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Topology of order complexes of intervals in subgroup lattices

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Abstract

We conjecture that the order complex of an open interval in the subgroup lattice of a finite group has the homotopy type of a wedge of spheres and prove that if (H,G) is a minimal counterexample to this conjecture then either G is almost simple or $G=HN$, where N is the unique minimal normal subgroup of G, N is non-Abelian and $H\cap N=1$.

1. Introduction

The question of whether for each finite lattice L there exist a finite group G and a subgroup H of G such that L is isomorphic to the lattice $[H,G]$ of subgroups of G which contain H is open. This question has its roots in universal algebra. Indeed, the question of whether every lattice is isomorphic to the lattice of congruences of a finite algebra (see, for example, [BuSa] for the appropriate definitions) is also open, and in the paper [PaPu] of P.P. Pálfy and P. Pudlák it is shown that these two questions have the same answer.

There has been significant progress towards proving that these questions have a negative answer. Beginning already in [PaPu], attention was focused on lattices of height two. For a positive integer n, let M_n be the lattice consisting of a minimum element $\hat{0}$, a maximum element $\hat{1}$ and n other elements, no two of which are related. It is believed that the set of n such that there exist finite G,H with $[H,G]$ isomorphic to M_n is quite sparse. Efforts

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to prove this is the case culminated in the paper [BaLu] of R. Baddeley and A. Lucchini, where the problem is reduced to the examination of almost simple groups.

Here we introduce a conjecture which says that in addition to the (conjectured) quantitative restrictions on intervals in subgroup lattices of finite groups described in the previous paragraph, there are qualitative restrictions on the topology of the order complex of such an interval. Recall that for a finite partially ordered set (poset) P, the *order complex* $\Delta\mathsf{P}$ is the abstract simplicial complex whose k-dimensional faces are chains $x_0 < x_1 < \cdots < x_k$ from P. Any such complex has a geometric realization in some Euclidean space and any two such realizations are homeomorphic. Thus to every partially ordered set P there is associated a topological space (which will also be denoted by $\Delta\mathsf{P}$). It is known (see [Qu]) that if P has a unique maximum element or a unique minimum element then $\Delta\mathsf{P}$ is contractible. Note that every finite lattice L has both a unique maximum element and a unique minimum element. It is standard practice in topological combinatorics to replace L by the poset $\bar{\mathsf{L}}$ obtained from L by removing the minimum and maximum elements before examining the order complex. So, it is natural in this context to examine the order complex of the open interval (H, G) of proper subgroups of a finite group G which properly contain the subgroup H. We can now state our main conjecture, along with two weaker versions which might be easier to prove.

Conjecture 1.1. *Let G be a finite group and let $H < G$. Then*

(A) *The complex $\Delta(H, G)$ has the homotopy type of a wedge of spheres.*
(B) *If $\Delta(H, G)$ is not connected then some connected component of $\Delta(H, G)$ is contractible.*
(C) *Let $\overline{B_3}$ be the poset obtained from the lattice of subsets of $\{1, 2, 3\}$ by removing the minimum element $\emptyset$ and the maximum element $\{1, 2, 3\}$. Let $\overline{2B_3}$ be the partially ordered set obtained by taking two disjoint copies of $\overline{B_3}$. Then (H, G) is not isomorphic to $\overline{2B_3}$.*

Before continuing, we make the following remarks.

(1) If a complex Γ which is not connected has the homotopy type of a wedge of spheres then every connected component of Γ is contractible (in which case Γ is homotopy equivalent to a wedge of 0-dimensional spheres). Therefore Conjecture (A) implies Conjecture (B). Also, each connected component of $\Delta\overline{2B_3}$ has the homotopy type of a circle, so Conjecture (B) implies Conjecture (C).
(2) If one adds a minimum element and a maximum element to $\overline{2B_3}$, a lattice is obtained, so Conjecture (C) is not empty of content. Moreover, $\overline{2B_3}$ is the smallest poset with this property whose order complex does not have the homotopy type of a wedge of spheres.
(3) We include Conjectures (B),(C) with the hope that they (at least (C)) will be easier to prove than (A). We hope that (C) will actually be easier than the conjecture that some M_n is not isomorphic to any interval $[H, G]$. Note that the results in [BaLu] apply only when $n > 50$ and that the three smallest n for which it is not known that M_n is isomorphic to some interval $[H, G]$ are 16, 23 and 35.

(4) It is shown in the paper [KrTh] of C. Kratzer and J. Thévenaz that if G is solvable then $\Delta(H, G)$ has the homotopy type of a wedge of spheres, so in any counterexample to (A) one has G nonsolvable.
(5) The most potent weapon currently available in topological combinatorics for showing that every interval in a given poset P has an order complex with the homotopy type of a wedge of spheres is the nonpure shellability theory of A. Björner and M. Wachs (see [BjWa1,BjWa2]). However, it is shown in [Sh] that for a finite group G, the complex $\Delta(1, G)$ is shellable if and only if G is solvable. Therefore shellability theory seems unlikely to provide any progress beyond what was already established in [KrTh].
(6) Using the homotopy complementation formula of Björner and J. Walker (see [BjWal]) and the classification of finite simple groups, one can show that $\Delta(1, G)$ is not connected if and only if G is a semidirect product CV, where V is elementary Abelian and C is cyclic of prime order and acts irreducibly on V. In this case every connected component of $\Delta(1, G)$ is contractible, so there is no counterexample to Conjecture (B) of the form $[1, G]$. Not much is known about the homotopy type of $\Delta(1, G)$ for an arbitrary finite group G.
(7) A conjectured set of combinatorial qualitative restrictions on intervals $[H, G]$ appears in the paper [Ba] of Baddeley.

In the next section, we will prove the following result.

Theorem 1.2. *Let (H, G) be a counterexample to one of the Conjectures* 1.1(A), (B), (C) *such that $|G|$ is minimal* (*with respect to the chosen conjecture*) *and, having fixed G, $[G : H]$ is also minimal. Then*

(1) *G is almost simple, or*
(2) *$G = HN$, where N is the unique minimal normal subgroup of G, N is non-Abelian and $H \cap N = 1$.*

Of course our eventual goal is to eliminate pairs (H, G) which satisfy the second condition of Theorem 1.2 (but are not almost simple) as possible counterexamples to any of Conjectures (A), (B), (C) and then use the classification of simple groups. It should be noted, though, that in the examination of the lattices M_n the elimination of pairs (H, G) satisfying $H \cap N = 1$ with (non-Abelian) N the unique minimal normal subgroup of G, which was the subject of the paper [BaLu], was the toughest part of the reduction to the almost simple case. On the other hand, the proof of Theorem 1.2 seems somewhat easier than the reduction to the case examined in [BaLu] for the lattices M_n, which is achieved in the papers [Kö,Lu] of P. Köhler and Lucchini, respectively.

2. Proving Theorem 1.2

In this section we prove Theorem 1.2. We will give the proof of the theorem with for Conjecture (A) in detail and then explain how to make minor adjustments to the given proof in order to prove the theorem for each of Conjectures (B), (C).

2.1. Conjecture (A)

Let (H, G) be a counterexample to Conjecture 1.1(A) satisfying the minimality conditions of the theorem. Since, by definition, $\Delta\emptyset = S^{-1}$, we know that H is not a maximal subgroup of G. Also, if $C = \mathrm{Core}_G(H)$ then

$$\Delta(H, G) \cong \Delta(H/C, G/C)$$

and the minimality of $|G|$ gives $C = 1$.

If L is a lattice and $x \in \mathsf{L}$ then $x^{\perp}$ is defined to be the set of lattice theoretic complements to x in L. So, for $K \in [H, G]$, $K^{\perp}$ consists of those $X \leqslant G$ such that $K \cap X = H$ and $\langle K, X\rangle = G$. Recall that an *antichain* in a poset is a set of elements, no two of which are related.

Lemma 2.1. *There is no $K \in (H, G)$ such that $K^{\perp}$ (in $[H, G]$) is an antichain.*

Proof. Assume for contradiction that $K^{\perp}$ is an antichain for some $K \in (H, G)$. By the homotopy complementation formula of Björner and Walker (see [BjWal]), we have

$$\Delta(H, G) \simeq \bigvee_{X \in K^{\perp}} \Sigma\big(\Delta(H, X) * \Delta(X, G)\big).$$

(Here $\simeq$ indicates homotopy equivalence, $\bigvee$ means wedge, Σ means suspension and $*$ means join.) By the minimality of $|G|$ and $[G : H]$, both $\Delta(H, X)$ and $\Delta(X, G)$ have the homotopy type of a wedge of spheres for each $X \in K^{\perp}$. It follows (see, for example, [BjWel, Lemma 2.5]) that $\Delta(H, G)$ has the homotopy type of a wedge of spheres, giving the desired contradiction. $\square$

Lemma 2.2. *For each $K \in [H, G)$ we have* $\mathrm{Core}_G(K) = 1$.

Proof. We proceed by induction in $[H, G)$, the base case $K = H$ having been settled above. Let $K \in (H, G)$ and let $C = \mathrm{Core}_G(K)$. Assume (for contradiction) that $C > 1$. By inductive hypothesis, we may assume that $\mathrm{Core}_G(L) = 1$ for all $L \in [H, K)$. It follows that $HC = K$. By Lemma 2.1 there exist $M_1, M_2 \in K^{\perp}$ such that $M_1 < M_2$. For $i = 1, 2$ we have

$$K = CH = C(M_i \cap K) = CM_i \cap K,$$

the last equality holding by the modular law for groups (see [As, 1.14]). Therefore, $K \leqslant CM_i$. Certainly $M_i \leqslant CM_i$, so $G = \langle K, M_i\rangle \leqslant CM_i$ and $CM_i = G$. Since $C \leqslant K$, we have $KM_i = G$. Therefore,

$$|G| = |KM_i| = \frac{|K||M_i|}{|K \cap M_i|} = [K : H]|M_i|$$

for $i = 1, 2$. This contradicts $M_1 < M_2$. $\square$

Lemma 2.3. *The group G has a unique minimal normal subgroup N. Moreover, $HN = G$, N is non-Abelian and $C_G(N) = 1$.*

Proof. Let N be any minimal normal subgroup of G and let M be a maximal subgroup of G which contains H. As noted above, $H \neq M$. By Lemma 2.2 we have $HN = MN = G$ and $\mathrm{Core}_G(M) = 1$, so G acts as a primitive permutation group on the set of cosets of M. Now

$$[M : M \cap N] = [G : N] = [H : H \cap N],$$

and since $H < M$ we have $1 \leqslant H \cap N < M \cap N$. It now follows (see [DiMo, Theorem 4.3B]) that N is the unique minimal normal subgroup of G, N is non-Abelian and $C_G(N) = 1$. □

Now we record some facts about N and G.

(1) Since N is non-Abelian and characteristically simple, we have

$$N = T_1 \times \cdots \times T_r,$$

where there is some non-Abelian simple group T such that $T_i \cong T$ for all $i \in [r] := \{1, \ldots, r\}$.

(2) The action of G on N by conjugation determines a homomorphism from G to $\mathrm{Aut}(N)$ with kernel $C_G(N)$=1. Therefore, G is (isomorphic to) a subgroup of $\mathrm{Aut}(N)$, which is in turn (isomorphic to) the wreath product $S_r[\mathrm{Aut}(T)]$.

(3) The minimal normal subgroups of N are $T_1, \ldots, T_r$ (see [DiMo, Theorem 4.3A(iv)]), so the action of H on N by conjugation determines an action of H on $\{T_1, \ldots, T_r\}$. Since $G = HN$ and N is a minimal normal subgroup of G, this action of H on $\{T_i\}$ is transitive.

If A is a group of automorphisms of a group B which stabilizes $C \leqslant B$, we write $[C, B]^A$ for the sublattice of $[C, B]$ consisting of all A-invariant groups in $[C, B]$.

(4) The maps $\phi : [H, G] \to [H \cap N, N]^H$ and $\psi : [H \cap N, N]^H \to [H, G]$ defined by $\phi(K) = K \cap N$ and $\psi(L) = HL$ are both order preserving and are inverses to each other. Therefore, we have

$$[H, G] \cong [H \cap N, N]^H.$$

We will work for the most part in $[H \cap N, N]^H$ from now on. For $i \in [r]$, let $\pi_i : N \to T_i$ be the natural projection. For $K \leqslant N$ and $i \in [r]$ let $K^i = \pi_i(K)$ and let $K_i = K \cap T_i$.

(5) Let $K \in [H \cap N, N]^H$. Then $K_i \trianglelefteq K^i$ for all $i \in [r]$. The transitivity of H on the T_i gives $K^i \cong K^j$, $K_i \cong K_j$, and $K^i/K_i \cong K^j/K_j$ for all $i, j \in [r]$.

(6) In particular, if $K \in [H \cap N, N]^H$ then for each $i \in [r]$ there is some $h \in H$ such that $K^i = (K^1)^h$, and if $h \in H$ maps T_i to T_j then $(K^i)^h = K^j$. Therefore,

$$K \leqslant \prod_{i=1}^{r} K^i = \prod_{h \in H} \left(K^1\right)^h \in [H \cap N, N]^H.$$

Notation. Let $R = H \cap N$.

Lemma 2.4. *If $r > 1$ then there exists some $K \in [R, N)^H$ such that $K^i = T_i$ for all $i \in [r]$.*

Proof. Assume the contrary. Let $X = N_H(T_1)$. For each $K \in [R, N)^H$ we have $K^1 \in [R^1, T_1)^X$. Conversely, if $S \in [R^1, T_1)^X$ then

$$K(S) := \prod_{h \in H} S^h \in [R, N)^H$$

with $K(S)^1 = S$. Moreover, if $L \in [R, N)^H$ and $L^1 = S$ then $L \leqslant K(S)$. Define $\phi : (R, N)^H \to [R^1, T_1)^X$ by $\phi(K) = K^1$. Then

- ϕ is order preserving, and
- for each $S \in \text{Image}(\phi)$, the poset

  $$\phi^{-1}_{\leqslant S} := \left\{K \in (R, N)^H \colon \phi(K) \leqslant S\right\}$$

  has a unique maximum element, namely, $K(S)$.

It follows from the Quillen fiber lemma (see [Qu, Definition 1.5, Proposition 1.6]) that

$$\Delta(R, N)^H \simeq \Delta\text{Image}(\phi).$$

If $R^1 \neq R_1$ then $\text{Image}(\phi) = [R^1, T_1)^X$ has a unique minimum element R^1. It follows (again [Qu, Definition 1.5]) that $\Delta\text{Image}(\phi)$ is contractible, and we conclude that

$$\Delta(H, G) \simeq \bigvee_0 S^0,$$

a contradiction. If $R^1 = R_1$ then there is no $K \in (R, N)^H$ with $K^1 = R^1$ so $\text{Image}(\phi) = (R^1, T_1)^X$. Here the Quillen fiber lemma gives

$$\Delta(H, G) \simeq \Delta\left(R^1, T_1\right)^X.$$

However, $(R^1, T_1)^X \cong (XR^1, XT_1)$, and we now have a contradiction to the minimality of $|G|$. $\square$

Corollary 2.5. *If $r > 1$ then $R_1 = 1$.*

Proof. By Lemma 2.4 there is some $K \in [R, N)^H$ such that $K^i = T_i$ for some, and therefore all, $i \in [r]$. Since each T_i is non-Abelian simple and $K_i \trianglelefteq K^i$, we have $K_i \in \{1, T_i\}$ for all i. Since $K \neq N$ we have $K_i = 1$ for some, and therefore all, $i \in [r]$. Since $R \leqslant K$ we have $R_i = 1$ for all i as claimed. □

For $K \leqslant N$ define an equivalence relation $\sim_K$ on $[r]$ by $i \sim_K j$ if and only if $\mathrm{Kernel}(\pi_i) \cap K = \mathrm{Kernel}(\pi_j) \cap K$. Note that $i \sim_K j$ if and only if there exists an isomorphism $\psi_{ij}^K : K^i \to K^j$ such that $\pi_j(k) = \psi_{ij}^K(\pi_i(k))$ for all $k \in K$. Let $\rho(K)$ be the partition of $[r]$ whose parts are the $\sim_K$-equivalence classes. For two partitions σ, ρ of $[r]$ we say σ *refines* ρ if each part of σ is a subset of some part of ρ.

Lemma 2.6. *If $K \leqslant L \leqslant N$ then*

(1) *$\rho(L)$ refines $\rho(K)$, and*
(2) *if $i \sim_L j$ then the restriction of ψ_{ij}^L to K^i is ψ_{ij}^K.*

Proof. We prove the first claim first. Say $i \not\sim_K j$. We may assume that there is some $x \in K$ such that $\pi_i(x) \neq 1$ but $\pi_j(x) = 1$. Since $x \in L$ we have $i \not\sim_L j$.

Now say $i \sim_L j$. For $y \in K^i$ there is some $x \in K \leqslant L$ such that $\pi_i(x) = y$, and we have

$$\psi_{ij}^L(y) = \pi_j(x) = \psi_{ij}^K(y). \qquad \square$$

For any $I \subseteq [r]$, let π_I be the projection of N onto $\prod_{i \in I} T_i$. For $K \leqslant N$ such that $\rho(K)$ has parts $I_1, \ldots, I_s$, define

$$K^+ := \prod_{j=1}^{s} \pi_{I_j}(K).$$

We now record the following key facts.

(1) If $K \in [1, N]^H$ then $\rho(K)$ is H-invariant, that is, H acts on the parts of $\rho(K)$.
(2) If $K \leqslant N$ with $K^i = T_i$ for each $i \in [r]$ then $K = K^+$. (This is well known and follows from [DiMo, Lemma 4.3A].)

Lemma 2.7. *If $r > 1$ then $R^i < T_i$ for all $i \in [r]$.*

Proof. Assume for contradiction that $R^i = T_i$ for some (and therefore all) $i \in [r]$. Let $K \in [R, N)^H$. Then $K^i = T_i$ for all i, so $K = K^+$. Moreover, $\rho(K)$ refines $\rho(R)$ (by Lemma 2.6(1)) and is H-invariant. Say $i \sim_K j$. Then $i \sim_R j$, so for each $x \in R \leqslant K$ we have

$$\psi_{ij}^K\big(\pi_i(x)\big) = \psi_{ij}^R\big(\pi_i(x)\big),$$

by Lemma 2.6(2). Since $R^i = T_i$ we have $\psi_{ij}^K = \psi_{ij}^R$. Since $K \neq N$, we have $K_i = 1$ for all i, and K is determined by $\rho(K)$ and the maps ψ_{ij}^R. Therefore, if $K, L \in [R, N]^H$ and $\rho(K) = \rho(L)$ then $K = L$.

Conversely, let Θ be any H-invariant partition of $[r]$ which refines $\rho(R)$. Let $\sim_\Theta$ be the equivalence relation determined by the parts of Θ. Define

$$K := K(\Theta) := \bigl\{x \in N\colon \pi_j(x) = \psi_{ij}^R\bigl(\pi_i(x)\bigr) \text{ whenever } i \sim_\Theta j\bigr\}.$$

Then $K \leqslant N$ and $K^i = T_i$ for all i. Since Θ refines $\rho(R)$, we have $R \leqslant K$. Say $a \in \mathrm{Aut}(N)$ induces the permutation σ on $[r]$. Then there exist $a_1, \ldots, a_r \in \mathrm{Aut}(T)$ such that for $x \in N$ we have

$$\pi_i(x^a) = \bigl(\pi_{i\sigma^{-1}}(x)\bigr)^{a_i}$$

for all $i \in [r]$. It is straightforward to show that K is a-invariant if and only if we have

(a) Θ is σ-invariant, and
(b) $\psi_{i\sigma^{-1},j\sigma^{-1}}^R a_j = a_i \psi_{ij}^R$ whenever $i \sim_\Theta j$.

Since R is H-invariant, we see that conditions (a), (b) are satisfied when $\Theta = \rho(R)$ and $a \in H$. Since any Θ under consideration refines $\rho(R)$, it follows that $K \in [R, N]^H$. Note also that if Φ is H-invariant and refines Θ then $K(\Phi) \leqslant K(\Theta)$.

We now see that if $\Pi_{H,R}$ is the set of H-invariant partitions of $[r]$ which refine $\rho(R)$, ordered by refinement, then

$$[H, G]^{\mathrm{op}} \cong \Pi_{H,R}.$$

(Here $[H, G]^{\mathrm{op}}$ is the set of subgroups sitting between H and G, ordered by reverse inclusion.) Now standard results on group actions show that if X is the stabilizer in H of any part of $\rho(R)$ then $\Pi_{H,R}$ is isomorphic with $[X, H]$ (see, for example, [DiMo, Theorem 1.5A]). Since $\Delta \mathsf{P}^{\mathrm{op}} = \Delta \mathsf{P}$ for any poset P, we have a contradiction to the minimality of G. □

To complete our proof we must examine the case where $r > 1$, $R_i = 1$ and $R^i \neq T_i$ for all i.

Lemma 2.8. *Say for $K \leqslant L \leqslant N$ we have*

(1) $K^i \neq 1$ *for all* $i \in [r]$, *and*
(2) $L = L^+$.

Let

$$X = \prod_{i=1}^{r} K^i.$$

Then $\rho(L \cap X) = \rho(L)$.

Proof. Let $Y = L \cap X$. Since $Y \leqslant L$ we know that $\rho(L)$ refines $\rho(Y)$ by Lemma 2.6(1), so it suffices to show that $\rho(Y)$ refines $\rho(L)$. Say $i \not\sim_L j$. Assumptions (1) and (2) of our lemma guarantee that we can pick some $x \in L$ such that $1 \neq \pi_i(x) \in K^i$ and $\pi_k(x) = 1$ whenever $i \not\sim_L k$. By Lemma 2.6(2), we have $\pi_l(x) \in K^l$ whenever $l \sim_L i$, so $x \in X$. We now have

$$x \in \big(\mathrm{Kernel}(\pi_j) \cap Y\big) \setminus \mathrm{Kernel}(\pi_i),$$

so $i \not\sim_Y j$. □

Lemma 2.9. *If $r > 1$ then $R = 1$.*

Proof. Assume for contradiction that $R^i \neq 1$ for some (and therefore all) $i \in [r]$. Set

$$X := \prod_{i=1}^{r} R^i.$$

By Corollary 2.5 and Lemma 2.7, we have $R < X < N$, so $X \in (R, N)^H$. We will show that $X^\perp$ is an antichain in $[R, N]^H$, thereby obtaining a contradiction to Lemma 2.1. Say $L \in X^\perp$, so $\langle X, L\rangle = N$. Since $L^1 \geqslant X^1 = R^1$, we must have $L^1 = T_1$ and therefore $L^i = T_i$ for all i. Since $L \neq N$ we have $L_i = 1$ for all i, and it follows that $L^+ = L$. So, the pair R, L satisfies conditions (1) and (2) of Lemma 2.8. Since $X \cap L = R$, we have $\rho(L) = \rho(R)$. Let I be a part of $\rho(R)$. For any $i \in I$, the restriction π_i^I of π_i to $\pi_I(L)$ is surjective (since π_i is surjective), and, by the definition of ρ, we see that π_i^I is also injective. therefore, $\pi_I(L) \cong T$. Assume $\rho(R)$ has s parts. Since $L = L^+$ we have $|L| = |T|^s$. Since every element of $X^\perp$ has the same order, $X^\perp$ is an antichain as claimed. □

2.2. Conjectures (B) *and* (C)

Here we discuss how to adjust the proof of Theorem 1.2 for Conjecture (A) to obtain proofs of the theorem for Conjectures (B) and (C). We examine each step in the proof.

Let P be a poset obtained from a finite lattice L by removing the minimum and maximum element, such that $\Delta\mathsf{P}$ is not connected and has no contractible connected component. (Note that $\overline{2B_3}$ is such a poset.) Then every connected component of $\Delta\mathsf{P}$ has at least two vertices and it follows that if $x \in \mathsf{P}$ then $x^\perp$ is not an antichain in L. Thus Lemma 2.1 holds with respect to Conjectures (B) and (C). Lemma 2.2 is proved using only group-theoretic arguments and Lemma 2.1, and Lemma 2.3 is proved using only group-theoretic arguments and Lemma 2.2. Thus Lemmas 2.2 and 2.3 hold with respect to Conjectures (B) and (C).

Lemma 2.4 uses group-theoretic arguments and the Quillen fiber lemma to produce a poset whose order complex is homotopy equivalent to that of $\Delta(H, G)$ and is either contractible or isomorphic to an interval in the subgroup lattice of the group XT_1 with $|XT_1| < |G|$. Since the topological properties used in formulating Conjecture (B) are homotopy invariant, we see that Lemma 2.4 holds with respect to Conjecture (B). Note also that for the map ϕ described in the proof, Image(ϕ) is isomorphic to a subposet of (H, G) (the isomorphism maps $\phi(L)$ to $K(\phi(L))$). It is straightforward to confirm that there is no proper subposet P of $\overline{2B_3}$ such that $\Delta\mathsf{P} \simeq \Delta\overline{2B_3}$. Therefore, if (H, G) is a counterexample to Conjecture (C) for which the conclusion of Lemma 2.4 does not hold, we conclude that Image(ϕ) is isomorphic to (H, G) and we obtain a contradiction to the minimality of $|G|$. Thus the lemma holds with respect to Conjecture (C).

Corollary 2.5 is proved using group-theoretic arguments and Lemma 2.4, and Lemma 2.6 is proved using only group theoretic arguments. Thus both of these results hold with respect to Conjectures (B) and (C).

The proof of Lemma 2.7 uses Lemma 2.6 and group theoretic arguments to show that $[H, G]^{\mathrm{op}}$ is isomorphic to $[X, H]$ and then concludes with the observation that $\Delta\mathsf{P}^{\mathrm{op}} = \Delta\mathsf{P}$ for any poset P. Thus the lemma holds with respect to Conjecture (B), which concerns the order complex. Since $\overline{2B_3}^{\mathrm{op}}$ is isomorphic with $\overline{2B_3}$, we see that the lemma also holds with respect to Conjecture (C).

The proof of Lemma 2.8 uses only group-theoretic arguments and the proof of Lemma 2.9 uses group-theoretic arguments, Corollary 2.5, and Lemmas 2.1, 2.7, and 2.8. Thus these results hold with respect to Conjectures (B) and (C).

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