Equivariant Homotopy of Posets and Some Applications to Subgroup Lattices

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In this paper we consider the action of a finite group G on the geometric realization $|CP|$ of the order complex CP of a poset P, on which a group G acts as a group of poset automorphisms. For special cases we give the G-homotopy type of $|CP|$. Moreover, we provide conditions which imply that the orbit space $|CP|/G$ is homotopy equivalent to the geometric realization of the order complex over the orbit poset P/G. The poset P/G is the set of orbits $[x] := \{x^g | g \in G\}$ of G in P ordered by $[x] \leq [y]: \Leftrightarrow \exists g \in G: x^g \leq y$. We apply all our results to the case $P = \Lambda(G)^0$ is the lattice of subgroups $H \neq 1, G$ of a finite group G. For finite solvable groups G we give the G-homotopy type of $\Lambda(G)^0$ and we show that $|C\Lambda(G)^0|/G$ and $|C(\Lambda(G)^0/G)|$ are homotopy equivalent. We do the same for a class of direct products of finite groups and for some examples of simple groups. Finally we show that for the Mathieu group $G = M_{12}$ the orbit space $|C\Lambda(G)^0|/G$ and $|C(\Lambda(G)^0/G)|$ are not homotopy equivalent.   © 1995 Academic Press, Inc.

1. Introduction

We investigate topological and combinatorial properties of finite partially ordered sets. In particular we consider the proper part $\Lambda(G)^0 := \Lambda(G) - \{1, G\}$ of the *lattice of subgroups* $\Lambda(G)$ of a finite group G. The group G acts on $\Lambda(G)$ by conjugation. The orbits of the action of G on $\Lambda(G)^0$ are the *conjugacy classes* $[H]$ of proper subgroups H of G. The orbits give rise to another partially ordered set. We denote the set of conjugacy classes of subgroups by $\Lambda(G)^0/G$ and order them by containment of representatives (i.e., $[H] \leq [U]: \Leftrightarrow \exists g \in G: H^g \leq U$). We will give some results on the relation of the topological behavior of $\Lambda(G)^0$ and $\Lambda(G)^0/G$.

More generally, let P be a finite partially ordered set (*poset* for short). We associate to P the *order complex* CP, which is the simplicial complex consisting of all non-empty chains $x_1 < \cdots < x_n$ in P. If a group G acts on the poset P as a group of poset automorphisms (i.e., $x \leq y \Leftrightarrow x^g \leq y^g$

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for all $x, y \in P$ and $g \in G$), then this action induces an action of G on the complex CP and thereby we obtain a representation of G as a group of homeomorphisms of the *geometric realization* $|CP|$ of CP. Hence we can speak of G-homotopy equivalent, G-contractible, and G-homeomorphic posets. For the basic notations in algebraic topology we refer the reader to the book of Munkres [Mu]. Two spaces are *G-homotopy equivalent* if they are homotopy equivalent in the usual sense [Mu, p. 108] and if all maps providing the equivalence are G-equivariant. If X and Y are spaces on which G acts as a group of homeomorphisms, then we call a map $f: X \to Y$ a *G-equivariant map* if $f(x)^g = f(x^g)$ for all $x \in X$ and $g \in G$. Analogously one defines the terms G-contractible and G-homeomorphic.

In several papers Björner, Walker, Kratzer, Thévenaz, and originally Quillen have classified the homotopy type of $|CP|$ for certain kinds of posets. Now we will do the same for the G-homotopy type of $|CP|$. We use these results to give the G-homotopy type of $\Lambda(G)^0$ and the homotopy type of $\Lambda(G)^0/G$ for finite solvable groups (extending results in [K-T]), some finite simple groups, and a class of direct products. We mention that in the work of Thévenaz and Webb [T-W] another set of techniques for the study of the G-homotopy type of a poset is developed and applied.

Moreover, our results allow us to investigate the orbits of the group action. There are two ways to divide out the G-operation from $|CP|$. The first and natural way is to look at the *orbit space* $|CP|/G$. The *orbit space* $|CP|/G$ is the topological space on the set of G-orbits in $|CP|$ whose topology is induced by the natural projection map $p: |CP| \to |CP|/G$ (see, for example, [Br]). The second possibility for dividing out the G-action is a generalization of the procedure, defined above for subgroup lattices, to an arbitrary poset P on which a group acts. Hence we impose a partial ordering on the orbits $[x] := \{x^g | g \in G\}$ of elements $x \in P$. The set of orbits $P/G = \{[x] | x \in P\}$ is ordered by $[x] \le [y]: \Leftrightarrow \exists g \in G: x^g \le y$. The poset P/G is called the *orbit poset* of P. We will state some results on the relations between $|CP|/G$ and $|C(P/G)|$ in special cases. But in the general case we see no way to describe the relation between the two spaces.

However, our results apply if $P = \Lambda(G)^0$. In this situation we are able to prove for some finite groups G that $|CP|/G = |C\Lambda(G)^0|/G$ and $P/G = \Lambda(G)^0/G$ are homotopy equivalent. In Section 4 we show this for finite solvable groups G. We will prove it by constructing a subposet Q of $P = \Lambda(G)^0$ for which P and Q are G-homotopy equivalent, P/G and Q/G are homotopy equivalent, and $|CQ|/G$ is homeomorphic to Q/G. In Section 5 we derive some conditions on direct products $G = U \times V$ of finite groups which imply that $|C\Lambda(G)^0|/G$ is homotopy equivalent to $\Lambda(G)^0/G$ if $|C\Lambda(U)^0|/U$ (resp. $|C\Lambda(V)^0|/V$) and $\Lambda(U)^0/U$ (resp.

$\Lambda(V)^0/V$) are homotopy equivalent. Finally we investigate in Section 6 some examples of non-abelian finite simple groups. We analyze the groups A_n, $n \leq 7$, M_{11}, M_{12}, and $PSL_2(\mathbf{F}_7)$. If G is one of the alternating groups or the first Mathieu group, we show that $|C\Lambda(G)^0|/G$ and $\Lambda(G)^0/G$ are homotopy equivalent. The same holds for the group $G = PSL_2(\mathbf{F}_7)$ but for this group $\Lambda(G)^0/G$ is contractible and $\Lambda(G)^0$ is not. This never happens for solvable groups and the other simple groups investigated here. The Mathieu group $G = M_{12}$ is an example where $|C\Lambda(G)^0|/G$ is not homotopy equivalent to $\Lambda(G)^0/G$. This generalizes results given in [B-G-V] on the Möbius numbers of $\Lambda(M_{12})^0$ and $\Lambda(M_{12})^0/M_{12}$.

2. EQUIVARIANT HOMOTOPY OF POSETS

In this section we will develop some results in equivariant homotopy theory of posets. Most of the results are equivariant versions of theorems of Björner and Walker [B-W1] and Kratzer and Thévenaz [K-T]. Some of them can be found in the paper of Thévenaz and Webb [T-W]. If the proofs are only easy modifications of the original ones we leave them to the reader. If we say that a group acts on a poset P, then we mean that G acts on the set P preserving the order relation (i.e., G acts as a group of poset automorphisms on P). If a group G acts on a set (resp. poset), (resp. topological space) X then we call X a G-set (resp. G-poset), (resp. G-space).

THEOREM 2.1 (Contractible Subcomplex lemma) [B-Wa, 2.2][We2, Satz I.1.14]. *Let P be a G-poset and let $P' \subseteq P$ be a subposet which is invariant under the action of G. If P' is G-contractible then P and the quotient space $|CP|/|CP'|$ are G-homotopy equivalent.*

For a poset P and an element $x \in P$ we denote by $P_{>x}$ the poset $\{y | y > x\}$. Analogously defined are the posets $P_{\geq x}$, $P_{<x}$, and $P_{\leq x}$. We write G_x for the stabilizer of x in G.

THEOREM 2.2 [Qu, Proposition 1.6] [T-W] [We2, Satz I.1.8]. *Let P be a G-poset and let $P' \subseteq P$ be a subposet which is invariant under G. If for all $x \in P - P'$ the G_x-poset $P_{>x}$ is G_x-contractible then P and P' are G-homotopy equivalent.*

By the previous theorems we see that it will become important to prove for a G-poset P that it is G-contractible. The following condition (adapted from [B-Wa]) gives a criterion which implies that P is G-contractible.

(C) Let P be a G-poset and let $\mathbf{a} \in P$ be an element which is invariant under G and satisfies the following two conditions.

(i) For all $x \in P$ either the infimum $\mathbf{a} \wedge x$ or the supremum $\mathbf{a} \vee x$ exists.

(ii) Let $x, y \in P$ be elements such that $x \leq y$. If the supremum $\mathbf{a} \vee x$ exists but $\mathbf{a} \vee y$ does not then the infimum $(\mathbf{a} \vee x) \wedge y$ exists.

THEOREM 2.3 [B-Wa, 3.2] [We2, Satz I.1.12]. *Let P be a G-poset and let $\mathbf{a} \in P$ be an element of P. If P and the element $\mathbf{a}$ fulfill the condition (C) then P is G-contractible.*

As a corollary we obtain a result which will be essential for the situation in the subgroup lattice of a finite solvable group. For a poset P with least element $\hat{0}$ and greatest element $\hat{1}$ we denote by P^0 the proper part $P - \{\hat{0}, \hat{1}\}$ of P. In particular, since all partially ordered sets in this paper are finite, every lattice occurring in this paper has a least element and a greatest element.

COROLLARY 2.4. *Let P be a G-lattice and let $\mathbf{a} \in P$ be an element which is invariant under G. Let $\mathbf{a}^\perp$ be the set $\{x \in P | x \wedge \mathbf{a} = \hat{0}$ and $x \vee \mathbf{a} = \hat{1}\}$. Then $P^0 - \mathbf{a}^\perp$ is G-contractible.*

Proof. Since $\hat{0}$ and $\hat{1}$ are invariant under G the poset $P^0 - \mathbf{a}^\perp$ is a G-poset. Because P is a lattice and by the choice of $\mathbf{a}^\perp$, condition (C) is fulfilled for $P^0 - \mathbf{a}^\perp$ and the element $\mathbf{a}$. ∎

For a subposet $P' \subseteq P$ of a G-poset P we will now study the following condition.

(I) $P - P'$ is an antichain which is invariant under G and the subposet P' is G-contractible.

Note that if P is a G-poset and if for a subposet P' the difference $P - P'$ is G-invariant then P' is a G-poset as well. Before we can give results on the G-homotopy type in situation (I) we have to define the action of G on a suitable topological space.

For two topological spaces X, Y we denote by $X * Y$ the *join* of X and Y [Mu, p. 386]. Note that if X and Y are G-spaces then $X * Y$ is a G-space as well [Di]. If Σ is a two element antichain then $|C\Sigma| * X$ is the topological *suspension* of the space X. We write as usual ΣX for $|C\Sigma| * X$. If $(X_i)_{i \in J}$ is a family of topological spaces then we denote by $\bigvee_{i \in J} X_i$ the *wedge* of the spaces X_i. Note that in general we have to define a wedgepoint $p_i \in X_i$ for all i in order to have the wedge $\bigvee_{i \in J} X_i$ well defined.

In the sequel we will describe a construction of topological spaces which will turn out to be important in our particular context.

Let P be a G-poset and P' be a subposet which fulfill (I). We choose a new point $p \notin P$ and form antichains $\Sigma_x = \{x, p\}$ for all $x \in A := P - P'$. We define an operation of G on the wedge $\bigvee_{x \in A} \Sigma_x |CP_{<x}| * |CP_{>x}|$ with wedge point p. An element $g \in G$ fixes the point p and permutes the elements $x \in A$ of the antichains Σ_x according to its operation on A. For an $x \in A$ and $y \in P_{<x}$ the image y^g of y under g is the corresponding element of P in $P_{<x^g}$. Analogously we define the operation for $y \in P_{>x}$. Now the described action induces an operation of G (as a group of homeomorphisms) on the topological space $\bigvee_{x \in A} \Sigma_x |CP_{<x}| * |CP_{>x}|$. We write $\bigvee^G_{x \in A} \Sigma_x |CP_{<x}| * |CP_{>x}|$ for the G-space with the operation specified above.

More abstractly one can construct a G-space (G-set) from an H-space (H-set) if H is a subgroup of G in the following way. Let X be an H-space (resp. H-set). Then $X \times G$ can be given the structure of a left H- and a right G-space (resp. -set) by the following definition $^h(x, g)^{g'} := (x^{h^{-1}}, hgg')$. Then the G-action on $X \times G$ induces a G-action [Di, (4.2)] on the orbit space $X \times_H G = H \backslash (G \times X)$. Now assume that the H-space X is actually a pointed H-space. More generally assume that there is a fixpoint $p \in X$. Then the image $\{p\} \times_H G$ of $\{p\} \times G$ in $X \times_H G$ is a G-orbit. Of course if X is an H-set then the G-action on $X \times_H G$ corresponds to the permutation representation induced from the permutation representation of H on X to G.

PROPOSITION 2.5 [B-W1]. *Let P be a G-poset and let $P' \subseteq P$ be a subposet which fulfill (I). Then P is G-homotopy equivalent to*

$$\bigvee^G_{x \in A} \Sigma_x |CP_{<x}| * |CP_{>x}|, \qquad A := P - P'.$$

Moreover assume that the set A is (as a G-set) isomorphic to the disjoint union $\Sigma^n_{i=1} G/H_i$ of coset spaces G/H_i for not necessarily different subgroups H_i of G. Then

$$\bigvee^G_{x \in A} \Sigma_x |CP_{<x}| * |CP_{>x}|, \qquad A := P - P',$$

and

$$\bigvee^n_{i=1} \left(\left(\left(\Sigma_{x_i} |CP_{<x_i}| * |CP_{>x_i}| \right) \times_{H_i} G \right) / \left(\{p\} \times_{H_i} G \right) \right)$$

are G-homeomorphic. Here the image of $(\{p\} \times_{H_i} G)$ is the wedge point and the x_i are chosen such that $H_i = G_{x_i}$ and $A = \bigcup^n_{i=1} \{x_i^g | g \in G\}$.

Proof. By (I) we deduce from the Contractible Subcomplex lemma (Theorem 2.1) that P and $|CP|/|CP'|$ are G-homotopy equivalent. Now

we choose the image of $|CP'|$ in $|CP|/|CP'|$ as the wedge point p. Since P' is invariant under G the point p is also invariant under G. For each $y \in |CP|$ we have either $y \in |CP'|$ or y can be identified with a point in $\{x\} * |CP_{<x}| * |CP_{>x}| * \{p\} - \{p\}$ for an element $x \in A$ which is uniquely determined by y since A is an antichain. In the second case (i.e., $y \notin |CP'|$) we have $y^g \in \{x^g\} * |CP_{<x^g}| * |CP_{>x^g}| * \{p\} - \{p\}$. This shows that the action of G on $|CP|/|CP'|$ is compatible with the action of G on $\bigvee_{x \in A}^{G} \Sigma_x |CP_{<x}| * |CP_{>x}|$. Now it is routine to show that $|CP|/|CP'|$ and $\bigvee_{x \in A}^{G} \Sigma_x |CP_{<x}| * |CP_{>x}|$ are G-homeomorphic. It remains to be shown that $\bigvee_{x \in A}^{G} \Sigma_x |CP_{<x}| * |CP_{>x}|$ and

$$\bigvee_{i=1}^{n} \left(\left(\left(\Sigma_{x_i} |CP_{<x_i}| * |CP_{>x_i}| \right) \times_{H_i} G \right) / \left(G \times_{H_i} \{p\} \right) \right)$$

are G-homeomorphic. For the sake of completeness, we will actually establish a G-homeomorphism between the topological sum $X = \coprod_{x \in A} \Sigma_x |CP_{<x}| * |CP_{>x}|$ and $Y = \coprod_{i=1}^{n} ((\Sigma_{x_i} |CP_{<x_i}| * |CP_{>x_i}|) \times_{H_i} G)$. This will prove the assertion since taking the various copies of p as the wedge point in the first topological sum and identifying the copies of p to a wedge point in the second topological sum will preserve a G-homeomorphism. We may assume that A is a transitive G-set. In particular $n = 1$, $A = G/H$ for a subgroup H of G, and $Y = (\Sigma_x |CP_{<x}| * |CP_{>x}|) \times_H G$ for some fixed $x \in A$ with $G_x = H$. Now we map $(y, g) \in Y$ to $f(y, g) = y^g \in \Sigma_{x^g} |CP_{<x^g}| * |CP_{>x^g}| \subseteq X$. Assume $(y^h, h^{-1}g)$ is another representative of the class of (y, g) in Y for some $h \in H$. Then $(y^h)^{h^{-1}g} = y^g$ and $x^{hg} = (x^h)^g = x^g$. Hence $f(y, g)$ is well defined. Moreover $f(y, g)^{g'} = (y^g)^{g'} = y^{gg'} = f((y, g)^{g'})$ shows that f is G-equivariant. Now let y be an element of $\Sigma_x |CP_{<x}| * |CP_{>x}|$. Then define $l(y^g) := (y, g) \in Y$. If $y^g = y^{g'}$ for two elements g, g' of G then $h = gg'^{-1}$ is an element of H. In particular $(y, g') = (y^{h^{-1}}, hh^{-1}g) = ((y^{g'})^{g^{-1}}, g) = (y, g)$, which proves that l is well defined. Analogously as for f one shows that l is G-equivariant. Moreover one verifies that $f \circ l = id_X$ and $l \circ f = id_Y$, which proves the assertion. ∎

In the situation (I) all posets $P_{<x}$ and $P_{>x}$ are G_x-posets for the stabilizer G_x of the element x in G. Our next aim is to deduce the G-homotopy type of P from the G_x-homotopy type of the posets $P_{<x}$ and $P_{>x}$. It will turn out that we do not need to look at $P_{<x}$ and $P_{>x}$ separately. We will therefore regard the union $P_{<x} \cup P_{>x}$ (actually a disjoint union) as a subposet of P. The reader is reminded of the fact that $|C(P_{<x} \cup P_{>x})|$ and $|CP_{<x}| * |CP_{>x}|$ are G_x-homeomorphic. Hence we would like to analyze the following situation.

(II) The G-posets P and P' satisfy (I). We set $A := P - P'$. For every $x \in A := P - P'$ there is a G_x-poset R_x which is G_x-homotopy equivalent to $P_{<x} \cup P_{>x}$.

But this does not suffice to deduce a suitable G-action on $V_{x \in A} \Sigma_x |CR_x|$, which is the space of concern. Therefore we need the following condition which assures the compatibility of the G_x-homotopy equivalences. Below in Remark 2.6 (iii) we will see that (II) and (III) are actually equivalent. But we will continue to use condition (III) for the sake of easy formulations.

(III) The G-posets P and P' satisfy (I). There is a G-poset Q and a subposet Q' which fulfill the condition (I). Furthermore the following three conditions hold:

(a) The set $Q - Q'$ is as a G-set isomorphic to the set $A := P - P'$ (Therefore in the sequel we can identify A and $Q - Q'$ in a suitable manner). Every $x \in A$ is maximal in Q and hence $Q_{>x} = \varnothing$ for all $x \in A$.

(b) For an $x \in A$ the G_x-posets $P_{<x} \cup P_{>x}$ and $Q_{<x}$ are G_x-homotopy equivalent.

(c) There exist G_x-homotopy equivalences $f_x : |CP_{<x}| * |CP_{>x}| \to |CQ_{<x}|$ such that $f_x(y)^g = f_{x^g}(y^g)$. (Here we use the action of G on the space $V^G_{x \in A} \Sigma_x |CQ_{<x}|$ defined above.)

In condition (III) the elements of the poset Q which are not included in some $Q_{<x}$ are of no importance. Therefore the following construction (see also Fig. 1) will provide the suitable model for our topological context. Assume the situation of condition (III).
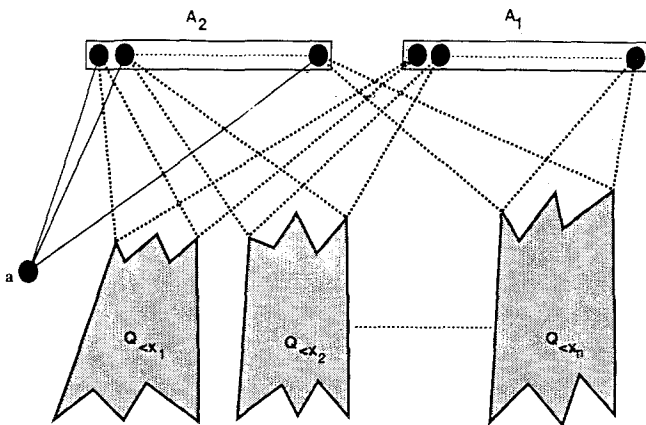

FIGURE 1

(i) Let Q' be the disjoint union of the posets $Q_{<x}$ for $x \in A$. An element g of the group G acts on Q' by mapping $y \in Q_{<x}$ to $y^g \in Q_{x^g}$.

(ii) Let A_1 and A_2 be two copies of the antichain A (regarded as a G-set). For an element x in A_i and an element $y \in Q'$ we define $x > y$ if and only if $y \in Q_{<x}$. The elements of A_1 and A_2 are defined to be pairwise incomparable.

(iii) Let **a** be an element which is not comparable with any element in the antichain A_1 and any element in Q'. Now we add the relation $\mathbf{a} \leq x$ for all $x \in A_2$. We define **a** to be a fixpoint of the action of G.

If we denote by $\mathscr{P}[Q_{<x}|x \in A]$ the poset constructed in (i)–(iii) then $\mathscr{P}[Q_{<x}|x \in A] - A_1$ and the element **a** fulfill the condition (C). Therefore by Theorem 2.3 and Proposition 2.5 the G-posets Q and $\mathscr{P}[Q_{<x}|x \in A]$ have the same G-homotopy type. Furthermore this construction justifies the concentration on the union $P_{<x} \cup P_{>x}$ in condition (III).

Remark 2.6. Let P_0 and P'_0 be G-posets which satisfy (I). For all $x \in A = P_0 - P'_0$ let R_x and T_x be G_x-posets.

(i) If the pair of G-posets $P = \mathscr{P}[(P_0)_{<x}|x \in A]$ (resp. $P = \mathscr{P}[(P_0)_{>x}|x \in A]$) and $Q = \mathscr{P}[R_x|x \in A]$ (resp. $Q = \mathscr{P}[T_x|x \in A]$) fulfills (III) for $P' = P - A_1$ and $Q' = Q - A_1$ then P_0 and

$$Q_0 := \mathscr{P}[R_x \cup T_x|x \in A]$$

fulfill (III) as well (the definition of A_1 is according to the construction depicted in Fig. 1, the order on $R_x \cup T_x$ is induced by the order R_x and T_x and by $y < y'$ for $y \in R_x$ and $y' \in T_x$).

(ii) Let $A = \sum_{i=1}^{n} G/H_i$ be a decomposition of the G-set A into coset spaces. Then we obtain the following decomposition of G-sets

$$\mathscr{P}[R_x|x \in A] = 2 \cdot \sum_{i=1}^{n} G/H_i \uplus \sum_{i=1}^{n} G \times_{H_i} R_{x_i} \uplus G/G,$$

where we have chosen x_i such that $H_i = G_{x_i}$ and $A = \bigcup_{i=1}^{n} \{x_i^g | g \in G\}$.

(iii) The conditions (II) and (III) are equivalent.

Proof. The first assertion follows immediately from Proposition 2.5. The second assertion is an immediate consequence of the construction of $\mathscr{P}[R_x|x \in A]$. For assertion (iii) one has to recall [Di, Proposition 4.3] the fact that an H-map $f: X \to Y$ between H-spaces X and Y has a unique extension to a G-map $f^G: X \times_H G \to Y \times_H G$. Moreover if f is an H-homotopy equivalence then f^G is a G-homotopy equivalence. This shows that given (II) condition (III) follows after choosing representatives
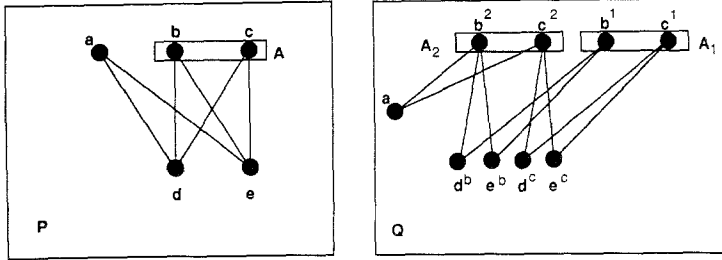
FIGURE 2

$x_1, \ldots, x_n$ of the G-orbits on A and setting $Q_{<x_i^g} := R_{x_i} \times_{G_{x_i}} g$ and $Q = \mathscr{P}[Q_{<x} | x \in A]$. Hence (II) implies (III). The other direction is trivial. ∎

Now we will give an example of the construction depicted in Fig. 1. Let G be the cyclic group of order 2 generated by the element $(bc)(de)$ acting on the poset P in Fig. 2. The poset Q on the right hand side of Fig. 2 is the poset $\mathscr{P}[P_{<b}, P_{<c}]$. The generator of the group G acts on Q by $(b^1 c^1)(b^2 c^2)(d^b e^c)(d^c e^b)$.

It is easily seen that both posets P and Q in Fig. 2 have the same G-homotopy type. The following proposition shows that this is actually a general fact.

PROPOSITION 2.7. *Let P and P' be two G-posets which fulfill* (III) *for the posets Q and Q'. Then for $A := P - P'$ the spaces*

$$\bigvee_{x \in A}^{G} \Sigma_x |CP_{<x}| * |CP_{>x}| \qquad \text{and} \qquad \bigvee_{x \in A}^{G} \Sigma_x |CQ_{<x}|$$

are G-homotopy equivalent.

Proof. For all $x \in A$ let f_x be a G_x-homotopy equivalence between $P_{<x} \cup P_{>x}$ and $Q_{<x}$ which satisfies (III)(c). By the equivariant Simplicial Approximation theorem [Br, Chap. I, Exercise 6] we may assume that each f_x is induced by an equivariant simplicial map $g_x: \Delta_x \to CQ_{<x}$ of a certain subdivision Δ_x of $|CP_{<x}| * |CP_{>x}|$, which is a G_x-complex. By the assumptions on f_x and by choosing the simplicial approximations uniformly for all $x \in A$ we may assume that $g_x(\sigma)^g = g_{x^g}(\sigma^g)$ for all $g \in G$ and $\sigma \in \Delta_x$. As usual we regard $\biguplus_{x \in A} CQ_{<x}$ as a G-set where G acts by mapping $\sigma \in CQ_{<x}$ to $\sigma^g \in CQ_{<x^g}$. We define a mapping f from

$\bigvee_{x \in A} \Sigma_x |CP_{<x}| * |CP_{>x}|$ to $\bigvee_{x \in A} \Sigma_x |CQ_{<x}|$ as a continuous extension of

$$g(y) := \begin{cases} y & \text{if } y \in \Sigma_x \text{ for some } x \\ g_x(y) & \text{if } y \in \Delta_x \end{cases}$$

defined on the vertices of the complexes $\Sigma_x \Delta_x$. Here we identify the antichains Σ_x which occur in $\Sigma_x \Delta_x$ and $\Sigma_x CQ_{<x}$. By the construction of the wedge and since the element x of A for which $y \in \Delta_x$ is unique, the map $g(y)$ is well defined. The extension f exists and can be made G-equivariant by the assumptions on the mappings f_x (see (III) (c)) and g_x. The same construction applies to the homotopy inverses of the maps f_x. We leave the verification of the claim that this construction preserves the property of being homotopy inverse to the reader. This completes the proof of the assertion that the two spaces are G-homotopy equivalent. ∎

We now turn to a more general situation. Provided condition (III) we analyze the reason why the action of G on P and on Q are compatible in our topological sense. Here the following construction of a group operation proves to be useful.

Let G be a group and let E be a G-set.

  (i) We decompose $E = \sum_{i=1}^{n} G/H_i$ into coset spaces (i.e., transitive G-sets) for not necessarily different subgroups H_i of G. For each $1 \le i \le n$ let F_i be a fixed H_i-set. Now we split $F_i = \sum_{j=1}^{n_i} H_i/K_j^i$ into transitive H_i sets for not necessarily different subgroups K_j^i of H_i.

  (ii) We form the G-set $G \times_{[H_1, \dots, H_n]} [F_1, \dots, F_n] := \sum_{i=1}^{n} \sum_{j=1}^{n_i} G/K_j^i$. For each x^g in the orbit of $x \in G/H_i \subseteq E$ there are $|H_i/K_j^i|$ cosets of K_j^i contained in gH_i. Therefore we can naturally partition G/K_j^i into $|G/G_i|$ subsets of cardinality $|H_i/K_j^i|$. Hence every element of $G \times_{[H_1, \dots, H_n]} [F_1, \dots, F_n]$ can be identified with a pair (x, y) for some $x \in E$ and y in some F_i. In particular, by an easy verification one shows that this identification actually establishes the isomorphism of G-sets from $G \times_{[H_1, \dots, H_n]} [F_1, \dots, F_n]$ to $\sum_{i=1}^{n} \sum_{j=1}^{n_i} H_i/K_j^i \times_{H_i} G$.

Thus if $E = G/H$ and $F_1 = H/K$ then $G \times_{[H_1]} [F_1]$ is a G-set isomorphic to $G/H = H/K \times_H G$.

The most interesting case for us is when the H_i-sets F_i are actually H_i-posets. Thus we have to impose an order relation on $G \times_{[H_1, \dots, H_n]} [F_1, \dots, F_n]$. Let y, y' be two elements of F_i. Then we define $y^g \le y'^g : \Leftrightarrow y \le y'$ (the second order relation taken in the H_i-poset F_i). We have to show that this actually defines a partial order (this actually follows from the fact that $G \times_H \cdot$ is a functorial construction, but we will verify this easy fact briefly). We may assume $n = 1$, $H = H_1$, and $F = F_1$. Now assume $y \le y'$, $x \nleq x'$ are elements of F and there are g_x, g_y in G such

that $y^{g_y} = x^{g_x}$ and $y'^{g_y} = x^{g_x}$. Then $g_y g_x^{-1}$ is an element of the stabilizer of y in G. By [Di, (1.14) Exercise 4] we infer that $G_y = H_y$, which implies $g_y g_x^{-1} \in H_y$. Now $x = y^{g_y g_x^{-1}}$, $x' = y'^{g_y g_x^{-1}}$ yields $x \leq x'$ contradicting the assumptions.

PROPOSITION 2.8. *Let P be a G-poset and let P' be a subposet which satisfies* (I). *We set $A := P - P'$. Let us assume the following situation*:

(i) *Let $A = \sum_{i=1}^{n} G/H_i$ be a decomposition of A into transitive G-sets. For every $1 \leq i \leq n$ let F_i be an H_i-set.*

(ii) *For $x \in G/H_i \subseteq A$ such that $G_x = H_i$ the space $|CP_{<x}| * |CP_{>x}|$ is H_i-homotopy equivalent to $\bigvee_{y \in F_i}^{H_i} \Sigma_y |CQ_y|$.*

Then P is G-homotopy equivalent to

$$\bigvee_{y \in G \times_{[H_1, \ldots, H_n]}[F_1, \ldots, F_n]}^{G} \Sigma_y \Sigma Q_y.$$

Proof. For $x \in A$ we set $R_i := \mathscr{P}[Q_y | y \in F_i]$. Now it is routine to show that P and its subposet P' (resp. R and its subposet $R' := R - A$) satisfy (III) (see also Remark 2.6 (iii)). Hence by Proposition 2.7 the poset P is G-homotopy equivalent to

$$\bigvee_{i=1}^{n} \left(G \times_{H_i} \left(\Sigma_{x_i} |CR_i| \right) \right) / \left(G \times_{H_i} \{p\} \right)$$

for a suitable choice of the $x_i \in G/H_i$. Now from Remark 2.6 (ii) and the construction of R_i we deduce that P is G-homotopy equivalent to

$$\bigvee_{i=1}^{n} \left(G \times_{H_i} \left(\Sigma_{x_i} \bigvee_{y \in F_i}^{H_i} \Sigma_y Q_y \right) \right) / \left(G \times_{H_i} \{p\} \right).$$

It is a well known fact from algebraic topology that suspension and wedge commute modulo homotopy equivalence (see for example [B-We]). Some technical computations show that this can be done G-equivariantly. Hence P is G-homotopy equivalent to $\bigvee_{i=1}^{n} \bigvee_{y \in F_i}^{H_i} \Sigma_{x_i} \Sigma_y Q_y$. By definition of the G-set $G \times_{[H_1, \ldots, H_n]}[F_1, \ldots, F_n]$ we get that P is G-homotopy equivalent to

$$\bigvee_{y \in G \times_{[H_1, \ldots, H_n]}[F_1, \ldots, F_n]}^{G} \Sigma_y \Sigma Q_y. \qquad \blacksquare$$

Before we apply our results to subgroup lattices we will have a look on the orbit space $|CP|/G$ for a G-poset P. We wish to relate this space to the order complex of the poset P/G.

3. ORBIT SPACES AND ORBIT POSETS

In this section we wish to discuss the relation between the homotopy type of $|CP|/G$ and the homotopy type of $|C(P/G)|$. In general the spaces seem to be almost unrelated. The one trivial fact which is true in general is that if CP is an n-dimensional simplicial complex, then $|CP|/G$ (resp. $C(P/G)$) is an n-dimensional regular CW-complex (resp. simplicial complex). In Fig. 3 we depict the space $|C(P/G)|$, $|C(Q/G)|$, $|CP|/G$, and $|CQ|/G$, where P and Q are taken from Fig. 2. Note that although P and Q are by Proposition 2.7 G-homotopy equivalent their orbit posets are not homotopy equivalent. Q/G has the homotopy type of a 1-sphere and P/G is contractible. On the other hand Q/G, $|CQ|/G$, and $|CP|/G$ are homotopy equivalent. The reason for this behavior will become clear later.

Our first approach is well known. We impose a rather strong condition on the operation of G on P.

(RE) For every chain $x_1 < \cdots < x_n$ in P and for every sequence of elements $g_1, \ldots, g_n \in G$ for which elementwise conjugation of the x_i
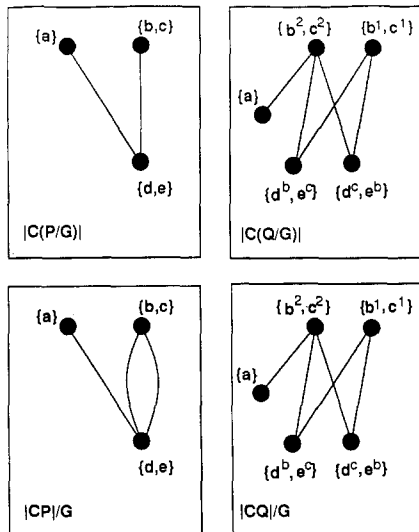

FIGURE 3

gives another chain $x_1^{g_1} < \cdots < x_n^{g_n}$ there exists a $g \in G$ such that $x_i^{g_i} = x_i^{g}$.

If (RE) is fulfilled for a G-poset P then we call P a *regular* G-poset.

PROPOSITION 3.1 [Br]. *Let P be a regular G-poset; then $|CP|/G$ and $|C(P/G)|$ are homeomorphic.*

Proof. By condition (RE) we know the inclusion relation of simplices in CP behaves well under the operation of the group G. Hence the orbits of the simplices in CP provide a suitable triangulation of the orbit space $|CP|/G$. ∎

Now, although we can verify (RE) for an adequate subposet of the poset of subgroups of a finite solvable group, for subgroup lattices of more general types of groups it is of little use for us. We will apply this criterion to some specific examples, but we have no approach to the general problem. In order to treat the case of solvable groups, we consider a suitable restriction of condition (RE). We return to the situation of condition (I) introduced in the last section.

(RE)$_I$ Condition (I) holds for the G-poset P and its subposet P'. Furthermore the poset $\mathscr{P}[P_{<x} \cup P_{>x} | x \in P - P']$ is a regular G-poset.

Provided condition (I), it is trivial that condition (RE)$_I$ is a consequence of (RE) but not vice versa.

Condition (RE)$_I$ has the following sloppy interpretation:

```
All "essential" identifications induced on P_{<x} ∪
P_{>x} by the action of G on P can be realized by the
operation of G_x.
```

We will show in the next section that condition (RE)$_I$ and the given interpretation is of group theoretical interest in the case where $P = A(G)^0$ is the proper part of the subgroup lattice of a group G and G_x is the normalizer of the subgroup x in an antichain A.

Before we can apply condition (RE)$_I$ we prove the following general lemma.

LEMMA 3.2. *Let P and P' satisfy the condition (III) for the G-posets Q and Q'. Then $|CP|/G$ is homotopy equivalent to*

$$\bigvee_{[x] \in A/G} \Sigma_{[x]} |CQ_{<x}|/G_x.$$

Proof. From Proposition 2.7 we know that P and Q are G-homotopy equivalent to

$$\bigvee_{x \in A}^{G} \Sigma_x |CQ_{<x}|.$$

Hence $|CP|/G$ and $|CQ|/G$ are homotopy equivalent to

$$\left(\bigvee_{x \in A}^{G} \Sigma_x |CQ_{<x}| \right) \Big/ G.$$

From [Di, (4.14) Exercise 1] we know that for an H-space X and a group G containing H as a subgroup the spaces $(X \times_H G)/G$ and X/H are homeomorphic. Therefore the assertion follows from the fact that the wedge points are fixed under the G-action and the second part of Proposition 2.5. ∎

PROPOSITION 3.3. *Let P and P' satisfy the condition* (III) *for the G-posets Q and Q'. If for $x \in A$ the poset $Q_{<x}/G_x$ is homotopy equivalent to $|CQ_{<x}|/G_x$ then $|CP|/G$ is homotopy equivalent to*

$$\bigvee_{[x] \in A/G} \Sigma_{[x]} |C(Q_{<x}/G_x)|.$$

In particular, the conditions of this proposition are fulfilled if Q and Q' satisfy $(RE)_I$.

Proof. This follows immediately from Lemma 3.2 and a repeated application of Proposition 2.7. ∎

In the next section we will apply the results of Section 2 and Section 3 to subgroup lattices of finite solvable groups.

4. SUBGROUP LATTICES OF FINITE SOLVABLE GROUPS

In this section all groups are finite and solvable. We actually would like to verify the conditions (III) and (RE) considered in Section 2 and Section 3 for the G-poset $\Lambda(G)^0$. We will see that condition (III) holds, but (RE) fails even for p-groups in simple examples [We2]. Therefore we turn our interest to a subset of $\Lambda(G)^0$.

DEFINITION 4.1. Let G be a group.

   (i) We call a subgroup H of G a C-subgroup if the interval $[H, G]$:= $\{U | H \leq U \leq G\}$ is a complemented lattice.

   (ii) By $\kappa(G)$ we denote the poset $\{U | U \in \Lambda(G)$ and U is a C-subgroup$\} \cup \{1\}$.

Obviously if U is a C-subgroup of G then U^g is also a C-subgroup. Hence $\kappa(G)$ is a G-poset. The poset $\kappa(G)$ is different from $\{1\}$ for $G \neq 1$ since G and all maximal subgroups are always C-subgroups. The group G

is the largest element of $\kappa(G)$ and 1 is the least element of $\kappa(G)$. In particular $\kappa(G)$ is a bounded poset. But $\kappa(G)$ is not a lattice in general [We2].

LEMMA 4.2.   *The posets $\Lambda(G)^0$ and $\kappa(G)^0$ are G-homotopy equivalent.*

Proof.   An interval $[H, G]$ in G is not complemented if and only if there is a normal subgroup N of G such that the element HN is not complemented in $[H, G]$ [K-T, Proposition 4.13]. Since HN is invariant under the normalizer $N_G(H)$ of H in G we deduce from Theorem 2.3 that $[H, G]^0 = (\Lambda(G)^0)_{>H}$ is $N_G(H)$-contractible. Now Theorem 2.2 applies and shows that $\Lambda(G)^0$ and $\kappa(G)^0$ are G-homotopy equivalent.   ∎

In the sequel we need a group theoretical characterization of C-subgroups. For this purpose we denote by

$$\mathscr{R}: 1 = N_0 < N_1 < \cdots < N_{l-1} < N_l < \cdots < N_{k-1} < N_k = G$$

a chief series of G. By I we denote the set of indices i of chief factors N_i/N_{i-1}. A complement of a chief factor N_i/N_{i-1} is a subgroup M of G such that $M \cap N_i = N_{i-1}$ and $MN_i = G$. For $i \in I$ we write $\mathscr{M}_{\mathscr{R}}(i)$ for the complements of the chief factors N_i/N_{i-1}. This allows us to formulate the following characterization.

PROPOSITION 4.3 [We3, Proposition 3.2].   *Let G be a group. A subgroup H of G is a C-subgroup if and only if there is a set $J \subseteq I$ and complements $M_j \in \mathscr{M}_{\mathscr{R}}(j)$ of N_j/N_{j-1} such that $H = \bigcap_{j \in J} M_j$. The set J is uniquely determined by H.*

COROLLARY 4.4.   *For any maximal chain $1 < H_1 < \cdots < H_k = G$ in $\kappa(G)$ there are elements $l_1 < \cdots < l_k$ of I and maximal subgroups $M_{l_j} \in \mathscr{M}_{\mathscr{R}}(l_j)$ such that $H_i = \bigcap_{j=1}^{k-i} M_{l_j}$.*

Proof.   This follows immediately from the uniqueness of the index set J in the representation of C-subgroups in Proposition 4.3.   ∎

COROLLARY 4.5.   *Let M be a complement of the minimal normal subgroup N of the group G. Then for a C-subgroup H of G we have either $N \leq H$ or $N \cap H = 1$. In the first case $H \cap M$ is a C-subgroup of M.*

COROLLARY 4.6 [Ku, (1.3)].   *Let M be the complement of the minimal normal subgroup N in G. Then $\kappa(M) = \{H \in \kappa(G) | H \leq M\}$.*

PROPOSITION 4.7.   *The G-poset $\kappa(G)^0$ is a regular G-poset.*

Proof.   We may assume that $H_k > \cdots > H_1 > 1$ is a chain in $\kappa(G)$ such that $H_k = G$.

(i) If H_1 and therefore all groups in the chain contain the minimal normal subgroup N then $H_1/N < \cdots < H_k/N$ is a chain in $\kappa(G/N)$. This can be derived from the definition of $\kappa(G)$ and the fact that the intervals $[H_i, G]$ and $[H_i/N, G/N]$ are isomorphic. In this case the assertion follows by induction on the order of the group.

By Corollary 4.6 we know that for a minimal normal subgroup N of G either $N \leq H_i$ or $N \cap H_i = 1$.

(ii) Therefore if N is not contained in H_1, it follows from $H_k = G$ that there is an index j such that $N < H_j$ and $N \cap H_{j-1} = 1$. In the representation of C-subgroups in Proposition 4.3 the index set J is determined by the C-subgroup. Thereby we infer that there is a complement M of N in G such that $M \cap H_j \geq H_{j-1}$. By Corollary 4.6 the groups $M \cap H_i$ for $i \geq j$ and H_i for $i \leq j - 1$ are C-subgroups of M. Hence

$$(*) \quad M = H_k \cap M > H_{k-1} \cap M > \cdots > H_j \cap M$$

$$\geq H_{j-1} > \cdots > H_2 > H_1 > 1$$

is a chain of C-subgroups of M. Now let $g_1, \ldots, g_k$ be elements of G such that $H_k^{g_k} > \cdots > H_1^{g_1} > 1$ is another chain in $\kappa(G)$. By the choice of M there is an element h of N such that $M^{g_k} = M^h$. Hence the chain

$$(**) \quad M = H_k^{g_k h^{-1}} \cap M > M \cap H_{k-1}^{g_{k-1} h^{-1}} > \cdots > H_j^{g_j h^{-1}} \cap M$$

$$\geq H_{j-1}^{g_{j-1} h^{-1}} > \cdots > H_1^{g_1 h^{-1}}$$

is another chain of C-subgroups in M. From $G = NM$ we deduce that there are elements $n_1, \ldots, n_k \in N$ and $m_1, \ldots, m_k \in M$ such that $g_i h^{-1} = n_i m_i$. For $i \geq j$ we have $H_i^{g_i h^{-1}} \cap M = (H_i \cap M)^{m_i}$, by the fact $N \leq H_i$. For $i \leq j - 1$ the subgroups H_i and $H_i^{g_i h^{-1}}$ are subgroups M which are conjugate under G. Therefore by Lemma 5.1 (see [H-I-Ö, Lemma 7.1]) they are conjugate under the operation of M. Now we apply the induction hypothesis to M and the chains $(*)$ and $(**)$. We find an element $m \in M$ such that

$$H_i^m = H_i^{g_i} \quad \text{for } i \leq j - 1$$

and

$$(H_i \cap M)^m = (H_i^{g_i} \cap M) = \left(H_i^{g_i h^{-1}} \cap M\right) \quad \text{for } i \geq j.$$

Since

$$H_i^m = \left(N(H_i \cap M) \right)^m = N(H_i \cap M)^m = N(H_i^m \cap M) = H_i^{g_i} \quad \text{for } i \geq j$$

the identity $H_i^{g_i} = H_i^g$ holds for all i. ∎

Now we know that $\kappa(G)^0$ is a regular G-poset. From this we deduce the following surprising fact.

THEOREM 4.8. *The topological spaces $|C\Lambda(G)^0|/G$ and $|C(\Lambda(G)^0/G)|$ are homotopy equivalent.*

Proof. From Lemma 3.2 we know that $\Lambda(G)^0$ and $\kappa(G)^0$ are G-homotopy equivalent. Now Proposition 4.7 and Proposition 3.1 show that $|C\kappa(G)^0|/G$ and $|C(\kappa(G)^0/G)|$ are homeomorphic. Therefore $|C\Lambda(G)^0|/G$ and $|C(\kappa(G)^0/G)|$ are homotopy equivalent. Since we have shown [We3, Proposition 2.2 and Theorem 5.5] that $|C(\kappa(G)^0/G)|$ and $|C(\Lambda(G)^0/G)|$ are homotopy equivalent the assertion follows. ∎

Our next aim is to give the G-homotopy type of $\Lambda(G)^0$.

Remark 4.9. Let $A := \mathscr{M}_{\mathscr{R}}(1)$ be the set of all complements of the minimal normal subgroup N_1 of G. Then $\Lambda(G)^0$ and $\Lambda(G)^0 - A$ satisfy (I).

Proof. Condition (I) follows from Theorem 2.3 and the fact that all complements of N are maximal subgroups. ∎

For the formulation of the next theorem we define for a solvable group G the constant $C(G)$ as the product $\prod_{i=1}^k c_i$ where c_i is the number of conjugacy classes in $\mathscr{M}_{\mathscr{R}}(i)$. Actually, this number is independent from the choice of the chief series since it is the absolute value of the Möbius number of $\Lambda(G)/G$ [We1]. For a complement M of the minimal normal subgroup one easily deduces the recurrence formula $C(M) = C(G)/c_1$. In the proof of the theorem we use the following easy remark on complements of commutator subgroups.

Remark 4.10. If G is a group such that $\Lambda(G)$ is complemented then there exists a complement H_G of G' which is a C-subgroup. Furthermore:

(i) All complements of the commutator subgroup G' which are C-subgroups are conjugate. Therefore the coset spaces G/H_G are isomorphic G-sets for all complements H_G of G' which are C-subgroups.

(ii) Let M be a complement of the minimal normal subgroup N. Then for a complement H_G of the commutator subgroup G' the group $H_M = M \cap H_G$ is a complement of the commutator subgroup M' of M. If H_G is a C-subgroup of G then M_G is a C-subgroup of M.

(iii) In the situation of (ii) the G-sets G/H_G and $M/H_M \times_M G$ are isomorphic.

The concept of the complements of the commutator subgroup which are C-subgroups is equivalent to the concept of infiltrated complements introduced by Thévenaz in [Th]. We are grateful to J. Thévenaz for pointing out an error in a previous formulation of Remark 4.10.

For the formulation of the following theorem we recall that ΣS^{k-1} and S^k are homeomorphic spaces. The result given in the theorem implies the results in [Th] about the representation of G on the homology groups of $|C\Lambda(G)^0|$.

THEOREM 4.11.   *Let G be a group. If $\Lambda(G)$ is not complemented then $\Lambda(G)^0$ is G-contractible. If $\Lambda(G)$ is complemented and if k is the length of a chief series then $\Lambda(G)^0$ is G-homotopy equivalent to $\bigvee_{x \in A(G)}^{G} S_x^{k-2}$. Here $A(G)$ is as a G-set isomorphic to the sum of $c(G)$ copies of the coset space G/H_G for a complement H_G of the commutator subgroup G' which is a C-subgroup.*

Proof.   If $\Lambda(G)$ is not complemented then by [K-T, Proposition 4.13] there is a normal subgroup N which is not complemented. Therefore by Corollary 2.4 the poset $\Lambda(G)^0$ is G-contractible.

Hence we may assume that $\Lambda(G)$ is a complemented lattice. Now let N_1 be the minimal normal subgroup in the chief series $\mathcal{R}$. If M is a complement of N then $\Lambda(M) \cong \Lambda(G/N)$. Since N is a C-subgroup the lattice $\Lambda(M)$ is complemented. By induction hypothesis we know that $\Lambda(M)$ has the M-homotopy type of $\bigvee_{x \in A(M)}^{M} S_x^{k-3}$. Here $A(M)$ is the $C(M) = C(G)/c_1$-fold disjoint union of M-sets M/H_M for a complement H_M of M' in M which is a C-subgroup in M. Since $M' = G' \cap M$ the group $M \cap H_G$ is such a complement.

If M is a normal subgroup of G then M is centralized by N. This shows that $A(M)$ can be regarded as a G-set and the G-homotopy type of $\Lambda(M)^0$ is $\bigvee_{x \in A(M)}^{G} S_x^{k-3}$.

If M is not normal then it is self-normalizing. Hence in all cases the $N_G(M)$-homotopy type of $\Lambda(M)^0$ is $\bigvee_{x \in A(M)}^{G} S_x^{k-3}$.

Now we can apply Proposition 2.7. Therefore $\Lambda(G)^0$ is G-homotopy equivalent to

$$\bigvee_{M \in \mathcal{M}_{\mathcal{R}}(1)}^{G} \Sigma_M \bigvee_{x \in A(M)}^{M} S_x^{k-3}.$$

Hence Proposition 2.8 and Remark 4.10 (iii) show that $\Lambda(G)^0$ is G-homo-

topy equivalent to

$$\bigvee_{x \in A(G)}^{G} S_x^{k-2}. \quad \blacksquare$$

COROLLARY 4.12 [We3, Theorem 6.5 (ii)]. *If $\Lambda(G)$ is a complemented lattice then the poset $\Lambda(G)^0/G$ and the topological space $|C\Lambda(G)^0|/G$ are homotopy equivalent to* $\bigvee_{i=1,\ldots,C(G)} S^{k-2}$. *Here k is the length of a chief series of G.*

Proof. This follows from Theorems 4.8 and 4.11 since $|A(G)/G| = C(G)$. $\blacksquare$

5. DIRECT PRODUCTS OF FINITE GROUPS

In this section we wish to investigate the G-homotopy type of $\Lambda(G)^0$ when G is the direct product of two non-trivial groups U and V.

LEMMA 5.1. *Let N be a non-trivial normal subgroup of a group G and let M be a complement of N. Then for a chain $H_1 < \cdots < H_n$ in $\Lambda(G)_{\leq M}$ and an element $g \in N_G(M)$ there is an $m \in M$ such that $H_i^g = H_i^m$. In particular*

   (i) *$\Lambda(M)^0/M$ and $(\Lambda(G)_{\leq M})^0/N_G(M)$ are isomorphic and,*

   (ii) *$|C\Lambda(M)^0|/M$ and $|C\Lambda(G)_{\leq M}|/N_G(M)$ are homeomorphic.*

Proof. The first part is proved in [Ha], [Lemma 1.5]. The assertions (i) and (ii) are trivial consequences. $\blacksquare$

LEMMA 5.2 [H-I-Ö, Lemma 8.1]. *Let $G = U \times V$ be a direct product of two non-trivial groups U and V. We denote by C the set of complements of $U \times 1$ in G. Then the mapping*

$$\theta: \begin{cases} HOM(V,U) \to C \\ \phi \qquad\qquad \mapsto \{(\phi(v),v)|v \in V\} \end{cases}$$

is bijective.

Using the result of the previous lemma we write V_ϕ for the complement $\{(\phi(v),v)|v \in V\}$ of $U \times 1$ in G determined by $\phi \in HOM(V,U)$. Before we can state some results about the G-homotopy type of direct products we have to restrict ourselves to a special class of products. Following Hawkes [Ha, Definition 2.1] we call a group U *weakly V-free* if $|Hom(V, N_U(T)/T)| = 1$ for all $1 \neq T \leq U$.

LEMMA 5.3.   *Let* $G = U \times V$ *be the direct product of two non-trivial groups* U *and* V. *Let* U *be weakly* V-*free and let* V_ϕ *be a complement of* $U \times 1$ *in* G. *Then the mapping*

$$\gamma: \begin{cases} \Lambda(G)_{\geq V_\phi} \to \Lambda(U)_{\geq \phi(V)} \\ H \qquad\quad \mapsto H \cap \phi(V) \end{cases}$$

is a lattice isomorphism. Furthermore for $(u, v) \in N_G(V_\phi)$ *we have* $\gamma(H^{(u, v)}) = \gamma(H)^u$.

Proof.   This is a direct consequence of [H-I-Ö, Lemma 8.3].   ∎

Immediately we obtain the following proposition.

PROPOSITION 5.4.   *Let* $G = U \times V$ *be the direct product of two non-trivial groups* U *and* V. *Let* U *be weakly* V-*free and let* V_ϕ *be a complement of* $U \times 1$ *in* G. *Then one of the three following cases holds*:

(i) $V_\phi = 1 \times V$. *Here* $\Lambda(G)_{\geq V_\phi}$ *is* U-*isomorphic to* $\Lambda(U)$ *and* $\Lambda(G)_{\geq V_\phi}/N_G(V_\phi)$ *is isomorphic to* $\Lambda(U)/U$.

(ii) ϕ *is not surjective. Then* $\phi(V) \times V$ *is the smallest element in* $(\Lambda(G)_{\geq V_\phi})^0$ *and* $(\Lambda(G)_{\geq V_\phi})^0$ *is* $N_G(V_\phi)$-*contractible.*

(iii) ϕ *is surjective and* $(\Lambda(G)_{\geq V_\phi})^0$ *is empty.*

THEOREM 5.5.   *Let* U *be weakly* V-*free and let* A *be the set of complements of* $U \times 1$ *in* $G = U \times V$.

(i) $\Lambda(G)^0$ *is* G-*homotopy equivalent to*

$$\left(\Sigma_{1 \times V} \left| C\Lambda(U)^0 \right| * \left| C\Lambda(V)^0 \right| \right) \vee \bigvee_{x \in A - \{1 \times V\}}^{G} \Sigma_x \left| C\Lambda(U)^0 \right|.$$

(ii) $\Lambda(G)^0/G$ *is homotopy equivalent to*

$$\left(\Sigma_{[1 \times V]} \left| C\Lambda(U)^0 \right|/U * \left| C\Lambda(V)^0 \right|/V \right) \vee \bigvee_{[x] \in A/G - \{[1 \times V]\}} \Sigma_{[x]} \left| C\Lambda(U)^0 \right|/U.$$

Proof.   (i) By Corollary 2.4 the G-poset $\Lambda(G)^0 - A$ is G-contractible. Hence the result follows from Proposition 5.4 and Proposition 2.5.

(ii) Before we can apply Proposition 2.5 we have to check condition (C) for $\mathbf{a} = [U \times 1]$ in $P = \Lambda(G)^0/G - A/G$. By the choice of A either $[U \times 1] \vee [H] = [(U \times 1)H]$ or $[U \times 1] \wedge [H] = [(U \times 1) \cap H]$ exists in $\Lambda(G)^0/G$. Let $[H] \leq [L]$ be elements of $\Lambda(G)^0/G$ such that $[U \times 1] \vee [H]$ exists but $[U \times 1] \vee [L]$ does not exist. Since $(U \times 1)L = G$ we deduce from [H-I-Ö, Lemma 8.3] that $L = T \times V$ for a proper subgroup

of U. If we denote by S the projection of H on V then $[U \times 1] \vee [H] = [(U \times 1)H] = [U \times S]$. Therefore we have $([U \times 1] \vee [H]) \wedge [L] = [U \times S] \wedge [T \times V]$. The last infimum exists in $\Lambda(G)^0/G$ and is equal to $[T \times S]$. Now an application of Theorem 2.3 and Proposition 5.4 completes the proof. ∎

COROLLARY 5.6. *Let U be weakly V-free and let G be the group $U \times V$. If $\Lambda(U)^0/U$ (resp. $\Lambda(V)^0/V$) and $|C\Lambda(U)^0|/U$ (resp. $|C\Lambda(U)^0|/G$) are homotopy equivalent, then $\Lambda(G)^0$ and $|C\Lambda(G)^0|/G$ are homotopy equivalent.*

Proof. This follows immediately from Theorem 5.5 and Proposition 2.7. ∎

6. SOME EXAMPLES FOR FINITE SIMPLE GROUPS

Since direct products of solvable groups are solvable, the results of the preceding section become useful only when we can analyze the G-homotopy type of finite simple groups. Here we cannot present any general result, but we analyze some cases which will hopefully give some insight into what could happen. All results about G-contractibility, which cannot be derived from the theorems of Section 2, were checked by computer. In order to formulate the problem algorithmically and reduce the amount of computation, such that it will become suitable for computers, we proceeded as follows. Using Theorem 2.2 we remove by hand as much as possible from the investigated poset P. Then we apply the computer to prove a G-equivariant version of combinatorial collapsibility [Gl] for CP. As collapsibility implies contractibility, we have chosen the definition of G-collapsibility such that it implies G-contractibility.

DEFINITION 6.1. Let K, K' be two simplicial complexes on which a group G acts as a group of simplicial automorphisms.

(i) We say thaat K' is an *elementary G-collapse* of K if there is a simplex $\sigma \in K$ such that for the sets $\mathrm{Star}_{\sigma^g}(K) = \{\tau \in K | \sigma^g \leq \tau\}$ the following conditions hold:

(a) For two $g, g' \in G$ the sets $\mathrm{Star}_{\sigma^g}(K)$ and $\mathrm{Star}_{\sigma^{g'}}(K)$ either coincide or are disjoint.

(b) Each $\mathrm{Star}_{\sigma^g}(K)$ contains a unique maximal simplex.

(c) $K' = K - \bigcup_{g \in G}(\mathrm{Star}_{\sigma^g}(K) - \{\sigma^g\})$.

(ii) We say that K is *G-collapsible* if there is a sequence of elementary G-collapses which starts in K and ends with the empty complex.

Note that if $K = CP$ and $\sigma = \{x_1 < \cdots < x_r\}$ is a chain in the poset P, then condition (a) of Definition 6.1 (i) is always satisfied.

We have applied this criterion in the investigation of suitable parts of the subgroup lattices of A_6, A_7, $PSL_2(\mathbf{F}_7)$, $PSL_2(\mathbf{F}_{11})$, M_{11}, and M_{12}.

At first we analyze the two smallest non-abelian simple groups.

A_5. First we consider the intervals in $\Lambda(A_5)$. The only $N_{A_5}(H)$-contractible intervals $[H, A_5]^0$ in $\Lambda(A_5)$ are $[Z_5, A_5]^0$ for the various copies of the cyclic group of order 5 and $[V_4, A_5]^0$ for the Kleinian four groups. Let Q_5 be the lattice obtained from $\Lambda(A_5)$ after the removal of the conjugacy classes of cyclic groups of order 5 and the Kleinian four groups. We deduce by Theorem 2.2 that $\Lambda(A_5)^0$ and Q_5^0 are A_5-homotopy equivalent. Now it is easy to verify that Q_5 satisfies (RE). Hence the orbit space $|C\Lambda(A_5)^0|/A_5$ is homotopy equivalent to Q_5^0/A_5. Obviously the latter is a 1-sphere (see also Fig. 4). Moreover CQ_5^0 is a connected 1-dimensional complex and therefore homotopy equivalent to a wedge of 1-spheres. Since $\mu(\Lambda(A_5)) = -60$ we deduce that $\Lambda(A)^0$ and Q_5^0 have the homotopy type of a wedge of 60 spheres S^1. In the next step we look at contractible subposets of Q_5^0 which satisfy the conditions of the Contractible Subcomplex lemma (Theorem 2.1). Here we find the subposets $[Z_3, S_3]$ and $[1, A_4] - \{Z_3, Z_3, Z_3, Z_3, 1\}$. Dividing out the associated topological spaces from $|CQ_5^0|$ we derive from the Contractible Subcomplex lemma (or by an easy verification) that $\Lambda(A_5)^0$ is A_5-homotopy equivalent to the space depicted in Fig. 5. The A_5-operation is indicated by the labeling of the vertices. But $\Lambda(A_5)^0$ is not A_5-homotopy equivalent to a wedge of 60 spheres S^1. The fact that this is not true can be easily derived from Fig. 5, since there is no point fixed by the action of A_5. Passing to the quotient space we see that $|C\Lambda(A_5)^0|/A_5$ is homotopy equivalent to a 1-sphere.

Now let us have a look at $\Lambda(A_5)^0/A_5$ in Fig. 4. Obviously this space is homotopy equivalent to a 1-sphere. Therefore $\Lambda(A_5)^0/A_5$ and $|C\Lambda(A_5)^0|/A_5$ are homotopy equivalent.

$PSL_2(7)$. Contrary to the situation in solvable groups we have the following behavior of $\Lambda(G)^0$ and $\Lambda(G)^0/G$. The subgroup lattice
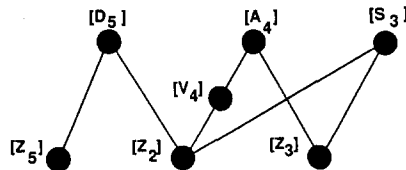

FIGURE 4

$\Lambda(PSL_2(\mathbf{F}_7))^0$ is not contractible but the poset of conjugacy classes $\Lambda(PSL_2(\mathbf{F}_7))^0/PSL_2(\mathbf{F}_7)$ is. After removing the subgroups H such that $[1, H]^0$ or $[H, PSL_2(\mathbf{F}_7)]^0$ is $N_{PSL_2(\mathbf{F}_7)}(H)$-contractible, we are left with the poset Q consisting of the subgroups of order 21, the cyclic groups of order 2 and 3, the groups isomorphic to S_3, and the two conjugacy classes of subgroups isomorphic to S_4. Some calculations show that Q is not (see also [K-T] for the computation of the homology groups) contractible. A maximal chain in Q is a maximal chain of C-subgroups of the top element in the chain. Since all elements of Q are solvable, Proposition 4.7 implies that Q is a regular $PSL_2(\mathbf{F}_7)$-poset. Hence $|C\Lambda(PSL_2(\mathbf{F}_7))^0|/PSL_2(\mathbf{F}_7)$ and $Q/PSL_2(\mathbf{F}_7)$ are homotopy equivalent. Easy computations show that $\Lambda(PSL_2(\mathbf{F}_7))^0/PSL_2(\mathbf{F}_7)$ and $Q/PSL_2(\mathbf{F}_7)$ are homotopy equivalent. Therefore $|\Lambda(PSL_2(\mathbf{F}))^0|/PSL_2(\mathbf{F}_7)$ and $\Lambda(PSL_2(\mathbf{F}_7))^0/PSL_2(\mathbf{F}_7)$ are both contractible and in particular homotopy equivalent. The space $|\Lambda(PSL_2(\mathbf{F}_7))|$ itself is homotopy equivalent to a wedge of 2-spheres and 1-spheres (again see [K-T] for computations of the homology groups). Again the homotopy equivalence cannot be G-equivariant.

We return to the alternating groups A_n, $n \geq 5$. The case $n = 5$ has been investigated above.

Next we turn to $n = 6$. By A we denote the set of subgroups of A_6 which are isomorphic to A_5. The poset $\Lambda(A_6)^0 - A$ is A_6-contractible. By Proposition 2.7 one shows that $\Lambda(A_6)^0$ is A_6-homotopy equivalent to a wedge of 12 copies of suspensions of spaces A_5-homeomorphic to that in Fig. 5. Hence $|C\Lambda(A_6)^0|/A_6$ is homotopy equivalent to a wedge of two 2-spheres. Each of these spheres corresponds to a conjugacy class of subgroups isomorphic to A_5. Surprisingly $\Lambda(A_6)^0/A_6 - A/A_6$ is contractible. Therefore $\Lambda(A_6)^0/A_6$ is also homotopy equivalent to a wedge of two 2-spheres. Here we use the fact that $(\Lambda(A_6)/A_6)_{\leq [A_5]}$ and $\Lambda(A_5)/A_5$ are isomorphic posets. Thus we have proved that $\Lambda(A_6)^0/A_6$ and $|C\Lambda(A_6)^0|/A_6$ are homotopy equivalent.

The case $n = 7$ is a little bit more difficult. Here we collect in the set A all subgroups isomorphic to A_6 and all subgroups H isomorphic to A_5 which act transitively on the set of 6 letters permuted by $A_6 \geq H$. The poset $\Lambda(A_7)^0 - A$ is A_7-contractible. But here we cannot apply Proposition 2.7 since A is not an antichain. We use an equivariant version of a theorem of Björner (private communication) which generalizes the classical version of Proposition 2.7 to the case where A is a convex subposet (one can use some elementary argumentation here as well, since the convex subset in consideration is very simple). Thereby we show that $\Lambda(A_7)^0$ is A_7-homotopy equivalent to a wedge of 7 copies of suspensions of spaces A_6-homotopic to $\Lambda(A_6)^0$. Dividing out the action of A_7 we obtain a 3-sphere. Again $\Lambda(A_7)^0/A_7 - A/A_7$ is contractible. Applying the theorem of Björner mentioned above we show that $\Lambda(A_7)^0/A_7$ is
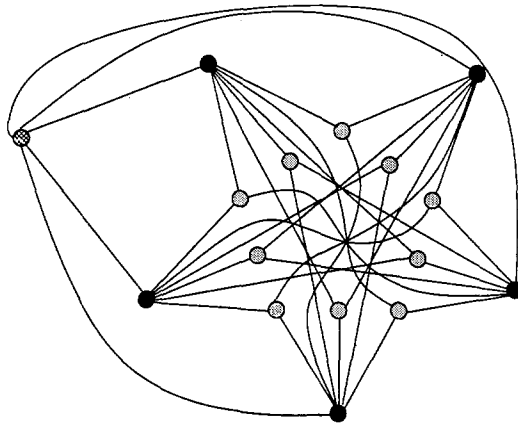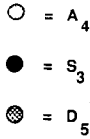
○ = A_4

● = S_3

◉ = D_5

FIGURE 5

homotopy equivalent to a 3-sphere. Analogously to the case $n = 6$ we used the fact that $(\Lambda(A_7)/A_7)_{\leq \lceil A_6 \rceil}$ and $\Lambda(A_6)/A_6$ are homotopy equivalent. So $\Lambda(A_7)^0/A_7$ and $|C\Lambda(A_7)^0|/A_7$ are homotopy equivalent.

Since $\Lambda(A_5)^0$ is homotopy equivalent to a wedge of 60 spheres S^1 we have shown that $\Lambda(A_6)^0$ is homotopy equivalent to a wedge of 720 spheres S^2 and that $\Lambda(A_7)$ is homotopy equivalent to a wedge of 2040 spheres S^3. Since the absolute value of the Möbius number of a poset which is homotopy equivalent to a wedge of n-spheres counts those spheres, a generalization of a conjecture on the Möbius number of A_n [H-I-Ö] (see also [St] for the analogous conjecture for $\mu(\Lambda(S_n))$) would be:

Conjecture 6.2. $\Lambda(A_n)$ is homotopy equivalent to a wedge of $n!/2$ spheres S^{n-4} for $n > 6$.

By our methods it would suffice to show that the poset $\Lambda(A_n)^0 - \{A_{n-1}^g | g \in A_n\}$ is A_n-contractible. It is well known that two subgroups of A_n contained in A_{n-1} are conjugate if and only if they are conjugate in A_n. Hence if we would show that $\Lambda(A_n)^0/A_n - \{[A_{n-1}]\}$ is contractible then we could inductively prove the original conjecture [H-I-Ö]:

Conjecture 6.3. For the Möbius number μ of $\Lambda(A_n)$ and $\Lambda(A_n)/A_n$ the following equation holds in the case $n > 6$:

$$\mu(\Lambda(A_n)) = |A_n| \cdot \mu(\Lambda(A_n)/A_n).$$

Applying the methods used for A_5, A_6, and A_7 to corresponding symmetric groups we obtain similar results. Now we turn to the first two Mathieu groups M_{11} and M_{12}.

M_{11}. In M_{11} there is one conjugacy class of subgroups isomorphic to $PSL_2(\mathbf{F}_{11})$. Setting $A := [PSL_2(\mathbf{F}_{11})]$ we obtain that $\Lambda(M_{11})^0 - A$ is M_{11}-contractible. Hence $\Lambda(M_{11})^0$ is M_{11}-homotopy equivalent to

$$\bigvee_{x \in A}^{M_{11}} \sum_x \left| C\Lambda(PSL_2(\mathbf{F}_{11}))^0 \right|.$$

By inspection one shows that for the $PSL_2(\mathbf{F}_{11})$-poset $P = \Lambda(PSL_2(\mathbf{F}_{11}))^0$ the orbit space $|CP|/PSL_2(\mathbf{F}_{11})$ is homotopy equivalent to $P/PSL_2(\mathbf{F}_{11})$. Now Proposition 3.3 implies that

$$\left| C\Lambda(M_{11})^0 \right|/M_{11} \quad \text{and} \quad \Sigma \left| C\left(\Lambda(PSL_2(\mathbf{F}_{11}))\right)^0/PSL_2(\mathbf{F}_{11}) \right|$$

are homotopy equivalent. But the poset $\Lambda(M_{11})^0/M_{11} - [PSL_2(\mathbf{F}_{11})]$ is contractible. Hence we have shown that the poset $\Lambda(M_{11})^0/M_{11}$ and $|C\Lambda(M_{11})^0|/M_{11}$ are homotopy equivalent.

It turns out that M_{12} behaves exceptionally. By using CAYLEY it has already been shown [B-G-V] that for the Möbius numbers the equation $\mu(\Lambda(G)) = |G'| \cdot \mu(\Lambda(G)/G)$ does not hold. Note that this equation holds for all other groups we have investigated above and in previous sections.

M_{12}. For the computational analysis of $\Lambda(M_{12})$ we refer to [B-R] and [B-G-V]. In $\Lambda(M_{12})^0$ there is one conjugacy class of subgroups isomorphic to $PSL_2(\mathbf{F}_{11})$ which acts transitive on the 12 letters in the usual permutation representation of M_{12}. If we denote by A this conjugacy class then $\Lambda(M_{12})^0 - A$ is M_{12}-contractible. By Proposition 2.5 we deduce that $\Lambda(M_{12})^0$ is M_{12}-homotopy equivalent to

$$\bigvee_{x \in A}^{M_{12}} \sum_x \left| C\Lambda(PSL_2(\mathbf{F}_{11}))^0 \right|.$$

But the poset $\Lambda(M_{12})^0$ is not a regular M_{12}-poset. Although this is also true for other cases investigated above, the failure occurs in this case in a crucial part of the subgroup lattice. The two conjugacy classes in $PSL_2(\mathbf{F}_{11})$ of subgroups isomorphic to A_5 are fused under the action of M_{12}. For a subgroup $H \in A$ the poset $(Q/M_{12})_{<[H]}$ is contractible but $\Lambda(H)^0/H$ is not. Hence some additional computation shows that

$$\left(\bigvee_{x \in A}^{M_{12}} \sum_x \left| C\Lambda(PSL_2(\mathbf{F}_{11}))^0 \right| \right) \Big/ M_{12}$$

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is contractible. This shows that $|C\Lambda(M_{12})^0|/M_{12}$ is contractible. But $|C(\Lambda(M_{12})^0/M_{12})|$ is not contractible since its Möbius number is not 0 [B-G-V, Appendix].

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