one-dimensional subspace of V. This completes the proof of (2) and with it the proof of Lemma 4.4. □

The following corollary comes immediately from Lemma 4.4.

Corollary 4.5. $|\overline{\mathcal{A}} \setminus \overline{\mathcal{A}}_S| \leq |\mathcal{T}|$.

We show now that this is false, under our continuing assumption that Theorem 4.3 is not true.

To this end consider all un-ordered pairs $\sigma = \{U, W\}$ of non-zero, complementary subspaces of V. Define an element a_σ of $\mathrm{SL}_n(q)$ as follows (supposing, for convenience, that $\dim U = 1$): a_σ is to act completely reducibly on V with $a_\sigma|_W$ being a Singer element of $\mathrm{GL}(W)$ and $a_\sigma|_U$ the scalar needed to make $\det a_\sigma = 1$.

LEMMA 4.6. *If a_σ is in $\overline{A} \in \overline{\mathcal{A}}$ then $\overline{A} \notin \overline{\mathcal{A}}_S$.*

PROOF. Suppose, on the contrary, that $\overline{A} \in \overline{\mathcal{A}}_S$. By Lemma 4.4 either $\overline{A}$ is a Singer cycle or contains a unique Singer cycle S with index 2. The first case does not occur as otherwise, by Clifford's Theorem, $V|_{\langle a_\sigma \rangle}$ would be a direct sum of isomorphic irreducible modules which it is not. In the second case $n = 2$ and a_σ^2 generates the unique subgroup of S of order $(q-1)/2$. But this is a subgroup of Z, a contradiction since $4 \nmid q - 1$. $\square$

LEMMA 4.7. *If σ, τ are distinct pairs of complementary subspaces of V then there is no member of $\overline{\mathcal{A}}$ containing both a_σ, a_τ.*

PROOF. Suppose, to the contrary, that $a_\sigma, a_\tau \in \overline{A}$, a member of $\overline{\mathcal{A}}$, and write $H := \langle a_\sigma, a_\tau \rangle$.

First suppose that $V|_H$ is irreducible. Since $V|_{\langle a_\sigma \rangle}$ is reducible, and since H is nilpotent, there is a composition factor K/L of H for which $a_\sigma \in L$, $V|_L$ is reducible whilst $V|_K$ is not. By Clifford's Theorem $V|_L$ is completely reducible and, since $V|_{\langle a_\sigma \rangle}$ has unique decomposition $U \oplus W$, this is also a decomposition of $V|_L$. Moreover U, W are conjugate submodules for L, a contradiction if $n = 3$.

So suppose $n = 2$. Let $|K : L| = r$, a prime. The elements of $\ker_L(U)$ and $\ker_L(W)$ all have determinant one so they are the identity. Consequently L is cyclic, indeed $L = \langle a_\sigma \rangle$ since its order is $q - 1$, the largest possible. $W = U\lambda$ for some r-element $\lambda \in K \setminus L$. Since a_σ acts as different scalars ℓ^{-1}, ℓ on U, W respectively λ, a_σ do not commute. Writing $U = \mathrm{GF}(q)u$:

$$(ua_\sigma^{-1})\lambda = (\ell u)\lambda = (u\lambda)a_\sigma = u(\lambda a_\sigma \lambda^{-1})\lambda$$

whence $u = u(\lambda a_\sigma \lambda^{-1} a_\sigma)$. That is $\lambda a_\sigma \lambda^{-1} a_\sigma = 1$ so $\lambda a_\sigma \lambda^{-1} = a_\sigma^{-1}$. But then λ^2 and a_σ commute entailing $r = 2$ or else λ and a_σ commute. Also the nilpotence of K demands $q-1=2^\gamma$ for some $\gamma \geq 2$. The order of $\mathrm{SL}_2(q)$ is therefore $2^{\gamma+1}(2^{\gamma-1}+1)(2^\gamma+1)$ so K is a Sylow 2-subgroup of $\mathrm{SL}_2(q)$ and therefore of H also. K is generalised quaternion when $\gamma > 2$. In this case $\langle a_\sigma \rangle$ is the unique cyclic subgroup of index 2 in the unique Sylow 2-subgroup of H and, similarly, so is $\langle a_\tau \rangle$. Therefore $\langle a_\sigma \rangle = \langle a_\tau \rangle$ whence $\sigma = \tau$. When $\gamma = 2$, G is either $\mathrm{PSL}_2(5) \cong \mathrm{Alt}_5$ or $\mathrm{PGL}_2(5) \cong \mathrm{Sym}_5$ but in neither case does G have a nilpotent minimal cover, by Lemma 4.1.

It remains to treat the case when $V|_H$ is reducible. We show first that it is completely reducible. Let us write $q = s^\delta$ where s is a prime. It will be enough to show that $s \nmid |H|$ and, that is, that $O_s(H) = 1$ since $\overline{A}$ is nilpotent. Suppose that X is a proper, non-zero submodule of $V|_H$. Then V/X is irreducible for H since it is irreducible for $\langle a_\sigma \rangle$. Therefore $O_s(H)$ is in the kernel of both $X, V/X$ but not in the kernel of V. If $1 \neq h \in O_s(H)$, $vX \mapsto v(h-1)$ is a well-defined $\langle a_\sigma \rangle$-isomorphism $V/X \to X$, a contradiction. Hence $O_s(H) = 1$ so, by Maschke's Theorem, $V|_H = X \oplus Y$ for some H-submodule Y. But X and Y admit both a_σ, a_τ each of which gives a unique splitting of V. Therefore $\sigma = \tau$. The proof of Lemma 4.7 is complete. □

There are more unordered pairs of complementary subspaces of V than there are subspaces of dimension one. Hence $|\overline{\mathcal{A}} \setminus \overline{A}_S| > |\mathcal{T}|$, a contradiction to Corollary 4.5. Thus our assumption that Theorem 4.3 was false is incorrect and Theorem 4.3 is established.

4.4. Sporadic simple groups In this section we give references from the literature to show that the sporadic simple groups do not have a nilpotent minimal cover. The idea is to indicate in each a Sylow subgroup P of prime order for which either $N_G(P)$ is not maximal or $C_G(P)$ is not nilpotent so as to invoke Corollary 3.4. In what follows C_n indicates a cyclic Sylow subgroup of order n in the group under consideration.

1. The Mathieu group $G = M_{11}$. $|G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$. $N_G(C_5)$ is a Frobenius group of order 20 and is not maximal (see p. 211 in [16]).

2. The Mathieu group $G = M_{12}$. $|G| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$ and $N_G(C_5) = C_2 \times F$ where F is a Frobenius group of order 20; it is not maximal (see p. 212 in [16]).

3. The Mathieu group $G = M_{22}$. $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$. $N_G(C_5)$ is a Frobenius group of order 20 and is not maximal (see p. 212 in [16].)

4. The Mathieu group $G = M_{23}$. $|G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ and $N_G(C_5)$ is a semi-direct product of C_{15} by C_4 and is not maximal (see p. 213 in [16]).

5. The Mathieu group $G = M_{24}$. $|G| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$. $N_G(C_{11})$ is a Frobenius group of order 110 and it is not maximal (see p. 213 in [16]).

6. The Janko group $G = J_1$. $|G| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$. $C_G(C_5) = C_5 \times D_6$ is the direct product of C_5 and a dihedral group of order 6 and so it is not nilpotent (see p. 213 in [16]).

7. The Hall-Janko group $G = J_2$. $|G| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$. $N_G(C_7)$ is a Frobenius group of order 42 and it is not maximal (see p. 214 in [16]).

8. The Janko group $G = J_3$. $|G| = 2^7 \cdot 3^5 \cdot 5 \cdot 7 \cdot 17 \cdot 19$. $N_G(C_{17})$ is a Frobenius group of order $17 \cdot 8$ and it is not maximal (see p. 214 in [16]).

9. The Janko group $G = J_4$. $|G| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$. $C_G(C_7) = C_7 \times \mathrm{Sym}_5$ is not nilpotent (see p. 215 in [16]).

10. The Conway group $G = Co_3$ $|G| = 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$. $C_G(C_7) \cong C_7 \times \mathrm{Sym}_3$ is not nilpotent (see pp.216 in [16]).

11. The Conway group $G = Co_2$. $|G| = 2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$. $N_G(C_{11})$ is a Frobenius group of order 110 (see p. 216 in [16]); it is not maximal (see p.154 in [7]).

12. The Conway group $G = Co_1$. $|G| = 2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$. $C_G(C_{11}) = C_{11} \times \mathrm{Sym}_3$ is not nilpotent (see p.302 in [1]).

13. The Fischer group $G = F_{22}$. $|G| = 2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. $C_G(C_7) \cong C_7 \times \mathrm{Sym}_3$ is not nilpotent (see p. 251 in [2]).

14. The Fischer group $G = F_{23}$. $|G| = 2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$. $C_G(C_{13}) \cong C_{13} \times \mathrm{Sym}_3$ is not nilpotent (see p. 252 in [2]).

15. The Fischer group $G = F'_{24}$. $|G| = 2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$. $C_G(C_{13}) \geq C_{13} \times \mathrm{Sym}_3$ so is not nilpotent (see p. 252 in [2]).

16. The Baby Monster $G = F_2$.

$$|G| = 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47.$$

$C_G(C_{11}) \cong C_{11} \times \mathrm{Sym}_5$ is not nilpotent (see p.217 in [7]).

17. The Fischer group (Monster) $G = F_1$.

$$|G| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71.$$

$C_G(C_{23}) \cong C_{23} \times \mathrm{Sym}_4$ is not nilpotent (see p.234 in [7]).

18. The Higman-Sims group $G = HS$. $|G| = 2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$. $N_G(C_7)$ is a Frobenius group of order 42 and it is not maximal (see p.220 in [16]).

19. The Held group $G = He$. $|G| = 2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$. $N_G(C_{17})$ is a Frobenius group of order $17 \cdot 8$ and it is not maximal (see p.221 in [16]).

20. The Suzuki group $G = Suz$. $|G| = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$. $C_G(C_7) \geq C_7 \times \mathrm{Alt}_4$ is not nilpotent (see p.303 in [1]).

21. The McLaghlin group $G = Mc$. $|G| = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$. $N_G(C_{11})$ is not maximal (see p.100 in [7]).

22. The Lyons group $G = Ly$. $|G| = 2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$. $C_G(C_7) \simeq C_7 \times SL_2(3)$ is not nilpotent (see p.223 in [16]).

23. The Rudvalis group $G = Ru$. $|G| = 2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$. $N_G(C_{29})$ is a Frobenius group of order $29 \cdot 14$ (see p. 224 in [16]) and it is not maximal (see p.126 in [7]).

24. The O' Nan-Sims group $G = O'N$. $|G| = 2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$. $N_G(C_{11})$ has order 110 (see p.225 in [16]) and it is not maximal (see p.132 in [7]).

25. The Thompson group $G = Th$. $|G| = 2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$. $N_G(C_{19})$ is a Frobenius group of order $19 \cdot 18$ (see p.225 in [16]) and it is not maximal (see p.79 in [10]).

26. The Harada group G. $|G| = 2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$. $C_G(C_7) = C_7 \times \mathrm{Alt}_5$ is not nilpotent (see p.226 in [16]).

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