Lattices in the Frame of a Finite Soluble Group

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The character table and the subgroup lattice of a finite group are both examples of structures that contain significant information about the group in a condensed form. Another interesting combinatorial object associated with a finite group is its frame, the retract of its subgroup lattice under the action of its inner automorphism group. Specifically, the *frame* $\mathcal{F}r(G)$ of a group G is the partially-ordered set (poset, for short) consisting of the conjugacy classes[(*)] $[H]$ of subgroups H of G with the relation $\preceq$ of partial order defined by the rule

$$[H] \preceq [L] \text{ if and only if } H \leq L^g \text{ for some } g \in G.$$

This investigation grew out of an attempt to distinguish soluble from insoluble finite groups by properties of their frames. It revealed several interesting lattice-like structures inside the frame of a finite soluble group, and we felt they merited exposure in a separate paper, even though they played only a minor role in our attempted characterization.

In the light of Hall's celebrated characterization of finite soluble groups by the existence of Sylow p-complements, it seems natural to begin such an investigation by looking at the Hall systems inside the frames of such groups. After setting the scene in Section 1 with some easy examples of frames of soluble and insoluble groups, we discuss the conjugacy classes of Hall subgroups in the frame of a finite soluble group G in Section 2. These form a well-behaved sublattice in $\mathcal{F}r(G)$ isomorphic with the lattice of subsets of $\sigma(G)$, the set of prime divisors of $|G|$. This sublattice evidently coincides with $\mathcal{F}r(G)$ if and only if G has square-free order, in which case $\mathcal{F}r(G)$ is a hypercube (which is the name we give to the lattice of subsets of some set). We show that these are the only groups whose frames are hypercubes.

The subgroups that are Hall subgroups of a *normal* subgroup generalise the notion of Hall subgroups; they are called normally embedded subgroups and inside the subgroup lattice of a finite soluble group, they have some striking properties. In Section 3 we show that the conjugacy classes of normally-embedded subgroups form another well-behaved lattice inside the frame of a soluble group. It is an open question whether this lattice, or indeed the lattice generated by conventional Hall subgroups, can be identified solely from knowledge of the abstract poset of a frame without reference to the underlying finite soluble group.

Since our work on Hall subgroups and the closely-related normally-embedded subgroups proved to be inconclusive in settling the soluble/insoluble divide, we turned our attention to the behaviour of maximal subgroups, whose behaviour is well known to be distinctive in soluble groups. It turns out that the conjugacy classes of maximal subgroups are indeed embedded in a 'soluble' frame in a special way. How they are embedded is described in

(*) We use $[H]$ to denote the conjugacy class $\{H^g \mid g \in G\}$ of H in G.

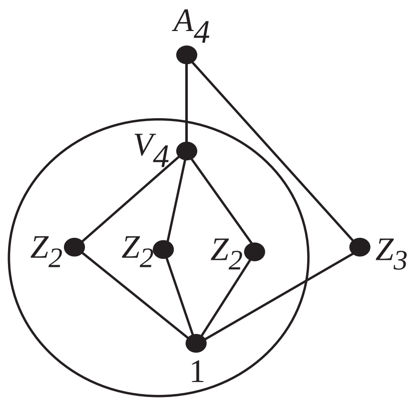
Section 4, where we show, in particular, that they generate a sublattice of $\mathcal{F}r(G)$ when G is soluble.

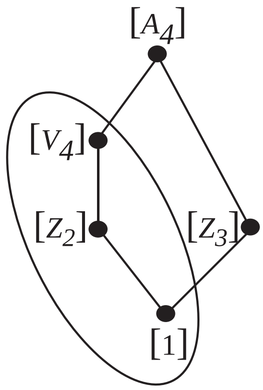
Our investigation raised more questions than it solved, and we devote the concluding Section 5 to discussing some of these. In a separate paper we will exploit some of our results to characterize soluble groups by their frames (ref. ?).

All groups considered are finite.

1. Some Examples of Frames of Finite Groups

If H is a subgroup of a group G, the interval $(1, H)$ in the subgroup lattice of G coincides with the subgroup lattice of H. But because of subgroup fusion, the same cannot be said of intervals $([1], [H])$ in the frames of groups. For instance the frame of an elementary abelian group V_4 of order four coincides with its subgroup lattice and in particular has three minimal elements (see the encircled part of the left-hand figure below). But within the poset $\mathcal{F}r(A_4)$, the interval $([1], [V_4])$ is a chain with only one minimal element (as shown inside the oval in the right-hand figure below).





The subgroup lattice of A_4 The frame of A_4

Thus, in order to study subgroup structure in frames, we need to take account of the way the frame of a group is modified by fusion when it is viewed as a subgroup of another group.

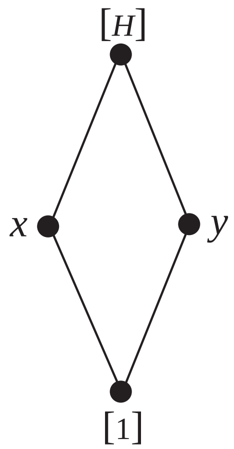
(1.1) Definitions.

(a) If $(P, \preceq)$ is a poset, a subset S of P is called an *ideal* of P provided that

$$\text{if } s \in S \text{ and } x \in P \text{ with } x \preceq s, \text{ then } x \in S.$$

(b) Let H be a subgroup of a group G. The interval $([1], [H])$ of $\mathcal{F}r(G)$, consisting of all G-conjugacy classes $[L]$ with $1 \leq L \leq H$, is evidently an ideal of $\mathcal{F}r(G)$. We will call it the *G-frame* of H and denote it by $\mathcal{F}r_G(H)$. We will use the term *relative frame* as a generic description of the G-frames of H for various groups G.

(1.2) Example. *Let H be a subgroup of a finite group G. Assume that the G-frame $\mathcal{F}r_G(H)$ of H is a square as shown in the diagram to the right. Then $|H| = pq$, for suitable primes p and q, and if $p = q$, then H is elementary abelian.*



Proof. Since the nodes x and y are minimal elements, they correspond to subgroups of prime orders p and q.

Case 1: $p = q$. In this case H is a p-group, and since a group of order p^n has subgroups of order p^i for all $i \in \{1, 2, \ldots, n\}$, it follows that $n = 2$. If H were cyclic, it would contain a unique subgroup of order p, which is not the case here. Therefore H is elementary abelian of order p^2.

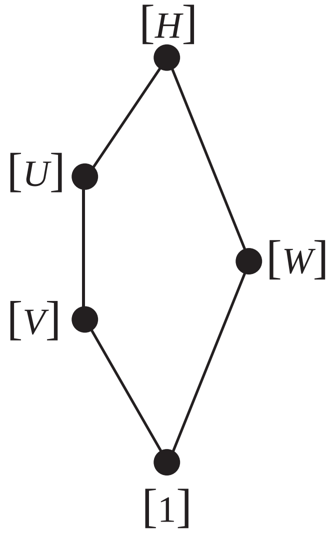
Case 2: $p \neq q$. In this case $|H| = p^a q^b$. If a or b were bigger than 1, H would have a proper Sylow subgroup S which would itself have a proper non-trivial subgroup T. These would form part of a chain $1 \prec [T] \prec [S] \prec [H]$ of length ≥ 3 in $\mathcal{F}r_G(H)$. Since $\mathcal{F}r_G(H)$ patently has no such chain, we have $a = b = 1$. ∎

(1.3) Example. *Let H be a subgroup of a finite group G, and assume that the G-frame $\mathcal{F}r_G(H)$ of H is the non-modular poset with 5 elements shown in the diagram to the right. Then*

> *(a) $|H| = p^2q$, where p and q are distinct primes with q an odd divisor of $p+1$, and*
>
> *(b) H has a normal elementary-abelian Sylow p-subgroup U (of order p^2) and a Sylow q-subgroup V (of order q) acting faithfully and irreducibly upon U.*

Furthermore, if $G = H$, then $G \cong A_4$.

For all choices of primes p and q satisfying the stated conditions, there exists a group G with a subgroup H such that $\mathcal{F}r_G(H)$ is the given poset.



Proof. Inspection of the pictured poset shows that $\mathcal{F}r_G(H)$ has two G-conjugacy classes of minimal subgroups $[V]$ and $[W]$ (with V and W subgroups of H), and so $|H|$ has at most two prime divisors, p and q say. Since the maximal chains of subgroups of a p-group have equal length, H cannot be a p-group; therefore we can suppose that $|V| = p \neq q = |W|$. The poset diagram shows that the subgroup U has no subgroups of order q; it is therefore a p-group of order p^2 (since it has a maximal chain of subgroups of length 2) and is necessarily abelian. Moreover, because it is maximal, U is a Sylow p-subgroup of H. Likewise W is a Sylow subgroup of order q, and it follows that $|H| = p^2q$, as claimed.

Next we exploit the fact, also evident from the poset diagram, that H has no subgroup of order pq. Since H is soluble, it has a minimal normal subgroup, N say. Observe that we cannot have $|N| = p$ or q, for then one of NW and NV would have order pq. Therefore $|N| = p^2$, and it follows that $N = U$ is a normal elementary-abelian subgroup of H acted upon irreducibly by W. It follows from Theorem B,9.8 of [2] that q divides $p^2 - 1$ but does not divide $p - 1$. Therefore the prime q is odd and divides $p + 1$, as asserted. Finally, if $G = H$, the subgroup W must permute the $p + 1$ subgroups of U order p in a transitive orbit; therefore $q = p + 1$. Consequently $p = 2$ and $q = 3$, and it follows that $G \cong A_4$.

Now let p and q be distinct primes, with q an odd divisor of $p + 1$. We describe two examples of pairs (G, H) with H a subgroup of G for which the poset diagram for $\mathcal{F}r_G(H)$ has the pictured form. First let G denote the semidirect product of the additive group $U = \mathrm{GF}(p^2)^+$ of the Galois field of p^2 elements by its multiplicative group $B = \mathrm{GF}(p^2)^\times$ of order $p^2 - 1$ (see Proposition B,12.9 of [2] for details). Let W denote the subgroup of order q in the cyclic group B, and set $H = UW$, a proper subgroup of G. Since B acts transitively on the non-identity elements of U, it permutes the subgroups of order p in G in a single orbit, and so, if V is a subgroup of U of order p, it follows that V and W are the only minimal elements of $\mathcal{F}r_G(H)$. Since W acts irreducibly on U, there are no subgroups of order pq in H. Hence the poset diagram for $\mathcal{F}r_G(H)$ has the desired form.

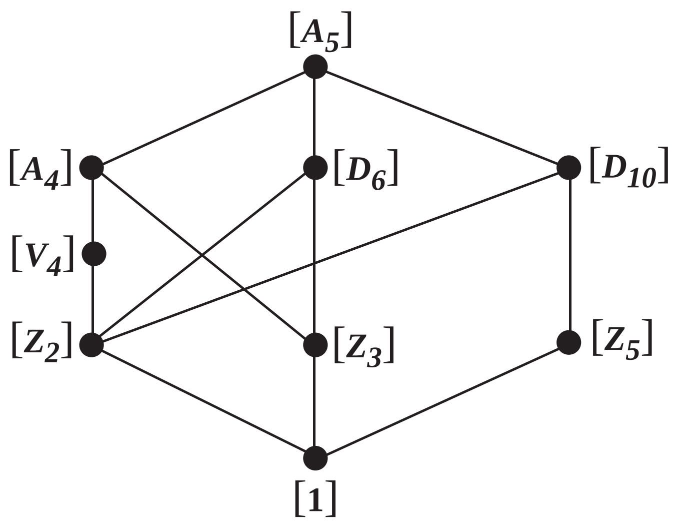
In the above example, G is soluble and $H \trianglelefteq G$. We now describe a second example with G insoluble and $H \ntrianglelefteq G$. Let A be a vector space of dimension $n\,(\geq 3)$ over $\mathrm{GF}(p)$, and form the semidirect product G of A by $B = \mathrm{GL(n,p)}$, with the natural action the general linear group B on A. Let W be a subgroup of B of order q. Since B acts non-trivially on A, it follows from Maschke's theorem and the analysis of Theorem B,9.8 of [2] that the restricted module A_W has a simple submodule U of dimension 2. Set $H = UW$. Since B permutes the one-dimensional subspaces of A transitively, it is not hard to see that the poset diagram for $\mathcal{F}r_G(H)$ again has the desired form. ∎

(1.4) Remark. If isomorphic subgroups of a group H are conjugate in H, no fusion can take place when H is embedded as a subgroup of a group G and so its G-frame coincides with its frame.

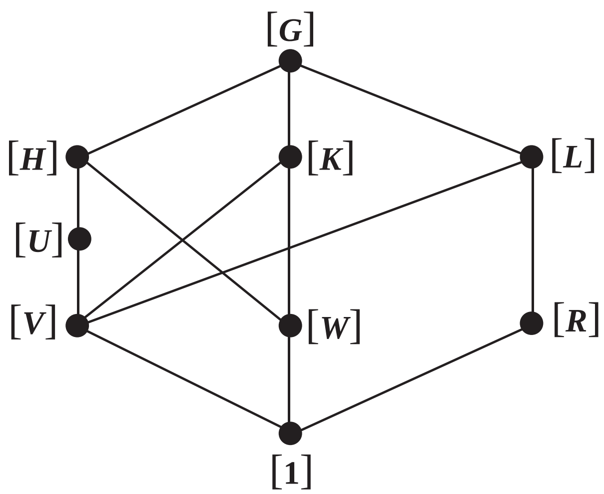
As can be seen from the picture of its frame on the right, the alternating group A_5 is an example of a group with this property. Thus

$$\mathcal{F}r_G(A_5) \cong \mathcal{F}r(A_5)$$

for all groups G that contain a copy of A_5 as a subgroup.



(1.5) Example. *Let G be a group with $\mathcal{F}r(G)$ isomorphic to $\mathcal{F}r(A_5)$, so that it has the Hasse diagram shown below. Then $G \cong A_5$.*



The nodes of $\mathcal{F}r(A_5)$ have been labelled in the style $[X]$ where X denotes a subgroup of the unknown group G. Let p, q and r denote the prime orders of the minimal subgroups V, W, and R respectively. We break the proof into a number of short steps.

Step 1: G is simple.

Proof. First observe that the maximal elements $[H]$ and $[K]$ have no infimum, and so G cannot be soluble by Theorem 5.7 below. Inspection of the above Hasse diagram shows that, for every proper non-trivial subgroup X of G, the posets of the intervals $(1, [X])$ and $([X], G)$ are either chains, or squares, or copies the non-modular poset of Example 1.3 and these three types of posets, as relative frames, all belong to soluble groups; this follows from the analysis of Examples 1.2 and 1.3 and the obvious fact that chains belong to cyclic groups of prime-power order. If X were normal in G, it would follow that both X and G/X are soluble and hence that G itself is soluble, a possibility we have just ruled out. Hence G is simple. ∎

Step 2: $H \cong A_4$, $p = 2$, and $q = 3$.

Proof. Note that the G-frame $\mathcal{F}r_G(H)$ of H is the non-modular poset of Example 1.3 and by the analysis of that Example, it follows that $|H| = p^2q$ and that H has a normal elementary-abelian Sylow p-subgroup U of order p^2 complemented by a Sylow q-subgroup W of order q.

We claim that the prime r is distinct from both p and q. If r were equal to p, the subgroup L would be a p-group, and in the frame of G, the elements $[U]$ and $[L]$ would each sit below the conjugacy class of Sylow p-subgroups of G. But $[G]$ is the least upper bound of $[U]$ and $[L]$, and this would imply that G is a p-group, which is not the case. Therefore $r \neq p$, and a similar argument shows that $r \neq q$. It follows from Example 1.2 that L has order pr and hence that $R \in \mathrm{Syl}_r(G)$. Consequently $|G| = p^2qr$.

Next we show that p is the smallest prime dividing $|G|$. Suppose, for a contradiction, that q is the smallest. Since the automorphism group $\mathrm{Aut}(\mathbb{Z}_q)$ of a cyclic group of order q has order $q - 1$, it follows that the Sylow q-subgroup W of G is contained in the centre of its normalizer; then, by a theorem of Burnside, G has a normal q-complement, contradicting Step 1. Consequently q is not the smallest prime divisor of $|G|$. Likewise neither is r, and so p is indeed the smallest prime divisor; in particular, $p + 1 \leq q$. However, from the analysis of Example 1.3 we know that q divides $p + 1$ and hence that $q \leq p + 1$. Consequently $q = p + 1$, and it follows that $p = 2$ and $q = 3$. Therefore the subgroup H has order 12 and is evidently a copy of A_4.

∎

Step 3: $|G| = 60$.

Proof. It is clear from the Hasse diagram that $U \in \mathrm{Syl}_2(G)$, $W \in \mathrm{Syl}_3(G)$, and $R \in \mathrm{Syl}_r(G)$. Since $|G| = 2^2 \cdot 3 \cdot r$, the subgroup L, which has order $2r$ by Example 1.2, therefore has index 6. Let x be an element of order r in L. Under the permutation representation of G by left multiplication on the six left cosets of L (faithful by Step 1), the element x fixes L and permutes the other five cosets with at least one non-trivial cycle. Since r is prime and does not divide 6, it follows that $r = 5$. ■

Step 4: $G \cong A_5$.

Proof. We now obtain a permutation representation of G on the 5 cosets of H, which is again faithful by Step 1. Since A_5 is the only subgroup of order 60 in S_5, the desired conclusion now follows. ■

This raises the question whether all non-abelian finite simple groups are characterized by their frames.

2. Hall Systems in the Frames of Finite Soluble Groups

Philip Hall's well-known characterization states that a finite group G is soluble if and only if it has a Sylow p-complement for each prime $p \in \sigma(G)$, the set of primes dividing $|G|$. If $|\sigma(G)| = s$, the set of all intersections(†) of a complete set of p-complements of G (one for each $p \in \sigma(G)$) forms a so-called Hall system of G consisting of 2^s Hall π-subgroups G_π of G, one for each subset π of $\sigma(G)$. Furthermore, G has a unique conjugacy class of Hall π-subgroups, and every π-subgroup of G is contained in some Hall π-subgroup of G. It follows that the map

$$\mu_G : \pi \mapsto [G_\pi] \tag{2.α}$$

is an injection from the power set $\mathcal{P}(\sigma(G))$ of all subsets of $\sigma(G)$ into $\mathcal{F}r(G)$. Moreover, when $\mathcal{P}(\sigma(G))$ is seen as a poset partially ordered by inclusion, the map $\mu = \mu_G$ is order-preserving because evidently

$$[G_\pi] \preceq [G_\rho] \text{ if and only if } \pi \subseteq \rho.$$

The minimal elements of $\mathcal{P}(\sigma(G))$ are mapped to the conjugacy classes $[G_p]$ of Sylow p-subgroups of G. Hence, for the map μ_G to be surjective, each of these $[G_p]$ must be a minimal element of $\mathcal{F}r(G)$, and the Sylow subgroups of G must all have prime order, or equivalently, G must have square-free order. On the other hand, if $|G|$ is square-free, every subgroup of G is a Hall subgroup, and then the map μ_G is surjective. Thus we have proved:

(2.1) Lemma. *Let G be a finite soluble group. The map μ_G defined above in (2.α) is surjective if and only if G has square-free order. In this case $\mathcal{F}r(G)$ is order-isomorphic to the poset of subsets of the set $\sigma(G)$.*

(2.2) Definition. A *hypercube* is a poset that is order-isomorphic to the poset (in fact, lattice) of subsets of some set, partially ordered by inclusion.

(†) including the empty intersection, which by convention is equal to G.

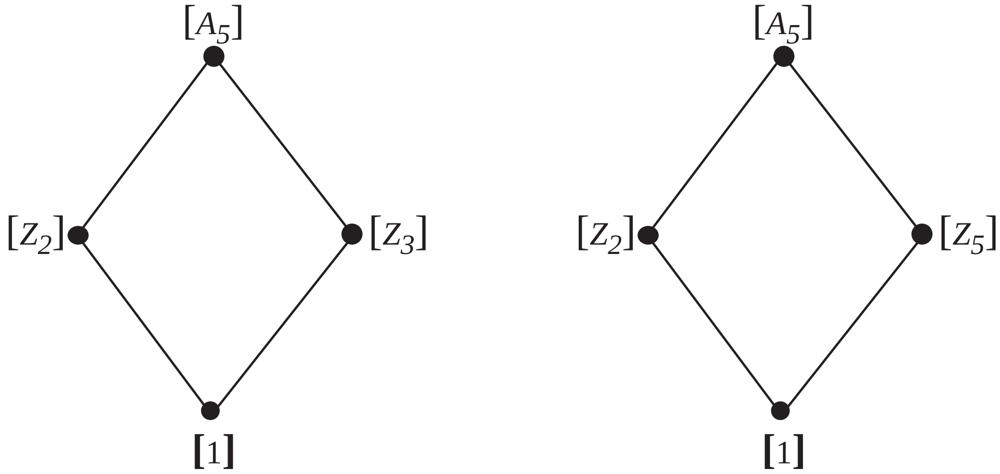
(2.3) Proposition. *The frame of a finite group G is a hypercube if and only if G has square-free order.*

Proof. The sufficiency is clear from Lemma 2.1. To prove the necessity, suppose that G is a finite group for which $\mathcal{F}r(G)$ is a hypercube, noting for induction purposes that every interval of a hypercube is also a hypercube. Among the minimal subgroups of G, let K be one of maximal (prime) order, p say. If L is a minimal subgroup of order q such that $[L] \neq [K]$, then $[K]$ and $[L]$ form two nodes of a square in $\mathcal{F}r(G)$; therefore K and a suitable conjugate of L are contained in a subgroup H of order pq by Example 1.2. Since $p \geq q$, an easy application of Sylow's theorem shows that $K \trianglelefteq H$. It follows that K is normalised by a conjugate of each minimal subgroup of G. But since $\mathcal{F}r(G)$ is a hypercube, any set of representatives of the conjugacy classes of minimal subgroups must generate G, and therefore $K \trianglelefteq G$.

Since $\mathcal{F}r(G/K)$ is a hypercube, we can argue by induction on $|G|$ that G/K has square-free order, and if p does not divide $|G/K|$, we are done. If, on the other hand, p divides $|G|$, then K is contained in a Sylow p-subgroup P of G of order p^2 whose G-frame $\mathcal{F}r_G(P)$ is a square; P therefore contains a second minimal subgroup $\overline{K}$ of order p and by the argument of the previous paragraph, $\overline{K} \trianglelefteq G$. If $K = \langle x \rangle$ and $\overline{K} = \langle y \rangle$, then $\langle xy \rangle$ is a subgroup of P which is conjugate neither to K nor to $\overline{K}$, and this possibility is ruled out by the fact that $\mathcal{F}r_G(P)$ does not contain 3 minimal elements. Therefore G has square-free order. ∎

Remark: In [3], Philip Hall shows that groups all of whose subgroups are complemented are characterized as groups isomorphic to subgroups of direct products of groups of square-free order; in particular, groups of square-free order are soluble.

Terminology. Let $(P, \preceq)$ be a poset. If S is a subset of P, the axioms of a partially-ordered set ensure that $(S, \preceq)$ is also a poset, although not necessarily an ideal in the sense of Definition 1.1 (a). If $(S, \preceq)$ has an additional property as a poset, we shall say the subset S has this property in P; for example, if $(S, \preceq)$ is a lattice, we shall say that 'S is a lattice in P'. Now it is quite possible that the supremum $s \vee t$ in S of two elements $s, t \in S$ is not the supremum of s and t in P: for the supremum of s and t in P may either not exist or it may exist and be different from $s \vee t$. For example, consider the two squares depicted below, which are both instances of lattices in $\mathcal{F}r(A_5)$ (see the diagram in Remark 1.4). In the left-hand lattice the join $[Z_2] \vee [Z_3]$ is equal to $[A_5]$, whereas $[Z_2]$ and $[Z_3]$ have no supremum in $\mathcal{F}r(A_5)$, and in the right-hand lattice the join $[Z_2] \vee [Z_5]$ is again $[A_5]$, while the supremum of $[Z_2]$ and $[Z_5]$ in $\mathcal{F}r(A_5)$ is $[D_{10}]$.



To exclude such possibilities, we introduce the idea of ‘rigid embedding’

(2.5) Definitions. Recall that an *upper (lower) semi-lattice* is a poset in which every pair of elements has a supremum (infimum), and that a *lattice* is a poset which is both an upper and a lower semi-lattice. Let S be an upper (lower) semi-lattice in a poset P. We say that S is *rigidly-embedded* in P if all pairs s and $t \in S$ have a supremum (infimum) in P which coincides with the supremum $s \vee t$ (infimum $s \wedge t$) in S. For a *rigidly-embedded lattice* both these conditions should hold.

Let π and ρ be subsets of $\sigma(G)$. Since the supremum (respectively infimum) of the pair $[G_\pi]$ and $[G_\rho]$ in $\mathcal{F}r(G)$ is $[G_{\pi\cup\rho}]$ (respectively $[G_{\pi\cap\rho}]$), the image of μ_G is a sublattice of $\mathcal{F}r(G)$ with unions and intersections preserved. Thus, in the terminology of Definitions 2.5 we have the following proposition:

(2.6) Proposition. *The conjugacy classes of Hall subgroups of a finite soluble group G form a rigidly-embedded sublattice of $\mathcal{F}r(G)$ order-isomorphic to the hypercube $\mathcal{P}(\sigma(G))$.*

(2.7) Question. Can the Hall subgroups be identified in the frame of a finite soluble group?

The meaning of this question needs to be made more precise. Although the image of the map μ_G defined in (2.α) is always a rigidly-embedded hypercube in $\mathcal{F}r(G)$, there may be several distinct hypercubes of the right size in $\mathcal{F}r(G)$. For instance, inspection of the poset drawn in Example 1.3 shows that there are two distinct rigidly-embedded squares in the frame of A_4. Can both of these arise as the image of μ_G for suitable groups G having the same frame as A_4? In fact, the answer is ‘no’ in this case, as the analysis of Example 1.3 shows, but we need to be careful. We will therefore interpret Question 2.7 as follows. If there is an order-isomorphism

$$\theta : \mathcal{F}r(G) \to \mathcal{F}r(H), \text{ do the maps } \theta \circ \mu_G = \mu_H \text{ have the same image?}$$

A stronger version of the question might call for a systematic method, an algorithm depending only on the isomorphism type of the frame, for locating the image of μ_G within the frame of G, for it seems that the only hope of answering the question affirmatively would be to find such an algorithm.

If the Hall systems of a soluble group were discernable in the poset of its frame, the conjugacy classes of its Sylow subgroups could be identified as the minimal elements of the rigidly-embedded hypercube that is the image of μ. If $P \in \mathrm{Syl}_p(G)$ and $|P| = p^a$, the exponent a is the length of a maximal chain in $\mathcal{F}r_G(P)$, an invariant of this relative poset. Thus, for a soluble group G of order $p_1^{a_1} p_2^{a_2} \dots p_s^{a_s}$, we would be able to read off from its frame the sequence $(a_1, a_2, \dots, a_s)$ of exponents in the prime decomposition of its order.

Terminology. If G is a finite group of order $p_1^{a_1} p_2^{a_2} \dots p_s^{a_s}$ whose distinct prime divisors $p_1, p_2, \dots, p_s$ have been numbered so that $a_1 \geq a_2 \geq \cdots \geq a_s \geq 1$, then we define the *order exponent* of G to be the sequence:

$$\mathrm{oe}(G) = (a_1, \dots, a_s).$$

A positive answer to Question 2.7 would therefore imply that two groups G and H with the isomorphic frames must satisfy $\mathrm{oe}(G) = \mathrm{oe}(H)$. This situation prompts another useful notion for studying the influence of frames.

(2.8) Definitions. (a) The *frame closure* $\mathcal{F}r\mathcal{C}l(\mathfrak{X})$ of a class $\mathfrak{X}$ of finite groups is defined to be the class of all groups whose frames are among the frames of $\mathfrak{X}$-groups; thus

$$\mathcal{F}r\mathcal{C}l(\mathfrak{X}) = (G \mid \exists\ X \in \mathfrak{X} \text{ such that } \mathcal{F}r(G) \cong \mathcal{F}r(X)).$$

If $\mathfrak{X} = \mathcal{F}r\mathcal{C}l(\mathfrak{X})$, we will say that the class $\mathfrak{X}$ is *framed*. The analysis of Example 1.5 shows that the class (A_5) of groups isomorphic to A_5 is framed.

(b) Let $\mathbf{a} = (a_1, \dots, a_s)$ be a sequence (an ordered s-tuple) of natural numbers a_i satisfying $a_1 \geq a_2 \geq \cdots \geq a_s \geq 1$. We define an associated class $\mathfrak{F}^{\mathbf{a}}$ of finite groups as follows:

$$\mathfrak{F}^{\mathbf{a}} = (G \text{ is a finite group} \mid \mathrm{oe}(G) = \mathbf{a}).$$

A positive answer to Question 2.7 would imply, in the terminology of Definition 2.8(a), that each class $\mathfrak{F}^{\mathbf{a}}$ of groups with a given exponential order $\mathbf{a}$ is framed. Although we have only limited evidence, we hazard the following conjecture.

(2.9) Conjecture. *Let $\mathbf{a} = (a_1, \dots, a_s)$ be an ordered s-tuple of natural numbers a_i satisfying $a_1 \geq a_2 \geq \cdots \geq a_s \geq 1$. Then $\mathfrak{F}^{\mathbf{a}}$ is framed.*

Proposition 2.3 shows that the conjecture is true for sequences of the form $\mathbf{a} = (1, 1, \dots, 1)$. It has also been proved for sequences with $s = 1$; indeed, in [1] Rolf Brandl proves the stronger result that, for any prime p, the class of non-cyclic finite p-groups is framed. Since the frame closure of the class $(\mathbb{Z}_{p^n})$ consists of all cyclic groups of order q^n for some prime q, it follows from Brandl's theorem that the class

$$\bigcup_{\text{primes } p} \mathfrak{S}_p$$

of groups of prime power order is also framed. (Here $\mathfrak{S}_p$ denote the class of p-groups.)

We bring this section to a close by proving Conjecture 2.9 for the special case $\mathbf{a} = (2,1)$, in other words, for groups of order p^2q for distinct primes p and q.

(2.10) Proposition. *Let p and q be distinct primes, and let G be a group of order p^2q. Let P denote the poset $\mathcal{F}r(G)$.*

(a) P has a unique element x satisfying the following conditions:

- *x is a maximal element of P;*
- *all the maximal chains of P joining $[1]$ to x have length two;*
- *there is a unique minimal element y of P not lying below x.*

(b) The image of $\mu = \mu_G$ is uniquely determined by $\mu(p) = x$ and $\mu(q) = y$.

(c) If $\mathcal{F}r(H) = \mathcal{F}r(G)$, there exist primes $\overline{p}$ and $\overline{q}$ such that $|H| = \overline{p}^2\overline{q}$.

Proof. First observe that G is soluble by Burnside's p^aq^b-Theorem.

(a) The possible orders of maximal subgroups of G are p^2, pq, and q. It is easy to see that the conjugacy class of Sylow p-subgroups of G satisfies the conditions described in all three bullet points and is therefore a candidate for x. We proceed to rule out the other two possibilities, first noting that a subgroup of order q is minimal and therefore cannot sit at the top of a chain of length two.

We show next that if G has a subgroup V of order pq, the condition described in the third bullet point fails to be satisfied when $x = [V]$. Since $\mathcal{F}r_G(V)$ contains a unique G-conjugacy class of subgroups of order p, it will suffice to show the following.

> **Assertion** $(\star)$: Assume that G has a subgroup V of order pq. Then G has either precisely one or at least three conjugacy classes of subgroups of order p.

Let $U \in \mathrm{Syl}_p(G)$. If U is cyclic, G evidently has just one conjugacy class of subgroups of order p. Therefore suppose that U is elementary abelian.

Case 1: $U \trianglelefteq G$. In this case, the subgroup $K = U \cap V$ is a normal Sylow p-subgroup of V, and since U is abelian, $K \trianglelefteq UV = G$. Since U is completely reducible by Maschke's theorem, there exists a second normal subgroup $\overline{K}$ of G such that $U = K \times \overline{K}$. If $K = \langle k \rangle$ and $\overline{K} = \langle \overline{k} \rangle$, then $[K]$, $[\overline{K}]$, and $[\langle k\overline{k} \rangle]$ are three distinct G-conjugacy classes of subgroups of order p, assertion $(\star)$ holds in this case.

Case 2: $U \ntrianglelefteq G$. Let N be a minimal normal subgroup of G, and first suppose that N is a p-group, necessarily contained in U. Since U non-normal and is abelian, we have $|N| = p$ and the normal subgroup $C_G(N)$ properly contains the maximal subgroup U; therefore $C_G(N) = G$ in this case. Since U is non-cyclic, it follows that $G = N \times W$, where W is a normal subgroup of G of order pq with no normal Sylow p-subgroup. Let $N = \langle k \rangle$. If $\langle \overline{k} \rangle \in \mathrm{Syl}_p(W)$, then the G-conjugacy class $[\langle \overline{k} \rangle]$ is contained in W, and it follows as before that $[N]$, $[\langle \overline{k} \rangle]$, and $[\langle k\overline{k} \rangle]$ are three distinct G-conjugacy classes of subgroups of order p.

We consider finally in Case 2 the possibility that N has order q, and note that $C_U(N)$, being centralized by N and U, lies in the centre of G. If $C_U(N) = 1$, the subgroup U acts faithfully on N and is therefore cyclic, which wehave supposed not to be the case. Therefore $C_U(N) \neq 1$ and so G contains a normal subgroup of order p, the situation already dealt with in the preceding paragraph.

Thus we have justified Assertion ($\star$) and ruled out $x = [V]$. Therefore $x = [U]$ is the unique solution satisfying all three bullet points. This proves Part (a).

(b) It follows at once from the analysis of Part (a) that x and y correspond respectively to the conjugacy classes of Sylow p-subgroups and Sylow q-subgroups of G, and Assertion (b) is clear.

(c) Let X be a subgroup of H for which $[X] = x$. If the minimal elements of $\mathcal{F}r(H)$ below x are conjugacy classes of subgroup of the same prime order, $\overline{p}$ say, then X is a $\overline{p}$-group, necessarily of order $\overline{p}^2$ by the chain-length hypothesis. On the other hand, if X has subgroups of orders $\overline{p}$ and $\overline{r}$ with $\overline{p} \neq \overline{r}$, these must be Sylow subgroups of X, and the chain-length hypothesis forces $|X| = \overline{pr}$. In any case, the order of X is the product of two not-necessarily-distinct primes $\overline{p}$ and $\overline{r}$. Suppose the element y corresponds to a conjugacy class of subgroups of prime order $\overline{q}$. If $\overline{p}$, $\overline{q}$, and $\overline{r}$ were distinct, then G would have square-free order, and by Proposition 2.3 its frame would be a hypercube; but then each of its three maximal elements would be squares and none of them would satisfy the third bullet condition. Therefore at least two of the primes $\overline{p}$, $\overline{q}$, and $\overline{r}$ are the same. The possibility that all three primes are the same is ruled out by Brandl's theorem cited above; for if H had prime-power order, the hypothesis $\mathcal{F}r(H) = \mathcal{F}r(G)$ would imply that G also had prime-power order, which is not the case.

Having exhausted all other possibilities, we are left with the conclusion that $|H| = \overline{p}^2\overline{q}$ with $\overline{p}$ and $\overline{q}$ distinct primes, as claimed. ∎

(2.11) Corollary. *The class $\mathfrak{F}^{(2,1)}$ is framed.*

3. A Lattice for Normally-Embedded Subgroups

In this section we aim to show that the conjugacy classes of normally-embedded subgroups of a finite soluble group G form a rigidly-embedded lattice in $\mathcal{F}r(G)$. But first we want to show that frames of groups are not themselves lattices in general. It is clear from the

Hasse diagram of A_5 displayed in (1.3) that $[A_4]$ and $[D_6]$ have no infimum (greatest lower bound), and so $\mathcal{F}r(A_5)$ is certainly not a lattice. We will now describe an example of a *soluble* group whose frame is also not a lattice.

(3.1) Example. *There exists a soluble group of order* $2^4 \cdot 5$ *whose frame is not a lattice.*

Proof. Let V denote the additive group of the Galois field $\mathrm{GF}(2^4)$ viewed as an $\mathbb{F}_2$-vector space of dimension 4. Let b be an element of order 5 in the multiplicative group $\mathrm{GF}(2^4)^\times$, which is cyclic of order 15. Since $0 = b^5 - 1 = (b-1)(b^4+b^3+b^2+b+1)$ and $b \neq 1$, we have $b^4 + b^3 + b^2 + b = -1 = 1$. Setting $c = b + b^{-1}$, we see that

$$c^3 = (b + b^{-1})^3 = b^3 + 3b + 3b^{-1} + b^{-3} = b^3 + b + b^4 + b^2 = 1,$$

and hence that c is an element of $\mathrm{GF}(2^4)$ of multiplicative order 3. We regard V as an $\mathbb{F}_2B$-module with $B = \langle b\rangle$ acting fixed-point-freely by right multiplication. Let $G = VB$, the semi-direct product of V by B with this action. The formula

$$S(n,r) = \frac{(p^n-1)(p^n-p)\cdots(p^n-p^{r-1})}{(p^r-1)(p^r-p)\cdots(p^r-p^{r-1})}$$

for the number $S(n,r)$ of subspaces of dimension r in a vector space of dimension n over the Galois field $\mathbb{F}_p = \mathrm{GF}(p)$, and the fact that B acts both irreducibly and fixed-point-freely on V, together show that:

- G has 3 conjugacy classes of one-generator subgroups in V, each with 5 elements;
- G has 7 conjugacy classes two-generator subgroups in V, each with 5 elements.

Let v be a non-identity element of V. Since c evidently permutes the 3 conjugacy classes of one-generator subgroups in V cyclically, a set of class representatives can be chosen to be $\langle v\rangle$, $\langle vc\rangle$, and $\langle vc^2\rangle$, using additive notation for the module V. We then obtain the following representatives of the 7 conjugacy classes two-generator subgroups in V:

1. $\langle v, vb\rangle$	2. $\langle v, vb^2\rangle$	3. $\langle v, vc\rangle$	4. $\langle v, vb^2c\rangle$
5. $\langle v, vbc^2\rangle$	6. $\langle vc, vb^2c\rangle$	7. $\langle vc, vb^2c^2\rangle$	

The element c fixes the conjugacy class of the subgroup numbered 3 (since right multiplication by c sends $\langle v, vc\rangle$ to $\langle vc, vc^2\rangle = \langle vc, v(c+1)\rangle = \langle vc, vc+v\rangle = \langle vc, v\rangle$), and permutes the remaining 6 classes of subgroups in two orbits of length 3. For example, c sends the subgroup numbered 1 to $\langle vc, vbc\rangle$ which belongs to the same conjugacy class as $\langle vc, vbc\rangle b^2 = \langle vb^2c, vb^3c\rangle = \langle v(b^3+b), v(b^4+b^2)\rangle$, since $c = b+b^{-1}$. But $\langle v(b^3+b), v(b^4+b^2)\rangle$ contains $v(b^3+b) + v(b^4+b^2) = v(b^4+b^3+b^2+b) = v$, and so $\langle vc, vbc\rangle b^2 = \langle v, vb^2c\rangle$. It follows that c sends the class of subgroup 1 to the class of subgroup 4. Similar calculations show that the action of c permutes these classes as follows:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 6 & 3 & 7 & 2 & 5 & 1 \end{pmatrix}.$$

Using this permutation and the obvious fact that, in the frame of G, the conjugacy class of $\langle v \rangle$ lies below the conjugacy classes of just the first five 2-generator subgroups in V, we obtain the following incidence relations for the conjugacy classes of $\langle v \rangle$, $\langle vc \rangle$, $\langle vc^2 \rangle$ and the two-generator subgroups 1 – 7:

	1	2	3	4	5	6	7
v	1	1	1	1	1	0	0
vc	0	1	1	1	0	1	1
vc^2	1	0	1	0	1	1	1

Inspection of this incidence table reveals that, in $\mathcal{F}r(G)$, each pair of minimal elements has 3 conjugacy classes of two-generator subgroups lying over it; therefore $\mathcal{F}r(G)$ is not a lower semi-lattice and so certainly not a lattice. A dual calculation shows that $\mathcal{F}r(G)$ is not an upper semi-lattice either. ∎

We now recall the definitions of some standard embedding properties of subgroups that are especially fruitful in the study of finite soluble groups.

(3.2) Definitions. Let H be a subgroup of a finite group G.

(a) We say that H is *normally embedded* in G (and write $H \,\mathrm{ne}\, G$) if every Sylow subgroup of H is a Sylow subgroup of some normal subgroup of G.

(b) We say that H is *pronormal* in G (and write $H \,\mathrm{pr}\, G$) if for all $g \in G$, the subgroup $\langle H, H^g \rangle$ contains an element x such that $H^g = H^x$, that is to say, if H and any conjugate of H are already conjugate in their join.

(c) If G is soluble and Σ is a Hall system of G, we say that Σ *reduces into* H (and write $\Sigma \searrow H$) if, for all $\pi \subseteq \sigma(G)$, the Hall π-subgroup G_π in Σ satisfies $H \cap G_\pi \in \mathrm{Hall}_\pi(H)$.

Here is a summary of some of the salient facts about pronormal and normally-embedded subgroups of a finite soluble group. A fuller account of their properties can be found in Sections 6 and 7 of Chapter 1 on [2].

(3.3) Some Known Facts. *Let G be a finite soluble group, and let Σ be a Hall system of G.*

(a) A normally-embedded subgroup of G is pronormal in G (7.2(b) of [2]).

(b) A conjugate of a pronormal subgroup is pronormal (immediate from definition).

(c) A subgroup H is pronormal in G if and only if Σ reduces into exactly one conjugate of H (6.6 of [2]).

(d) The set $\mathcal{N}e(\Sigma)$ defined as follows:

$$\mathcal{N}e(\Sigma) = \{U \leq G \mid U \,\mathrm{ne}\, G \text{ and } \Sigma \searrow U\}$$

forms a lattice (a sublattice of the subgroup lattice of G) in which the join and meet operations are respectively 'permutable product' and 'intersection' of subgroups (7.9 of [2]).

(e) The set $\mathcal{N}e\,(\Sigma)$ contains a unique member of each conjugacy class of normally-embedded subgroups of G (by Parts (a) and (c)).

We will also need two important results due to Fischer that are stated in full and proved in I,6.9 and I,6.12 of [2]. Here we amalgamate them in a format tailored to our later needs.

(3.4) Fischer's Theorem. *Let $H_1\,,\,H_2\,,\,\ldots\,,\,H_s$ be a set of pronormal subgroups of a finite soluble group G into each of which a given Hall system Σ of G reduces.*

(a) Let $\mathcal{S}$ denote the set of all subgroups of the form

$$\langle H_1^{g_1}\,,\,H_2^{g_2}\,,\,\ldots\,,\,H_s^{g_s}\rangle$$

for all choices of $g_1\,,\,g_2\,,\,\ldots\,,\,g_s \in G$. Then the minimal elements of the poset $\mathcal{S}$, partially ordered by inclusion, form a conjugacy class of pronormal subgroups of G; furthermore, the join $\langle H_1\,,\,H_2\,,\,\ldots\,,\,H_s\rangle$ is the unique member of this class into which Σ reduces and is, in particular, pronormal.

(b) Suppose there exist elements $g_1\,,\,g_2\,,\,\ldots\,,\,g_s \in G$ such that each pair of the subgroups $H_1^{g_1}\,,\,H_2^{g_2}\,,\,\ldots\,,\,H_s^{g_s}$ is permutable, and set

$$L = H_1^{g_1}H_2^{g_2}\ldots H_s^{g_s} \quad \text{and} \quad J = \langle H_1\,,\,H_2\,,\,\ldots\,,\,H_s\rangle.$$

Then $J = H_1H_2\ldots H_s$ and J is conjugate to L in G.

Our aim is to show that the conjugacy classes of normally-embedded subgroups of a finite soluble group form a rigidly-embedded lattice in its frame.

(3.5) Lemma. *Let Σ be a Hall system of a finite soluble group G. Then the map $\nu : \mathcal{N}e\,(\Sigma) \to \mathcal{F}r(G)$ sending a normally-embedded subgroup H of G to its G-conjugacy class $[H]$ is an order-preserving injection (in other words, an order-monomorphism).*

Proof. We first show that ν is injective. Let H and K be subgroups in $\mathcal{N}e\,(\Sigma)$ such that $\nu(H) = \nu(K)$. The $[H] = [K]$, in other words, H is conjugate to K. Since $\Sigma \searrow H$ and $\Sigma \searrow K$, and since H and K are pronormal subgroups of G by (3.3)(a), it follows from (3.3)(c) that $H = K$. Therefore ν is injective.

It is clear from the definition of the order $\preceq$ in $\mathcal{F}r(G)$ that $\nu(H) = [H] \preceq [J] = \nu(J)$ whenever a subgroup H is contained in a subgroup J, and so ν preserves orders, as claimed. ∎

(3.6) Theorem. *Let Σ be a Hall system of a finite soluble group G. The set of conjugacy classes of normally-embedded subgroups of G (that is to say, the image $\nu(\mathcal{N}e\,(\Sigma))$) forms a rigidly-embedded lattice in $\mathcal{F}r(G)$.*

Proof. Let U, V be normally-embedded subgroups of G such that $\Sigma \searrow U$ and $\Sigma \searrow V$. We begin by showing that, for all choices of elements x and y in G

(3.α) $$U^x \cap V^y \text{ is conjugate to a subgroup of } U \cap V.$$

Since $U^x \cap V^y$ is conjugate to $U \cap V^{yx^{-1}}$, it will suffice to show that

(3.β) $$U \cap V^g \text{ is conjugate to a subgroup of } U \cap V.$$

for all $g \in G$.

Let $p \mid |G|$ and let P be the Sylow p-subgroup in Σ. Set

$$P_1 = P \cap U, \quad P_1 = P \cap U, \quad \text{and} \quad P_3 = P_1 \cap P_2 = P \cap (U \cap V).$$

Since Σ reduces into U and V, by (3.3)(d) it also reduces into $U \cap V$. Therefore

$$P_1 \in \mathrm{Syl}_p(U), \quad P_2 \in \mathrm{Syl}_p(V), \quad \text{and} \quad P_3 \in \mathrm{Syl}_p(U \cap V).$$

Let $\Sigma^\star$ be a Hall system of G that reduces into $U \cap V^g$, and let $P^\star$ denote the Sylow p-subgroup in $\Sigma^\star$. The key step in the proof is to show that the Sylow p-subgroup $(U \cap V^g) \cap P^\star$ of $U \cap V^g$ is contained in a conjugate of a Sylow p-subgroup of $U \cap V$ lying in $P^\star$, in other words that

(3.γ) $$\text{if } P_0 = U \cap V^g \cap P^\star, \text{ then } \exists\, x \in G \text{ such that } P_0 \leq P_3^x \leq P^\star.$$

Since U and V are normally embedded, there exist normal subgroups N_1 and N_2 of G such that $P_i \in \mathrm{Syl}_p(N_i)$ for $i = 1, 2$. Set $N = N_1 \cap N_2$. Since $N \trianglelefteq G$, we have $P_1 \cap N \in \mathrm{Syl}_p(N)$, and therefore $P_1 \cap N = P \cap N$; similarly $P_2 \cap N = P \cap N$. It follows that $P_1 \cap N = (P_1 \cap N) \cap (P_2 \cap N) = (P_1 \cap P_2) \cap N = P_1 \cap P_2 = P_3$ because $P_1 \cap P_2 \subseteq N_1 \cap N_2 = N$. We have therefore shown that

(3.δ) $$P_3 \in \mathrm{Syl}_p(N_1 \cap N_2).$$

As a p-subgroup of U, our P_0 is contained in a conjugate of P_1 and therefore lies in N_1. Furthermore, as a p-subgroup of V^g, our P_0 is also contained in a conjugate of P_2^g and therefore lies in N_2. Consequently P_0 is contained in the Sylow p-subgroup $P^\star \cap (N_1 \cap N_2)$ of $N_1 \cap N_2$, and this is conjugate to P_3 by (3.δ). We have justified the assertion labelled (3.γ).

We will now refresh our notation. Let $\sigma(G) = \{p_1, p_2, \ldots, p_s\}$, and let P_i be the Sylow p_i-subgroup contained in the Hall system $\Sigma \cap U \cap V$ of $U \cap V$. (These newly defined P_i should not be confused with the symbols P_i used earlier in the proof.) Then $U \cap V = P_1 P_2 \ldots P_s$, the permutable product of the subgroups in its Sylow basis. It follows from (3.γ) that

$$\exists\, x_1, x_2, \ldots, x_s \in G \text{ such that } U \cap V^g \subseteq W = \langle P_1^{x_1}, P_2^{x_2}, \ldots, P_s^{x_s} \rangle,$$

and where the Hall system $\Sigma^\star$ of G reduces into each $P_i^{x_i}$ for $i \in \{1, 2, \ldots, s\}$. Since each of the subgroups $P_i^{x_i}$ is a Sylow subgroup of a normal subgroup of G, it is pronormal in G. We can now apply Fischer's Theorem 3.4 (b) with $H_i = P_i^{x_i}$ and $g_i = x_i^{-1}$ for $i = 1, 2, \ldots, s$ to conclude that W is conjugate to $U \cap V$. This justifies the assertion labelled $(3.\beta)$ and with it also $(3.\alpha)$.

If $[L]$ is a lower bound for $[U]$ and $[V]$ in $\mathcal{F}r(G)$, then $L \le U^x$ and $L \le V^y$ for suitable $x, y \in G$. By $(3.\alpha)$ there exists an element $z \in G$ such that $L^z \le U \cap V$, whence $[L] \preceq [U \cap V]$. Therefore $[U \cap V]$ is the infimum of $[U]$ and $[V]$ in $\mathcal{F}r(G)$, and, of course, $[U \cap V]$ belongs to the image of ν by (3.3)(d).

We now turn our attention to the existence of a supremum (least upper bound) for $[U]$ and $[V]$ in $\mathcal{F}r(G)$. If $[U] \preceq [T]$ and $[V] \preceq [T]$, then $\langle U^x, V^y\rangle \le T$ for suitable $x,\ y \in G$. Since $\Sigma \searrow U$ and $\Sigma \searrow V$ and U and V are pronormal in G by (3.3)(a), we can deduce from Fischer's theorem 3.4 (a) that some conjugate of $\langle U, V\rangle$ is contained in $\langle U^x, V^y\rangle$ and hence that $[\langle U, V\rangle] \preceq [T]$. By (3.3)(d) the product $UV = \langle U, V\rangle$ is a normally-embedded subgroup of G and $\Sigma \searrow UV$; it follows that $[UV]$ belongs to $\nu(\mathcal{N}e\,(\Sigma))$ and is the supremum for $[U]$ and $[V]$ in $\mathcal{F}r(G)$.

This completes the proof that the lattice $\nu(\mathcal{N}e\,(\Sigma))$ is rigidly embedded in the frame of G. ∎

We have been unable to decide whether the rigidly-embedded lattice of conjugacy classes of normally-embedded subgroups of G can be identified poset-theoretically in $\mathcal{F}r(G)$. For a p-group it can, because then 'normally-embedded' means 'normal', and Mainardis shows in [5] that the normal subgroup lattice of a finite p-group is determined poset-theoretically by its frame. In general, however, normal subgroups are not determined by the frame of a group . For instance, the two groups of order 6 both have square frames, and $\mathbb{Z}_6$ has four normal subgroups while S_3 has only three. However, all four subgroups of S_3 are normally embedded.

4. Making Ready

In the Section 5 we will study subgroups U of a finite soluble group G that are the intersections of certain maximal subgroups — the prefrattini subgroups are examples of such subgroups — with a view to showing that their image under the map $U \mapsto [U]$ behaves well in the frame of G. In this section we develop some preparatory machinery to pave the way.

(4.1) Definition. Let p be a prime. A maximal subgroup M of a group G is said to be *p-maximal* if the index $|G : M|$ is a power of p. It is well known that every maximal subgroup of a p-soluble is either p-maximal or else has index prime to p.

We recall the definition of the *socle* $\mathrm{Soc}(G)$ of a group G as the product of its minimal normal subgroups and note the fact that it is the direct product of a suitable subset of them. The notation $M <\cdot G$ will mean that M is a maximal subgroup of G, and as usual $\phi(G)$ denotes the Frattini subgroup, the intersection of all the maximal subgroups of G.

(4.2) Proposition. *Let p be a prime, and let G be a p-soluble group with $O_{p'}(G) = \phi(G) = 1$. Let $Q \in \mathrm{Hall}_{p'}(G)$, and put $Q_0 = Q \cap O_{p,p'}(G)$ and $N = N_G(Q_0)$. Let*

$$K = [\mathrm{Soc}(G), Q_0],$$

and assume that $G \neq \mathrm{Soc}(G)$. Then the following conclusions hold:

(a) $1 \neq K \trianglelefteq G$ and N complements K in G.

(b) If M is a maximal subgroup of G containing Q but not K, then $N \leq M$.

(c) $N = \bigcap\{M <\cdot G \mid Q \leq M$ and $MK = G\}$.

(d) The complements to K in G form a conjugacy class of G.

Proof. We first note some immediate consequences of the hypotheses. Since $O_{p'}(G) = 1$, the Fitting subgroup is a p-group, and since $\phi(G) = 1$, it coincides with $\mathrm{Soc}(G)$, which is therefore elementary abelian and completely reducible as an $\mathbb{F}_pG$-module. Furthermore, since $O_{p',p}(G)$, which coincides with $\mathrm{Soc}(G)$, contains its centralizer, $\mathrm{Soc}(G)$ is a self-centralizing normal subgroup of G. The hypothesis that $G \neq \mathrm{Soc}(G)$ implies that $\mathrm{Soc}(G)$ is a proper subgroup of $O_{p,p'}(G)$ and therefore that $Q_0 \neq 1$. Since $\mathrm{Soc}(G)$ is abelian and is not centralized by Q_0, we have

$$1 \neq [\mathrm{Soc}(G), Q_0] = [\mathrm{Soc}(G), \mathrm{Soc}(G)Q_0] = [\mathrm{Soc}(G), O_{p,p'}(G)] \trianglelefteq G.$$

It follows that K is a non-trivial normal subgroup of G which is a product of non-central minimal normal p-subgroups of G.

(a) Observe that $Q_0 \in \mathrm{Hall}_{p'}(KQ_0)$ and that $KQ_0 = O^p(O_{p,p'}(G))$ by A,12.4 of [2]; thus the subgroup KQ_0 is normal in G, and we can apply the Frattini argument to conclude that $G = N_G(Q_0)Q_0K = NK$. Furthermore,

$$N \cap K = N_K(Q_0) \leq C_K(Q_0) \leq C_{\mathrm{Soc}(G)}(Q_0) \cap [\mathrm{Soc}(G), Q_0] = 1$$

by A,12.4 of [2] again, and therefore N complements K in G.

(b) Since K is a product of minimal normal subgroups of G, the hypothesis $MK = G$ implies that one of these, U say, in not contained in M. Therefore $MU = G$ and $M \cap U = 1$. Since $U \leq K = [K, Q_0]$, we have $C_U(Q_0) = 1$.

Let $n \in N$ and write $n = mu$ with $m \in M$ and $u \in U$. Then $Q_0 = (Q_0)^{mu}$, and therefore $(Q_0)^{u^{-1}} = (Q_0)^m \leq M$ since $Q_0 \leq Q \leq M$ by hypothesis. Let $g \in Q_0$. Then $ugu^{-1} \in u(Q_o)u^{-1} \leq M$ and so $[g, u^{-1}] = g^{-1}(ugu^{-1}) \in M$. However, the normality of U implies that $(g^{-1}ug)u^{-1} \in U$, and therefore $[u^{-1}, g] \in M \cap U = 1$. Since g was an arbitrary element of Q_0, it follows that $u \in C_U(Q_0) = 1$. Therefore $n = m \in M$ and we have shown that $N \leq M$, as claimed.

(c) Set

$$I = \bigcap \{ M <\!\cdot\, G \mid Q \leq M \text{ and } MK = G \}. \tag{4.α}$$

Let U be a minimal normal subgroup of G contained in K. Since $\phi(G) = 1$, there is a maximal subgroup M of G not containing U; such an M satisfies $MK = G$, and by the conjugacy of p-complements in p-soluble groups, we can assume that $Q \leq M$. Since $N \leq I$ by Part (b), we have $IK \geq NK = G$ by Part (a), and as K is abelian, it follows that $I \cap K \trianglelefteq KI = G$. By definition of I, the normal subgroup $I \cap K$ is contained in every maximal subgroup of G satisfying $MK = G$. Since maximal subgroups of G either contain K or satisfy $MK = G$, it follows that $I \cap K$ is contained in every maximal subgroup of G. The hypothesis $\phi(G) = 1$ implies that $I \cap K = 1$, and by the Dedekind law we then have $I = I \cap G = I \cap NK = N(I \cap K) = N$, as asserted.

(d) Let C be a complement to K in G. The intersection $S = C \cap O_{p,p'}(G)$ is a p-complement of $O_{p,p'}(G)$ and is therefore conjugate to Q_0. Since $C \leq N_G(S)$, a conjugate C^g of C lies in $N_G(Q_0) = N$, and by order considerations $C^g = N$. ■

The next lemma is an analogue for p-soluble groups of Theorem A,16.6 of [D-K].

(4.3) Lemma. *Let p be a prime, and let L and M be inconjugate p-maximal subgroups of a p-soluble group G. Then*

(i) $LM = G$, and

(ii) if $\mathrm{Core}_G(L) \not\subseteq M$, then $L \cap M$ is a p-maximal subgroup of M.

Proof. Recall that the *core* of a subgroup of G is the intersection of its conjugates in G. If X is a p-maximal subgroup of a p-soluble group G, then $G/\mathrm{Core}_G(X)$ has a unique minimal normal subgroup $R/\mathrm{Core}_G(X)$, which is a p-chief factor of G complemented by X. Furthermore, all complements of $R/\mathrm{Core}_G(X)$ are conjugate to X in G.

Set $T = \mathrm{Core}_G(L)$, and let R/T be the unique minimal normal subgroup of G/T. Since M is not conjugate to L, it does not complement R/T, and so either $T \not\leq M$ or $R \leq M$. In the first case, $LM \leq TM = G$, while in the second case, $LM \leq LR = G$. This justifies Part (i).

If $T \not\leq M$, then $TM = G$, and there exists a p-chief factor U/V of G complemented by M with $U \leq T$. By the Dedekind law, $(L \cap M)U = L$, and so under the isomorphism $mV \mapsto mU$ from M/V to G/U, the subgroup $(L \cap M)/V$ is mapped to $(L \cap M)U/U = L/U$, which is a p-maximal subgroup of G/U. It follows that $(L \cap M)/V$ is a p-maximal subgroup of M/V, and therefore $L \cap M$ is p-maximal in M, as asserted in Part (ii). ■

(4.4) Lemma. *Let K be a product of minimal normal p-subgroups of a p-soluble group G. Let $M_1, M_2, \ldots, M_r$ be a set of p-maximal subgroups of G which satisfy the condition:*

$$(4.\beta) \qquad (\textstyle\bigcap_{i=1}^{r} M_i) \cap K = 1.$$

Then the set $\{M_1, M_2, \ldots, M_r\}$ contains a subset whose intersection complements K in G.

Proof. We argue by induction on r, the number of maximal subgroups. If one of the maximal subgroups M_i contains K, we can omit it from the list without changing the hypotheses; therefore assume that $M_iK = G$ for $i = 1, \ldots, r$. Let $M_i \cap K = R_i$, a normal subgroup of G, and label the M_i's so that set $\{R_i\}_{i=1}^{s}$ is minimal subject to the condition:

$$\textstyle\bigcap_{i=1}^{s} R_i = 1.$$

This minimal requirement implies that $R_i \neq R_j$ when $i \neq j$ and therefore that $M_iR_j = G$. If $s = 1$, then K is a minimal normal subgroup of G complemented by M_1 and we are done. Therefore suppose that $s \geq 2$.

Evidently $(\bigcap_{i=1}^{s} M_i) \cap K = \bigcap_{i=1}^{s} R_i = 1$ and so the set $M_1, M_2, \ldots, M_s$ also satisfies Condition $(4.\beta)$. Moreover, setting $M = M_1$ and $R = R_1$, we see from Lemma 4.3 that $\{M \cap M_i\}_{i=2}^{s}$ is a set of $s-1$ maximal subgroups of M satisfying the condition:

$$(4.\gamma) \qquad (\textstyle\bigcap_{i=2}^{s} (M_i \cap M)) \cap R = 1.$$

Since R is a completely-reducible normal subgroup of G, it is a product of certain minimal normal subgroups of G, and because $G = MK$ and K is abelian, these are also minimal normal subgroups of M. Therefore we can apply induction with M in place of G and R in place of K to conclude that the intersection of a suitably-labelled subset $\{M \cap M_i\}_{i=2}^{t}$ of the set $\{M \cap M_i\}_{i=2}^{s}$ complements R in M and hence also complements K in G. Since $\bigcap_{i=2}^{t} (M \cap M_i) = \bigcap_{i=1}^{t} M_i$, we have the desired conclusion. ■

(4.5) Corollary. *Let $\{M_i\}_{i=1}^{r}$ be a set of p-maximal subgroups of a p-soluble group G, each containing a given p-complement Q of G, and assume that $Q \neq 1$. Assume further that $\bigcap_{i=1}^{r} M_i$ contains no non-trivial normal subgroup of G. This assumption implies, in particular, that $O_{p'}(G)\phi(G) = 1$, and so in accordance with the notation of Proposition 4.2, we set*

$$Q_0 = Q \cap O_{p,p'}(G), \quad K = [\mathrm{Soc}(G), Q_0], \quad \textit{and} \quad N = N_G(Q_0).$$

Then the subscripts for the M_i's can be so chosen that

(a) $\bigcap_{i=1}^{t} M_i = N$, *and*

(b) $M_j \cap N$ *is a p-maximal subgroup of N for $j = t+1, \ldots, r$.*

Proof. Our assumptions imply that $K \neq 1$ and therefore that K is not contained in all the M_i's. Hence we can renumber the M_i's so that for some integer $t \geq 1$ we have

$$M_iK = G \text{ for } i = 1, \ldots, t \quad \text{and} \quad K \leq M_i \text{ for } i = t+1, \ldots, r.$$

It follows from the assumption that 1 is the only normal subgroup of G common to all the M_i's that

$$\left(\bigcap_{i=1}^t M_i\right) \cap K = 1, \tag{4.δ}$$

and so by Lemma 4.4 the intersection of a suitable subset of $\{M_i\}_{i=1}^t$ complements K in G. But by Part (b) of Proposition 4.2, the subgroup $N = N_G(Q_0)$ is contained in each of $M_1, \ldots, M_t$. Since N complements K by Part (a) of Proposition 4.2, Assertion (a) of this Corollary now follows.

Since $(M_j \cap N)K/K = M_j/K \lessdot G/K = NK/K$ for $j = t+1, \ldots, r$, the isomorphism between G/K and N implies that $M_j \cap N$ is a p-maximal subgroup of N, which is Assertion (b). ■

Since maximal subgroups are pronormal, Part (a) of the next result can be seen as an analogue of the fact that, in finite soluble groups, a Hall system reduces into a unique conjugate of a pronormal subgroup.

(4.6) Lemma. *(a) Let p be a prime and let M be a maximal subgroup of a p-soluble group G. If $g \in G$ and $M \cap M^g$ contains a p-complement Q of G (or equivalently, if $|G : (M \cap M^g)|$ is a power of p), then $M = M^g$.*

(b) If Σ is a Hall system of a soluble group G and $M \lessdot G$, there is a unique conjugate of M into which Σ reduces.

Proof. (a) If $Q \leq M \cap M^g$, then Q and $Q^{g^{-1}}$ are p-complements of M. By the conjugacy of p-complements, there exists an element $m \in M$ such that $Q^m = Q^{g^{-1}}$, and so $gm \in N_G(Q)$. If $M \trianglelefteq G$, certainly $M = M^g$. On the other hand, if $M \ntrianglelefteq G$, then $N_G(Q) \leq M$ by Lemma I,6.5 of [2] — for although the lemma is stated and proved only for soluble groups, the proof given there works equally well for p-maximal subgroups M of a p-soluble group. It follow that gm, and hence g itself, is in M, and we conclude that $M = M^g$.

(b) Since M is p-maximal for some prime p, it contains a Sylow p-complement of G; thus some conjugate M^g of M contains the p-complement in Σ. Since $\Sigma \searrow M^g$ if and only if M^g contains the p-complement of Σ, the assertion of uniqueness in Part (b) now follows from Part (a). ■

(4.7) Notation.

(a) If $\mathcal{M} = \{M_1, M_2, \ldots, M_t\}$ is a set of subgroups of G, we denote their intersection by

$$\mathcal{I}(\mathcal{M}) = \bigcap_{i=1}^t M_i \, .$$

(b) Let $\mathcal{M} = \{M_1, M_2, \ldots, M_t\}$ be a set of p-maximal subgroups of a p-soluble group G, and let Q be a p-complement of G. For $i = 1, \ldots, t$ let $(M_i)^{g_i}$ be the unique conjugate of M_i that contains Q (assured by Lemma 4.6). Then $\mathcal{M}_Q$ will denote the set

$$\mathcal{M}_Q = \{(M_1)^{g_1}, (M_2)^{g_2}, \ldots, (M_t)^{g_t}\}.$$

(c) Likewise, if $\mathcal{M} = \{M_1, M_2, \ldots, M_t\}$ is a set of maximal subgroups, and Σ a Hall system, of a soluble group G, then $\mathcal{M}_\Sigma$ will denote the uniquely determined set

$$\mathcal{M}_\Sigma = \{(M_1)^{g_1}, (M_2)^{g_2}, \ldots, (M_t)^{g_t}\}s$$

of conjugates into each of which Σ reduces.

(4.8) Proposition. *Let p be a prime. Let $\mathcal{M} = \{M_1, M_2, \ldots, M_t\}$ be a set of pairwise-inconjugate p-maximal subgroups of a p-soluble group G, and let $Q \in \mathrm{Hall}_{p'}(G)$. Then $\mathcal{I}(\mathcal{M})$ is conjugate to a subgroup of $\mathcal{I}(\mathcal{M}_Q)$.*

Proof. We argue by induction on $|G|$. Let R denote the core of $\mathcal{I}(\mathcal{M})$ in G. Then $\{M_i/R\}$ is a set of pairwise-inconjugate p-maximal subgroups of G/R, and if $R \neq 1$, it follows by induction that $\mathcal{I}(\mathcal{M})/R = \left(\bigcap_{i=1}^t (M_i/R)\right)$ is conjugate to a subgroup of $\left(\bigcap_{i=1}^t (M_i/R)^{g_i R}\right) = \mathcal{I}(\mathcal{M}_Q)/R$, where the elements g_i are those derived in (4.7)(b). Lifting this statement to G yields the desired conclusion in this case. Therefore suppose that $R = 1$ and, in particular, that $O_{p'}(G)\phi(G) = 1$ since $O_{p'}(G)\phi(G)$ is contained in all p-maximal subgroups of G.

If $G = \mathrm{Soc}(G)$, then G is an elementary abelian p-group and the conclusion of the Lemma is obvious. We can therefore suppose that G satisfies the hypotheses of Proposition 4.2 and adopt the notation of that result; in particular, we set $Q_0 = Q \cap O_{p,p'}(G)$, $K = [\mathrm{Soc}(G), Q_0]$, and $N = N_G(Q_0)$, recalling that (i) $K \neq 1$, (ii) $Q \leq N$, and (iii) N complements K in G.

Since $\mathcal{I}(\mathcal{M})$ is core-free, we can suppose, for some $s \geq 1$, that the M_i have been labelled so that

$$M_i K = G \text{ for } i = 1, \ldots, s \quad \text{and} \quad K \leq M_i \text{ for } i = s+1, \ldots, t.$$

Since $(\bigcap_{i=1}^s M_i) \cap K$ is a normal subgroup of G, it is contained in the core of $\mathcal{I}(\mathcal{M})$ and is therefore trivial. Hence $M_1, M_2, \ldots, M_s$ is a set of maximal subgroups of G satisfying the hypotheses of Lemma 4.4. Since K is completely reducible, the conclusion of that Lemma tells us that $(\bigcap_{i=1}^s M_i)$ is contained in a complement to K in G and is therefore conjugate to a subgroup of N by Part (d) of Proposition 4.2. Thus we can find an element $x \in G$ such that

$$\left(\bigcap_{i=1}^s (M_i)\right)^x = \bigcap_{i=1}^s (M_i)^x \leq N.$$

For $i = s+1, \ldots, t$ we have $K \leq (M_i)^x = ((M_i)^x \cap N)K$ by the Dedekind law. The isomorphism between N and G/K implies that $(M_i)^x \cap N$ is a p-maximal subgroup of N and that $(M_i)^x \cap N$ is not conjugate to $(M_j)^x \cap N$ when $i \neq j$. If, according to (4.7)(b), elements g_i are chosen so that $Q \leq (M_i)^{g_i}$ for $i = 1, 2, \ldots, t$, observe that $(M_i)^{g_i} \cap N$ is the unique N-conjugate of $(M_i)^x \cap N$ which contains Q. By induction

$$\left(\bigcap_{i=s+1}^t (M_i)^x\right) \cap N = \bigcap_{i=s+1}^t ((M_i)^x \cap N)$$

is conjugate by an element $n \in N$ to a subgroup of

$$\bigcap_{i=s+1}^t ((M_i)^{g_i} \cap N) = N \cap \left(\bigcap_{i=s+1}^t (M_i)^{g_i}\right) = \bigcap_{i=1}^t (M_i)^{g_i}$$

since the subgroup

$$(M_1)^{g_1} \cap (M_2)^{g_2} \cap \ldots \cap (M_s)^{g_s}$$

not only contains $N = N_G(Q_0)$ by Part (c) of Proposition 4.2, but actually equals N by Lemma 4.4. Thus

$$\left(\bigcap_{i=1}^{t} M_i\right)^{xn} = (\bigcap_{i=1}^{s}(M_i)^x)^n \cap \left(\bigcap_{i=s+1}^{t}(M_i)^x\right)^n leN^n \cap \left(\bigcap_{i=s+1}^{t}(M_i)^x\right)^n \leq \bigcap_{i=1}^{t}(M_i)^{g_i}$$

in other words, $(\mathcal{I}(\mathcal{M}))^{xn} \subseteq \mathcal{I}(\mathcal{M}_Q)$, which is the desired conclusion. ∎

(4.9) Proposition. *Let p be a prime, and let Q be a p-complement of a p-soluble group G. Let $\mathcal{M} = \{M_1\,, M_2\,, \ldots, M_t\}$ be a set of distinct p-maximal subgroups of G containing Q (so that $\mathcal{M} = \mathcal{M}_Q$ in the above notation of 4.9 (a)). Then $\mathcal{I}(\mathcal{M})$ is a pronormal subgroup of G containing Q.*

Proof. If $\mathcal{I}(\mathcal{M})$ contains a non-trivial normal subgroup R of G, the result follows easily by induction on the group order applied to the set $\{M_1/R,\, M_2/R,\, \ldots,\, M_t/R\}$ of p-maximal subgroups of G/R, for the conclusion that $\mathcal{I}(\mathcal{M})/R$ is pronormal in G/R implies that $\mathcal{I}(\mathcal{M})$ is pronormal in G. Since normal subgroups are certainly pronormal the conclusion of the Proposition is trivially true if G is abelian We may therefore assume that

(i) $O_{p'}(G)\phi(G) = 1$,

(ii) $K \neq 1$ in the notation of Proposition 4.2, and

(iii) that the labels have been chosen (with $s \geq 1$) so that

$$\bigcap_{i=1}^{s} M_i = N = N_G(Q_0) \quad \text{and} \quad K \leq M_j \text{ for } j = s+1, \ldots, t.$$

As in the proof of Proposition 4.8, the subgroups $M_i \cap N$ are p-maximal in N. If $H = \bigcap_{i=1}^{t} M_i$, then $H = \bigcap_{i=s+1}^{t}(M_i \cap N)$, and by induction H is a pronormal subgroup of N. We will now deduce that H is pronormal in G.

Let $g \in G$. We must find an element $x \in \langle H, H^g\rangle$ such that $H^g = H^x$. Write $g = nk$ with $n \in N$ and $k \in K$. The subgroup $Q_0 = Q \cap O_{p,p'}(G)$, as a Hall subgroup of a normal subgroup, is certainly pronormal in G, and so there exists an element $z \in \langle Q_0, (Q_0)^k\rangle$ such that $(Q_0)^k = (Q_0)^z$. Since

$$\langle Q_0, (Q_0)^k\rangle = \langle Q_0, (Q_0)^{nk}\rangle \leq \langle H, H^g\rangle,$$

we have $z \in \langle H, H^g\rangle$ and therefore

$$\langle H, H^{gz^{-1}}\rangle \leq \langle H, H^g\rangle. \tag{4.ϵ}$$

Because $N^k = N_G((Q_0)^k) = N_G((Q_0)^z) = N^z$, we have $kz^{-1} \in N_G(N) = N$, and therefore $k = n_0 z$ for some $n_0 \in N$. But $gz^{-1} = nkz^{-1} = nn_0zz^{-1} = nn_0 \in N$, and from (4.$\epsilon$) can deduce that

$$\langle H, H^{nn_0}\rangle \leq \langle H, H^g\rangle.$$

But our inductive conclusion that H is pronormal in N then tells us that $\langle H, H^{nn_0}\rangle$ contains an element y such that $H^{nn_0} = H^y$. Setting $x = yz$, we conclude that $x \in \langle H, H^g\rangle$ and that

$$H^x = H^{yz} = H^{nn_0 z} = H^{nk} = H^g,$$

as desired. ■

We recall that a subgroup U of a finite soluble group G is called *system permutable* if G has a Hall system Σ such that $US = SU$ for all $S \in \Sigma$. A brief discussion of their properties can be found on pages 230 – 34 of [2]. The following characterization of system permutable subgroups is due to Avinoam Mann:

(4.10) Lemma. *A subgroup U of a finite soluble group G is system permutable if and only if it can be written as an intersection of the form*

$$U = \textstyle\bigcap_{i=1}^{r} U_i \,, \tag{4.ζ}$$

where U_i is a subgroup of p_i-power index in G and $\{p_1\,,\, p_2\,,\,\ldots\,,\, p_r\}$ is a set of distinct primes dividing $|G|$.

Proof. First suppose that U is a Σ-permutable subgroup of G. If $\sigma(G) = \{p_1\,,\, p_2\,,\,\ldots\,,\, p_s\}$, let G^{p_i} be the p_i-complement in Σ, and set $U_i = UG^{p_i}$, which is a subgroup of G by our supposition. Evidently $U \le U_i$, and so

$$U \subseteq \textstyle\bigcap_{i=1}^{s} U_i \,.$$

If n_p denotes the highest power of the prime p dividing a natural number n, we have

$$|U_i|_p = \left(\frac{|U||G^{p_i}|}{|U \cap G^{p_i}|}\right)_p = \frac{|U|_p|G^{p_i}|_p}{|U \cap G^{p_i}|_p} = |U|_p$$

and it follows that the order of a Sylow p_i-subgroup of $\bigcap_{i=1}^{r} U_i$ is at most $|U|_p$. Hence

$$|U| \ge |\textstyle\bigcap_{i=1}^{r} U_i| \quad \text{and therefore} \quad U = \textstyle\bigcap_{i=1}^{r} U_i \,.$$

Conversely, let U be a subgroup of the form indicated by Equation $4.\iota$, and by adjoining additional $U_i = G$ to the intersection if necessary, suppose without loss of generality that $\sigma(G) = \{p_1\,,\, p_2\,,\,\ldots\,,\, p_r\}$. Because the subgroups U_i have pairwise coprime indices, we have $|U|_{p_i} = |U_i|_{p_i}$. For $1 = 1, 2, \ldots, r$ let G^{p_i} be a Sylow p_i-complement of U_i, and note that G^{p_i} is also a Sylow p_i-complement of G. Since U and G^{p_i} have coprime indices in U_i, the subgroup U permutes with G^{p_i} for each value of i. From I,4.26 of [2] we can then deduce that U permutes with the Hall system of G generated by the complement basis $\{G^{p_i}\}_{i=1}^{r}$ of G. ■

5. Maximal subgroups in the Frame of a Soluble Group

Some striking properties of their maximal subgroups set soluble finite groups apart from insoluble ones. The tractable structure of the primitive soluble groups is one reason for this (see Section 15 of Chapter A of [2]). In this section we study subgroups that are the intersections of certain maximal subgroups of a finite soluble group and investigate how their conjugacy classes behave in its frame.

(5.1) Proposition. *Let $\mathcal{M} = \{M_1, M_2, \ldots, M_r\}$ and $\overline{\mathcal{M}} = \{\overline{M}_1, \overline{M}_2, \ldots, \overline{M}_s\}$ be two sets of p-maximal subgroups of a p-soluble group G, each of them containing a given p-complement Q of G. Let $H = \bigcap_{i=1}^r M_i$ and $\overline{H} = \bigcap_{i=1}^s \overline{M}_i$. Then*

(a) $\langle H, \overline{H}\rangle = H\overline{H}$, and

(b) $H\overline{H} = \bigcap\{M <\cdot\, G \mid H\overline{H} \leq M\}$.

Proof. We argue by induction on $|G|$.

Case 1: There exists a minimal normal subgroup T of G contained in $\overline{H}$.

If T is also contained in H, induction applied to the quotient group G/T yields the statement (a) $\langle H, \overline{H}\rangle/T = (H\overline{H})/T$, and (b) $(H\overline{H})/T = \bigcap\{M/T <\cdot\, G/T \mid (H\overline{H})/T \leq M/T\}$. Lifting these statements to G gives the desired conclusions.

If, on the other hand, T is not contained in H, it is not contained in one of the maximal subgroups in $\mathcal{M}$, say $T \not\leq M_1$. In this case, T is an elementary abelian p-subgroup complemented by M_1 for some prime p. Since the distinct subgroups $M_1, M_2, \ldots, M_r$ all contain Q, by Lemma 4.6(a) they are pairwise inconjugate. It then follows from Lemma 4.4(b) that the subgroup $M_i \cap M_1$ is a p-maximal subgroup of M_1 containing Q for $i = 2, \ldots, r$, and evidently $\cap_{i=2}^r (M_i \cap M_1) = H$. Likewise, $\overline{M}_i \cap M_1$ is a p-maximal subgroup of M_1 containing Q for $i = 1, \ldots, s$ and $\bigcap_{i=1}^s (\overline{M}_i \cap M_1) = \overline{H} \cap M_1$. Since $|M_1| < |G|$, by induction we have

(a) $\langle H, (\overline{H} \cap M_1)\rangle = H(\overline{H} \cap M_1)$, and

(b) $H(\overline{H} \cap M_1) = \bigcap\{L <\cdot\, M_1 \mid H(\overline{H} \cap M_1) \leq L\}$.

Since $H(\overline{H} \cap M_1)$ is a subgroup of G, so also is $H(\overline{H} \cap M_1)T$, which is equal to $H\overline{H}$ by the Dedekind law. Therefore $\langle H, \overline{H}\rangle = H\overline{H}$. Under the isomorphism between M_1 and G/T, we see that $LT <\cdot\, G$ when $L <\cdot\, M_1$, and that $H\overline{H}$ is equal to

$$H(\overline{H} \cap M_1)T = (\bigcap\{L <\cdot\, M_1 \mid H(\overline{H} \cap M_1) \leq L\})T = \bigcap\{LT \mid H(\overline{H} \cap M_1) \leq L <\cdot\, M_1\}.$$

Thus $H\overline{H}$ is the intersection of the maximal p-subgroups of G that contain it, and this completes the proof in Case 1.

A symmetric argument applies when H contains a non-trivial normal subgroup of G. Therefore we are left with the following situation.

Case 2: $\mathrm{Core}_G(H) = \mathrm{Core}_G(\overline{H}) = 1$.

In this case, $O_{p'}(G)\phi(G) = 1$ and so $F(G) = \mathrm{Soc}(G)$ is a product of minimal normal p-subgroups of G. Adopting the notation of Proposition 4.2 with $Q_0 = Q \cap O_{p,p'}(G)$, $N = N_G(Q_0)$ and $K = [\mathrm{Soc}(G), Q_0]$, we also have

$$(5.\alpha) \qquad \left(\bigcap_{i=1}^{r} M_i\right) \cap K = 1 \quad \text{and} \quad \left(\bigcap_{i=1}^{s} \overline{M}_i\right) \cap K = 1$$

By Corollary 4.5 the maximal subgroups can be renumbered so that

$$(5.\beta) \qquad \bigcap_{i=1}^{t} M_i = \bigcap_{i=1}^{\overline{t}} \overline{M}_i = N,$$

and so that for $i = t+1, \ldots, r$ and $j = \overline{t}+1, \ldots, s$ the subgroups $M_i \cap N$ and $\overline{M}_j \cap N$ are p-maximal subgroups of N all containing the p-complement Q of N. Moreover, we have

$$H = \bigcap_{i=t+1}^{r}(M_i \cap N) \quad \text{and} \quad \overline{H} = \bigcap_{i=\overline{t}+1}^{s}(\overline{M}_i \cap N)$$

Since $|N| < |G|$, we can apply induction to the two sets

$$\{(M_{t+1} \cap N),\, (M_{t+2} \cap N), \ldots,\, (M_r \cap N)\} \text{ and } \{(\overline{M}_{\overline{t}+1} \cap N),\, (\overline{M}_{\overline{t}+2} \cap N), \ldots,\, (\overline{M}_s \cap N)\}$$

of p-maximal subgroups of N to deduce that

$$(\mathrm{a}^*) \ \langle H, \overline{H}\rangle = H\overline{H} \quad \text{and} \quad (\mathrm{b}^*) \ H\overline{H} = \bigcap \{L \lessdot N \mid H\overline{H} \le L\}$$

By the Dedekind law, it follows from (b*) that $H\overline{H}$ is equal to

$$(5.\gamma) \qquad N \cap \left(\bigcap \{LK \mid H\overline{H} \le L \lessdot N\}\right) = \left(\bigcap_{i=1}^{t} M_i\right) \cap \left(\bigcap \{LK \mid H\overline{H} \le L \lessdot N\}\right)$$

and since $LK \lessdot G$ when $L \lessdot N$, we conclude from $(5.\gamma)$ that $H\overline{H}$ is the intersection of p-maximal subgroups of G. ∎

(5.2) Notation. Let Σ be a Hall system of a finite soluble group G. We will use $\mathcal{MS}_\Sigma(G)$ to denote the set of all maximal subgroups of G into which Σ reduces, and $\mathcal{IMS}_\Sigma(G)$ denote the set of all intersections of subsets of these maximal subgroups; thus

$$\mathcal{IMS}_\Sigma(G) = \{J \le G \mid \exists\ M_1, \ldots, M_j \in \mathcal{MS}_\Sigma(G) \text{ such that } J = \bigcap_{i=1}^{j} M_i\}.$$

Evidently $\mathcal{MS}_\Sigma(G) \subseteq \mathcal{IMS}_\Sigma(G)$, and the convention that G is the 'empty' intersection means that $G \in \mathcal{IMS}_\Sigma(G)$.

(5.3) Lemma. *Let Σ be a Hall system of a finite soluble group G, and let $J \in \mathcal{IMS}_\Sigma(G)$. If $T \trianglelefteq G$, then $JT \in \mathcal{IMS}_\Sigma(G)$*

Proof. By induction we can suppose without loss of generality that T is a minimal normal subgroup of G. By hypothesis G has maximal subgroups $M_1, M_2, \ldots, M_j$ into each of which Σ reduces such that

$$J = M_1 \cap M_2 \cap \ldots \cap M_j\,,$$

If $T \leq J$, the result is trivially true. We can therefore suppose that one of the M_i's, without loss of generality say M_1, does not contain T. Since M_1 complements T in G, it follows from Lemma 4.6 that

$$M_i \cap M_1 \text{ is a maximal subgroup of } M_1 \text{ for } i = 2, \ldots, j.$$

and hence that each $T(M_i \cap M_1)$ is a maximal subgroup of G. Using the fact that

$$T \cap ((M_i \cap M_1)(M_k \cap M_1)) = 1 = (T \cap (M_i \cap M_1))(T \cap (M_k \cap M_1)),$$

for all $1 \leq i, k \leq j$, we can repeatedly apply of Lemma A,1.2 of [2] to show that

$$\bigcap_{i=2}^{j} T(M_i \cap M_1) = T\left(\bigcap_{i=2}^{j} M_i \cap M_1\right) = TJ = JT.$$

By I,4.22 of [2] we know that Σ reduces into each maximal subgroup $T(M_i \cap M_1)$ of G, and so $JT \in \mathcal{IMS}_\Sigma(G)$, as claimed. ■

(5.4) Proposition. *Let p be a prime, let Σ be a Hall system of a finite soluble group G, and let Q be the p-complement in Σ. Assume that $\mathcal{IMS}_\Sigma(G)$ contains two subgroups of the form QP and $Q\overline{P}$, where P and $\overline{P}$ are p-groups such that $P^g \leq \overline{P}$ for some $g \in G$. Then $QP \leq Q\overline{P}$.*

Proof. We proceed by induction on $|G|$, following the well-trodden paths of earlier proofs. By hypothesis there exist p-maximal subgroups $M_1, M_2, \ldots, M_r$ and $\overline{M}_1, \overline{M}_2, \ldots, \overline{M}_{\overline{r}}$ of G containing Q such that

$$QP = M_1 \cap M_2 \cap \ldots \cap M_r \quad \text{and} \quad Q\overline{P} = \overline{M}_1 \cap \overline{M}_2 \cap \ldots \cap \overline{M}_{\overline{r}}.$$

Let $R = O_{p'}(G)\phi(G)$ and note that $R \leq QP \cap Q\overline{P}$. If $R \neq 1$, the result follows by an easy induction argument applied to G/R. Therefore we can suppose that $O_{p'}(G)\phi(G) = 1$. In accordance with the notation of Proposition 4.2, we set

$$Q_0 = Q \cap O_{p,p'}(G), \quad K = [\mathrm{Soc}(G), Q_0], \quad \text{and} \quad N = N_G(Q_0),$$

and since the result is trivially true when $G = \mathrm{Soc}(G)$, we can also suppose that $K \neq 1$.

If the core of QP is non-trivial, it contains a minimal normal subgroup A of G, which is necessarily a p-group because $O_{p'}(G) = 1$ and which is therefore contained in P. Then $A \leq P^g \leq \overline{P}$, and we can apply induction to the subgroups QP/A and $\overline{P}/A$ in G/A to conclude that $QP/A \leq Q\overline{P}/A$; in this case the result is clear.

If the normal subgroup $\mathrm{Core}_G(Q\overline{P}) \neq 1$, it contains a minimal normal p-subgroup B of G, and B is therefore contained in $\overline{P}$. By Lemma 5.3, the subgroup QPB belongs to

$\mathcal{IMS}_\Sigma(G)$, and evidently $(PB)^g = (P)^g B \le \overline{P}$. Applying induction to the subgroups QPB/B and $\overline{P}/B$ in G/B, we conclude that $QPB/B \le Q\overline{P}/B$, and the result again follows. Hence we can suppose that $\mathrm{Core}_G(Q\overline{P}) = 1$. By Corollary 4.5 the subscripts for the M_i's and $\overline{M}_i$'s can be chosen, with t, $\overline{t} \ge 1$, so that

(a) $\bigcap_{i=1}^{t} M_i = \bigcap_{i=1}^{\overline{t}} \overline{M}_i = N$, and

(b) $M_j \cap N$ and $\overline{M}_k \cap N$ are p-maximal subgroups of N for $j = t+1, \ldots, r$ and $k = \overline{t}+1, \ldots, \overline{r}$.

It follows that QP and $Q\overline{P}$ both belong to $\mathcal{IMS}_{\Sigma\cap N}(N)$ and, in particular, that $P^g \le N$.

Now recall that N complements K in G, and write $g = kn$ with $n \in N$ and $k \in K$. Since $P \le N \cap N^{g^{-1}} = N \cap N^{k^{-1}}$, the elementary fact that

$$k^{-1} \in C_G(N \cap N^{k^{-1}})$$

implies that k centralizes P and hence that $P^g = P^n$. The hypotheses of the Proposition are now satisfied by the subgroups QP and $Q\overline{P} \in \mathcal{IMS}_{\Sigma\cap N}(N)$, and since $|N| < |G|$, we conclude by induction that $QP \le Q\overline{P}$, as desired. ∎

(5.5) Proposition. *Let Σ be a Hall system of a finite soluble group G, and let J and $\overline{J} \in \mathcal{IMS}_\Sigma(G)$. Then*

(a) J is a system permutable subgroup of G,

(b) $J \cap \overline{J} \in \mathcal{IMS}_\Sigma(G)$,

(c) $\langle J, \overline{J}\rangle = J\overline{J}$, and

(d) $J\overline{J} \in \mathcal{IMS}_\Sigma(G)$.

(e) The prefrattini subgroup of G associated with Σ is the unique smallest element of $\mathcal{IMS}_\Sigma(G)$.

Proof. Let J and $\overline{J} \in \mathcal{IMS}_\Sigma(G)$. By definition there exist two sets of maximal subgroups $\mathcal{M} = \{M_1, M_2, \ldots, M_j\}$ and $\overline{\mathcal{M}} = \{\overline{M}_1, \overline{M}_2, \ldots, \overline{M}_{\overline{j}}\}$ of G into each of which Σ reduces such that

$$J = M_1 \cap M_2 \cap \ldots \cap M_j \quad \text{and} \quad \overline{J} = \overline{M}_1 \cap \overline{M}_2 \cap \ldots \cap \overline{M}_{\overline{j}}.$$

Let $p \in \sigma(G)$, the set of primes dividing $|G|$, and let $\mathcal{M}(p)$ (respectively $\overline{\mathcal{M}}(p)$) denote the set of p-maximal subgroups in $\mathcal{M}$ (respectively $\overline{\mathcal{M}}$); they are the ones that contain the p-complement Q in Σ. Let P^* denote the Sylow p-subgroup in Σ, and set

$$J(p) = \bigcap_{M\in\mathcal{M}(p)} M \quad \text{and} \quad P = P^* \cap J(p),$$

with the usual convention that $J(p) = G$ when $\mathcal{M}(p) = \emptyset$; define $\overline{J}(p)$ and $\overline{P}$ analogously. By I,4.22(a) of [2] a Hall system that reduces into a set of subgroups also reduces into their intersection; therefore Σ reduces into $J(p)$ and, in particular, $P \in \mathrm{Syl}_p(J(p))$. Furthermore, since

$$J = \bigcap_{p\in\sigma(G)} J(p) \quad \text{and} \quad \overline{J} = \bigcap_{p\in\sigma(G)} \overline{J}(p),$$

Σ reduces into J and $\overline{J}$. It also reduces into

$$J \cap \overline{J} = M_1 \cap M_2 \cap \ldots \cap M_j \cap \overline{M}_1 \cap \overline{M}_2 \cap \ldots \cap \overline{M}_{\overline{j}},$$

whence $J \cap \overline{J} \in \mathcal{IMS}_\Sigma(G)$, as claimed in Assertion (b).

Since the subgroups $J(p)$ have p-power index in G, Mann's characterization in Proposition 4.10 implies that J and $\overline{J}$ are system permutable subgroups of G and thus justifies Assertion (a).

Since Σ reduces into $J(p)$ and $\overline{J}(p)$, we have $J(p) = PQ = QP$ and $\overline{J}(p) = \overline{P}Q = Q\overline{P}$. Therefore $J(p)\overline{J}(p) = P\overline{P}Q$ and by Proposition 5.1 (a) the subset $J(p)\overline{J}(p)$ is a subgroup of G. Since $P\overline{P} \subseteq P^*$, we have $P\overline{P} \cap Q \subseteq P^* \cap Q = 1$; therefore $|P\overline{P}Q| = |P\overline{P}||Q|$ and it follows that $P\overline{P}$ is a Sylow p-subgroup of $P\overline{P}Q = J(p)\overline{J}(p)$. Since $P\overline{P} \leq P^* \leq J(q)$ for all $q \neq p$, we have

$$P\overline{P} \in \mathrm{Syl}_p\left(\textstyle\bigcap_{s\in\sigma(G)}(J(s)\overline{J}(s))\right). \tag{5.δ}$$

But evidently $P\overline{P} \leq \langle J, \overline{J}\rangle \leq \bigcap_{s\in\sigma(G)}(J(s)\overline{J}(s))$, and so $P\overline{P} \in \mathrm{Syl}_p(\langle J, \overline{J}\rangle)$. Let n_p denote the highest power of the prime p dividing a natural number n. Since $\Sigma \searrow (J \cap \overline{J})$, we have $P \cap \overline{P} \in \mathrm{Syl}_p(J \cap \overline{J})$, and therefore

$$|J\overline{J}|_p = \frac{|J|_p|\overline{J}|_p}{|J \cap \overline{J}|_p} = \frac{|P||\overline{P}|}{|P \cap \overline{P}|} = |P\overline{P}| = |\langle J, \overline{J}\rangle|_p.$$

It follows that $|\langle J, \overline{J}\rangle| = |J\overline{J}|$, and Assertion (c) is proved.

It now follows from (5.δ) that the two subgroups $J\overline{J}$ and $\bigcap_{s\in\sigma(G)}(J(s)\overline{J}(s))$ have the same order and are therefore equal. Hence by Proposition 5.1 (b) we have

$$J\overline{J} = \textstyle\bigcap_{s\in\sigma(G)}(J(s)\overline{J}(s)) = \bigcap_{s\in\sigma(G)}\left(\bigcap\{M \lessdot G \mid J(s)\overline{J}(s) \leq M\}\right)$$

and since $J(s) \leq M \lessdot G$ implies that $\Sigma \searrow M$, we conclude that $J\overline{J} \in \mathcal{IMS}_\Sigma(G)$, as claimed in Assertion (d).

(e) This final assertion follows from V, 5.17 of [2], which characterizes the prefrattini subgroup of G associated with Σ as the intersection of all maximal subgroups of G into which Σ reduces. ■

The following theorem follows immediately from Proposition 5.5.

(5.6) Theorem. *Let Σ be a Hall system of a finite soluble group G. The set $\mathcal{IMS}_\Sigma(G)$ is a sublattice of the subgroup lattice of G; the join of two subgroups in this sublattice is their permutable product; its smallest element is the prefrattini subgroup associated with Σ.*

This theorem has significant consequences for the structure of the frame of a finite soluble group. Recall that the map [ ] sends a subgroup U of G to the conjugacy class $[U] = \{U^g \mid g \in G\}$ of G; [ ] can be regarded as a map from a given set S of subgroups of G to the frame $\mathcal{F}r(G)$ of G, and if S is partially ordered by inclusion, the definition of the partial order $\preceq$ ensures [ ] that the map

$$[\,] : (S, \subseteq) \to (\mathcal{F}r(G), \preceq)$$

is order-preserving.

(5.7) Theorem. *Let Σ be a Hall system of a finite soluble group G. The map*

$$[\,] : \mathcal{IMS}_\Sigma(G) \to \mathcal{F}r(G)$$

is injective and its domain is order-isomorphic to its image, which is therefore a lattice in $\mathcal{F}r(G)$. As a lower semi-lattice, this image in rigidly embedded in $\mathcal{F}r(G)$ and has the conjugacy class of prefrattini subgroups as its unique minimal element.

Proof. Let $J, \overline{J} \in \mathcal{IMS}_\Sigma(G)$. We begin by showing that

$$(5.\epsilon) \qquad J \le \overline{J} \quad \text{if and only if} \quad [J] \preceq [\overline{J}].$$

noting that if $J \le \overline{J}$, then certainly $[J] \preceq [\overline{J}]$. Suppose that $[J] \preceq [\overline{J}]$. Then $J^g \le \overline{J}$ for some $g \in G$ by definition of $\preceq$. Let $p \mid |G|$, let $P \in \mathrm{Syl}_p(J)$, and choose $\overline{P} \in \mathrm{Syl}_p(\overline{J})$ such that $P^g \le \overline{P}$.

By definition of J and $\overline{J}$ there exist a sets

$$\mathcal{M} = \{M_1\,,\, M_2\,,\, \dots\,,\, M_j\} \quad \text{and} \quad \overline{\mathcal{M}} = \{\overline{M}_1\,,\, \overline{M}_2\,,\, \dots\,,\, \overline{M}_{\overline{j}}\}$$

of maximal subgroups M_i and $\overline{M}_i$ of G into each of which Σ reduces such that

$$(5.\zeta) \qquad J = M_1 \cap M_2 \cap \ldots \cap M_j \quad \text{and} \quad \overline{J} = \overline{M}_1 \cap \overline{M}_2 \cap \ldots \cap \overline{M}_{\overline{j}}\,.$$

When $p \mid |G|$, let $\mathcal{M}(p)$ (respectively $\overline{\mathcal{M}}(p)$) denote the set of p-maximal subgroups in $\mathcal{M}$ (respectively $\overline{\mathcal{M}}$); they are the ones that contain the p-complement Q of Σ. Set

$$J(p) = \bigcap_{M \in \mathcal{M}(p)} M \quad \text{and} \quad \overline{J}(p) = \bigcap_{\overline{M} \in \overline{\mathcal{M}}(p)} \overline{M},$$

with the usual convention that $J(p) = G$ when $\mathcal{M}(p) = \emptyset$. Of course, $J(p)$ and $\overline{J}(p)$ both belong to $IMS_\Sigma(G)$. Since J is the intersection of the subgroups $J(p)$ over the distinct primes p dividing $|G|$, and since $|G : J(p)|$ is coprime with $|G : J(q)|$ when $p \neq q$, we have $P \in \mathrm{Syl}_p(J(p))$ and therefore $J(p) = QP$, where Q is the p-complement in Σ. Likewise, $\overline{J}(p) = Q\overline{P}$, and since $P^g \leq \overline{P}$, we can deduce from Proposition 5.4 that $J(p) \leq \overline{J}(p)$. Since this holds for all prime divisors p of $|G|$, it follows that

$$J = \bigcap_{p\in\sigma(G)} J(p) \ \leq \ \bigcap_{p\in\sigma(G)} \overline{J}(p) = \overline{J},$$

and the assertion labelled $(5.\epsilon)$ above is justified. It follows immediately that if J and $\overline{J}$ are elements of $\mathcal{IMS}_\Sigma(G)$ with $[J] = [\overline{J}]$, then $J = \overline{J}$ and so the map [ ] is injective. Assertion $(5.\epsilon)$ shows further that the partial order induced on the image of [ ] in $\mathcal{F}r(G)$ is exactly corresponds to the partial order on its domain, which means that the subposet of $\mathcal{F}r(G)$ generated by this image is isomorphic to $\mathcal{IMS}_\Sigma(G)$ and is therefore a lattice, as claimed in the Theorem.

We show next that $[J]$ (as defined in $(5.\zeta)$ above) is the infimum of the set

$$\{[M_1], [M_2], \ldots, [M_j]\}$$

in $\mathcal{F}r(G)$, first noting that $[J]$ certainly is a lower bound for this set. Let $[L]$ be an arbitrary lower bound for this set. By definition of the partial order on $\mathcal{F}r(G)$, the condition $[L] \preceq [M_i]$ means that $L \leq (M_i)^{x_i}$ for some $x_i \in G$, and by Proposition 4.8 there exists an element $y(p) \in G$ such that L is contained in $J(p)^{y(p)}$ for each $p \in \sigma(G)$. Let $G_{p'}$ (called Q above) denote the p-complement in Σ and note that $(G_{p'})^{y(p)} \leq J(p)^{y(p)}$. The conjugacy of Hall systems ensures the existence of an element $g \in G$ such that $(G_{p'})^{y(p)} \in \Sigma^g$ for all $p \in \sigma(G)$; since $(G_{p'})^{y(p)} \in \mathrm{Hall}_{p'}(J(p))$, we have $\Sigma^g \searrow J(p)^{y(p)}$. Now $J(p)$ is pronormal in G by Proposition 5.1, and so $J(p)^g$ is the unique conjugate into which $(\Sigma)^g$ reduces by I,6.6(b) of [2]. Therefore $J(p)^{y(p)} = J(p)^g$ for all $p \in \sigma(G)$, and we have

$$L \leq \bigcap_{p\in\sigma(G)} J(p)^{y(p)} = \bigcap_{p\in\sigma(G)} J(p)^g = J^g.$$

Thus $[L] \preceq [J]$, and $[J]$ is the desired infimum in $\mathcal{F}r(G)$ of the given set of maximal subgroups.

If $\overline{J} = \overline{M}_1 \cap \overline{M}_2 \cap \ldots \cap \overline{M}_{\overline{j}}$ is a second subgroup in $\mathcal{IMS}_\Sigma(G)$, the argument above shows that a lower bound $[L]$ for $[J \cap \overline{J}]$ in $\mathcal{F}r(G)$ is a lower bound for the set $[M_1], \ldots, [M_j]$, $[\overline{M}_1], \ldots, [\overline{M}_{\overline{j}}]$ and that L therefore contained in a conjugate of

$$M_1 \cap M_2 \cap \ldots \cap M_j \cap \overline{M}_1 \cap \overline{M}_2 \cap \ldots \cap \overline{M}_{\overline{j}} = J \cap \overline{J}.$$

Hence $[J \cap \overline{J}]$ is the infimum of $[J]$ and $[\overline{J}]$ in $\mathcal{F}r(G)$, and this shows that the image of $\mathcal{IMS}_\Sigma(G)$ under [ ] is rigidly embedded as a lower semi-lattice in $\mathcal{F}r(G)$. ∎

(5.8) Corollary. *When G is soluble, the sublattice of $\mathcal{F}r(G)$ generated by the image of $\mathcal{IMS}_\Sigma(G)$ under the map $[\;]$, in particular, the conjugacy class of prefrattini subgroups, is uniquely determined by the poset structure of $\mathcal{F}r(G)$.*

6. Open Questions

We have already mentioned several unresolved questions: for instance, using the poset structure of $\mathcal{F}r(G)$ alone, can we locate (i) the conjugacy classes of Hall subgroups and (ii) the conjugacy classes of normally-embedded subgroups of a finite soluble group G? An obvious question arising from Theorem 5.7 is the following:

(6.1) Open Question. *Is the image of $\mathcal{IMS}_\Sigma(G)$ in $\mathcal{F}r(G)$ under the map $[\,]$ a rigidly-embedded lattice?*

According to I,4.29 of [2], the set of Σ-permutable subgroups of a finite soluble group G is a sublattice of the subgroup lattice of G. What can be said about the image of this set in $\mathcal{F}r(G)$ under the map $[\,]$? To be specific, we can ask:

(6.2) Open Question. *Let Σ be a Hall system of a finite soluble group G, and let $\Pi = \Pi(\Sigma)$ denote the lattice of all Σ-permutable subgroups of G.*

(a) Is the map $[\,] : \Pi \to \mathcal{F}r(G)$ injective?

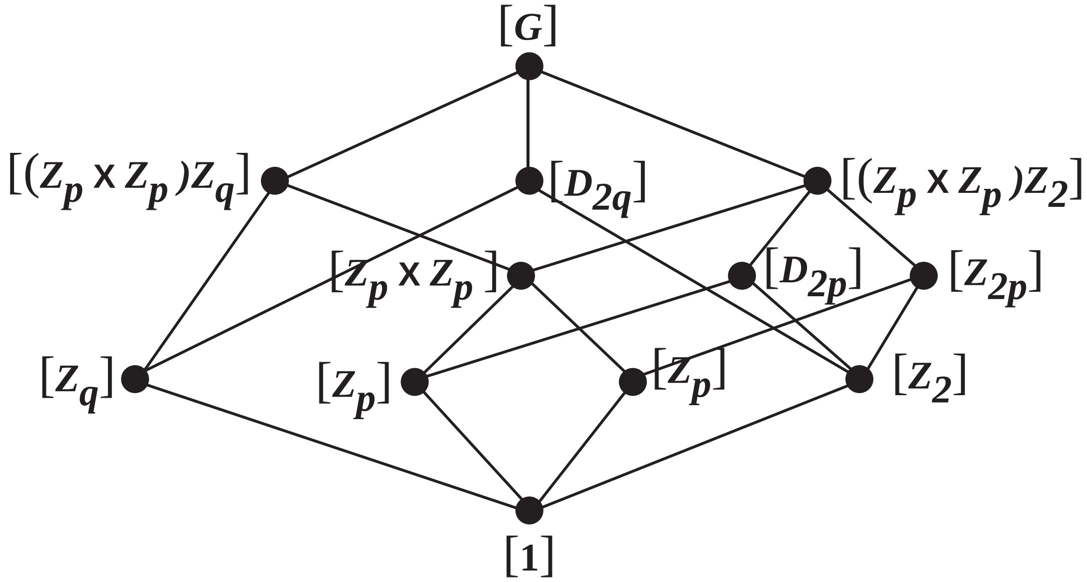
(b) Is Π order-isomorphic to the poset inherited by its image $[\Pi]$ in $\mathcal{F}r(G)$ (and therefore a lattice)?

(c) Is this image a rigidly-embedded lower semi-lattice (upper semi-lattice) in $\mathcal{F}r(G)$?

As we saw in Section 1, certain groups (such as A_5) are uniquely determined by their frames. On the other hand, all groups whose orders are the product of two distinct primes have the same frame (namely, a square), and, as the following example shows, even quite complicated groups can have the poset-isomorphic frames.

(6.3) Example. Let p and q be primes such that $p + 1 = 2q$ (for instance, take $(p, q) = (5, 3)$, $(13, 7), (37, 19), (61, 31)$, and so on). Then the extended affine group (described in Proposition B,12.9 on page 195 of [2]) has a primitive subgroup G of order $2p^2q$ which is the semidirect product of an elementary abelian group P of order p^2 by a dihedral group of order $2q$.

Since the centralizer in P of a Sylow 2-subgroup has order q (corresponding to the fixed field of the involutary field automorphism on $\mathrm{GF}(p^2)^+ \cong P$), the group G has exactly one conjugacy class of dihedral subgroups of order $2p$, one conjugacy class of cyclic subgroups of order $2p$, and just two conjugacy classes of subgroups of order p. It is not difficult to see that the frame of G has the form shown below and is therefore independent of the choice of primes p and q satisfying $p + 1 = 2q$. (As far as we know, the question whether there exist infinitely-many such pairs of primes p and q is open.)



(6.4) Open Question. *If a given poset is the frame of two non-isomorphic groups , do there exist infinitely-many distinct groups with this poset as their frame?*

In the above example we have two non-isomorphic groups of nilpotent length 3 with the same frame. However, there is some experimental evidence to suggest that, as the chief-factor rank of a group increases, so its frame carries steadily more information about its structure and, in particular, about its order. Since chief factor rank of a soluble group increases with its nilpotent length, we have the following related questions

(6.5) Open Questions. *Do there exist an natural numbers d and l such that:*

(i) the order of a primitive soluble group of degree at least d is uniquely determined by its frame?

(ii) the order of a primitive soluble group of nilpotent length at least l is uniquely determined by its frame?

The frame-closure of several natural classes of finite groups have been characterized. For example, Mainardis shows in [6] that the frame closure of the class of finite abelian groups is the class of supersoluble T-groups with abelian Sylow subgroups (a T-group being a group whose subnormal subgroups are all normal). Moreover, the class of supersoluble groups is itself framed; this is clear from Iwasawa's well-known characterization of supersoluble groups by the property that their maximal chains of subgroups all have the same length (see, for example, Satz VI,9.7 on page 719 of [4]). The class $\mathfrak{N}$ of nilpotent groups is certainly not framed; morevoer, it is easy to see that if p and q are primes satisfying

$p+1 = 2q$, then the frame of the primitive group $H = (\mathbb{Z}_p \times \mathbb{Z}_p)\mathbb{Z}_{2q}$, (which may be viewed as the semidirect product of $\mathrm{GF}(p^2)^+$ by the $\mathbb{Z}_{2q}$ in $\mathrm{GF}(p^2)^\times$) is poset-isomorphic to the frame pictured shown above in Example 6.3. Since H has nilpotent length 2 while the group G in Example 6.3 has nilpotent length 3, it follows that the class $\mathfrak{N}^2$ is not framed.

(6.6) Open Questions.

(a) Is $\mathcal{F}r\mathcal{C}l(\mathfrak{N}^l) \subseteq \mathfrak{N}^{l+1}$*?*

(b) Do their exist primitive saturated formations that are framed?

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