Contemporary Mathematics

On the nonabelian tensor squares of free nilpotent groups of finite rank

Russell D. Blyth, Primož Moravec, and Robert Fitzgerald Morse

ABSTRACT. We determine the nonabelian tensor squares and related homological functors of the free nilpotent groups of finite rank.

1. Introduction

Let G be any group. Then the group $G \otimes G$ generated by the symbols $g \otimes h$, where $g, h \in G$, subject to the relations

$$gh \otimes k = ({}^g h \otimes {}^g k)(g \otimes k) \quad \text{and} \quad g \otimes hk = (g \otimes h)({}^h g \otimes {}^h k)$$

for all g, h, and k in G, where ${}^x y = xyx^{-1}$ for $x, y \in G$, is called the *nonabelian tensor square of* G. Let $\nabla(G)$ be the subgroup of $G \otimes G$ generated by the set $\{g \otimes g \mid g \in G\}$. The group $\nabla(G)$ is a central subgroup of $G \otimes G$ [**8**]. The factor group $G \otimes G/\nabla(G)$ is called the *nonabelian exterior square* of G, denoted by $G \wedge G$. For elements g and h in G, the coset $(g \otimes h)\nabla(G)$ is denoted $g \wedge h$.

In his paper [**13**] C. Miller gives a group theoretic interpretation of the Schur multiplier of a group G or, equivalently, $H_2(G)$, the second integral homology group of G. Miller shows that $H_2(G)$ is the group that contains all relations satisfied by the commutators in G modulo those commutator relations which are trivially, or universally, satisfied by G. He interprets $H_2(G)$ to be a measure of the extent to which relations among commutators in G fail to be consequences of universal commutator relations. A relation is universally satisfied if it holds in the free group. We list some of these in (2.1).

For the free group F_n of rank n, the group $H_2(F_n)$ is trivial, since F_n does not satisfy any relations other than the universal commutator relations. Let $\mathcal{N}_{n,c} \cong F_n/\gamma_{c+1}(F_n)$ be the free nilpotent group of class c and rank n. By Theorem 1 of [**17**], $H_2(G)$ is isomorphic to the free abelian group of rank $M(n, c+1)$, where $M(n, c)$ is the number of basic commutators in n symbols of weight c. This matches

---

2000 *Mathematics Subject Classification.* 20F05, 20F12, 20F18, 20J06.

Key words and phrases. Free nilpotent groups, Nonabelian tensor squares, Schur multiplier.

The second and third authors thank the Institute for Global Enterprise in Indiana for its financial support of this research. The third author thanks the Ministry of Science of Slovenia for supporting his postdoctorial leave to visit the University of Evansville and the Institute for Global Enterprise in Indiana for its generous hospitality while visiting there.

Miller's interpretation of $H_2(G)$, as it captures the commutator relations of $\mathcal{N}_{n,c}$ that are not consequences of universal commutator relations.

Implicit in Miller's work is that $H_2(G)$ is the kernel of the commutator mapping

$$1 \longrightarrow H_2(G) \longrightarrow G \wedge G \xrightarrow{\kappa'} G' \longrightarrow 1, \tag{1.1}$$

where $\kappa'(g \wedge h) = [g, h]$ for all g, h in G. R. K. Dennis in his preprint "In Search of New 'Homology' Functors Having a Close Relationship to K-theory" [**9**] makes note of (1.1) and extends the nonabelian exterior square to the nonabelian tensor square. Dennis considers the commutator map

$$1 \longrightarrow J_2(G) \longrightarrow G \otimes G \xrightarrow{\kappa} G' \longrightarrow 1,$$

where $\kappa(g \otimes h) = [g, h]$ for all g, h, in G, and investigates its kernel $J_2(G)$. Brown and Loday in [**8**] show that $J_2(G)$ is isomorphic to $\pi_3 SK(G, 1)$, the third homotopy group of the suspension of an Eilenberg MacLane space $K(G, 1)$.

In the same paper, Brown and Loday introduce the nonabelian tensor product $G \otimes H$ of two groups G and H. This product is defined if the two groups act on each other in a compatible way. The nonabelian tensor square $G \otimes G$ can be considered a specialization of the nonabelian tensor product, where the actions are taken to be conjugation. The nonabelian tensor square of a group is always defined.

The study of $G \otimes G$ from a group theoretic point of view was started by Brown, Johnson and Robertson in their seminal paper "Some Computations of Non-Abelian Tensor Products of Groups" [**7**]. One focus of their paper is to "compute" the nonabelian tensor square for various groups. By computing the nonabelian tensor square of a group G, we mean finding a simple or standard form for expressing $G \otimes G$. The definition of the nonabelian tensor square gives no insight as to the group it describes or its structure. Starting in [**7**], methods in computational group theory have been invoked to investigate this problem.

In [**7**] the approach to computing the nonabelian tensor square for a finite group G is to form the finite presentation given in the definition and to use a computer program to perform Tietze transformations to simplify the presentation. This simplified presentation is then examined to determine the isomorphism type of $G \otimes G$. This technique was applied to all of the nonabelian groups of order up to 30. This method becomes impractical for large finite groups since one starts with $|G|^2$ generators and $2|G|^3$ relations.

To compute some examples of the nonabelian tensor product for finite groups, Ellis and Leonard [**11**] construct a group in which the nonabelian tensor product naturally embeds. In the specialized case of the nonabelian tensor square, we denote this group by $\nu(G)$, following Rocco [**15**], who independently investigated its properties.

In the following, we fix G to be an arbitrary group with presentation $\langle \mathcal{G} | \mathcal{R} \rangle$.

Definition 1.1. Let G be a group with presentation $\langle \mathcal{G} | \mathcal{R} \rangle$ and let G^φ be an isomorphic copy of G via the mapping $\varphi : g \to g^\varphi$ for all $g \in G$. We define the group $\nu(G)$ to be

$$\nu(G) = \langle \mathcal{G}, \mathcal{G}^\varphi | \mathcal{R}, \mathcal{R}^\varphi, {}^x[g, h^\varphi] = [{}^x g, ({}^x h)^\varphi] = {}^{x^\varphi}[g, h^\varphi], \forall x, g, h \in G \rangle.$$

The motivation for considering $\nu(G)$ relative to the nonabelian tensor square is the following theorem given in [**11**] and [**15**].

THEOREM 1.2. *Let G be a group. The map $\phi : G \otimes G \to [G, G^\varphi] \lhd \nu(G)$ defined by $\phi(g \otimes h) = [g, h^\varphi]$ for all g and h in G is an isomorphism.*

It is clear from the definition of $\nu(G)$ that it is generated by $2|\mathcal{G}|$ elements, which is significantly smaller than the number of generators given in the definition of the nonabelian tensor square. Ellis and Leonard [**11**] show that the relations for $\nu(G)$ can be significantly pruned depending on the size and structure of the center of G. Hence the computational strategy is to construct a relatively small finite presentation of $\nu(G)$, compute a concrete presentation for $\nu(G)$, and then apply standard computational group theory methods to find the subgroup $[G, G^\varphi]$. Ellis and Leonard were able to compute the nonabelian tensor squares for some large finite p-groups, such as the Burnside group of exponent 4 and rank 2, which has order 2^{12}, by applying a p-quotient algorithm to find a power-conjugate presentation of $\nu(G)$, from which the subgroup $[G, G^\varphi]$ can easily be determined. This computation is essentially impossible using the Tietze transformations method.

For an infinite group the definition of the nonabelian tensor square leads to an infinite presentation. The standard technique for computing the nonabelian tensor square for infinite groups is to find a mapping $\Phi : G \times G \to L$ for some group L. If Φ satisfies certain conditions then we call Φ a crossed pairing. If Φ is a crossed pairing then it lifts to a unique homomorphism $\Phi^* : G \otimes G \to L$. Hence to compute the nonabelian tensor square one proposes a group L that one intends to show is isomorphic to $G \otimes G$, devises a crossed pairing Φ, and shows that the lift Φ^* is actually an isomorphism. This method has been used to compute the nonabelian tensor squares for the free nilpotent groups of class 2 of finite rank [**1**] and the infinite metacyclic groups [**3**]. In each of these cases the nonabelian tensor square is abelian. The crossed pairing method was also used to compute the nonabelian tensor squares of the free 2-Engel groups of finite rank (see [**2**] and [**6**]). An appropriate group L for the free 2-Engel group of rank n was suggested by using the computational techniques of Ellis and Leonard [**11**] to compute the nonabelian tensor square of a finite image of the free 2-Engel group of rank n, namely the Burnside group of exponent 3 and rank n. In the free 2-Engel case, where the nonabelian tensor squares are not abelian, the computations were overwhelming, and the viability of this method for general use seems limited.

To overcome the limitations of the crossed pairing method when the nonabelian tensor square is not abelian, Blyth and Morse [**5**] extend the method used by Ellis and Leonard [**11**] to infinite groups and, in particular, to polycyclic groups. If G is a polycyclic group then $G \otimes G$ is polycyclic and so is $\nu(G)$ [**5**]. Hence for finite and infinite polycyclic groups both $G \otimes G$ and $\nu(G)$ are finitely presented. A finite presentation of $\nu(G)$ can be described in terms of a polycyclic generating sequence of G. Using a polycyclic quotient algorithm, one is able to compute a polycyclic representation for $\nu(G)$ and use standard algorithms for polycyclic groups to compute the subgroup $[G, G^\varphi]$. Such standard algorithms are implemented in the GAP [**12**] package `Polycyclic` [**10**]. For nilpotent groups this method works well since there exist fast and effective nilpotent quotient algorithms, for example, `nq` [**14**], for computing a polycyclic presentation for $\nu(G)$. A simple GAP program that computes the nonabelian tensor square for nilpotent groups is given in [**5**]. This program creates a finite presentation for $\nu(G)$ using the polycyclic presentation of G and then organizes a series of function calls to compute a polycyclic presentation for $\nu(G)$ and to compute the subgroup $[G, G^\varphi]$ of $\nu(G)$.

Rocco [**15**] initiates the development of a commutator calculus associated with the subgroup $[G, G^\varphi]$ of $\nu(G)$ that allows for general computations in this subgroup. This commutator calculus is extended in [**5**]. In this paper we extend it further by providing some new identities. (See Lemma 2.1, identities (ii) and (iii).) The commutator calculus is used in [**5**] to compute the nonabelian tensor square of the free nilpotent group of class 3. We use this calculus in proving Theorem 1.6, the main theorem of this paper.

In [**5**] a group $\tau(G)$ is defined that is analogous to the group $\nu(G)$ in that the subgroup $[G, G^\varphi]$ of $\tau(G)$ is isomorphic to the nonabelian exterior square, $G \wedge G$, of G.

Definition 1.3. Let G be any group. Then we define $\tau(G)$ to be the quotient group $\nu(G)/\phi(\nabla(G))$, where $\phi : G \otimes G \to [G, G^\varphi]$ is as defined in Theorem 1.2.

Since ϕ isomorphically embeds $\nabla(G)$ into $[G, G^\varphi]$, it follows that

$$[G, G^\varphi]/\phi(\nabla(G)) \cong G \wedge G.$$

We henceforth denote $[G, G^\varphi]/\phi(\nabla(G))$ by $[G, G^\varphi]_{\tau(G)}$. The following proposition is now evident.

Proposition 1.4. *Let G be any group. The map*

$$\hat{\phi} : G \wedge G \to [G, G^\varphi]_{\tau(G)} \lhd \tau(G)$$

defined by $\hat{\phi}(g \wedge h) = [g, h^\varphi]_{\tau(G)}$ is an isomorphism.

Assembling the maps together we obtain the following sequence of mappings:

$$G \otimes G \xrightarrow{\ \phi\ } [G, G^\varphi] \xrightarrow{\ \sigma\ } [G, G^\varphi]_{\tau(G)} \xrightarrow{\ \hat{\phi}^{-1}\ } G \wedge G \tag{1.2}$$

where ϕ and $\hat{\phi}^{-1}$ are isomorphisms and σ is an epimorphism that is the restriction of the canonical epimorphism $\nu(G) \to \tau(G)$ to the subgroup $[G, G^\varphi]$. Using (1.2) an arbitrary generator $g \otimes h$ of $G \otimes G$ is mapped to the element $g \wedge h$ in $G \wedge G$ by

$$\hat{\phi}^{-1}(\sigma(\phi(g \otimes h))) = g \wedge h. \tag{1.3}$$

We use this composition of homomorphisms in the proof of Theorem 1.6 below.

To date only the nonabelian tensor squares of the free nilpotent groups of class 2 and 3 with finite rank have been computed.

Theorem 1.5 ([**1**],[**5**]). *Denote the free nilpotent group of class c and rank $n > 1$ by $\mathcal{N}_{n,c}$ and the free abelian group of rank n by F_n^{ab}.*

(i) For $c = 2$, $\mathcal{N}_{n,2} \otimes \mathcal{N}_{n,2} \cong F_{f(n)}^{\mathrm{ab}}$, where

$$f(n) = \frac{n(n^2 + 2n - 1)}{3};$$

(ii) For $c = 3$, $\mathcal{N}_{n,3} \otimes \mathcal{N}_{n,3}$ is the direct product of W_n and $F_{h(n)}^{\mathrm{ab}}$, where W_n is a nilpotent of class 2 group minimally generated by $n(n-1)$ elements and

$$h(n) = \frac{n(3n^3 + 14n^2 - 3n + 10)}{24}.$$

The following commutative diagram is found in [**7**]:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
H_3(G) \arrow[r] & \Gamma(G^{\mathrm{ab}}) \arrow[r, "\psi"] \arrow[d] & J_2(G) \arrow[r] \arrow[d] & H_2(G) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \nabla(G) \arrow[r] & G \otimes G \arrow[r] \arrow[d, "\kappa"'] & G \wedge G \arrow[r] \arrow[d, "\kappa'"'] & 1 \\
 & & G' \arrow[r, equal] \arrow[d] & G' \arrow[d] & \\
 & & 1 & 1 &
\end{tikzcd}
\tag{1.4}
$$

All sequences in this diagram are exact and the short exact sequences are central. The group $\Gamma(G^{\mathrm{ab}})$ is the Whitehead quadratic functor found in [**18**].

The purpose of this paper is to compute the nonabelian tensor squares of the free nilpotent groups of class c and rank n as well as most of the other homological functors in Diagram (1.4). Our main theorem is the following:

THEOREM 1.6. *Let $G = \mathcal{N}_{n,c}$ be the free nilpotent group of class c and rank $n > 1$. Then*

$$G \otimes G \cong \Gamma(G^{ab}) \times G \wedge G.$$

A covering group $\hat{G}$ of a group G is a central extension

$$1 \longrightarrow H_2(G) \xrightarrow{\iota} \hat{G} \longrightarrow G \longrightarrow 1,$$

where the image of ι is a subset of $\hat{G}'$. If G is a finitely generated group then $\hat{G}'$ is isomorphic to $G \wedge G$ by Corollary 2 of [**7**].

Suppose $G = \mathcal{N}_{n,c}$. Then $\hat{G} \cong \mathcal{N}_{n,c+1}$ is a covering group for G. Since G is a finitely presented group, $G \wedge G$ is isomorphic to $\mathcal{N}'_{n,c+1}$. In Section 3 we prove that $\nabla(G)$ is isomorphic to $\Gamma(G^{\mathrm{ab}})$ (Corollary 3.2). Since G^{ab} is isomorphic to F_n^{ab}, by a result of Whitehead ([**18**], Section 5), $\Gamma(G^{\mathrm{ab}}) \cong F^{\mathrm{ab}}_{\binom{n+1}{2}}$.

From these observations we obtain the following corollary.

COROLLARY 1.7. *Let $G = \mathcal{N}_{n,c}$ be the free nilpotent group of class c and rank $n > 1$. Then*

$$G \otimes G \cong \mathcal{N}'_{n,c+1} \times F^{\mathrm{ab}}_{\binom{n+1}{2}}.$$

In the case when $c = 3$, the subgroup W_n of Theorem 1.5 is not isomorphic to $\mathcal{N}'_{n,4}$, as the free abelian factor has rank larger then $\binom{n+1}{2}$. The group W_n is in fact a direct product of a free nilpotent group of class 2 and rank $M(n,2)$, and a free abelian group of rank $\binom{n}{2}$. This case suggests an investigation into the structure of $\mathcal{N}'_{n,4}$. We will give a detailed structural description of $\mathcal{N}'_{n,c+1}$, the derived subgroup of the free nilpotent group of class $c+1 > 3$ and rank n, in a companion paper [**4**]. However, to illustrate the application of Corollary 1.7, we give a complete structure description for the $c = 3$ case in Section 3.

Theorem 1.6 is motivated by exploring examples computed using GAP [**12**]. An illustrative description of how these examples were computed is given in Section 5. Our proof of Theorem 1.6 given in Section 4 relies on knowledge of $\nabla(G)$. The structure of the group $\nabla(G)$ and most of the groups in Diagram 1.4, except the nonabelian tensor square, is given in Section 3.

2. A commutator calculus

In this section we introduce and extend a commutator calculus for the subgroup $[G, G^{\varphi}]$ of $\nu(G)$. An account of this calculus can be found in [**5**], which is based in part on [**15**]. The identities found in [**15**] use right actions and are restated using left actions both in [**5**] and this paper. The identities listed for the tensor square in Proposition 3 of [**7**], which use left actions, are now naturally reflected in the identities found in this calculus. Since all conjugation and commutation in this paper is done using left actions, we include a few basic commutator identities for the convenience of the reader. Let G be any group and x, y and z be elements of G. Then

$$\begin{aligned} {}^{x}y &= [x,y]\cdot y; \\ [xy,z] &= {}^{x}[y,z]\cdot[x,z]; \\ [x,yz] &= [x,y]\cdot {}^{y}[x,z]; \\ [x^{-1},y] &= {}^{x^{-1}}[x,y]^{-1} = [x^{-1},[x,y]^{-1}]\cdot[x,y]^{-1}; \\ [x,y^{-1}] &= {}^{y^{-1}}[x,y]^{-1} = [y^{-1},[x,y]^{-1}]\cdot[x,y]^{-1}; \end{aligned} \tag{2.1}$$

and $[x^{-1},y^{-1}] = [x^{-1},[y^{-1},[x,y]]]\cdot[y^{-1},[x,y]]\cdot[x^{-1},[x,y]]\cdot[x,y].$

The following lemma records some basic identities used in this paper.

LEMMA 2.1. *Let G be a group. The following relations hold in $\nu(G)$:*

(i) ${}^{[g_3,g_4^{\varphi}]}[g_1,g_2^{\varphi}] = {}^{[g_3,g_4]}[g_1,g_2^{\varphi}]$ *and* ${}^{[g_3^{\varphi},g_4]}[g_1,g_2^{\varphi}] = {}^{[g_3,g_4]}[g_1,g_2^{\varphi}]$ *for all* g_1, g_2, g_3, g_4 *in* G*;*
(ii) $[g_1^{\varphi},g_2,g_3] = [g_1,g_2,g_3^{\varphi}] = [g_1^{\varphi},g_2,g_3^{\varphi}] = [g_1,g_2^{\varphi},g_3] = [g_1^{\varphi},g_2^{\varphi},g_3] = [g_1,g_2^{\varphi},g_3^{\varphi}]$ *for all* g_1, g_2, g_3 *in* G*;*
(iii) $[g_1,[g_2,g_3]^{\varphi}] = [g_2,g_3,g_1^{\varphi}]^{-1}$*;*
(iv) $[g,g^{\varphi}]$ *is central in* $\nu(G)$ *for all* g *in* G*;*
(v) $[g_1,g_2^{\varphi}][g_2,g_1^{\varphi}]$ *is central in* $\nu(G)$ *for all* g_1, g_2 *in* G*;*
(vi) $[g,g^{\varphi}] = 1$ *for all* g *in* G'*.*

PROOF. All of the identities can be found in [**5**] except (ii) and (iii). In [**15**] it was shown that

$$[g_1^{\varphi},g_2,g_3] = [g_1,g_2,g_3^{\varphi}] = [g_1^{\varphi},g_2,g_3^{\varphi}] \tag{2.2}$$

and

$$[g_1,g_2^{\varphi},g_3] = [g_1^{\varphi},g_2^{\varphi},g_3] = [g_1,g_2^{\varphi},g_3^{\varphi}]. \tag{2.3}$$

Using these identities we have

$$\begin{aligned}
[g_1, g_2^\varphi, g_3] &= [[g_2^\varphi, g_1]^{-1}, g_3] \\
&= {}^{[g_2^\varphi, g_1]^{-1}}[g_2^\varphi, g_1, g_3]^{-1} \\
&= {}^{[g_2, g_1]^{-1}}[g_2^\varphi, g_1, g_3]^{-1} && \text{by Lemma 2.1(i)} \\
&= {}^{[g_1, g_2]}[g_2, g_1, g_3^\varphi]^{-1} && \text{by (2.2)} \\
&= {}^{[g_1, g_2][g_1, g_2]^{-1}}[g_1, g_2, g_3^\varphi] \\
&= [g_1, g_2, g_3^\varphi].
\end{aligned}$$

Hence it follows that all six commutators in (2.2) and (2.3) are equal.

Identity (iii) is a simple consequence of (ii):

$$[g_1, [g_2, g_3]^\varphi] = [g_2^\varphi, g_3^\varphi, g_1]^{-1} = [g_2, g_3, g_1^\varphi]^{-1}.$$

□

We represent a generator $g \otimes h$ of $G \otimes G$ as $[g, h^\varphi]$ using the isomorphism ϕ of Theorem 1.2. The commutator identities of Lemma 2.1 allow us to make nonabelian tensor computations with familiar commutator calculations.

3. Structure of Homological Functors

In this section we determine the groups $\nabla(\mathcal{N}_{n,c})$, $\Gamma(\mathcal{N}_{n,c}^{\mathrm{ab}})$, and $J_2(\mathcal{N}_{n,c})$. Let A be an abelian group. The Whitehead quadratic functor, $\Gamma(A)$, is an abelian group with generators $\gamma(a)$, where $a \in A$, with the following relations:

$$\gamma(a^{-1}) = \gamma(a),$$
$$\gamma(abc)\gamma(a)\gamma(b)\gamma(c) = \gamma(ab)\gamma(bc)\gamma(ca),$$

for $a, b, c \in A$. There is a well defined homomorphism

$$\psi : \Gamma(G^{ab}) \to G \otimes G$$

such that $\psi(\gamma(g)G') = g \otimes g$ ([**7**], page 181). The image of ψ is $\nabla(G)$.

The projection map

$$\pi : G \otimes G \to G^{\mathrm{ab}} \otimes G^{\mathrm{ab}},$$

where $G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}$ is an ordinary tensor product ([**7**], Remark 2) abelianizes $G \otimes G$. Suppose G^{ab} has a basis $\{a_1, \ldots, a_n\}$. Then $G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}$ is an abelian group with a basis

$$\{a_i \otimes a_i, a_i \otimes a_j, (a_i \otimes a_j)(a_j \otimes a_i) \mid 1 \le i, j \le n, i < j\}. \tag{3.1}$$

Proposition 3.1. *Let G be a group whose abelianization is free abelian of finite rank. Then $\nabla(G) \cong \Gamma(G^{\mathrm{ab}})$.*

Proof. The abelianization G^{ab} of G is isomorphic to F_n^{ab} for some n. Let $\{a_i \mid 1 \le i \le n\}$ be a basis for G^{ab}. By a result of Whitehead ([**18**], page 62) there is a basis B for $\Gamma(G^{\mathrm{ab}})$ consisting of $\gamma(a_i)$ and $(a_i, a_j) = \gamma(a_i a_j)\gamma(a_i)^{-1}\gamma(a_j)^{-1}$ for

$1 \leq i, j \leq n$ and $i < j$. Define $\Phi = \pi\psi$, where ψ and π are defined above. Then we have that $\Phi(\gamma(a_i)) = a_i \otimes a_i$ and

$$\begin{aligned}\Phi((a_i, a_j)) &= \Phi(\gamma(a_i a_j))\Phi(\gamma(a_i)^{-1})\Phi(\gamma(a_j)^{-1}) \\ &= \Phi(\gamma(a_i a_j))\Phi(\gamma(a_i))^{-1}\Phi(\gamma(a_j))^{-1} \\ &= (a_i a_j \otimes a_i a_j)(a_i \otimes a_i)^{-1}(a_j \otimes a_j)^{-1} \\ &= (a_i \otimes a_j)(a_j \otimes a_i)(a_j \otimes a_j)(a_i \otimes a_i)(a_i \otimes a_i)^{-1}(a_j \otimes a_j)^{-1} \\ &= (a_i \otimes a_j)(a_j \otimes a_i).\end{aligned}$$

The last two equalities hold as we are computing in the usual abelian tensor product, $G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}$. The images of the elements of the basis B for $\Gamma(G^{\mathrm{ab}})$ under Φ are part of the basis (3.1) for $G^{\mathrm{ab}} \otimes G^{\mathrm{ab}}$. We conclude that ψ is injective. Since ψ is also surjective, it is bijective and $\Gamma(G^{\mathrm{ab}}) \cong \nabla(G)$. □

Corollary 3.2. *Let $G = \mathcal{N}_{n,c}$ be the free nilpotent group of class c and rank $n > 1$. Then $\nabla(G) \cong \Gamma(G^{\mathrm{ab}})$.*

Using Corollary 1.7 we can give a complete structure description of $\mathcal{N}_{n,3} \otimes \mathcal{N}_{n,3}$.

Proposition 3.3. *Let $G = \mathcal{N}_{n,3}$ be the free nilpotent group of class 3 and rank $n > 1$. Then $G \otimes G \cong \mathcal{N}_{M(n,2),2} \times F^{\mathrm{ab}}_{f(n)}$, where*

$$f(n) = \frac{n(n+1)(3n^2 + 11n - 2)}{24}.$$

Proof. By Corollary 1.7, $G \otimes G \cong \mathcal{N}'_{n,4} \times F^{\mathrm{ab}}_{\binom{n+1}{2}}$. In [**5**], Lemma 32, it was shown that

$$\mathcal{N}'_{n,4} \cong \mathcal{N}_{M(n,2),2} \times F^{\mathrm{ab}}_{g(n)},$$

where $g(n) = M(n,3) + M(n,4) - M(M(n,2),2)$. Using the Witt-Hall identity, $M(n,2) = n(n-1)/2$, $M(n,3) = n(n^2-1)/3$, and $M(n,4) = n^2(n^2-1)/4$. Therefore,

$$\begin{aligned}g(n) &= \frac{n(n^2-1)}{3} + \frac{n^2(n^2-1)}{4} - \frac{n^4 - 2\,n^3 - n^2 + 2\,n}{8} \\ &= \frac{3\,n^4 + 14\,n^3 - 3\,n^2 - 14\,n}{24}.\end{aligned}$$

It follows that

$$\begin{aligned}f(n) &= \binom{n+1}{2} + g(n) \\ &= \binom{n+1}{2} + \frac{3\,n^4 + 14\,n^3 - 3\,n^2 - 14\,n}{24} \\ &= \frac{n(n+1)(3n^2 + 11n - 2)}{24}.\end{aligned}$$

□

The description of $\mathcal{N}_{n,3} \otimes \mathcal{N}_{n,3}$ in Theorem 1.5 (ii) fails to include all of $\nabla(\mathcal{N}_{n,3})$ in the free abelian factor of the direct product. The group $\nabla(\mathcal{N}_{n,c})$ has n independent generators of the form $g_i \otimes g_i$ and $\binom{n}{2}$ independent generators of the form $(g_i \otimes g_j)(g_j \otimes g_i)$. This matches the result of Corollary 3.2: $\nabla(\mathcal{N}_{n,c}) \cong \Gamma(\mathcal{N}^{\mathrm{ab}}_{n,c})$, which is free abelian of rank $\binom{n+1}{2} = n + \binom{n}{2}$. The direct factor W_n of the direct product in Theorem 1.5 (ii) includes the $\binom{n}{2}$ generators of $\nabla(\mathcal{N}_{n,3})$ of the form $(g_i \otimes g_j)(g_j \otimes g_i)$. Hence the rank of the free abelian direct factor of Theorem 1.5 is

$$\begin{aligned} n + g(n) &= n + M(n,3) + M(n,4) - M(M(n,2),2) \\ &= \frac{n(3n^3 + 14n^2 - 3n + 10)}{24}. \end{aligned} \tag{3.2}$$

Adding in the generators of $\nabla(\mathcal{N}_{n,3})$ included in W_n to (3.2), we obtain $f(n)$ from Proposition 3.3.

The discussion above depends, of course, on the proof of Theorem 1.6. We begin with some preliminary results that are used in the proof of Theorem 1.6. We start by formally introducing the notion of a basic sequence of commutators. Our exposition follows Sims [**16**].

A basic sequence of commutators in the free group F_n of rank n is an infinite sequence $c_1, c_2, \ldots$ of elements of F_n, where each c_i has associated with it a positive integer w_i called its weight. The sequence is defined as follows. The c_i are ordered by weight i.e. if $j > i$ then $w_j \geq w_i$. The $c_1, \ldots, c_n$ are the free generators of F_n arranged in some order. If $w_k > 1$ then c_k is described explicitly by $[c_j, c_i]$, where $j > i$ and $w_j + w_i = w_k$. If $w_j > 1$, so that c_j is described by $[c_q, c_p]$ with $q > p$, then $p \leq i$. Lastly, for each $j > i$ such that either $w_j = 1$ or $w_j > 1$ and c_j is described as $[c_q, c_p]$ with $p \leq i$, there is a unique index k such that c_k is described as $[c_j, c_i]$. We fix one basic sequence of commutators in the free group F_n and denote it by $\mathcal{C}_n$. The elements of $\mathcal{C}_n$ are referred to as basic commutators. We denote the subsequence of commutators of $\mathcal{C}_n$ whose weight is at most w by $\mathcal{C}_{n,w}$. The elements of $\mathcal{C}_{n,c}$ map to $\mathcal{N}_{n,c}$, the free nilpotent group of class c and rank n, via the natural homomorphism $F_n \to F_n/\gamma_{c+1}(F_n) \cong \mathcal{N}_{n,c}$. We will consider elements of $\mathcal{C}_{n,c}$ as the same as their images in $\mathcal{N}_{n,c}$.

The following proposition found in [**16**] will be used in the next section.

PROPOSITION 3.4. *The subsequence $\mathcal{C}_{n,c}$ of $\mathcal{C}_n$ forms a polycyclic generating sequence for $\mathcal{N}_{n,c}$.*

We conclude this section by showing that $J_2(\mathcal{N}_{n,c})$ splits. This follows from the following more general statement.

PROPOSITION 3.5. *Let G be a polycyclic group whose abelianization and second homology are both free abelian groups. Then $J_2(G) \cong \Gamma(G^{\mathrm{ab}}) \times H_2(G)$.*

PROOF. Let G be a polycyclic group, generated by $\{g_i \mid 1 \leq i \leq n\}$. The central subgroup $J_2(G)$ of $G \otimes G$ is the kernel of the commutator mapping $\kappa : G \otimes G \to G'$, defined by $\kappa(g \otimes h) = [g, h]$. The tensor square $G \otimes G$ is a polycyclic group [**5**]. Hence $J_2(G)$ is a finitely generated abelian group, and therefore is a direct product $F \times T$, where F is a free abelian group of finite rank and T is a finite abelian group. The group $\Gamma(G^{\mathrm{ab}})$ is free abelian of rank $\binom{n+1}{2}$, while the Schur multiplier $H_2(G)$ is free abelian of finite rank, say d. In the exact sequence

$$\Gamma(G^{\mathrm{ab}}) \xrightarrow{\ \psi\ } J_2(G) \xrightarrow{\ \beta\ } H_2(G) \longrightarrow 0 \tag{3.3}$$

from Diagram (1.4), the kernel of β is isomorphic to $\nabla(G)$, which in turn is isomorphic to $\Gamma(G^{ab})$ by Proposition 3.1. Hence the kernel of β is free abelian of rank $\binom{n+1}{2}$. But $\beta(T) = 1$, since $H_2(G)$ is torsion free, so $T \subset \ker(\beta)$. We conclude that $T = 1$, and that $J_2(G)$ is free abelian of rank $\binom{n+1}{2} + d$ and hence is isomorphic to $\Gamma(G^{\mathrm{ab}}) \times H_2(G)$. $\square$

Corollary 3.6. *Let $G = \mathcal{N}_{n,c}$ be the free nilpotent group of class c and rank $n > 1$. Then $J_2(G) \cong \Gamma(G^{\mathrm{ab}}) \times H_2(G)$ is a free abelian group of rank $\binom{n+1}{2} + M(n, c+1)$.*

4. Structure of the Tensor Square

In this section we prove Theorem 1.6. Our proof relies extensively on the commutator calculus given in Section 2.

If G is nilpotent of class c then $\nu(G)$ is nilpotent of class at most $c+1$ [**15**]. We make use of a general observation about nilpotent groups that we apply to $\nu(G)$. If X is a set of elements of a nilpotent group of class c, then any commutator of any weight at least 2 with entries from $X \cup X^{-1}$ can be written as a product of commutators all of whose entries lie in X. This fact is proved by induction on the weight k of a commutator, with base case $k = c$, using the commutator identities (2.1). Consequently, the following result for $\nu(G)$ holds.

Lemma 4.1. *Let G be a nilpotent group of class c. Let u and v be commutators of weight $i \geq 1$ and $j \geq 1$ respectively. Then in $\nu(G)$ the commutators $[u^{-1}, v^{\varphi}]$, $[u, v^{-\varphi}]$ and $[u^{-1}, v^{-\varphi}]$ can all be expressed as products of commutators whose entries are positive words in u, u^{φ}, v and v^{φ}.*

By Proposition 3.4 in Section 3, the sequence $C_{n,c} = \{c_1, \ldots, c_t\}$ is a polycyclic generating sequence for $\mathcal{N}_{n,c}$. We denote the elements $c_1, \ldots, c_n$ in $\mathcal{C}_{n,c}$ of weight 1 by $g_1, \ldots, g_n$. By Proposition 25 of [**5**] and Lemma 4.1 the subgroup $[\mathcal{N}_{n,c}, \mathcal{N}_{n,c}^{\varphi}]$ of $\nu(\mathcal{N}_{n,c})$ is generated by the elements

$$\{[c_i, c_j^{\varphi}] \mid c_i, c_j \in \mathcal{C}_{n,c}\}. \tag{4.1}$$

Our goal is to prune this set of generators for $[\mathcal{N}_{n,c}, \mathcal{N}_{n,c}^{\varphi}]$ so that a one-to-one correspondence between a generating set for $[\mathcal{N}_{n,c}, \mathcal{N}_{n,c}^{\varphi}]$ and the set of generators of the factors in the direct product of Theorem 1.6 can be realized.

Lemma 4.2. *Let $G = \mathcal{N}_{n,c}$ be the free nilpotent group of class c and rank $n > 1$, with polycyclic generating sequence $\mathcal{C}_{n,c} = \{c_1, \ldots, c_t\}$. Then $[G, G^{\varphi}]$ is generated by*

(i) $[g_i, g_i^{\varphi}]$ *for* $i = 1, \ldots, n$;
(ii) $[g_i, g_j^{\varphi}]$ *for* $1 \leq i < j \leq t$;
(iii) $[c_j, c_i^{\varphi}]$ *for* $1 \leq i < j \leq t$, *where* $w_j + w_i \leq c + 1$.

Proof. All generators of $[G, G^{\varphi}]$ of the form $[c_i, c_i^{\varphi}]$ for $i > n$ are trivial by Lemma 2.1 (vi). This leaves only the generators of the form $[g_i, g_i^{\varphi}]$ for $i = 1, \ldots, n$ as possibly nontrivial generators of the form $[c_i, c_i^{\varphi}]$. These generators are listed in (i).

Suppose $i < j$ and $w_i + w_j \geq 3$ with $w_i \geq 2$. Then $c_i = [c_q, c_p]$ for some q, p such that $q > p$. By Lemma 2.1 (iii) we have

$$[c_i, c_j^{\varphi}] = [[c_q, c_p], c_j^{\varphi}] = [c_j, [c_q, c_p]^{\varphi}]^{-1} = [c_j, c_i^{\varphi}]^{-1}.$$

Similarly, the equality holds if $w_j \geq 2$. Hence all generators of the form $[c_i, c_j^{\varphi}]$ with $w_i + w_j \geq 3$ and $i < j$ can be expressed in terms of elements of (iii). However, this argument does not eliminate those generators with $w_i = w_j = 1$ and $i < j$; these generators are listed in (ii).

Since $\nu(G)$ is nilpotent of class at most $c+1$ all generators of the form $[c_j, c_i^{\varphi}]$, where $j > i$ and $w_j + w_i > c+1$ are trivial. Hence the upper weight restriction of (iii) holds. ☐

Our analysis of $\mathcal{N}_{n,c} \otimes \mathcal{N}_{n,c}$ now focuses on the subgroup generated by the elements listed in Lemma 4.2 (iii).

PROPOSITION 4.3. *Let $G = \mathcal{N}_{n,c}$ be the free nilpotent group of class c and rank $n > 1$, with the basic sequence of commutators $\mathcal{C}_{n,c} = \{c_1, \ldots, c_t\}$ as its polycyclic generating sequence. The subgroup*

$$N = \langle [c_j, c_i^{\varphi}] \mid j > i, w_j + w_i \leq c+1 \rangle$$

is a normal subgroup of $[G, G^{\varphi}]$ isomorphic to $G \wedge G$.

PROOF. The elements $[g_i, g_i^{\varphi}]$ for $i = 1, \ldots, n$ are in the center of $\nu(G)$. Hence we need only show that

$$^{[g_i, g_j^{\varphi}]}[c_k, c_m^{\varphi}]$$

is an element of N when $i < j$ and $k > m$. Now

$$\begin{aligned}
^{[g_i, g_j^{\varphi}]}[c_k, c_m^{\varphi}] &= {}^{[g_i^{\varphi}, g_j]}[c_k, c_m^{\varphi}] \qquad \text{Lemma 2.1(i)} \\
&= {}^{[g_j, g_i^{\varphi}]^{-1}}[c_k, c_m^{\varphi}] \\
&= [g_j, g_i^{\varphi}]^{-1} \cdot [c_k, c_m^{\varphi}] \cdot [g_j, g_i^{\varphi}],
\end{aligned}$$

which is an element of N.

To show that N is isomorphic to $G \wedge G$, we recall that $G \wedge G$ is isomorphic to the derived subgroup of $\mathcal{N}_{n,c+1}$, which is generated by the commutators

$$\mathcal{C} = \{c_i \in \mathcal{C}_{n,c+1} \mid w_i > 1\}.$$

Every element c_k in $\mathcal{C}$ is uniquely expressed by $c_k = [c_j, c_i]$ for some $c_j, c_i \in \mathcal{C}_{n,c}$, where $j > i$. The isomorphism from $G \wedge G$ to $\mathcal{N}'_{n,c+1}$ is realized by $c_j \wedge c_i \mapsto [c_j, c_i] = c_k$. The isomorphism from $[G, G^{\varphi}]_{\tau(G)}$ to $\mathcal{N}'_{n,c+1}$ is defined by $\hat{\phi}^{-1}([c_j, c_i^{\varphi}]_{\tau(G)}) = [c_j, c_i]$. Similarly we can set up a mapping of generators from N to $\mathcal{N}'_{n,c+1}$ by $[c_j, c_i^{\varphi}] \mapsto [c_j, c_i]$.

We have now the following version of a short exact sequence from Diagram (1.4):

$$0 \longrightarrow \nabla(G) \xrightarrow{\ \iota\ } [G, G^{\varphi}] \xrightarrow{\ \sigma\ } \mathcal{N}'_{n,c+1} \longrightarrow 0 \tag{4.2}$$

Suppose $x \in \iota(\nabla(G)) \cap N$. Then using the Hall collection process, x may be written as a product of powers of the generators of N in order of increasing commutator weight. Let d be the least weight of a factor that appears nontrivially in this expression for x. Then the induced map $\sigma^* : [G, G^{\varphi}] \mapsto \gamma_d(\mathcal{N}_{n,c+1})/\gamma_{d+1}(\mathcal{N}_{n,c+1})$ maps x to a product of powers of weight d basic commutators. Since $\gamma_d(\mathcal{N}_{n,c+1})/\gamma_{d+1}(\mathcal{N}_{n,c+1})$ is a free abelian group with basis the set of basic commutators of weight d, and $\sigma(x) = 1$ (as $x \in \iota(\nabla(G)) = \ker(\sigma)$) we obtain a contradiction unless $x = 1$. Hence $\iota(\nabla(G)) \cap N = 1$.

The mapping from N to $\mathcal{N}'_{n,c+1}$ is an epimorphism, and since $\ker(\sigma) \cap N = 1$, we conclude that N is isomorphic to $\mathcal{N}'_{n,c+1}$, and hence the result follows. ☐

PROOF OF THEOREM 1.6. Since $\nabla(G)$ is a central subgroup of $[G, G^{\varphi}]$ it is normal in $[G, G^{\varphi}]$. The subgroup $N \cong G \wedge G$ is normal in $[G, G^{\varphi}]$ by Proposition 4.3. As was shown in the proof of Proposition 4.3, $\nabla(G) \cap N = 1$. Now $[g_i, g_j^{\varphi}]$

for $i < j$ are the only generators of $[G, G^{\varphi}]$ not obviously in either N or $\nabla(G)$. However

$$[g_i, g_j^{\varphi}] = ([g_i, g_j^{\varphi}][g_j, g_i^{\varphi}]) \cdot [g_j, g_i^{\varphi}]^{-1}$$

is a product of elements of $\nabla(G)$ and N. Hence $[G, G^{\varphi}] = \nabla(G)N$. Therefore we conclude that

$$G \otimes G \cong [G, G^{\varphi}] = \nabla(G) \times N \cong \nabla(G) \times G \wedge G \cong \Gamma(G^{ab}) \times G \wedge G.$$

The last isomorphism holds by Corollary 3.2. □

5. Computational Interplay

In this section we provide an account of how Theorem 1.6 was motivated by our use of computational methods, some of which are outlined in Section 1. Specifically, the computed examples provided evidence that the nonabelian tensor square of the free nilpotent group $\mathcal{N}_{n,c}$ is a direct product of its nonabelian exterior square and a free abelian group whose rank depends on the rank n of the group. This observation is not immediately obvious. This fact was missed in three earlier publications: Bacon, Kappe and Morse [**2**]; Blyth, Morse and Redden [**6**]; and Blyth and Morse [**5**]. All three of these papers also used computer examples to help formulate their final general results.

One problem with making computer calculations is interpreting the output given by the computer. Moreover, relating this output to the symbolic manipulations required can be a challenge. Our strategy is to provide a GAP representation of the symbolic or abstract objects we are working with and to then map the GAP symbolic objects to the computer generated output. In our particular case we were interested in mapping a basic sequence of commutators to the generators of $G \otimes G$. So we first represent basic commutators in GAP and then relate them to the polycyclic groups we construct. The purpose of this section is to explicitly demonstrate how we accomplished this.

We start with the following GAP functions that create objects that symbolically represent a basic sequence of commutators. These functions are straightforward to implement:

```
BasicSeq(<symset>,<maxweight>);
ComEval(<comm>);
Weight(<comm>);
```

The function `BasicSeq` generates a basic sequence of commutators in the symbol set `<symset>` of weights 1 to `<maxweight>`. The output of `BasicSeq` is a list of lists. Each list in the list represents a fully bracketed commutator. If s is in the symbol set then $[s]$ is a fully bracketed commutator of length 1. Suppose c and d are fully bracketed commutators of weights w_c and w_d respectively. Then $[c, d]$ is a fully bracketed commutator of weight $w_c + w_d$. If the elements from the symbol set are group elements then we can form the element of the group represented by a fully bracketed commutator. The function `ComEval` forms this group element from a fully bracketed commutator represented as a list. The `Weight` function computes the weight of the commutator one of these lists represents. Below is a example whose symbol set consists of the generators of a free group of rank 3.

```
gap> F := FreeGroup(3);;
gap> b := BasicSeq(GeneratorsOfGroup(F),3);;
gap> PrintArray(b);
```

```
[ [                  f1 ],
  [                  f2 ],
  [                  f3 ],
  [              [ f2 ],            [ f1 ] ],
  [              [ f3 ],            [ f1 ] ],
  [              [ f3 ],            [ f2 ] ],
  [  [ [ f2 ], [ f1 ] ],            [ f1 ] ],
  [  [ [ f2 ], [ f1 ] ],            [ f2 ] ],
  [  [ [ f2 ], [ f1 ] ],            [ f3 ] ],
  [  [ [ f3 ], [ f1 ] ],            [ f1 ] ],
  [  [ [ f3 ], [ f1 ] ],            [ f2 ] ],
  [  [ [ f3 ], [ f1 ] ],            [ f3 ] ],
  [  [ [ f3 ], [ f2 ] ],            [ f2 ] ],
  [  [ [ f3 ], [ f2 ] ],            [ f3 ] ] ]

gap> ## Compute the weights of each commutator represented.
gap> ## The number of commutators of each weight
gap> ## corresponds to the Witt-Hall formula.
gap> ##
gap> List(b,Weight);
[ 1, 1, 1, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3 ]

gap> ## Evaluate each commutator as an element in the free
gap> ## group F
gap> ##
gap> List(b,ComEval);
[ f1, f2, f3, f2^-1*f1^-1*f2*f1, f3^-1*f1^-1*f3*f1,
  ...
  ...
  ...
  f2^-1*f3^-1*f2*f3^-1*f2^-1*f3*f2*f3 ]
```

The goal is to map the elements of a basic sequence of commutators to elements of $G \otimes G$. To do this we need the subgroups G and G^{φ}, which are the left and right isomorphic embeddings of G in $\nu(G)$. The GAP program listed in [**5**] to compute the nonabelian tensor square computes these values and returns $[G, G^{\varphi}] \cong G \otimes G$. This GAP program can be modified to return a record with the fields `lbg` (left base group), and `rbg` (right base group). These GAP variables correspond to the mathematical objects G and G^{φ} respectively. The nonabelian tensor square is computed by the command `CommutatorSubgroup(lbg,rbg)`. Nothing from the program listed in [**5**] is modified except we are returning different computed values in the form of a record. We rename this function `BaseGroups` from the name `TensorSquare` given in [**5**] to reflect the different returned values. We fix our example group to be $\mathcal{N}_{3,3} = G$. The GAP object `G` below corresponds to G. We compute `lbg`, and `rbg` for `G` and compute the tensor square, `ts`, from them.

```
gap> ## Create the free nilpotent of group of class 3
gap> ## and rank 3 and compute the base groups.
gap> ##
gap> G   := NilpotentQuotient(FreeGroup(3),3);;
```

```
gap> r   := BaseGroups(G);;

gap> ## Save the base groups for later use
gap> lbg := r.lbg;
Pcp-group with orders [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
                        0, 0, 0 ]
gap> rbg := r.rbg;
Pcp-group with orders [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
                        0, 0, 0 ]

gap> ## Compute the tensor square from the returned values
gap> ##
gap> ts := CommutatorSubgroup(lbg,rbg);
Pcp-group with orders [ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
                        0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
                        0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
                        0, 0 ]

gap> ## List the induced generating sequence of ts
gap> ##
gap> Igs(ts);
[ g10, g11, g12, g13, g14, g15, g17, g18, g19, g30, g31,
  g34, g35, g36, g40, g41, g42, g46, g47, g48, g49, g50,
  g51, g52, g53, g54, g55, g56, g57, g58, g59, g60, g61,
  g62, g63 ]
```

How can we show that ts above is a direct product of two subgroups? How can we identify the generators of the two subgroups as specific elements in ts that can be used to guide us in our proof? The following is one method for doing this using the basic commutator routines introduced above.

```
gap> ## Record the minimal generators of F, lbg and rbg
gap> ##
gap> Fm := GeneratorsOfGroup(F);;
gap> Lm := MinimalGeneratingSet(lbg);;
gap> Rm := MinimalGeneratingSet(rbg);;

gap> ## Create the basic sequence of commutators of weight
gap> ## at most four and then prune them to those of
gap> ## weight greater than 1.
gap> ##
gap> b2 := Filtered(BasicSeq(GeneratorsOfGroup(F),4),
                        x->Weight(x)>1);;
gap> PrintArray(b2);
[ [                      [ f2 ],            [ f1 ] ],
  [                      [ f3 ],            [ f1 ] ],
  [                      [ f3 ],            [ f2 ] ],
  [            [ [ f2 ], [ f1 ] ],          [ f1 ] ],
  [            [ [ f2 ], [ f1 ] ],          [ f2 ] ],
  [            [ [ f2 ], [ f1 ] ],          [ f3 ] ],
```

```
[             [ [ f3 ], [ f1 ] ],            [ f1 ] ],
[             [ [ f3 ], [ f1 ] ],            [ f2 ] ],
[             [ [ f3 ], [ f1 ] ],            [ f3 ] ],
[             [ [ f3 ], [ f2 ] ],            [ f2 ] ],
[             [ [ f3 ], [ f2 ] ],            [ f3 ] ],
[             [ [ f3 ], [ f1 ] ], [ [ f2 ], [ f1 ] ] ],
[             [ [ f3 ], [ f2 ] ], [ [ f2 ], [ f1 ] ] ],
[             [ [ f3 ], [ f2 ] ], [ [ f3 ], [ f1 ] ] ],
[  [ [ [ f2 ], [ f1 ] ], [ f1 ] ],           [ f1 ] ],
[  [ [ [ f2 ], [ f1 ] ], [ f1 ] ],           [ f2 ] ],
[  [ [ [ f2 ], [ f1 ] ], [ f1 ] ],           [ f3 ] ],
[  [ [ [ f2 ], [ f1 ] ], [ f2 ] ],           [ f2 ] ],
[  [ [ [ f2 ], [ f1 ] ], [ f2 ] ],           [ f3 ] ],
[  [ [ [ f2 ], [ f1 ] ], [ f3 ] ],           [ f3 ] ],
[  [ [ [ f3 ], [ f1 ] ], [ f1 ] ],           [ f1 ] ],
[  [ [ [ f3 ], [ f1 ] ], [ f1 ] ],           [ f2 ] ],
[  [ [ [ f3 ], [ f1 ] ], [ f1 ] ],           [ f3 ] ],
[  [ [ [ f3 ], [ f1 ] ], [ f2 ] ],           [ f2 ] ],
[  [ [ [ f3 ], [ f1 ] ], [ f2 ] ],           [ f3 ] ],
[  [ [ [ f3 ], [ f1 ] ], [ f3 ] ],           [ f3 ] ],
[  [ [ [ f3 ], [ f2 ] ], [ f2 ] ],           [ f2 ] ],
[  [ [ [ f3 ], [ f2 ] ], [ f2 ] ],           [ f3 ] ],
[  [ [ [ f3 ], [ f2 ] ], [ f3 ] ],           [ f3 ] ] ]
```

The elements in the GAP object b2 represent all of the commutators in the basic sequence of the form $[c_i, c_j]$, where c_i and c_j are elements in the sequence and $i > j$. These commutators generate $\mathcal{N}'_{3,4} \cong \gamma_2(F_3/\gamma_5(F_3))$, which we know is isomorphic to $G \wedge G$.

Our objective now is to map the elements of this basic sequence of commutators to elements in ts and see if the resulting subgroup is normal in ts and, if so, whether or not it is a direct factor.

We evaluate each c_i and c_j of $[c_i, c_j]$ to obtain pairs of words in the free generators Fm.

```
gap> ## The elements of b2 are lists that represent
gap> ## a fully bracketed commutator of weight at least
gap> ## two. Hence it is a list of the form [left, right],
gap> ## where left and right are lists representing
gap> ## commutators in our basic sequence.
gap> ## We create a word in the free group F for the left
gap> ## and right commutator using our ComEval function
gap>
gap> e2 := List(b2,x->[ComEval(x[1]),ComEval(x[2])]);
[ [ f2, f1 ], [ f3, f1 ], [ f3, f2 ],
  [ f2^-1*f1^-1*f2*f1, f1 ], [ f2^-1*f1^-1*f2*f1, f2 ],
  [ f2^-1*f1^-1*f2*f1, f3 ], [ f3^-1*f1^-1*f3*f1, f1 ],
  ...
  ...
  ...
  [ f2^-1*f3^-1*f2*f3*f2^-1*f3^-1*f2^-1*f3*f2^2, f3 ],
```

```
  [ f2^-1*f3^-1*f2*f3^-1*f2^-1*f3*f2*f3, f3 ] ]
```

For each element of e2, we substitute the free generator symbols of the left element with the generators of the left base group whose GAP object is Lm and we substitute the free generators of the right element with the generators of the right base group Rm.

```
gap> t2 := List(e2,x->
        [MappedWord(x[1],Fm,Lm), MappedWord(x[2],Fm,Rm)]);
[ [ g2, g4 ],  [ g3, g4 ],  [ g3, g5 ],  [ g7, g4 ],
  [ g7, g5 ],  [ g7, g6 ],  [ g8, g4 ],  [ g8, g5 ],
  [ g8, g6 ],  [ g9, g5 ],  [ g9, g6 ],  [ g8, g16 ],
  [ g9, g16 ], [ g9, g20 ], [ g22, g4 ], [ g22, g5 ],
  [ g22, g6 ], [ g23, g5 ], [ g23, g6 ], [ g25*g27^-1, g6 ],
  [ g24, g4 ], [ g24, g5 ], [ g24, g6 ], [ g25, g5 ],
  [ g25, g6 ], [ g26, g6 ], [ g28, g5 ], [ g28, g6 ],
  [ g29, g6 ] ]
```

We now create the subgroup ts generated by the commutators represented by the elements of t2 and check to see that it is normal in ts.

```
gap> N := Subgroup(ts,List(t2,LeftNormedComm));;
gap> IsNormal(ts,N);
true
```

By Lemma 21 of [**5**], $\nabla(G)$ is generated by the elements $[g_i, g_i^{\varphi}]$ for $i = 1, \ldots, n$ and $[g_i, g_j^{\varphi}][g_j, g_i^{\varphi}]$, where $1 \leq i, j \leq n$ and $i \neq j$. In our example, $n = 3$. Hence $\nabla(G)$ is generated by only six generators. We enumerate these generators and create the subgroup generated by them.

```
gap> A := Subgroup(ts,[Comm(Lm[1],Rm[1]),Comm(Lm[2],Rm[2]),
                       Comm(Lm[3],Rm[3]),
                       Comm(Lm[1],Rm[2])*Comm(Lm[2],Rm[1]),
                       Comm(Lm[1],Rm[3])*Comm(Lm[3],Rm[1]),
                       Comm(Lm[2],Rm[3])*Comm(Lm[3],Rm[2])]
                  );;
gap> IsSubgroup(Centre(ts),A);
true
```

The subgroups N and A are normal in ts. They also have trivial intersection:

```
gap> IsTrivial(Intersection(N,A));
true
```

The nonabelian tensor square ts is generated by N and A:

```
gap> ts = Subgroup(ts,Concatenation(GeneratorsOfGroup(N),
                                    GeneratorsOfGroup(A)));
true
```

We conclude that ts is the direct product of N and A.

Finally we show that N is isomorphic to $\mathcal{N}'_{3,4} \cong G \wedge G$.

```
gap> ## Form the derived subgroup of the free nilpotent group
gap> ## of class 4 and rank 3.
gap> ##
gap> H := DerivedSubgroup(NilpotentQuotient(FreeGroup(3),4));;
```

```
gap> ## Form a mapping from N to H and check to see if it is
gap> ## a homomorphism
gap> ##
gap> hom := GroupGeneralMappingByImages(N,H,
              MinimalGeneratingSet(N),MinimalGeneratingSet(H));;
gap> IsPcpGroupHomomorphism(hom);
true
gap> ## Check to see if hom is an isomorphism
gap> ##
gap> IsTrivial(Kernel(hom));
true
```

While computing examples like this does not constitute a proof, such examples gave us direction in formulating and proving Theorem 1.6.

References

[1] Michael R. Bacon. On the nonabelian tensor square of a nilpotent group of class two. *Glasgow Math. J.*, 36(3):291–296, 1994.
[2] Michael R. Bacon, Luise-Charlotte Kappe, and Robert Fitzgerald Morse. On the nonabelian tensor square of a 2-Engel group. *Arch. Math. (Basel)*, 69(5):353–364, 1997.
[3] James R. Beuerle and Luise-Charlotte Kappe. Infinite metacyclic groups and their nonabelian tensor squares. *Proc. Edinburgh Math. Soc. (2)*, 43(3):651–662, 2000.
[4] Russell D. Blyth, Primož Moravec, and Robert Fitzgerald Morse. On the derived subgroup of the free nilpotent group and its applications. To appear.
[5] Russell D. Blyth and Robert Fitzgerald Morse. Computing the nonabelian tensor square of polycyclic groups. Submitted.
[6] Russell D. Blyth, Robert Fitzgerald Morse, and Joanne L. Redden. On computing the nonabelian tensor squares of the free 2-Engel groups. *Proc. Edinb. Math. Soc. (2)*, 47(2):305–323, 2004.
[7] R. Brown, D. L. Johnson, and E. F. Robertson. Some computations of nonabelian tensor products of groups. *J. Algebra*, 111(1):177–202, 1987.
[8] Ronald Brown and Jean-Louis Loday. Van Kampen theorems for diagrams of spaces. *Topology*, 26(3):311–335, 1987. With an appendix by M. Zisman.
[9] R. Keith Dennis. In Search of New "Homology" Functors having a Close Relationship to K-theory. Unpublished preprint.
[10] B. Eick and W. Nickel. *Polycyclic – Computing with polycyclic groups*, 2002. A GAP Package, see [**12**].
[11] Graham Ellis and Frank Leonard. Computing Schur multipliers and tensor products of finite groups. *Proc. Roy. Irish Acad. Sect. A*, 95(2):137–147, 1995.
[12] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.4*, 2005. (`http://www.gap-system.org`).
[13] Clair Miller. The second homology group of a group; relations among commutators. *Proc. Amer. Math. Soc.*, 3:588–595, 1952.
[14] W. Nickel. *nq – Nilpotent Quotients of Finitely Presented Groups*, 2003. A GAP Package, see [**12**].
[15] N. R. Rocco. On a construction related to the nonabelian tensor square of a group. *Bol. Soc. Brasil. Mat. (N.S.)*, 22(1):63–79, 1991.
[16] Charles C. Sims. *Computation with finitely presented groups*, volume 48 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994.
[17] Ursula Martin Webb. The Schur multiplier of a nilpotent group. *Trans. Amer. Math. Soc.*, 291(2):755–763, 1985.
[18] J. H. C. Whitehead. A certain exact sequence. *Ann. of Math. (2)*, 52:51–110, 1950.

Department of Mathematics and Computer Science, Saint Louis University, St. Louis, MO 63103, USA

E-mail address: blythrd@slu.edu

Fakulteta za matematiko in fiziko, Univerza v Ljubljani, Jadranska 19, 1000 Ljubljana, Slovenia

E-mail address: primoz.moravec@fmf.uni-lj.si

Department of Electrical Engineering and Computer Science, University of Evansville, Evansville IN 47722 USA

E-mail address: rfmorse@evansville.edu

URL: faculty.evansville.edu/rm43