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PARTITIONS AND OTHER COVERINGS OF FINITE GROUPS

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I have a great memory of Reinhold Baer. I admired his high scientific values and his deep human qualities. He visited Italy many times, giving courses of lectures and seminars. He very kindly invited me to Germany to give lectures. Consequently, I am very glad that with this paper I may pay homage to his memory.

This paper is a survey of the theory of partitions of finite groups to which Baer made many contributions of the greatest importance. We will also consider other coverings connected with partitions, but we will not examine geometric applications of partitions. All groups are assumed to be finite.

1. Some basic definitions

Let G be a group. A set $\Pi = \{H_1, \dots, H_n\}$ of subgroups H_i $(i = 1, \dots, n)$ is said to be a *partition* of G if every element $x \in G,\ x \neq 1$, belongs to one and only one subgroup $H_i \in \Pi$. If $n = 1$, the partition is said to be trivial.

Let S be a subgroup of G. A set $\Pi = \{H_1, \dots, H_n\}$ of subgroups of G is said to be a *strict S-partition* of G if $S \leq H_i$ $(i = 1, \dots, n)$ and every element of $G \setminus S$ belongs to one and only one of the subgroups H_i $(i = 1, \dots, n)$. If $n = 1$ the strict S-partition is said to be trivial. If $\Pi = \{H_1, \dots, H_n\}$ is a partition (S-partition) of G, the subgroups H_i $(i = 1, \dots, n)$ are said to be the *components* of Π.

2. A paper by G. A. Miller

The first paper concerning partitions of groups was published in 1906 by G. A. Miller [20]. He proved that any abelian group G having a non trivial partition Π is an elementary abelian p-group of order $\geq p^2$. Moreover, if $|G| = p^m$ and $|H_i| = p^a$ for every $H_i \in \Pi$, then a divides m. Conversely, if G is an elementary abelian group of order p^m and a divides m, then G has a partition $\Pi = \{H_1, \dots, H_r\}$ with $|H_i| = p^a,\ i = 1, \dots, r$. Miller also proved that if Π is a non trivial partition of a non abelian p-group G, then all elements

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of G having order $> p$ belong to the same component of Π. Moreover, if all the components of a non trivial partition of a group G have the same order, then no element of G has order p^2, p being a prime.

3. Covering by centralizers

Starting in 1907, M. Cipolla [8] gave an interesting covering of a non abelian group. Let G be such a group and Z its center. Cipolla called "sottogruppo fondamentale" (fundamental subgroup) a subgroup of G which is the centralizer of some element of $G \setminus Z$. The center of a fundamental subgroup is called the "normocentro" (normocenter). The set of all fundamental subgroups and the set of all normocenters of G are coverings of G.

G has at least three fundamental subgroups (and three normocenters); moreover, it has exactly three fundamental subgroups (normocenters) if and only if G/Z is the Klein group of order 4. In 1926, G. Scorza [28] proved that a group G is covered by exactly three proper subgroups if and only if it contains a normal subgroup N such that G/N is the Klein group.

Let H be the set of all fundamental subgroups of G and K the set of its normocenters. H and K are partially ordered sets with respect to inclusion and they have the same length. Cipolla called the length of H (and of K) the "rango" (rank) of G. A group G has rank 1 if and only if K is a strict Z-partition.

4. J. W. Young and the primitive partition

In 1927 J. W. Young [31] published a paper concerning the construction of a particular partition in (finite or infinite) groups. We consider only the finite case.

Let G be a non trivial group and s an element of G, $s \neq 1$. We define the sets $H_0, K_0, H_1, K_1, \ldots, H_i, K_i, \ldots$ in the following way:

$$\begin{array}{lcl} H_0 & = & \langle s \rangle, \\ K_0 & = & \{x \in G \mid x^t \in H_0, x^t \neq 1 \text{ for some } t\}, \\ H_1 & = & \langle K_0 \rangle, \\ K_1 & = & \{x \in G \mid x^t \in H_1, x^t \neq 1 \text{ for some } t\}, \\ \ldots & .. & \ldots., \\ H_i & = & \langle K_{i-1} \rangle, \\ K_i & = & \{x \in G \mid x^t \in H_i, x^t \neq 1 \text{ for some } t\}, \\ \ldots & .. & \ldots. \end{array}$$

If G is finite, $H_n = K_n$ for some integer n. Put $H_n = L(s)$.

The set of all distinct subgroups $L(s)$ for $s \in G$, $s \neq 1$, is a partition Π_0 of G, the *primitive partition*. Let Π be a different partition of G. Then every component of Π_0 is contained in some component of Π. If $L(s) = G$ for some $s \in G$, $s \neq 1$, G has only the trivial partition.

Young gave also some theorems concerning partitions of a (finite or infinite) abelian group. In the case of a finite abelian group G he proved again the following result of Miller [20]: a finite abelian group has a non trivial partition if and only if it is a p-group of exponent p and order $> p$.

5. Kontorovich and completely decomposable groups

In the years 1939 and 1940 P. Kontorovich published two papers ([18], [19]) concerning partitions of groups. Following Kontorovich, a group G is said to be *completely decomposable* if it has a partition Π such that every component of Π is cyclic. A cyclic subgroup H of a group G is said to be *maximal-cyclic* if it is not contained in some cyclic subgroup $K \neq H$.

A group G necessarily satisfies one of the following conditions:

(1) the center of G is not cyclic;
(2) the center of G is $\neq 1$ and maximal cyclic;
(3) the center of G is cyclic, $\neq 1$, but it is not maximal cyclic;
(4) the center of G is 1.

Kontorovich in [18] proved that in cases (1) and (2), G is completely decomposable if and only if it is a p-group of exponent p. In case (3) G is completely decomposable if and only if the center of G has order p and is contained in a maximal-cyclic normal subgroup; in case (4) G is completely decomposable if and only if every proper subgroup of G is completely decomposable.

In [19] Kontorovich gave a condition for the direct product of two completely decomposable groups to be completely decomposable. He also studied groups which are not completely decomposable but such that every proper subgroup is completely decomposable.

In 1950 M. Suzuki [29] proved that a non simple, non-solvable group is completely decomposable if and only if it is isomorphic with a projective general linear group $\mathrm{PGL}(q)$, $q = p^h$, p odd.

6. Baer and the classification of groups with non trivial partitions

Until 1960 the results concerning partitions of groups were partial and almost fragmentary. The paper [2] of Reinhold Baer, published in 1961, introduced new ideas which made it possible to obtain complete classifications of partitions. This paper, of almost forty pages, was dedicated to Richard Brauer.

Following Baer, a partition Π of the group G is said to be *normal* if $a^{-1}Xa \in \Pi$ for every $X \in \Pi$ and $a \in G$. The primitive partition is normal. If a group has a non trivial partition, it also has a normal non trivial partition.

A normal non trivial partition Π of a group G is said to be a *Frobenius partition* if there exists a component $X \in \Pi$ such that $N_G(X) = X$. If $a \in G \setminus X$, then $a^{-1}Xa \in \Pi$, so $(a^{-1}Xa) \cap X = 1$ and G is a Frobenius group. The components of Π which are not conjugate to X are contained in the Frobenius kernel K of G. The set consisting of K and the subgroups conjugate to X is a non trivial partition of G, the *minimal* Frobenius partition. Baer proved that a group G can have at most one minimal Frobenius partition.

A non trivial partition Π of a group G is said to be *elementary* if G has a normal subgroup K such that all cyclic subgroups which are not contained in K have order p prime and are components of Π. All normal non trivial partitions of a p-group of exponent $> p$ are elementary.

A non trivial partition K of a group G is said to be *non-simple* if there exists a proper normal subgroup N of G such that for every component $X \in \Pi$, either $X \leq N$ or $X \cap N = 1$. Let G be a group and Π a normal non trivial partition. Suppose Π is not a Frobenius partition and is non-simple. Then G has a normal subgroup K of index p (p a prime) in G which is generated by all elements of G having order $\neq p$. So Π is elementary.

A non trivial partition K of a group G is said to be a *Hall partition* if every component of Π is a Hall subgroup of G. Baer proved the following theorem. Let Π be a non trivial, normal Hall partition of a group G. Then either G is a Frobenius group or G is simple, non abelian and the components of Π are exactly the maximal nilpotent subgroups of G.

In 1961 Baer published another paper [3] in which he proved the following proposition. Let the group G have a normal, simple non trivial partition Π. Suppose that the Fitting subgroup $F(G)$ of G is non trivial. Then G is isomorphic with S_4 and the components of Π are the cyclic subgroups of G which are not contained in $F(G)$.

7. Baer, Kegel and Suzuki complete the classification of partitions

In the same year 1961, O. Kegel, who was at that time a young student of Baer, completed the results of his teacher concerning partitions of groups with non trivial Fitting subgroup. In [13] he proved that a group G having a non trivial partition with all components subnormal is a p-group. He also proved that the following conditions for a non p-group G with a non trivial partition Π are equivalent:

(1) the center of G is $\neq 1$;
(2) every component of Π is contained in a proper normal subgroup of G;
(3) Π is normal non simple, $F(G)$ is a component of Π, $|G/F(G)| = p$ prime, p divides $|F(G)|$.

Finally, Kegel proved that if G is a group with $F(G) \neq 1$ having a non trivial partition, then one of the following conditions is satisfied:

(1) G is a p-group;
(2) G is a Frobenius group;
(3) $F(G) \in \Pi$, $|G/F(G)| = p$, p being a prime dividing $|F(G)|$;
(4) G is isomorphic to S_4.

The papers of Baer and Kegel described above provide a classification of groups G with $F(G) \neq 1$ having non trivial partitions. In 1961 Baer published a paper [4], dedicated to F. K. Schmidt, concerning partitions of groups G with $F(G) = 1$. Let G be a non simple group having only one minimal normal subgroup S. Suppose S is simple and non abelian. If G has a non trivial partition $[G : S] = 2$, the Sylow subgroups of G of odd order are abelian, and the 2-Sylow subgroups of G are non Dedekind.

The complete classification of groups with trivial Fitting subgroup having a non trivial partition is due to M. Suzuki. In a paper [30] from 1961 he proved the following theorem. Let G be a non solvable group and Π a normal non trivial partition of G. Suppose that G is not a Frobenius group. Then the components of Π are nilpotent. The greatest part of the paper is occupied by the proof of the following proposition. Let G be a non solvable group and Π a normal non trivial partition of G. If all components of G are nilpotent, then $|G|$ is even. Of course Suzuki does not use the result of Thompson and Feit concerning the solubility of groups of odd order; this result was known but not yet published. Instead Suzuki used character theory. He obtained the following main theorem. Let G be a non solvable group and Π a normal non trivial partition of G. Suppose that all components of G are nilpotent. If the centralizer of an involution of G is not nilpotent, then G is isomorphic with $\mathrm{PGL}(2, p^h)$, p odd. If this centralizer is nilpotent, then G is isomorphic with either $\mathrm{PSL}(2, p^h)$ or the Suzuki group $G(q)$.

In the same year 1961, O. Kegel and G. Wall [15] proved a theorem exploiting the results of the paper [4] of Baer. Let G be a non simple group with $F(G) = 1$, having a non trivial partition. Then G is isomorphic with $\mathrm{PGL}(2, q)$, where $q = p^h$, p prime, $q \geq 5$. This result is contained in that of Suzuki in [30], but was obtained independently and without using character theory.

Let G be a group and p a prime. We recall that the subgroup generated by all the elements of G whose order is not p is called the Hughes subgroup and denoted by $H_p(G)$. The group G is said to be a group of Hughes-Thompson type if it is not a p-group and $H_p(G) \neq G$ for some prime p. In such a group we have, by a theorem of Hughes and Thompson [11], $[G : H_p(G)] = p$, and, by a theorem of Kegel [14], $H_p(G)$ is nilpotent.

On the basis of the results of Baer, Kegel and Suzuki, which we have described, a group G has a non trivial partition if and only if it satisfies one of the following conditions:

(1) G is a p-group with $H_p(G) \neq G$ and $|G| > p$;
(2) G is a Frobenius group;
(3) G is a group of Hughes-Thompson type;
(4) G is isomorphic with $\mathrm{PGL}(2, p^h)$, p being an odd prime;
(5) G is isomorphic with $\mathrm{PSL}(2, p^h)$, p being a prime;
(6) G is isomorphic with a Suzuki group $G(q), q = 2^h, h > 1$.

In 1963 Baer published a paper [5] concerning partitions of abelian groups. Its interest is mainly geometric.

8. Classification of strict S-partitions

In 1964 G. Zappa [32] reduced the problem of determining the strict S-partitions to the analogous problem for partitions.

Let G be a group and S a subgroup of G. Let N be a normal subgroup of G such that $N \leq S$. Then the set $H_1, \dots, H_r$ is a strict S-partition of G if and only if the set $H_1/N, H_2/N, \dots, H_r/N$ is a strict S/N-partition of G/N. So the problem of determining the strict S-partitions of a group G is reduced to the case when S is *antinormal* in G, that is to say, when no normal subgroup N of G with $N \neq 1$ is contained in S.

Zappa [32] proved the following theorem.

Let G be a group and S an antinormal subgroup of G. Let $H_1, H_2, \dots, H_r$ be a set of subgroups of G with $S \leq H_i$ $(i = 1, \dots, r)$. Then the following conditions are equivalent:

(1) $H_1, H_2, \dots, H_r$ is a non trivial strict S-partition of G;
(2) G is a Frobenius group and S a Frobenius complement of G. The Frobenius kernel K is a p-group, $H_i = SK_i$, where $\{K_1, \dots K_r\}$ is a non trivial partition of K.

9. Roland Schmidt and the lattice of centralizers

In 1970 Roland Schmidt [24] studied the lattice of centralizers of subsets in a group G. If this lattice has length 2, G is said to be an M-group. It is easy to prove that a group G with center Z is an M-group if and only if it is non-abelian and the centralizer of any element of $G \setminus Z$ is abelian. Consequently every M-group has rank 1 (see Section 3). Moreover, the set of fundamental subgroups of G is a strict Z-partition. In his paper Schmidt gave a complete classification of M-groups. In every case G/Z is a p-group, a Frobenius group, or is isomorphic with A_4, $\mathrm{PGL}(2, p^n)$ or $\mathrm{PSL}(2, p^n)$.

10. Isaacs and the equally partioned groups

In the year 1973 I. M. Isaacs published a paper [12] concerning *equally partioned* groups, that is to say, groups with a non trivial partition Π such that all components of Π have the same order. He improved the result of Miller [20] proving that every equally partioned group is a p-group of exponent p. He also proved the following theorem. Let G be a group and Π a non trivial partition of G such that $HK = KH$ for every $H, K \in \Pi$. Then G is an elementary abelian p-group.

11. $\Pi-$automorphisms of a group with a partition Π

Let G be a group and Π a non trivial partition of G. An automorphism α of G is said to be a Π-*automorphism* if $H^\alpha = H$ for every $H \in \Pi$. In 1967 R. H. Schulz [26] proved that every non trivial Π-automorphism α of a group G with a non trivial partition is fix-point-free.

J. André [1] and M. Biliotti [6] proved the following theorem. Let G be a group with a non trivial partition Π. Then a group of non trivial Π-automorphisms of G is necessarily cyclic. M. Biliotti and A. Scarselli [7] in 1979 gave the following result. Let G be a group with a non trivial partition Π and a non trivial Π-automorphism. Then G is a p-group of exponent p, and the class of G is ≤ 2 for $p > 3$, and ≤ 3 for $p = 3$. In the same year A. Herzer [9] proved that for $p = 3$ the class of G is ≤ 2.

On the basis of the results of Biliotti, Scarselli and Herzer, the theorem of G. Zappa (see Section 8) concerning strict S-partitions can be extended in the following way.

Let S be an antinormal subgroup of the group G. Let $\{H_1, \ldots, H_s\}$ be a set of subgroups of G with $S \leq H_i$ $(i = 1, \ldots, s)$. Then the following conditions are equivalent:

(1) $\{H_1, \ldots, H_s\}$ is a non trivial strict S-partition of G;
(2) G is a Frobenius group and S is a cyclic Frobenius complement; the Frobenius kernel K is a p-group of exponent p and class ≤ 2, and $H_i = SK_i$ with $\{K_1, \ldots, K_s\}$ a non trivial partition of K.

12. On linear partitions

A non trivial partition Π of a group G is said to be *linear* if there exists a non empty set ε of subgroups of G such that:

(1) every element of ε is the union of at least 2 elements of Π;
(2) every 2 elements of Π lie in a unique element of ε;
(3) G is not an element of ε.

In 1986, A. Herzer [10] gave some examples of linear partitions. In the same year, B. H. Schulz [27] proved that every group with a linear partition is an abelian p-group or a Frobenius group.

13. Khukhro and partitions of p-groups

Let G be a p-group. An automorphism α of order p of G is said to be a splitting automorphism if for every $x \in G$, $xx^\alpha x^{\alpha^2} \ldots x^{\alpha^{p-1}} = 1$.

In 1990 E. I. Khukhro [16] proved that for a p-group G with $|G| > p$ the following conditions are equivalent:

(1) G has a non trivial partition;
(2) $H_p(G) \neq G$;

(3) $G = G_1\langle\alpha\rangle$ with G_1 a subgroup of index p in G, and a an element of G of order p inducing on G_1 a splitting automorphism.

He also proved that there exists a function $f(d,p)$ (d integer ≥ 1, p prime, $f(d,p)$ integer ≥ 1), such that every group having order a power of p and d generators with a non trivial partition has class $\leq f(d,p)$. He also gave similar bounds for the class of p-groups in which $H_p(G)$ has index $\geq p^2$.

14. Other results on partitions of p-groups

In the years 1988–1991, V. Pannone obtained further results on partitions of p-groups. A partition of a p-groups is said to be *atomic* if all components have order p. Let G be a p-group and let

$$G = \gamma_0(G) > \gamma_1(G) > \cdots > \gamma_m(G) = 1$$

be its lower central series. Then G is said to be of *submaximal class* if $|\gamma_i(G) : \gamma_{i+1}(G)| = p$ for $i = 1, \ldots, m-1$; it is said to be of *maximal class* if it is of submaximal class and $|\gamma_0(G) : \gamma_1(G)| = p^2$.

In the paper [21] published in 1988, Pannone proved that if G is a group of order p^n (p prime) and of maximal class with a non trivial partition Π, there are at least $p^n - p^{n-2} - p^{n-3}$ components of Π of order p. In particular in such a group every equipartition is atomic. The same author proved in [22] that there are exactly three non abelian groups of order p^6 having non atomic non trivial equipartitions. Other interesting results of Pannone are contained in [23].

15. Classification of groups of rank 1

After the classification of M-groups by of R. Schmidt [24], G. Zappa [33] in 1996 characterized the groups of rank 1 which are not M-groups.

Every extraspecial p-group G of order $> p^3$ has rank 1 and is not a M-group. The semidirect product $G\langle a\rangle$, G being an extraspecial p-group of order $> p^3$ and a an element of order 2 such that $a^{-1}ba = b^{-1}$ for every non central element $b \in G$, is also a group of rank 1 and it is not a M-group. A group G with center Z is a group of rank 1 and is not a M-group if and only if satisfies one of the following conditions:

(A) $G = P \times K$, P being a p-group of rank 1 which is not a M-group and K an abelian p'-group;

(B) G/Z is a Frobenius group with Frobenius kernel N/Z, Frobenius complement F/Z cyclic, $Z(N) = Z$ and N satisfying condition (A).

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