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On Representing Finite Lattices as Intervals in Subgroup Lattices of Finite Groups

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Let M_n be the lattice of length 2 with $n \geq 1$ atoms. It is an open problem to decide whether or not every such lattice (or indeed whether or not every finite lattice) can be represented as an interval in the subgroup lattice of some finite group. We complete the work of the second author, Lucchini, by reducing this problem to a series of questions concerning the finite non-abelian simple groups.

1. INTRODUCTION

For any natural number n, let M_n be the lattice of length 2 with n atoms. Let T be a subgroup of a finite group G. The interval $[G/T]$ in the subgroup lattice of G is isomorphic to M_n if and only if there are precisely n proper subgroups of G that strictly contain T, and moreover, each such subgroup K satisfies

$$T <_{\max} K <_{\max} G.$$

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(Here the notation $A <_{\max} B$ means that A is a maximal subgroup of B.) We use Ω to denote the set of all $n \in \mathbb{N}$ such that M_n is isomorphic to some $[G/T]$. The determination of Ω is an open question. (For some history of this question see [18].)

It is easy to see that $n \in \Omega$ if $n = 1$ or 2, or if $n - 1$ is a prime power. For a long while it was conjectured that these were the only elements in Ω. However, in 1983 Feit [6] showed that both 7 and 11 lie in Ω, and more recently the second author showed that $n \in \Omega$ if

$$n = q + 2 \qquad \text{or} \qquad n = \frac{q^t + 1}{q + 1} + 1$$

for any prime power q and for any odd prime t. Currently no other elements of Ω are known; for convenience we let K denote the set of known elements of Ω.

Köhler [10] has shown that if $n \notin \mathsf{K}$ and G is a finite group chosen so that $|G|$ is minimal subject to the existence of a subgroup $T \leq G$ with $[G/T] \cong \mathsf{M}_n$, then the socle $\operatorname{Soc} G$ of G is the unique minimal normal subgroup of G and is non-abelian. The second author [16] has shown that if in addition $n > 50$, then either $\operatorname{Soc} G$ is simple (in which case G is said to be *almost simple*), or the intersection $(\operatorname{Soc} G) \cap T$ is trivial.

Given the classification of finite simple groups (henceforth referred to as CFSG) it is to be hoped that

$$\mathsf{S} = \{n \in \mathbb{N} : \mathsf{M}_n \cong [G/T] \text{ for some } T \leq G \text{ with } G \text{ almost simple}\}$$

can be determined. (The case in which G is either alternating or symmetric is currently under consideration by Alberto Basile, a research student of the second author.) The present paper is concerned with the case $(\operatorname{Soc} G) \cap T = \{\mathrm{id}_G\}$. This falls naturally into two subcases, either $G \neq (\operatorname{Soc} G)T$ or $G = (\operatorname{Soc} G)T$, and we let $\Omega(2.\mathrm{D})$, $\Omega(2.\mathrm{E})$ be the subsets of Ω comprising those integers that arise in these subcases. (Formal definitions of $\Omega(2.\mathrm{D})$ and $\Omega(2.\mathrm{E})$ are given in Section 2.) We refer to these subcases respectively as the *not-T-complement* and the *T-complement* cases.

The statement of our results depends on some quite technical definitions and so we feel it is inappropriate to give a full statement in this introduction. We can however give some idea of the analysis used. We assume that G is a finite group with a subgroup T such that $\operatorname{Soc} G$ is non-abelian and is the unique minimal normal subgroup of G, and such that

$$[G/T] \cong \mathsf{M}_n, \qquad \text{and} \qquad \operatorname{Soc} G \cap T = \{\mathrm{id}\};$$

if appropriate, we may also assume that $n \notin K$ and that $|G|$ is minimal among such groups. It is easy to see that in all cases there exists a subgroup H of G containing T and complementing $\operatorname{Soc} G$. (This is a consequence of Lemma 4.3(i)–(ii) in the not-T-complement case, and is immediate in the T-complement case as we can take $H = T$.) The key step is then to show that we may assume that the socle of H is non-abelian and is the unique minimal normal subgroup of H. (In the T-complement case this follows from either Theorem 6.4(iii) or Lemma 7.1(iii) depending on which subcase applies, while in the not-T-complement case Proposition 4.9 shows that H is almost simple.) Let E, F be minimal normal subgroups of $\operatorname{Soc} H$, $\operatorname{Soc} G$, respectively, so that both E and F are non-abelian simple groups; let α, β denote respectively the maps $N_H(E) \to \operatorname{Aut} E$, $N_G(F) \to \operatorname{Aut} F$ induced by conjugation. We will see that F is isomorphic to a section of E; the bulk of the paper is then concerned with obtaining as far as possible a characterization of the pair (G, T) in terms of E, F, the representation of F as a section of E, and other information as appropriate—for example, in the not-T-complement case we additionally require knowledge of $\alpha(N_H(E))$ and $\alpha(N_T(E))$, while in the T-complement case the situation is more complex and also requires knowledge of the images under β of various subgroups of H. Our philosophy throughout is that the global properties of the pair (G, T) are largely controlled by the internal structure of E.

Thus the net effect of the paper is to reduce the problem of determining $\Omega(2.\mathrm{D})$ and $\Omega(2.\mathrm{E})$ to a series of problems concerning the finite non-abelian simple groups. Thus this paper can be seen as the final step in the reduction of the problem of determining Ω to one which can be tackled via CFSG. Here we must stress that we do believe that the questions raised concerning the finite non-abelian simple groups can be answered, and indeed hope to have some answers in the near future. At this point, we should perhaps abandon caution and remark that the results that we expect to obtain in this direction provide yet more evidence to support the conjecture that $\Omega \neq \mathbb{N}$.

The layout of the paper is as follows. Section 2 contains the "Results Diagram": this is a schematic representation of our results, and is designed as an aid to understanding the significance of each individual result. Section 3 sets out our notation and gives a variety of preliminary results. Of particular importance is the information given on the maximality of top groups in twisted wreath products; the latter play a central role in the paper. The subsets $\Omega(2.\mathrm{D})$ and $\Omega(2.\mathrm{E})$ of Ω are investigated in Section 4 and Sections 5–7, respectively. We finish in Section 8 by giving some final comments and some examples.

2. THE RESULTS DIAGRAM

As mentioned in the Introduction, our results depend on some quite technical definitions, and we see no point in giving these definitions any earlier than need be. However, we feel it helpful to give a schematic representation of our results as an aid in understanding the significance of each individual result.

Our results are phrased in terms of various subsets of $\mathbb{N}$. The Introduction referred to the subsets Ω, $\mathcal{K}$, $\mathcal{S}$, $\Omega(2.D)$, $\Omega(2.E)$ and for the sake of precision we now expand upon or repeat their definition.

$$\Omega = \left\{ n \in \mathbb{N} \colon \begin{array}{l} \text{there exists a group } G \\ \text{with a subgroup } T \\ \text{such that } M_n \cong [G/T] \end{array} \right\}, \tag{2.A}$$

$$\mathcal{K} = \left\{ n \in \mathbb{N} \colon \begin{array}{l} n = 1, 2, q+1, q+2, \text{ or } (q^t+1)/(q+1)+1 \\ \text{for some prime power } q \\ \text{and some odd prime } t \end{array} \right\}. \tag{2.B}$$

(Recall that $\mathcal{K}$ is precisely the set of currently known elements of Ω.)

$$\mathcal{S} = \left\{ n \in \mathbb{N} \colon \begin{array}{l} M_n \cong [G/T] \text{ for some } T \le G \\ \text{with } G \text{ almost simple} \end{array} \right\}, \tag{2.C}$$

$$\Omega(2.D) = \left\{ n \ge 16 \colon \begin{array}{l} M_n \cong [G/T] \text{ for some } T \le G \text{ with} \\ \operatorname{Soc} G \lhd_{\min} G, \operatorname{Soc} G \text{ non-abelian}, \\ T \cap \operatorname{Soc} G = \{\mathrm{id}\}, \text{ and } G \ne T(\operatorname{Soc} G) \end{array} \right\}, \tag{2.D}$$

$$\Omega(2.E) = \left\{ n \ge 16 \colon \begin{array}{l} M_n \cong [G/T] \text{ for some } T \le G \text{ with} \\ \operatorname{Soc} G \lhd_{\min} G, \operatorname{Soc} G \text{ non-abelian}, \\ T \cap \operatorname{Soc} G = \{\mathrm{id}\} \text{ and } G = T(\operatorname{Soc} G) \end{array} \right\}. \tag{2.E}$$

Note that we have restricted attention to integers $n \ge 16$ in defining $\Omega(2.D)$ and $\Omega(2.E)$ since 16 is the smallest positive integer not in $\mathcal{K}$, and so is the smallest positive integer not known to be in Ω. With this notation the results discussed in the introduction of Köhler [10] and the second author [16] show that

$$\Omega = \mathcal{K} \cup \{n \le 50 : n \in \Omega\} \cup \mathcal{S} \cup \Omega(2.D) \cup \Omega(2.E). \tag{2.F}$$

We now explain our results diagram, Fig. 1. Observe that the diagram is essentially a tree comprising nodes joined by a single bond or by a double bond (or not joined at all). Each node represents the subset of $\mathbb{N}$ as indicated by the (unboxed) label adjacent to the node. Those subsets that have not yet been defined are all denoted by notation of the form $\Omega(\text{—})$ or $\Delta(\text{—})$: the bracketed reference gives the number of the relevant definition. The diagram encodes our results according to the following conventions: firstly that if the "descendent" of a node is joined to its "parent" by a double bond, then the "descendent" is a subset of its

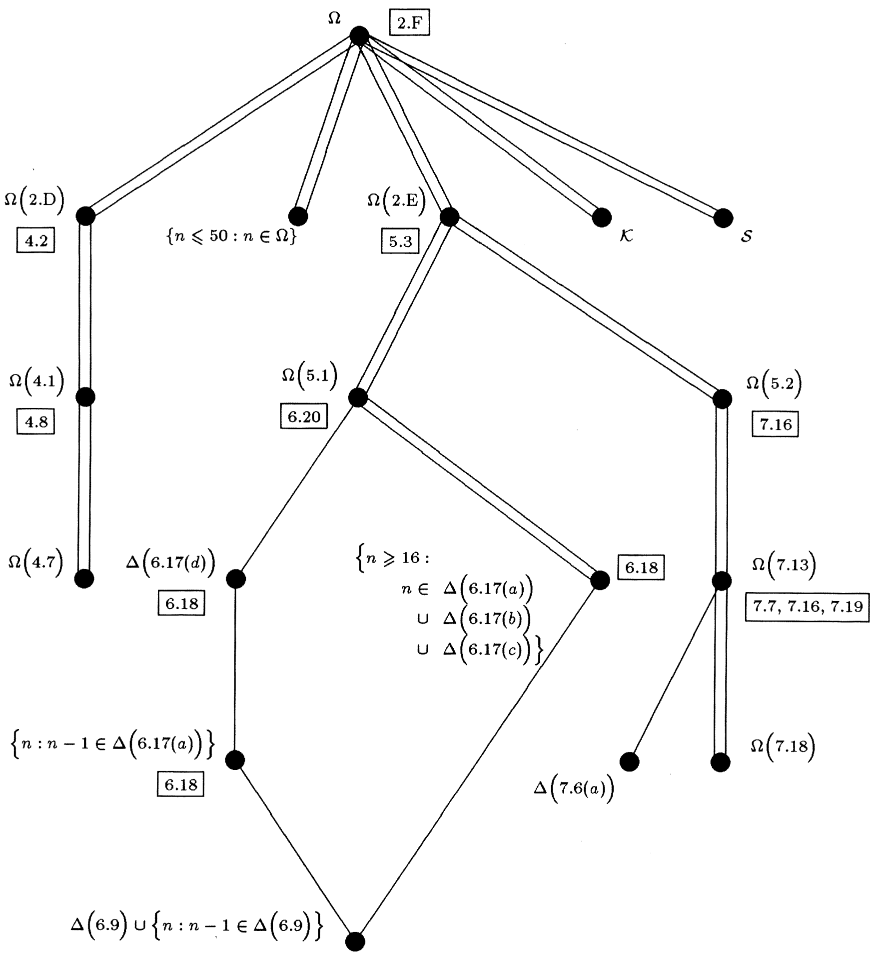


FIG. 1. The results diagram.

"parent"; and secondly that the "parent" is a subset of the union of all its "descendents." The reference of the result(s) thus encoded is given in a box adjacent to each "parent" node. Thus on considering the "descendents" of $\Omega(2.\mathrm{E})$ we see that

$$\Omega(2.\mathrm{E}) = \Omega(5.1) \cup \Omega(5.2)$$

and that this result is the content of Theorem 5.3.

3. PRELIMINARIES

3.1. *General Notation.* In the remainder of the paper all groups are assumed finite. We use id_G, or simply id if no confusion arises, to denote the identity element of a group G. The group of automorphisms of a group G is denoted $\mathrm{Aut}\, G$. We use θ for the natural map $G \to \mathrm{Aut}\, G$ given by

$$g \mapsto \text{“conjugation by } g\text{”} \qquad \text{for all } g \in G.$$

The image $\mathrm{Inn}\, G$ of θ is the subgroup of $\mathrm{Aut}\, G$ comprising *inner* automorphisms; we say that an automorphism is *outer* if it is not inner, and use $\mathrm{Out}\, G$ to denote the quotient $\mathrm{Aut}\, G/\mathrm{Inn}\, G$. If the centre $Z(G)$ of G is trivial, then θ gives rise to an isomorphism between G and $\mathrm{Inn}\, G$; in such cases we will often identify G and $\mathrm{Inn}\, G$ via θ. In particular, we will always use θ to identify F with $\mathrm{Inn}\, F$ whenever F is a non-abelian simple group, although we will normally remind the reader when doing so.

Maps are written on the left; in view of this, the composition $\beta \circ \alpha$ of maps α and β means "α followed by β." However, if α, β are automorphisms of a group G, then for $g \in G$ we often write g^{α} instead of $\alpha(g)$, and $\alpha\beta$ instead of $\beta \circ \alpha$.

The *socle* $\mathrm{Soc}\, G$ of a group G is the product of all minimal normal subgroups of G. If $H \leq G$ and K is either a subgroup of G or a subgroup of $\mathrm{Aut}\, G$, then the K-core of H in G is denoted by $\mathrm{Core}_K H$ and is the largest K-invariant subgroup of G that is contained in H; explicitly

$$\mathrm{Core}_K H = \bigcap_{x \in K} H^x;$$

if $K = G$ then we simplify terminology and refer to the *core* of H in G.

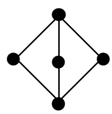
Given a subgroup H of G an *overgroup of H in G* is any subgroup K satisfying $H \leq K \leq G$; a *strict overgroup of H in G* is an overgroup of H in G that is not equal to H. Given also a homomorphism χ with domain H, then an *extension of χ in G* is any homomorphism whose domain is an overgroup of H in G and whose restriction to H is equal to χ; it is a *strict extension of χ in G* if, in addition, its domain is a strict overgroup of H in G.

We use standard notation for normalisers and centralizers, that is, $N_G(H)$, $C_G(H)$ denote respectively the normaliser, centralizer of H in G. For subgroups $H_1, \ldots, H_n$ of G we set

$$N_G(H_1, \ldots, H_n) = N_G(H_1) \cap \cdots \cap N_G(H_n).$$

We also use the same notation even when G is a subgroup of $\operatorname{Aut} K$ with $H_1, \ldots, H_n$ subgroups of K.

As noted in the Introduction, we use M_n to denote the lattice of length 2 with $n \geq 1$ atoms; for example, M_3 is the lattice



We also use M_0 to denote the lattice of length 1, namely

For a subgroup H of a group G, we use $[G/H]$ to denote the lattice whose elements are the overgroups of H in G with partial order given by inclusion. If S is a subgroup of $\operatorname{Aut} G$ such that H is S-invariant, then $[G/H]_S$ denotes the sublattice of $[G/H]$ comprising the S-invariant overgroups of H in G.

3.2. *Wreath Products*. Suppose that G is a subgroup of the symmetric group S_n. The wreath product $H \wr G$ of H by G is the semi-direct product $H^n \rtimes G$ of the direct product H^n of n copies of H by G in which the conjugation action of G on H^n is given by

$$(h_1, \ldots, h_n)^{g^{-1}} = (h_{1g}, \ldots, h_{ng}) \qquad \text{for all } h_1, \ldots, h_n \in H \text{ and } g \in G.$$

Wreath products arise in this paper from applications of the "embedding theorem" which is well known and goes back to Frobenius: for more recent expositions see [5, Sect. 5; 19, Sect. 4].

THEOREM 3.3 (Embedding Theorem). *Let H be a subgroup of the finite group G, let $x_1, \ldots, x_n$ be a right transversal for H in G, and let ξ be any homomorphism with domain H. Then the map $G \to \xi(H) \wr S_n$ given by*

$$x \mapsto \left(\xi\left(x_1 x x_{1\pi}^{-1}\right), \ldots, \xi\left(x_n x x_{n\pi}^{-1}\right)\right)\pi \qquad \textit{for all } x \in G,$$

where $\pi \in S_n$ *satisfies* $x_i x x_{i\pi}^{-1} \in H$ *for all* $i = 1, \ldots, n$ *is a well-defined homomorphism with kernel equal to* $\operatorname{Core}_G(\ker \xi)$.

We remark that the above is usually applied with ξ equal to the identity map on H, in which case $\ker \xi$ is trivial and the map $G \to H \wr S_n$ is a monomorphism—hence the name "Embedding Theorem." Also we often fail to distinguish between homomorphisms defined with respect to different choices of transversal: the relationship between such homomorphisms is made explicit by the "Uniqueness theorem" of [11].

3.4. *Twisted Wreath Products*. The concept of a twisted wreath product (originally due to B. H. Neumann [17]) plays a central role in this paper. Consequently we feel obliged to give a complete treatment of the construction, rather than refer the reader to a suitable reference such as [21] or [3]. We note however that the following treatment does use a slightly different notation for the base group of a twisted wreath product; this change is for the purposes of encoding more information in the notation.

The ingredients for the construction of a twisted wreath product are

$$\text{a group } F, \qquad \text{a group } T, \qquad \text{a subgroup } S \text{ of } T,$$
$$\text{and a homomorphism } \phi : S \to \operatorname{Aut} F.$$

Define the *base group* $B^F_{\phi, T}$ by

$$B^F_{\phi, T} = \left\{ f : T \to F : f(ts) = f(t)^{\phi(s)} \text{ for all } t \in T, s \in S \right\}.$$

We view $B^F_{\phi, T}$ as a group by defining multiplication in the natural way; note that $B^F_{\phi, T} \cong F^{|T:S|}$. Define an action of T on $B^F_{\phi, T}$ as follows: for $f \in B^F_{\phi, T}$ and $t \in T$ let f^t, the image of f under t, be defined by

$$f^t(x) = f(tx) \qquad \text{for all } x \in T.$$

It is easily verified that this does indeed give an action of T on $B^F_{\phi, T}$. The semi-direct product $X = B^F_{\phi, T} \rtimes T$ with respect to this action is called the *twisted wreath product of F by T* and we write

$$X = F \operatorname{twr}_\phi T$$

and refer to T as the *top group* of the twisted wreath product. (The subgroup S is recoverable from this notation as the domain of ϕ.)

In the situation where L is a subgroup of F invariant under $\phi(S)$ we often use ι to denote the natural map $\phi(S) \to \operatorname{Aut} L$ and consider the twisted wreath product $L \operatorname{twr}_{\iota \circ \phi} T$ with base group $B^L_{\iota \circ \phi, T}$. Given that

$$\begin{aligned} B^L_{\iota \circ \phi, T} &= \left\{ f : T \to L : f(ts) = f(t)^{\iota(\phi(s))} \text{ for all } t \in T, s \in S \right\} \\ &= \left\{ f : T \to L : f(ts) = f(t)^{\phi(s)} \text{ for all } t \in T, s \in S \right\} \end{aligned}$$

since $x^{\iota(\phi(s))} = x^{\phi(s)}$ for all $x \in L$ and $s \in S$, we sometimes abuse notation and write $B^L_{\phi,T}$ in place of $B^L_{\iota\circ\phi,T}$. The superscript in the base group notation then serves to distinguish this base group from the base group $B^F_{\phi,T}$. We also abuse notation by viewing maps $T \to L$ as maps $T \to F$ so that $B^L_{\phi,T}$ becomes a subgroup of $B^F_{\phi,T}$. In fact, with this viewpoint $B^L_{\phi,T}$, as a subgroup of $B^F_{\phi,T}$, is normalised by T, and moreover, the action of T on $B^L_{\phi,T} \leq B^F_{\phi,T}$ is identical to the action of T on $B^L_{\phi,T}$ as the base group of $L \operatorname{twr}_\phi T$. Hence this viewpoint identifies $L \operatorname{twr}_\phi T$ with the subgroup $B^L_{\phi,T}T$ of $F \operatorname{twr}_\phi T$.

In cases where no confusion arises we will omit the superscript and subscripts in the base group notation as appropriate.

Twisted wreath products occur naturally as the following result shows.

LEMMA 3.5 (Bercov [4], Lafuente [13]). *Let G be a group with a normal subgroup M complemented by a subgroup T. Suppose that F is a subgroup of M such that for some $t_1(= \mathrm{id}_T), t_2, \dots, t_k \in T$ we can write*

$$M = F^{t_1} \times \cdots \times F^{t_k},$$

where conjugation by T permutes the F^{t_i} among themselves, that is, $\{F^{t_1}, \dots, F^{t_k}\}$ is the set of T-conjugates of F. Set $S = N_T(F)$ and let $\phi : S \to \operatorname{Aut} F$ be the map induced by the conjugation action of S on F. Then there exists an isomorphism $G \to F \operatorname{twr}_\phi T$ which maps M to $B^F_{\phi,T}$ and which restricts to give the identity on T.

We shall often be concerned with the possible maximality of the top group in a twisted wreath product. Hence the following results are relevant. (The first two are given without proof: both are straightforward, and Corollary 3.7 follows from Lemmas 3.1 and 3.2 of [3], while Lemma 3.6 is the content of the proof of [3, 3.1]. However, we note that Corollary 3.7 does depend on CFSG, or more precisely on its consequence, the "Schreier conjecture.")

LEMMA 3.6. *Let F be a non-abelian simple group, and let $F \operatorname{twr}_\phi T$ be a twisted wreath product with twisting homomorphism $\phi : S \to \operatorname{Aut} F$. Suppose that X is a T-invariant proper subgroup of the base group $B^F_{\phi,T}$ of $F \operatorname{twr}_\phi T$. Then precisely one of the following holds*:

- (i) *there exists a proper $\phi(S)$-invariant subgroup L of F with $X \leq B^L_{\phi,T}$*;
- (ii) *there exists a strict extension ρ of ϕ in T such that $X = B^F_{\rho,T}$.*

We remark that conclusion (ii) is sensible; indeed, if given an arbitrary twisted wreath product $F \operatorname{twr}_\phi T$ and a strict extension ρ of ϕ in T, then the base group $B^F_{\rho,T}$ is a non-trivial proper subgroup of $B^F_{\phi,T}$ and is normalised by T. In fact, the action of T on $B^F_{\rho,T}$ as a subgroup of $B^F_{\phi,T}$ is identical to the action of T on $B^F_{\rho,T}$ as the base group of $F\operatorname{twr}_\rho T$; thus $F\operatorname{twr}_\rho T$ appears naturally as a subgroup of $F\operatorname{twr}_\phi T$.

COROLLARY 3.7. *Let F be a non-abelian simple group, and let $F\operatorname{twr}_\phi T$ be a twisted wreath product with twisting homomorphism $\phi : S \to \operatorname{Aut} F$. Then T is maximal in $F\operatorname{twr}_\phi T$ if and only if the following both hold*:

(i) $\phi(S) \geq \operatorname{Inn} F$;

(ii) *there does not exist a strict extension ρ of ϕ in T.*

LEMMA 3.8. *Let $F\operatorname{twr}_\phi T$ be a twisted wreath product with twisting homomorphism $\phi : S \to \operatorname{Aut} F$, and let R be any overgroup of S in T. Suppose that T is a maximal subgroup of $F\operatorname{twr}_\phi T$. Then R is a maximal subgroup of $F\operatorname{twr}_\phi R$.*

Proof. We suppose that R is not a maximal subgroup of $F\operatorname{twr}_\phi R$ and argue for a contradiction. Then there exists a non-trivial proper subgroup X of $B_{\phi,R}$ that is normalised by R. Let Y be the subset of $B_{\phi,T}$ comprising all maps $f \in B_{\phi,T}$ such that for all $t \in T$ the map $f_t : R \to F$ given by

$$f_t(r) = f(tr) \qquad \text{for all } r \in R$$

is in X.

We claim that Y is a non-trivial proper subgroup of $B_{\phi,T}$ normalised by T. The verification of this is straightforward and we leave it to the reader. Thus T is not a maximal subgroup of $F\operatorname{twr}_\phi T$, a contradiction.

Suppose that F is a non-abelian simple group and consider the base group B_ϕ of the twisted wreath product $F\operatorname{twr}_\phi T$. Now $B_\phi \cong F^{|T:S|}$ and it is clear that T is transitive on the simple direct factors of B_ϕ. Hence B_ϕ is a minimal normal subgroup of $F\operatorname{twr}_\phi T$. We shall often be concerned with whether or not B_ϕ is the unique minimal normal subgroup, and so equal to the socle of $F\operatorname{twr}_\phi T$; clearly this holds if and only if its centralizer in $F\operatorname{twr}_\phi T$ is trivial.

LEMMA 3.9. *Let $F\operatorname{twr}_\phi T$ be a twisted wreath product with twisting homomorphism $\phi : S \to \operatorname{Aut} F$, and suppose that the centre of F is trivial. Then*

$$C_T(B_\phi) = \operatorname{Core}_T(\ker \phi) \qquad \textit{and} \qquad C_{F\operatorname{twr}_\phi T}(B_\phi) \cong \operatorname{Core}_T(\phi^{-1}(\operatorname{Inn} F)).$$

Proof. This is straightforward, or can be seen as a particular instance of [3, 2.7(3)].

3.10. *Sections.* The ordered pair (C, D) is said to be a *section of E isomorphic* to a group T whenever C is a normal subgroup of D and D is a subgroup of E such that the quotient D/C is isomorphic (in the usual sense) to T. It is a *non-abelian simple section* if T is a non-abelian simple group, and is a *proper section* if $T \not\cong E$.

Suppose that P either is a subgroup of $\operatorname{Aut} E$ or normalises E inside some larger group. The section (C, D) of E is said to be *P-contained* in the section $(\tilde{C}, \tilde{D})$ of E if

$$C = \tilde{C} \cap D, \qquad \tilde{D} = \tilde{C}D, \qquad \text{and} \qquad N_P(C, D) \leq N_P(\tilde{C}, \tilde{D});$$

moreover, it is *strictly P-contained* in $(\tilde{C}, \tilde{D})$ if in addition $C < \tilde{C}$. If there does not exist a section of E strictly P-containing the section (C, D) of E, then (C, D) is said to be a *P-maximal* section of E; it is a *maximal section* of E if it is $\operatorname{Aut} E$-maximal.

We state without proof some easy consequences of the above definitions.

PROPOSITION 3.11. *Suppose that (C, D) and $(\tilde{C}, \tilde{D})$ are sections of E and that $P \leq \operatorname{Aut} E$. Then the following all hold.*

(1) *If (C, D) is P-contained in $(\tilde{C}, \tilde{D})$, then the quotients D/C and $\tilde{D}/\tilde{C}$ are isomorphic.*

(2) *The following statements are equivalent:*

- (i) *(C, D) is P-contained in $(\tilde{C}, \tilde{D})$;*
- (ii) *(C, D) is $(DN_P(C, D))$-contained in $(\tilde{C}, \tilde{D})$;*
- (iii) *(C, D) is $N_P(C, D)$-contained in $(\tilde{C}, \tilde{D})$.*

(3) *If $Q \leq P$ and (C, D) is P-contained in $(\tilde{C}, \tilde{D})$, then (C, D) is Q-contained in $(\tilde{C}, \tilde{D})$.*

Suppose that $\phi : S \to \operatorname{Aut} F$ is a homomorphism from a subgroup S of a group T to the automorphism group of a non-abelian simple group F with $\phi(S) \geq \operatorname{Inn} F$ and that N is a subgroup of T normalised by S. Then $\phi(N \cap S)$ is a normal subgroup of the almost simple group $\phi(S)$ and so is either trivial or contains $\operatorname{Inn} F$. The following results deal separately with these two cases and demonstrate the relevance of sections to the maximality of top groups of wreath products.

LEMMA 3.12. *Let N, T, ϕ, S, and F be as in the preceding paragraph. Suppose that $\phi(N \cap S)$ is trivial. Then the map $\rho : NS \to \operatorname{Aut} F$ given by*

$$\rho : xy \mapsto \phi(y) \qquad \text{for all } x \in N \text{ and } y \in S$$

is well-defined, is an extension of ϕ in $N_T(N)$, and contains N in its kernel.

Proof. This is straightforward.

LEMMA 3.13. *Let N, T, ϕ, S, and F be as in the paragraph immediately preceding the above lemma. Suppose that $\phi(N \cap S) \geq \operatorname{Inn} F$. Then*

$$\left(N \cap \ker \phi, N \cap \phi^{-1}(\operatorname{Inn} F)\right)$$

is a section of N isomorphic to F. Moreover, if $(N \cap \ker \phi, N \cap \phi^{-1}(\operatorname{Inn} F))$ is S-contained in the section (C, D) of N then the map $\sigma : N_T(C, D) \to \operatorname{Aut} F$, given by requiring that

$$\phi(x)^{\sigma(y)} = \phi(z)$$

whenever $x, z \in N \cap \phi^{-1}(\operatorname{Inn} F)$ and $y \in N_T(C, D)$ are such that $x^y z^{-1} \in C$, is well-defined and extends ϕ in T.

Conversely, if σ is an extension of ϕ in $N_T(N)$ then $(N \cap \ker \phi, N \cap \phi^{-1}(\operatorname{Inn} F))$ is S-contained in the section $(N \cap \ker \sigma, N \cap \sigma^{-1}(\operatorname{Inn} F))$ of N.

Remark 3.14. With the notation of the above lemma, observe that the section

$$(C, D) = \left(N \cap \ker \phi, N \cap \phi^{-1}(\operatorname{Inn} F)\right)$$

certainly S-contains itself. Now if $x, z \in D = N \cap \phi^{-1}(\operatorname{Inn} F)$ and $y \in N_T(C, D)$ are such that $x^y z^{-1} \in C = N \cap \ker \phi$ then $x^y \in D$ and furthermore

$$\phi(x^y) = \phi(z).$$

Hence by the lemma the map $\sigma : N_T(C, D) \to \operatorname{Aut} F$ given by requiring that

$$\phi(x)^{\sigma(y)} = \phi(x^y) \qquad \text{for all } x \in D \text{ and } y \in N_T(C, D)$$

is well-defined and extends ϕ.

Proof of Lemma 3.13. This follows by adapting the proofs of [3, 3.1 and 3.5].

Corollary 3.15. *Let N, T, ϕ, S, and F be as in the paragraph immediately preceding Lemma* 3.12. *Suppose that N is a normal subgroup of T and that $\phi(N \cap S) \geq \operatorname{Inn} F$. Then ϕ has no strict extensions in T if and only if both of the following hold*:

(i) $(N \cap \ker \phi, N \cap \phi^{-1}(\operatorname{Inn} F))$ *is an S-maximal section of N*;

(ii) $S = N_T(N \cap \ker \phi, N \cap \phi^{-1}(\operatorname{Inn} F))$.

Proof. This is straightforward.

Suppose that N, T, ϕ, S, and F are as above; suppose also that $\phi(S \cap N) \geq \operatorname{Inn} F$. Set

$$U = N \cap \ker \phi \qquad \text{and} \qquad V = N \cap \phi^{-1}(\operatorname{Inn} F).$$

By Lemma 3.13 the section (U, V) is a section of N isomorphic to F. Suppose further that N is a minimal normal subgroup of T. As $\phi(N \cap S) \geq \operatorname{Inn} F$ we see that N is non-abelian and so is isomorphic to the direct product of its minimal normal subgroups, each of which are isomorphic to a fixed non-abelian simple group. An easy argument (cf. [3, 5.3]) shows that there exists a minimal normal subgroup E of N with

$$\kappa(U) \neq \kappa(V),$$

where $\kappa : N_T(E) \to \operatorname{Aut} E$ is induced by conjugation. As κ induces an epimorphism from the quotient V/U onto the quotient $\kappa(V)/\kappa(U)$ we deduce that $(\kappa(U), \kappa(V))$ is a section of E also isomorphic to F (where we identify E with $\operatorname{Inn} E$). The following result relates possible S-maximality of (U, V) to possible $\kappa(N_S(E))$-maximality of $(\kappa(U), \kappa(V))$.

LEMMA 3.16. *Suppose that (U, V) and $(\tilde{U}, \tilde{V})$ are non-abelian simple sections of the minimal normal subgroup N of a group T with $L \leq T$ such that (U, V) is L-contained in $(\tilde{U}, \tilde{V})$. Then there exists a minimal normal subgroup E of N such that $(\kappa(\tilde{U}), \kappa(\tilde{V}))$ is a section of E isomorphic to the quotient $\tilde{V}/\tilde{U}$, where $\kappa : N_T(E) \to$ Aut E is induced by conjugation. Let E be so chosen; then $(\kappa(U), \kappa(V))$ is a section of E which is $\kappa(N_L(E, U, V))$-contained in $(\kappa(\tilde{U}), \kappa(\tilde{V}))$. Moreover (U, V) is an L-maximal section of N if and only if the following all hold:*

(i) *$(\kappa(U), \kappa(V))$ is a $\kappa(N_L(E, U, V))$-maximal section of E;*

(ii) *$\operatorname{Core}_N U$ of N contains every normal subgroup of N that is normalised by $N_L(U, V)$ and that does not contain E;*

(iii) *$U \cap E = \kappa(U)$.*

Remark 3.17. (1) The above lemma is essentially a rephrasing of Theorem 5.1 of [1]. However, we choose to give a proof of the lemma here, rather than refer to [1], as the method of proof used here is quite different from [1] and much more appropriate in the current context.

(2) We stress that condition (i) of the above lemma considers $\kappa(N_L(E, U, V))$-maximality, and does *not* consider $\kappa(N_L(E))$-maximality. This is important in that in general we only have

$$\kappa(N_L(E, U, V)) \leq N_{\kappa(N_L(E))}(\kappa(U), \kappa(V))$$

and not equality. For an example demonstrating this, see the final section of Remark 4.8 of [1].

(3) With the notation of the above lemma, conditions (ii) and (iii) hold if and only if

$$U \cap E^x = \begin{cases} \kappa(U)^x & \text{if } x \in N_L(U,V); \\ E^x & \text{if } x \in T \setminus N_T(E)N_L(U,V). \end{cases}$$

In turn the latter holds if and only if

$$U = \left(\prod_{i=1}^{m} \kappa(U)^{x_i} \right) \times \left(\prod_{i=m+1}^{l} E^{x_i} \right),$$

where $x_1, \ldots, x_m$ is a right transversal for $N_L(E,U,V)$, in $N_L(U,V)$ and $x_1, \ldots, x_l$ is a right transversal for $N_T(E)$ in T.

Proof of Lemma 3.16. The discussion immediately prior to the statement of the lemma may be adapted to show the existence of a minimal normal subgroup E of N such that

$$\big(\kappa(\tilde{U}), \kappa(\tilde{V})\big),$$

where $\kappa : N_T(E) \to \operatorname{Aut} E$ is induced by conjugation, is a section of E also isomorphic to $\tilde{V}/\tilde{U}$. We fix this choice of E and κ. Given that (U,V), $(\tilde{U},\tilde{V})$ are non-abelian simple sections with the former L-contained in the latter, we have that $\tilde{V} = \tilde{U}V$, and that V/U, $\tilde{V}/\tilde{U}$ are isomorphic non-abelian simple groups: we deduce that in the following diagram (in which the maps are the obvious ones) all homomorphisms are epimorphisms and moreover that all are isomorphisms with the possible exception of those in the bottom row.

$$\begin{array}{ccccc} \tilde{V}/\tilde{U} & \to & \kappa(\tilde{V})/\kappa(\tilde{U}) & \to & \kappa(\tilde{U})\kappa(V)/\kappa(\tilde{U}) \\ \downarrow & & & & \downarrow \\ V/U & \to & \kappa(V)/\kappa(U) & \to & \kappa(V)/\kappa(\tilde{U}) \cap \kappa(V) \end{array}$$

It follows from the commutativity of this diagram that all maps are in fact isomorphisms, and in particular that

$$\kappa(V)/\kappa(U) \cong V/U \qquad \text{and} \qquad \kappa(U) = \kappa(V) \cap \kappa(\tilde{U}).$$

We conclude that $(\kappa(U), \kappa(V))$ is indeed a section of E isomorphic to $\tilde{V}/\tilde{U}$ and, on noting that $\kappa(N_L(E,U,V))$ normalises both $\kappa(\tilde{U})$ and $\kappa(\tilde{V})$ since $N_L(U,V)$ is contained in $N_T(\tilde{U},\tilde{V})$, that $(\kappa(U), \kappa(V))$ is $\kappa(N_L(E,U,V))$-contained in $(\kappa(\tilde{U}), \kappa(\tilde{V}))$.

We turn to the "Moreover" statement. Set $S = N_L(U,V)$ and note that (U,V) is L-maximal if and only if (U,V) is S-maximal. Let $x_1,\dots,x_l$ be a right transversal for $N_T(E)$ in T such that $x_1,\dots,x_m$ is a right transversal for $N_S(E)$ in S. Let (C,D) be a section of E that $\kappa(N_S(E))$-contains $(\kappa(U),\kappa(V))$. It is straightforward to see that

$$X = \left(\prod_{i=1}^{m} C^{x_i}\right) \times \left(\prod_{i=m+1}^{l} E^{x_i}\right)$$

is normalised by both S and V. Thus $X \cap V$ is a normal subgroup of V that contains U, and so is equal to either U or V as V/U is simple. But $\kappa(X) = \kappa(U) \lneqq \kappa(V)$ and so $X \cap V = U$. Hence (U,V) is S-contained in the section (X, XV) of N, and the necessity of conditions (i)–(iii) follows. Conversely, suppose that conditions (i)–(iii) all hold and that (U,V) is S-contained in the section $(\tilde{U},\tilde{V})$ of N. Condition (i) together with the first part of the lemma implies that $\kappa(\tilde{U}) = \kappa(U)$. Given that S normalises $\tilde{U}$, it follows that

$$\tilde{U} \le \left(\prod_{i=1}^{m} \kappa(U)^{x_i}\right) \times \left(\prod_{i=m+1}^{l} E^{x_i}\right),$$

where by conditions (ii) and (iii) the right hand side equals U. Hence $\tilde{U} \le U$ whence the section (U,V) is equal to $(\tilde{U},\tilde{V})$ as required.

With the notation of the above lemma, it is clear that if E is a minimal normal subgroup of N such that $(\kappa(U),\kappa(V))$ is isomorphic to V/U, then the same holds with E replaced by any $N_T(U,V)$-conjugate of E (and κ redefined appropriately). Hence, if $N_T(U,V)$ is transitive on the minimal normal subgroups of N, equivalently if

$$T = N_T(E)N_T(U,V),$$

then E can be replaced by any minimal normal subgroup of N. As this is often true in the situations arising later in this paper it is worthwhile to consider this more carefully. For the sake of precision we have the following definition.

Definition 3.18. We say that (E,N,T,F,S,ϕ) is a (3.18)-tuple if the following all hold:

(i) E is a minimal normal subgroup of the group N which in turn is a minimal normal subgroup of the group T;

(ii) F is a non-abelian simple group;

(iii) S is a subgroup of T such that $T = N_T(E)S$;

(iv) ϕ is a homomorphism $S \to \operatorname{Aut} F$ such that $\phi(S \cap N) \ge \operatorname{Inn} F$.

Given a (3.18)-tuple (E, N, T, F, S, ϕ) we fix the notation κ, C, D, η, X, P, and σ as follows.

The map κ is the usual map $N_T(E) \to \operatorname{Aut} E$ induced by conjugation. We stress that we identify E with $\operatorname{Inn} E$ so that κ restricts to give the identity map on E. We set

$$C = \kappa(\ker \phi \cap N) \qquad \text{and} \qquad D = \kappa\big(\phi^{-1}(\operatorname{Inn} F) \cap N\big).$$

Note that the section $(N \cap \ker \phi, N \cap \phi^{-1}(\operatorname{Inn} F))$ of N is isomorphic to F. The remarks immediately prior to Lemma 3.16 coupled with the fact that $T = N_T(E)S$ (equivalently that S is transitive on the minimal normal subgroups of N), imply that (C, D) is a section of E isomorphic to F. In fact it is clear that the map $D/C \to F$ given by

$$C\kappa(x) \mapsto \phi(x) \qquad \text{for all } x \in N \cap \phi^{-1}(\operatorname{Inn} F),$$

where F is identified with $\operatorname{Inn} F$ in the usual way, is an isomorphism. We use this isomorphism to identify D/C with F, and let $\eta : N_{\operatorname{Aut} E}(C, D) \to \operatorname{Aut} F$ be induced by the conjugation action on the quotient D/C.

Let $x_1, \dots, x_l$ be any right transversal for $N_S(E)$ in S. We set

$$X = \prod_{i=1}^{l} C^{x_i} \le N.$$

Notice that X is independent of the transversal chosen and furthermore that the section $(N \cap \ker \phi, N \cap \phi^{-1}(\operatorname{Inn} F))$ of N is S-contained in $(X, X(N \cap \phi^{-1}(\operatorname{Inn} F)))$. We set $P = N_T(X, X(N \cap \phi^{-1}(\operatorname{Inn} F)))$, and let $\sigma : P \to \operatorname{Aut} F$ be defined by requiring that

$$\phi(x)^{\sigma(y)} = \phi(z)$$

whenever $x, z \in N \cap \phi^{-1}(\operatorname{Inn} F)$ and $y \in P$ are such that $x^y z^{-1} \in X$. By Lemma 3.13 the map σ is well-defined and extends ϕ. We leave the reader to verify that for $x \in N_P(E)$ we have $\kappa(x) \in N_{\operatorname{Aut} E}(C, D)$ and

$$\sigma(x) = \eta(\kappa(x)). \tag{3.A}$$

LEMMA 3.19. *Let (E, N, T, F, S, ϕ) be a (3.18)-tuple and let κ, C, D, η, X, P, and σ be as defined above. Let $x_1, \dots, x_l$ be any right transversal for $N_S(E)$ in S. Then the following all hold*:

(i) $N \cap \ker \sigma = \prod_{i=1}^{l} (E \cap \ker \eta)^{x_j}$;

(ii) $\sigma(N \cap P) = \operatorname{Core}_{\phi(S)}(\eta(N_E(C, D)))$;

(iii) $\kappa(N \cap P) = E \cap \eta^{-1}(\mathrm{Core}_{\phi(S)}(\eta(N_E(C, D))))$;

(iv) $(N \cap \ker \sigma, N \cap \sigma^{-1}(\mathrm{Inn}\, F))$ *is a P-maximal section of N if and only if*

$$\big(E \cap \ker \eta, E \cap \eta^{-1}(\mathrm{Inn}\, F)\big)$$

is a $\kappa(N_P(E))$*-maximal section of* E.

Proof. Part (i) follows by a straightforward calculation. We turn to (ii). Observe that $N \cap P$ is normalised by P, and so also by S, whence $\sigma(N \cap P)$ is normalised by $\sigma(S) = \phi(S)$. As $\kappa(N \cap P) \leq E$ we deduce from (3.A) that

$$\sigma(N \cap P) \leq \eta\big(N_E(C, D)\big)$$

and so $\sigma(N \cap P)$ is contained in the $\phi(S)$-core of $\eta(N_E(C, D))$. Conversely, if x is an element of $\mathrm{Core}_{\phi(S)}(\eta(N_E(C, D)))$ then

$$x \in \bigcap_{i=1}^{l} \big(\eta\big(N_E(C, D)\big)\big)^{\phi(x_i)}.$$

For each $i = 1, \dots, l$ choose $y_i \in N_E(C, D)$ with $\eta(y_i) = x^{\phi(x_i^{-1})}$, and set

$$y = y_1^{x_1} \cdots y_l^{x_l} \in N.$$

We claim that $y \in P$ and that $\sigma(y) = x$. Certainly y normalises X. To show that y also normalises $X(N \cap \phi^{-1}(\mathrm{Inn}\, F))$ we must show that for all u in the intersection $N \cap \phi^{-1}(\mathrm{Inn}\, F)$ there exists $v \in N \cap \phi^{-1}(\mathrm{Inn}\, F)$ with $u^y v \in X$, or equivalently, given that $X = \prod_{i=1}^{l} C^{x_i}$, we must show that for all $u \in N \cap \phi^{-1}(\mathrm{Inn}\, F)$ there exists $v \in N \cap \phi^{-1}(\mathrm{Inn}\, F)$ with

$$\kappa\Big((u^y v)^{x_i^{-1}}\Big) \in C \qquad \text{for all } i = 1, \dots, l.$$

So suppose $u \in N \cap \phi^{-1}(\mathrm{Inn}\, F)$. As $\phi(N \cap S) \geq \mathrm{Inn}\, F$ there exists $v \in N \cap \phi^{-1}(\mathrm{Inn}\, F)$ with $\phi(v^{-1}) = \phi(u)^x$. Since $N = \prod_{i=1}^{l} E^{x_i}$ and since $y = y_1^{x_1} \cdots y_l^{x_l}$ with each $y_i \in E$ we have for each $i = 1, \dots, l$

$$\kappa\Big((u^y)^{x_i^{-1}}\Big) = \kappa\big(u^{x_i^{-1}}\big)^{y_i},$$

whence

$$\kappa\Big((u^y v)^{x_i^{-1}}\Big) = \kappa\big(u^{x_i^{-1}}\big)^{y_i} \kappa\big(v^{x_i^{-1}}\big).$$

Recalling that $y_i \in N_E(C, D)$ and that $x_i \in S$ we see that both $\kappa(u^{x_i^{-1}})^{y_i}$ and $\kappa(v^{x_i^{-1}})$ are in D. Thus the right hand side above is in C if and only if its image under η is trivial. However,

$$\begin{aligned}
&\eta\Big(\kappa(u^{x_i^{-1}})^{y_i}\kappa(v^{x_i^{-1}})\Big)\\
&\quad= \Big(\eta\big(\kappa(u^{x_i^{-1}})\big)\Big)^{\eta(y_i)}\eta\big(\kappa(v^{x_i^{-1}})\big)\\
&\quad= \Big(\eta\big(\kappa(u^{x_i^{-1}})\big)\Big)^{\phi(x_i)x\phi(x_i^{-1})}\eta\big(\kappa(v^{x_i^{-1}})\big), \qquad \text{by the choice of } y_i,\\
&\quad= \big(\phi(u^{x_i^{-1}})\big)^{\phi(x_i)x\phi(x_i^{-1})}\phi(v^{x_i^{-1}}), \qquad \text{by (3.A)}\\
&\quad= \big(\phi(u)^x\phi(v)\big)^{\phi(x_i^{-1})}
\end{aligned}$$

which is trivial by the choice of v. Hence we have shown that $y \in P$. Moreover if $x_i \in N_S(E)$ (and this holds for precisely one $i = 1, \dots, l$) so that $\kappa(y) = \kappa(y_i^{x_i}) = y_i^{\kappa(x_i)}$, then by (3.A)

$$\sigma(y) = \eta(\kappa(y)) = \eta\big(y_i^{\kappa(x_i)}\big) = \eta(y_i)^{\eta(\kappa(x_i))} = \eta(y_i)^{\phi(x_i)} = x$$

and the claim holds. The reverse containment

$$\mathrm{Core}_{\phi(S)}\big(\eta\big(N_E(C,D)\big)\big) \le \sigma(N \cap P)$$

follows, whence (ii) holds.

We turn to (iii). By (i) we have $\kappa(N \cap P) \ge E \cap \ker \eta$ and so given that $\sigma(N \cap P) = \eta(\kappa(N \cap P))$ we have

$$\kappa(N \cap P) = E \cap \eta^{-1}(\sigma(N \cap P)).$$

Part (iii) now follows from (ii).

Finally, as $P \ge S$, whence P is also transitive on the minimal normal subgroups of N, every proper normal subgroup of N normalised by P is trivial. Part (iv) follows by assuming part (i) and by applying Lemma 3.16 to the section

$$(U, V) = \big(N \cap \ker \sigma, N \cap \sigma^{-1}(\mathrm{Inn}\, F)\big)$$

of N.

PROPOSITION 3.20. *Let (E, N, T, F, S, ϕ) be a* (3.18)*-tuple and let κ, C, D, and η be as defined above. Then ϕ has no strict extensions in NS if and only if the following all hold*:

(i) *(C, D) is a $\kappa(N_S(E))$-maximal section of E*;

(ii) *$S \cap E \ge C$*;

(iii) *$\mathrm{Core}_{\phi(S)}(\eta(N_E(C, D))) = \phi(S \cap N)$.*

Remark 3.21. We note that in the instance where the group C is trivial then condition (ii) of the lemma is trivially satisfied.

Proof of Proposition 3.20. It is easy to see that if (E, N, T, F, S, ϕ) is a (3.18)-tuple, then so is (E, N, NS, F, S, ϕ). Thus it is enough to prove the lemma under the additional assumption that $T = NS$.

Let X, P, and σ be as above. Note that the conclusions of Lemma 3.19 all hold.

Suppose that ϕ has no strict extensions in T. Then clearly $P = S$ and $\sigma = \phi$, whence (iii) follows from Lemma 3.19(ii). Also by Corollary 3.15

$$\left(N \cap \ker \phi, N \cap \phi^{-1}(\operatorname{Inn} F)\right)$$

is an S-maximal section of N, and (i) and (ii) follow from Lemma 3.16(i) and (iii), respectively.

Conversely, suppose that (i)–(iii) all hold. Condition (ii) implies that X (as defined above) is contained in S. As also $N \cap \ker \phi \le X \le N \cap \ker \sigma$ we have

$$N \cap \ker \phi \le X \le S \cap N \cap \ker \sigma = N \cap \ker \phi$$

and equality must hold. Now $(E \cap \ker \eta, (E \cap \ker \eta)D)$ is clearly a section of E that $\kappa(N_S(E))$-contains (C, D), whence (i) implies that $C = E \cap \ker \eta$. From Lemma 3.19(i) we deduce that

$$N \cap \ker \sigma = X = N \cap \ker \phi.$$

This coupled with Lemma 3.19(ii) and condition (iii) implies that $N \cap S = N \cap P$. Now by assumption $T = NS$ and so

$$P = P \cap (NS) = (N \cap P)S = S$$

which in turn implies that $\sigma = \phi$. Having already seen that $C = E \cap \ker \eta$, condition (i) together with Lemma 3.19(iv) shows that

$$\left(N \cap \ker \sigma, N \cap \sigma^{-1}(\operatorname{Inn} F)\right)$$

is a P-maximal section of N. Recall that P is defined as the normaliser in T of this section. We finish by applying Corollary 3.15 to deduce that σ, which we know is equal to ϕ, has no strict extensions in T as required.

COROLLARY 3.22. *Let* (E, N, T, F, S, ϕ) *be a* (3.18)-*tuple such that* ϕ *has no strict extensions in* NS. *Let* κ, C, D, *and* η *be as defined above, and* *let* $x_1, \ldots, x_l$ *be a right transversal for* $N_S(E)$ *in* S. *Then* $C = E \cap \ker \eta$,

$$N \cap \ker \phi = \prod_{i=1}^{l} C^{x_i} \tag{3.B}$$

and

$$Y \cap S = Y \cap \ker \phi = \prod_{\{i \,:\, E^{x_i} \leq Y\}} C^{x_i}, \tag{3.C}$$

where Y *is any proper normal subgroup of* N.

Proof. As in the proof of Proposition 3.20 we may immediately reduce to the case where $T = NS$. From Proposition 3.20(i) we see that (C, D) is a $\kappa(N_S(E))$-maximal section of E whence we certainly have $E \cap \ker \eta = C$. Let P and σ be as in the statement of Lemma 3.19. As ϕ has no strict extensions in T we have $\sigma = \phi$ whence (3.B) follows from Lemma 3.19(i).

Suppose that Y is a proper normal subgroup of N. As $T = N_T(E)S$ there exists $x \in S$ with $E \not\leq Y^x$, or equivalently, with $Y^x \leq C_N(E) \leq \ker \kappa$. Now by (3.A)

$$\phi(Y^x \cap S) = \eta(\kappa(Y^x \cap S)).$$

As $\kappa(Y^x)$ is trivial we deduce that $Y^x \cap S \leq \ker \phi$. Also as $\phi(Y \cap S)$ is $\phi(S)$-conjugate to $\phi(Y^x \cap S)$ (namely by $\phi(x)$), we see that $Y \cap S \leq \ker \phi$, whence $Y \cap S = Y \cap \ker \phi$ as required. The remainder of (3.C) follows from (3.B) via an easy calculation.

3.23. *Subgroup Lattices of Non-Abelian Characteristically Simple Groups*. We shall need two results on subgroup lattices of characteristically simple groups.

LEMMA 3.24. *Let* N *be a non-abelian characteristically simple group, and let* H *be a soluble subgroup of* Aut N. *Then*

$$\left[N/\{\mathrm{id}_N\}\right]_H \not\cong \mathsf{M}_m$$

for any $m \geq 0$.

Proof. We must show that there exist strictly comparable H-invariant non-trivial proper subgroups of N. We proceed by induction on $|N|$. Let Q be a minimal normal subgroup of H. Note that Q is an elementary abelian q-group for some prime q.

Suppose that $C_N(Q)$ is trivial. An easy argument (cf. [7, 6.2.3]) shows that N is a q'-group. A generalization of a conjecture due to Frobenius and proved by Wang using CFSG [22] forces N to be soluble—a contradiction. (Alternatively, we can avoid this use of CFSG as follows: the last part of the proof of the Aschbacher–O'Nan–Scott theorem as given in [14] shows that Q normalises a unique Sylow p-subgroup P of N where p is any prime dividing $|N|$, and by Burnside's normal p-complement theorem [7, 7.4.3] either $Z(P)$ and P, or P and $N_N(P)$ are strictly comparable H-invariant non-trivial proper subgroups of N.)

So we may assume that $C_N(Q)$ is non-trivial. If $C_N(Q)$ is not characteristically simple then the result clearly follows. If $C_N(Q)$ is non-abelian and characteristically simple, then we are done by induction. If $C_N(Q)$ is abelian and characteristically simple, then by Burnside's normal p-complement theorem [7, 7.4.3], $C_N(Q)$ and $N_N(C_N(Q))$ are strictly comparable and are certainly H-invariant non-trivial proper subgroups of N.

COROLLARY 3.25. *Let F be a non-abelian simple group, and let H be a subgroup of* Aut F. *Suppose that*

$$\left[F/\{\mathrm{id}_F\}\right]_H \cong \mathsf{M}_m$$

for some $m \geq 0$. Then one of the following holds:

(i) $m = 0$ *and* $H \geq \operatorname{Inn} F$;

(ii) $m = 1$ *and* $H \cap \operatorname{Inn} F$ *is a non-trivial non-abelian characteristically simple proper subgroup of* $\operatorname{Inn} F$.

Proof. The previous lemma shows that H is insoluble, and so by the "Schreier conjecture" $H \cap \operatorname{Inn} F$ is an insoluble, and so non-abelian, non-trivial subgroup of $\operatorname{Inn} F$. Clearly $m = 0$ if $H \cap \operatorname{Inn} F = \operatorname{Inn} F$, in which case (i) holds, and so we may assume that $H \cap \operatorname{Inn} F$ is a proper subgroup of $\operatorname{Inn} F$, whence $m \geq 1$. We must show that (ii) holds.

Suppose that $m > 1$. Identify F with $\operatorname{Inn} F$ in the usual way and let L be a non-trivial H-invariant subgroup of F distinct from $H \cap F$. Then $\langle H \cap F, L\rangle$ is an H-invariant subgroup of F strictly containing both $H \cap F$ and L and so equals F. However, L is normal in $\langle H \cap F, L\rangle$ which contradicts the simplicity of F. Thus $m = 1$. Finally observe that $H \cap F$ is characteristically simple as if not then it must strictly contain a non-trivial H-invariant subgroup.

4. THE NOT-T-COMPLEMENT CASE

In this section we are concerned with the determination of $\Omega(2.\mathrm{D})$. For convenience, we say that the pair (G, T) *is a* (2.D)-*pair of rank* n, if the following both hold:

(A) T is a subgroup of G such that

$$[G/T] \cong \mathsf{M}_n;$$

(B) the socle $\operatorname{Soc} G$ of G is a minimal normal subgroup of G, is non-abelian, and satisfies

$$(\operatorname{Soc} G) \cap T = \{\mathrm{id}\} \qquad \text{and} \qquad (\operatorname{Soc} G)T \neq G.$$

Thus

$$\Omega(2.\mathrm{D}) = \{n \geq 16\colon \text{there exists a } (2.\mathrm{D})\text{-pair of rank } n\}.$$

Our first step in the study of $\Omega(2.\mathrm{D})$ is to translate the problem into the language of twisted wreath products.

DEFINITION 4.1. We say that the tuple (H, T, F, Q, ϕ) satisfies (4.1), or is a (4.1)-*tuple*, if the following all hold:

(i) T is a maximal subgroup of the group H and T is core-free in H;

(ii) F is a non-abelian simple group;

(iii) Q is a core-free subgroup of H satisfying $H = QT$;

(iv) ϕ is a homomorphism $Q \to \operatorname{Aut} F$ such that $\phi(Q \cap T) \geq \operatorname{Inn} F$ and such that the restriction $\phi|_{Q \cap T}\colon Q \cap T \to \operatorname{Aut} F$ has no strict extensions in T.

Moreover, we say that the tuple (H, T, F, Q, ϕ) is a (4.1)-*tuple of rank n*, if it satisfies (4.1) and there exist precisely $n - 1$ homomorphisms $\phi_1, \dots, \phi_{n-1}\colon Q \to \operatorname{Aut} F$ (one of which is ϕ) such that for each $i = 1, \dots, n-1$

$$\phi_i|_{Q \cap T} = \phi|_{Q \cap T} \qquad \text{and} \qquad \tilde{\phi}_i = \tilde{\phi}, \tag{4.A}$$

where $\tilde{\phi}_i$, $\tilde{\phi}$ are the homomorphisms $Q \to \operatorname{Out} F$ obtained by composing ϕ_i, ϕ, respectively, with the natural quotient map $\operatorname{Aut} F \to \operatorname{Out} F$.

The subset $\Omega(4.1)$ of $\mathbb{N}$ is defined by

$$\Omega(4.1) = \{n \geq 16\colon \text{there exists a } (4.1)\text{-tuple of rank } n\}.$$

The significance of (4.1)-tuples is clear from the following result.

THEOREM 4.2. $\Omega(2.\mathrm{D}) = \Omega(4.1)$.

The proof of the theorem will be constructive; given a (4.1)-tuple of rank n, we will construct a (2.D)-pair of rank n, and conversely, given a (2.D)-pair of rank $n \geq 3$, we will construct a (4.1)-tuple of rank n. Before doing so we need to fix some notation concerning (2.D)-pairs and to give some preparatory lemmas.

Suppose that (G, T) is a (2.D)-pair of rank n so that there are precisely n proper overgroups of T in G each of which is maximal in G. Let

$M = \operatorname{Soc} G$. Set $H_n = MT$ which is then the unique maximal subgroup of G containing both T and M; let the other $n - 1$ distinct maximal subgroups of G containing T be $H_1, \ldots, H_{n-1}$. Let F be a fixed minimal normal subgroup of M; observe that F is a non-abelian simple group since M is non-abelian and is a minimal normal subgroup of G.

LEMMA 4.3. *Suppose that* (G, T) *is a* (2.D)*-pair of rank* n; *let* M, $H_1, \ldots, H_n$ *be as above. Then the following all hold*:

(i) $G = MH_i$ *for all* $i = 1, \ldots, n - 1$;

(ii) $M \cap H_i = \{\mathrm{id}\}$ *for all* $i = 1, \ldots, n - 1$;

(iii) $\operatorname{Core}_G H_i = \{id\}$ *for all* $i = 1, \ldots, n - 1$;

(iv) *if* $n \geq 3$, *then* $\operatorname{Core}_{H_i} T = \{\mathrm{id}\}$ *for all* $i = 1, \ldots, n - 1$;

(v) *M is the unique minimal normal subgroup of* H_n.

Proof. Suppose that $1 \leq i \leq n - 1$. Now MH_i is a subgroup of G containing M, H_i, and T. As H_n is the unique maximal subgroup of G containing both M and T, and as $H_n \neq H_i$, we have that $G = MH_i$ and so (i) holds. To see (ii) observe that $M \cap H_i$ is a subgroup of M normalized by T. As T is maximal in $H_n = MT$ and as $M \cap T$ is trivial, we deduce that the only non-trivial subgroup of M that is normalised by T is M itself. If $M \cap H_i = M$ then $H_i \geq MH_i = G$, a contradiction. Hence $M \cap H_i$ is trivial as required. Part (iii) now follows since M is the unique minimal normal subgroup of G.

Suppose now that $n \geq 3$. Let α be the natural quotient map $G \to G/M$. Parts (i) and (ii) of the lemma imply that the homomorphism $\alpha|_{H_i} : H_i \to G/M$ obtained by restricting α is an isomorphism. Let $\psi : H_1 \to H_2$ be the isomorphism obtained by composing $\alpha|_{H_1} : H_1 \to G/M$ with the inverse of $\alpha|_{H_2} : H_2 \to G/M$. Note that $\psi|_T$ is the identity map on T. Thus if N is a non-trivial normal subgroup of H_1 contained in T, then $\psi(N) = N$ is a normal subgroup of H_2 contained in T; it follows that N is a normal subgroup of $\langle H_1, H_2 \rangle$. The maximality of H_1 and H_2 in G implies that $G = \langle H_1, H_2 \rangle$. Hence N is a normal subgroup of G. But this is a contradiction as M is the unique minimal normal subgroup of G and $M \nleq T$. So we have shown that T is core-free in H_1. Likewise T is core-free in H_i for all $i = 1, \ldots, n - 1$ and (iv) holds.

Finally we verify (v). As noted above, M is the only non-trivial subgroup of M that is normalised by T, and so is certainly a minimal normal subgroup of $H_n = MT$. By definition M is the unique minimal normal subgroup of G and is non-abelian, whence $C_G(M)$ is trivial. Hence $C_{H_n}(M)$ is also trivial, and the uniqueness of M follows.

LEMMA 4.4. *Let H be a group with subgroups Q, T satisfying $H = QT$; let F be any group with a trivial centre, and let ϕ be a homomorphism*

$Q \to \operatorname{Aut} F$; *let* X *be the twisted wreath product* $F \operatorname{twr}_\phi H$. *Then the complements in* X *to the base group* B_ϕ *of* X *that contain* T *are in one-to-one correspondence with the homomorphisms* $\xi : Q \to \operatorname{Aut} F$ *satisfying*

$$\xi|_{Q \cap T} = \phi|_{Q \cap T} \qquad \text{and} \qquad \tilde{\xi} = \tilde{\phi}, \tag{4.B}$$

where $\tilde{\xi}$, $\tilde{\phi}$ *are the maps* $Q \to \operatorname{Out} F$ *obtained by composing* ξ, ϕ, *respectively, with the natural quotient map* $\operatorname{Aut} F \to \operatorname{Out} F$.

Remark 4.5. The proof of Lemma 4.4 could be executed in the following fashion: firstly, prove the special case in which $Q = H$, and, secondly, deduce the result from this by using the results of [8]. We choose instead to give a direct proof involving the construction of an explicit bijection between the two sets involved as we feel this to be more illuminating. However, we must stress that this is precisely the type of result that depends on the ideas investigated in [8].

Proof of Lemma 4.4. As mentioned above, the proof is constructive: we give an explicit bijection between the two sets. For convenience, let $\mathcal{S}_{\text{comp}}$ be the set of complements in X to B_ϕ that contain T, and let $\mathcal{S}_{\text{map}}$ be the set of homomorphisms $\xi: Q \to \operatorname{Aut} F$ satisfying (4.B).

We now set up some notation that will remain in force for the rest of this proof. Let

$$F_Q = \left\{ f \in B_\phi : f(x) = \mathrm{id}_F \text{ for all } x \notin Q \right\};$$

observe that F_Q is minimal among the normal subgroups of B_ϕ that are normalised by Q and that are isomorphic to F (cf. Lemma 2.3 of [3]). Identify F_Q with F via the map $f \mapsto f(\mathrm{id}_H)$ for all $f \in F_Q$, and let $\chi : N_X(F_Q) \to \operatorname{Aut} F$ be the map induced by conjugation. Recall that X can be written as the semi-direct product of B_ϕ by H; let $\alpha : X \to H$ be the map obtained by quotienting out B_ϕ so that $\alpha|_H$ is the identity map on H. Let $\beta : \operatorname{Aut} F \to \operatorname{Out} F$ be the natural quotient map.

We claim that the restriction $\chi|_Q$ of χ to Q, which is indeed a subgroup of $N_X(F_Q)$, is equal to ϕ. To see this note that the definition of χ ensures that for $x \in F$ and $q \in Q$

$$x^{\chi(q)} = f^q(\mathrm{id}_H),$$

where $f \in F_Q$ is such that $f(\mathrm{id}_H) = x$. However,

$$f^q(\mathrm{id}_H) = f(q) = f(\mathrm{id}_H)^{\phi(q)}$$

and the claim holds.

Suppose that $L \in \mathsf{S}_{\mathrm{comp}}$. Observe that $\alpha|_L$ is an isomorphism $L \to H$ that restricts to give the identity map on T and to give an isomorphism between $N_L(F_Q)$ and $N_H(F_Q)$. (The latter holds since $B_\phi \leq N_X(F_Q)$ and $X = B_\phi L = B_\phi H$ imply that

$$N_X(F_Q) = B_\phi N_L(F_Q) = B_\phi N_H(F_Q).)$$

Observe also that $Q = N_H(F_Q)$. We define $\xi_L : Q \to \operatorname{Aut} F$ to be the composition of $(\alpha|_{N_L(F_Q)})^{-1}$ with $\chi|_{N_L(F_Q)}$. We claim that $\xi_L \in \mathsf{S}_{\mathrm{map}}$. We consider first the condition $\xi_L|_{Q \cap T} = \phi|_{Q \cap T}$. Let $t \in Q \cap T$; then $\alpha(t) = t$, whence

$$\xi_L(t) = \chi(t) = \phi(t)$$

since $\chi|_Q = \phi$ as verified in the previous paragraph. We turn to the condition $\tilde{\xi}_L = \tilde{\phi}$. Since $\chi(B_\phi) = \operatorname{Inn} F = \ker \beta$ and since $\chi|_Q = \phi$ it is straightforward to see that the diagram

$$\begin{tikzcd} N_X(F_Q) \arrow[r, "\alpha|_{N_X(F_Q)}"] \arrow[d, "\chi"] & Q \arrow[d, "\tilde{\phi}"] \\ \operatorname{Aut} F \arrow[r, "\beta"] & \operatorname{Out} F \end{tikzcd}$$

commutes. Hence so does

$$\begin{tikzcd} N_L(F_Q) \arrow[r, "\alpha|_{N_L(F_Q)}"] \arrow[d, "\chi|_{N_L(F_Q)}"] & Q \arrow[d, "\tilde{\phi}"] \\ \operatorname{Aut} F \arrow[r, "\beta"] & \operatorname{Out} F \end{tikzcd}$$

Recalling that $\alpha|_{N_L(F_Q)}$ is an isomorphism $N_L(F_Q) \to Q$ we deduce that $\tilde{\phi}$ is equal to the composition $\beta \circ (\chi|_{N_L(F_Q)}) \circ (\alpha|_{N_L(F_Q)})^{-1}$. This composition is precisely $\tilde{\xi}_L$ and so $\tilde{\phi} = \tilde{\xi}_L$ as required.

Thus the map $\gamma : \mathsf{S}_{\mathrm{comp}} \to \mathsf{S}_{\mathrm{map}}$ given by

$$L \mapsto \xi_L \qquad \text{for all } L \in \mathsf{S}_{\mathrm{comp}}$$

is well-defined. We prove that γ is a bijection by exhibiting an inverse.

Suppose that $\xi \in \mathsf{S}_{\mathrm{map}}$. Given $h \in H$ we define $f_h \in B_\phi$ by requiring that for all $t \in T$

$$f_h(t) = \theta^{-1}\Big(\xi(t^{-1}hx)\phi(t^{-1}hx)^{-1} \Big), \tag{4.C}$$

where $\theta : F \to \operatorname{Inn} F$ is the natural map taking $y \in F$ to "conjugation by y" (which is an isomorphism as F has a trivial centre by assumption) and where x is any element of T such that $t^{-1}hx \in Q$. We must verify that f_h is a well-defined element of B_ϕ. We start by showing that the right hand side of (4.C) is an element of F well-defined in terms of t. Fix $t \in T$. Firstly, note that, as $H = QT$, elements $x \in T$ such that $t^{-1}hx \in Q$ do indeed exist. If $x, y \in T$ are such that both $t^{-1}hx$ and $t^{-1}hy$ are elements of Q, then $y = xq$ for some $q \in Q \cap T$, whence

$$\begin{aligned}\xi(t^{-1}hy)\phi(t^{-1}hy)^{-1} &= \xi(t^{-1}hxq)\phi(t^{-1}hxq)^{-1}\\ &= \xi(t^{-1}hx)\xi(q)\phi(q)^{-1}\phi(t^{-1}hx)^{-1}\end{aligned}$$

since both ξ and ϕ are homomorphisms $Q \to \operatorname{Aut} F$. Now $\xi|_{Q\cap T} = \phi|_{Q\cap T}$, and it follows that $\xi(t^{-1}hx)\phi(t^{-1}hx)^{-1}$ is a uniquely determined element of $\operatorname{Aut} F$. Moreover, the condition $\tilde{\xi} = \tilde{\phi}$ implies that $\xi(t^{-1}hx)\phi(t^{-1}hx)^{-1}$ lies in $\operatorname{Inn} F$ whence $\theta^{-1}(\xi(t^{-1}hx)\phi(t^{-1}hx)^{-1})$ is a well-defined element of F. Thus there exists some map $f_h : H \to F$ satisfying (4.C). It remains to show that f_h can be chosen so that $f_h \in B_\phi$, and moreover, that there is a unique such choice. If $t = sq$ for some $s \in T$ and $q \in Q$, then $q \in Q \cap T$ and

$$f_h(s)^{\phi(q)} = \theta^{-1}\left(\xi(s^{-1}hy)\phi(s^{-1}hy)^{-1}\right)^{\phi(q)},$$

where $y \in T$ is chosen so that $s^{-1}hy \in Q$. Here we can assume that $y = x$ where x is chosen so that $t^{-1}hx \in Q$ since $s^{-1}hx = qt^{-1}hx$. Thus

$$\begin{aligned}f_h(s)^{\phi(q)} &= \theta^{-1}\left(\xi(s^{-1}hx)\phi(s^{-1}hx)^{-1}\right)^{\phi(q)}\\ &= \theta^{-1}\left(\xi(q)\xi(t^{-1}hx)\phi(t^{-1}hx)^{-1}\phi(q)^{-1}\right)^{\phi(q)}\\ &= f_h(t)\end{aligned}$$

since $\xi(q) = \phi(q)$ and since conjugation by $\phi(q)$ followed by θ^{-1} is the same as θ^{-1} followed by the action of $\phi(q)$. It follows that there does indeed exist an element $f_h \in B_\phi$ satisfying (4.C); there is a unique such element since if $g \in H$, then we can write $g = sq$ for some $s \in T$ and $q \in Q$ whence

$$f_h(g) = f_h(s)^{\phi(q)},$$

and the element $f_h \in B_\phi$ is determined by its values on elements of T.

We define the subset $L_\xi \subseteq X$ by

$$L_\xi = \{ f_h h : h \in H\}.$$

We claim that $L_\xi \in \mathcal{S}_{\text{comp}}$ and, moreover, that the map $\xi \mapsto L_\xi$ is the inverse of γ.

We show firstly that L_ξ is a subgroup of X. To see this we must show that

$$f_h (f_g)^{h^{-1}} = f_{hg} \qquad \text{for all } h, g \in H.$$

Suppose $t \in T$ and that $x, y \in T$ are chosen so that both $t^{-1}hx$ and $t^{-1}hgy$ lie in Q. Then $(t^{-1}hx)^{-1}(t^{-1}hgy) = x^{-1}gy \in Q$ and

$$\begin{aligned}
(f_h(t))^{-1} f_{hg}(t) &= \theta^{-1}\Big(\phi(t^{-1}hx)\,\xi(t^{-1}hx)^{-1}\,\xi(t^{-1}hgy)\,\phi(t^{-1}hgy)^{-1}\Big) \\
&= \theta^{-1}\Big(\phi(t^{-1}hx)\,\xi(x^{-1}gy)\,\phi(x^{-1}gy)^{-1}\,\phi(t^{-1}hx)^{-1}\Big) \\
&= \theta^{-1}\Big(\xi(x^{-1}gy)\,\phi(x^{-1}gy)^{-1}\Big)^{\phi(t^{-1}hx)^{-1}} \\
&= f_g(x)^{\phi(x^{-1}h^{-1}t)} \\
&= f_g(h^{-1}t) \\
&= (f_g)^{h^{-1}}(t)
\end{aligned}$$

as required. Now L_ξ is clearly a complement to B_ϕ in X since $|L_\xi| = |H|$ and since $L_\xi \cap B_\phi$ is trivial. To see that $T \le L_\xi$ we take $h \in T$ and show that $f_h = \mathrm{id}_{B_\phi}$, that is, $f_h(t) = \mathrm{id}_F$ for all $t \in T$. To suppose that $t \in T$; then $x = h^{-1}t \in T$ satisfies $t^{-1}hx = \mathrm{id}_Q \in Q$ and so

$$f_h(t) = \theta^{-1}\Big(\xi(\mathrm{id}_Q)\,\phi(\mathrm{id}_Q)^{-1}\Big) = \mathrm{id}_F.$$

Hence $L_\xi \in \mathcal{S}_{\text{comp}}$.

We now consider $\gamma(L_\xi)$: we must show that this equals ξ. As $B_\phi \le N_X(F_Q)$ it is clear that

$$N_{L_\xi}(F_Q) = \{ f_q q : q \in Q\}.$$

Choose $f \in F_Q$; then

$$\begin{aligned}
f(\mathrm{id}_H)^{\chi(f_q q)} = f^{f_q q}(\mathrm{id}_H) &= \big(f_q^{-1} f f_q\big)^q (\mathrm{id}_H) \\
&= \big(f_q^{-1} f f_q\big)(q) \\
&= \Big(\big(f_q^{-1} f f_q\big)(\mathrm{id}_H)\Big)^{\phi(q)} \\
&= f(\mathrm{id}_H)^{\theta(f_q(\mathrm{id}_H))\phi(q)} = f(\mathrm{id}_H)^{\xi(q)}
\end{aligned}$$

since (4.C) with $t = x = \mathrm{id}_T$ and $h = q \in Q$ shows that $\theta(f_q(\mathrm{id}_H)) = \xi(q)\phi(q)^{-1}$. Thus $\chi(f_q q) = \xi(q)$ and it follows that $\gamma(L_\xi) = \xi$.

Finally we suppose that $\xi = \gamma(L)$ for some $L \in \mathcal{S}_{\mathrm{comp}}$. We must show that $L_\xi = L$. Since $T \le L_\xi$, $T \le L$ and $H = TQ$, which implies that $X = N_X(F_Q)T$, we have that $L_\xi = TN_{L_\xi}(F_Q)$ and $L = TN_L(F_Q)$. Given that L and L_ξ have the same order, it is sufficient to show that $N_L(F_Q) \le N_{L_\xi}(F_Q)$. So suppose that $l \in N_L(F_Q)$; clearly we can write $l = fq$ for some $f \in B_\phi$ and $q \in Q$. We will show that $f = f_q$ where f_q is defined with respect to $\xi = \gamma(L)$. Let $x, t \in T$ be such that $t^{-1}qx \in Q$. Note that $t^{-1}lx = f^t t^{-1}qx \in N_L(F_Q)$ as $T \le L$ and as B_ϕ normalises F_Q. Choose $g \in F_Q$ and consider $g(\mathrm{id}_H)^{\xi(t^{-1}qx)}$. By the definition of ξ we have

$$
\begin{aligned}
g(\mathrm{id}_H)^{\xi(t^{-1}qx)} &= g(\mathrm{id}_H)^{\chi(t^{-1}lx)} = g^{t^{-1}lx}(\mathrm{id}_H) \\
&= \left((f^t)^{-1} g f^t\right)^{t^{-1}qx}(\mathrm{id}_H) \\
&= \left(f(t)^{-1} g(\mathrm{id}_H) f(t)\right)^{\phi(t^{-1}qx)}
\end{aligned}
$$

whence $\theta(f(t)) = \xi(t^{-1}qx)\phi(t^{-1}qx)^{-1}$. As this holds for any $t \in T$ we have $f = f_q$, and so $l = f_q q \in L_\xi$.

We have now finished the proof of Lemma 4.4.

Proof of Theorem 4.2. As already observed, it is enough to show that given a (2.D)-pair of rank $n \ge 3$ there exists a (4.1)-tuple of the same rank, and conversely, that given a (4.1)-tuple of rank n there exists a (2.D)-pair of the same rank.

Suppose that (G, T) is a (2.D)-pair of rank $n \ge 3$. Let $M, H_1, \ldots, H_n$, and F be as defined immediately prior to Lemma 4.3. For each $i = 1, \ldots, n-1$ let $Q_i = N_{H_i}(F)$ and let $\phi_i : Q_i \to \operatorname{Aut} F$ be the homomorphism induced by the conjugation action of Q_i on F. Choose and fix an integer i such that $1 \le i \le n-1$; we claim that (H_i, T, F, Q_i, ϕ_i) is a (4.1)-tuple. To see this we must verify that conditions (i)–(iv) of Definition 4.1 all hold.

Certainly (ii) holds. Now T is a maximal subgroup of H_i by the definition of (2.D)-pairs, and T is core-free in H_i by Lemma 4.3(iv); hence (i) holds.

Lemma 4.3(v) implies that T is transitive by conjugation on the simple direct factors of M; as these are precisely the G-conjugates of F we deduce that $H_i = TQ_i$. Lemma 4.3(iii), together with the maximality of H_i in G, implies that G, in its action on the coset space $(G : H_i)$ by right multiplication, is a primitive permutation group. Furthermore, Lemma 4.3(i)–(ii) imply that M is a regular subgroup of G. (A permutation group is regular if and only if it is transitive and the stabilizer of any point is trivial.) Thus the primitive permutation group G has a non-abelian and regular socle; such primitive permutation groups are commonly referred to as primitive permutation groups of twisted wreath type. The results of [14] (see also [3, 8.4]) imply that Q_i, as the normaliser in a point-stabilizer of a simple direct factor of the socle, is core-free in H_i. (This deduction uses the "Schreirer conjecture" and hence depends on CFSG.) We have verified that (iii) holds.

We finally consider (iv). Lemma 3.5 shows that there exists an isomorphism $H_n \to F \operatorname{twr}_\sigma T$ that maps $T \leq H_n$ to the top group T of the twisted wreath product where σ is the restriction of ϕ_i to $Q_i \cap T = N_T(F)$. As T is maximal in H_n it follows that T is maximal in $\mathrm{F} \operatorname{twr}_\sigma T$ and by applying Corollary 3.7 we deduce that (iv) holds.

We have shown that (H_i, T, F, Q_i, ϕ_i) is a (4.1)-tuple; we claim moreover that it is a (4.1)-tuple of rank n. To see this, we once again use Lemma 3.5, this time to deduce that there exists an isomorphism $G \to F \operatorname{twr}_{\phi_i} H_i$ that maps M to the base group and H_i to the top group of the twisted wreath product. It follows from Lemma 4.3 that there are precisely $n - 1$ complements in $F \operatorname{twr}_{\phi_i} H_i$ to the base group that contain T, namely, the images of $H_1, \dots, H_{n-1}$ under this isomorphism. We can now apply Lemma 4.4 to find that the rank of the tuple is indeed n.

Conversely, suppose that (H, T, F, Q, ϕ) is a (4.1)-tuple of rank n. Let X be the twisted wreath product $F \operatorname{twr}_\phi H$. We claim that (X, T) is a (2.D)-pair of rank n. To verify this we must show that conditions (A) and (B) given at the start of this section both hold.

Certainly T is a subgroup of X. We consider (B). Let B_ϕ be the base group of the twisted wreath product $X = F \operatorname{twr}_\phi H$. By definition Q is core-free in H, whence so is $\phi^{-1}(\operatorname{Inn} F)$. Lemma 3.9, together with the remarks made before its statement, implies that B_ϕ is the unique minimal normal subgroup of X and so equals the socle of X. Now since T is a proper subgroup of H, and since H complements B_ϕ in X, it follows both that $B_\phi \cap T = \{\mathrm{id}\}$ and that $B_\phi T \neq X$. Hence (B) holds.

It remains only to show that $[X/T] \cong M_n$. Let L be any proper overgroup of T in X. We consider $L \cap B_\phi$ which is a subgroup of B_ϕ normalised by T. Set $\sigma = \phi|_{Q \cap T}$ and observe that there exists an isomorphism $B_\phi T \to F \operatorname{twr}_\sigma T$ given by

$$t \mapsto t \qquad \text{for all } t \in T$$

and

$$f \mapsto f|_T \in B_\sigma \qquad \text{for all } f \in B_\phi,$$

where B_σ is the base group of $F \operatorname{twr}_\sigma T$. Applying Corollary 3.7 to condition (iv) of Definition 4.1, we deduce that T is a maximal subgroup of $B_\phi T$, whence the only T-invariant subgroups of B_ϕ are B_ϕ and $\{\mathrm{id}\}$. Hence either $L \cap B_\phi$ is trivial or $L \geq B_\phi$.

If $L \geq B_\phi$ then $B_\phi T \leq L$. Now the proper subgroups of $X = B_\phi H$ containing $B_\phi T$ are in one-to-one correspondence with the proper subgroups of H containing T; hence $L = B_\phi T$ as T is maximal in H.

Suppose now that $L \cap B_\phi$ is trivial. Let α be the natural map $X \to H$ obtained by quotienting out B_ϕ. Observe that $\alpha(L) \cong L$, and so $\alpha(T)$ is a proper subgroup of $\alpha(L)$. However, $\alpha(T) = T$ and is a maximal subgroup of H; we deduce that $\alpha(L) = H$, whence L is a complement in X to B_ϕ that contains T. Lemma 4.4 implies that precisely $n - 1$ such complements exist.

We have shown that any non-trivial overgroup of T in X is either equal to $B_\phi T$, or is a complement to B_ϕ in X containing T, and also that precisely $n - 1$ such complements exist. Hence $[X/B_\phi] \cong M_n$ as required.

Remark 4.6. Let (H, T, F, Q, ϕ) be a (4.1)-tuple of rank n. Then, as shown in the above proof, $(F \operatorname{twr}_\phi H, T)$ is a (2.D)-pair of rank n with $B_\phi = \operatorname{Soc}(F \operatorname{twr}_\phi H)$. Using Lemma 4.3 in an argument analogous to that used in the fourth paragraph of the above proof, we deduce that both $F \operatorname{twr}_\phi H$ and $F \operatorname{twr}_{\phi|Q \cap T} T$ in their actions on respectively the coset spaces $(F \operatorname{twr}_\phi H : H)$ and $(F \operatorname{twr}_{\phi|Q \cap T} T : T)$ are primitive permutation groups of twisted wreath type.

Having reduced the problem of determining $\Omega(2.D)$ to that of determining $\Omega(4.1)$, it is not yet clear that we have made any advance in our attempt to reduce to questions concerning simple groups. We now correct this.

Definition 4.7. Let $\Omega(4.7)$ be the subset of $\mathbb{N}$ given by

$$\Omega(4.7) = \left\{ n \geq 16 : \begin{array}{l} \text{there exists a } (4.1)\text{-tuple} \\ (H, T, F, Q, \phi) \text{ of rank } n \\ \text{with } H \text{ almost simple} \end{array} \right\}.$$

Theorem 4.8. $\Omega(4.1) = \Omega(4.7)$.

This is an immediate consequence of the following proposition.

Proposition 4.9. *Suppose that* (H, T, F, Q, ϕ) *is a* (4.1)-*tuple of rank* $n \geq 3$, *and moreover that* $|H|$ *is minimal among all* (4.1)-*tuple of rank* n. *Then* H *is almost simple*.

Before proving the proposition we fix some notation and give some preparatory results.

Suppose that (H, T, F, Q, ϕ) is a (4.1)-tuple of rank n. Let $\phi_1, \dots, \phi_{n-1}$ be the $n-1$ homomorphisms $Q \to \operatorname{Aut} F$ satisfying (4.A); let $\sigma : Q \cap T \to \operatorname{Aut} F$ be the common restriction of $\phi_1, \dots, \phi_{n-1}$. Set $V = \phi^{-1}(\operatorname{Inn} F)$. The condition $\tilde{\phi}_i = \tilde{\phi}$ implies that $\phi_i^{-1}(\operatorname{Inn} F) = \phi^{-1}(\operatorname{Inn} F) = V$ for all $i = 1, \dots, n-1$.

Lemma 4.10. *Suppose that (H, T, F, Q, ϕ) is a (4.1)-tuple of rank n; let V, σ, and $\phi_1, \dots, \phi_{n-1}$ be as above. For a subgroup K of Q, let $\overline{K}$ denote the image of K under the quotient map*

$$Q \to Q \Big/ \bigcap_{i=1}^{n-1} \ker \phi_i .$$

Then $\overline{V} \cong F^{n-1}$, the images $\overline{\ker \phi_1}, \dots, \overline{\ker \phi_{n-1}}$ are the $n-1$ distinct maximal normal subgroups of $\overline{V}$, and $\overline{V \cap T} \cong F$ complements each $\overline{\ker \phi_i}$ in $\overline{V}$.

Proof. Since $\sigma(Q \cap T) \geq \operatorname{Inn} F$ by assumption, we have $\sigma(V \cap T) \cong F$. Let $1 \leq i \leq n-1$; since $\phi_i|_{Q \cap T} = \sigma$ we have $\phi_i(V) = \operatorname{Inn} F = \phi_i(V \cap T)$, whence $V = (\ker \phi_i)(V \cap T)$ and $V / \ker \phi_i \cong F$. As $\ker \sigma = (\ker \phi_i) \cap T$, it follows that $\overline{V \cap T} \cong F$, that $\overline{V \cap T}$ complements each $\overline{\ker \phi_i}$ in $\overline{V}$, and that $\overline{V} / \overline{\ker \phi_i} \cong F$. The latter implies that, for each $i = 1, \dots, n-1$, $\overline{\ker \phi_i}$ is a maximal normal subgroup of $\overline{V}$ with quotient F; we claim that they are all distinct. Suppose that $1 \leq i \leq j \leq n$ are such that $\overline{\ker \phi_i} = \overline{\ker \phi_j}$. Clearly this forces $\ker \phi_i = \ker \phi_j$. Choose $v \in V$. As $V = (\ker \phi_i)(V \cap T)$, we can write $v = ut$ where $u \in \ker \phi_i = \ker \phi_j$ and $t \in T$. Thus

$$\phi_i(v) = \phi_i(u)\phi_i(t) = \mathrm{id}_{\operatorname{Aut} F}\, \sigma(t) = \phi_j(u)\phi_j(t) = \phi_j(v),$$

whence $\phi_i|_V = \phi_j|_V$. As any automorphism of F is determined uniquely by its action on F, or equivalently by its conjugation action on $\operatorname{Inn} F$, this forces $\phi_i = \phi_j$, whence $i = j$ as required.

We have shown that $\overline{V}$ has $n-1$ distinct maximal normal subgroups that have quotient F and that have a trivial common intersection. This implies that $\overline{V} \cong F^{n-1}$ and we are finished.

Corollary 4.11. *Suppose that (H, T, F, Q, ϕ) is a (4.1)-tuple of rank n; let V be as above. Then the following all hold*:

(i) $\ker \phi_i = \ker \phi_j$ *if and only if* $i = j$;

(ii) *in any chief series of Q there are at least $n-1$ chief factors isomorphic to F*;

(iii) *if K is a normal subgroup of V satisfying $K(\ker \phi_i) = V$ for all $i = 1, \dots, n-1$, then*

$$K\left(\bigcap_{i=1}^{n-1} \ker \phi_i\right) = V.$$

Proof. Parts (i) and (ii) are immediate; (iii) follows by considering the normal subgroups of $F^{n-1} \cong V/(\bigcap_{i=1}^{n-1} \ker \phi_i)$.

LEMMA 4.12. *Suppose that (H, T, F, Q, ϕ) is a* (4.1)*-tuple of rank $n \geq 3$ and that L is a subgroup of H. Let $^{-}$ denote reduction modulo* $\mathrm{Core}_L Q$. *Suppose further that the following all hold*:

(a) $Q \leq L$;

(b) $\mathrm{Core}_L Q \leq \bigcap_{i=1}^{n-1} \ker \phi_i$;

(c) $\overline{L \cap T} <_{\max} L$.

Then $(\overline{L}, \overline{L \cap T}, F, \overline{Q}, \eta)$ is a well-defined (4.1)*-tuple of rank n, where η is the unique map satisfying $\eta(\overline{q}) = \phi(q)$ for all $q \in Q$.*

Remark 4.13. Recall from Remark 4.6 that for each $i = 1, \dots, n-1$ the twisted wreath product $F \operatorname{twr}_{\phi_i} H$ in its action on the coset space $(F \operatorname{twr}_{\phi_i} H : H)$ is a primitive permutation group of twisted wreath type. In the terminology of [3, Sect. 8] a subgroup L of H satisfying (a) and (b) of the above lemma is a ‘‘balanced subgroup’’ of each such twisted wreath product. The results of [3] show that non-trivial balanced subgroups exist if and only if the primitive permutation group $F \operatorname{twr}_{\phi_i} H$ possesses a non-trivial blow-up decomposition (in the sense of [12]). The above lemma works by essentially using the blow-up decomposition to reduce to a smaller primitive permutation group of twisted wreath type, namely $F \operatorname{twr}_{\eta} \overline{L}$.

Proof of Lemma 4.12. We start by showing that the tuple $(\overline{L}, \overline{L \cap T}, F, \overline{Q}, \eta)$ is well-defined. Note that $\mathrm{Core}_L Q$ is a normal subgroup of L and so both $\overline{L}$ and the map $L \to \overline{L}$, $l \mapsto \overline{l}$ are well-defined. Thus given (a) both $\overline{Q}$ and $\overline{L \cap T}$ are well-defined. By (b) we have $\mathrm{Core}_L Q \leq \ker \phi$ and so η is also well-defined.

We now show that $(\overline{L}, \overline{L \cap T}, F, \overline{Q}, \eta)$ is a (4.1)-tuple, i.e., that conditions (i)–(iv) of Definition 4.1 all hold.

Given (c) we need only verify that $\overline{L \cap T}$ is core-free in $\overline{L}$ to show that (i) holds. Suppose not, i.e., that $\mathrm{Core}_{\overline{L}}(\overline{L \cap T})$ is non-trivial. Then there exists a subgroup N of $(\mathrm{Core}_L Q)/(L \cap T)$ that is normalised by L and that strictly contains $\mathrm{Core}_L Q$. In particular, N is normalised by Q. As usual set $V = \phi_i^{-1}(\mathrm{Inn}\, F)$ which is independent of i by (4.A). Now the facts that $H = QT$, that $\phi_i(Q \cap T) \geq \mathrm{Inn}\, F$, and that $\phi_i|_{Q \cap T}$ has no strict extensions in T together imply that $\phi_i(Q) \geq \mathrm{Inn}\, F$ and that ϕ_i has no strict extensions in H. Lemma 3.12, together with the remarks immediately preceding it, implies that either $N \leq \ker \phi_i$ or $\phi_i(N \cap Q) \geq \operatorname{Inn} F$. If the former then certainly $N \leq Q$, whence $N \leq \operatorname{Core}_L Q$ as N is normalised by L. Thus we may assume that $\phi_i(N \cap Q) \geq \operatorname{Inn} F$ whence we have

$$V = (V \cap N)(\ker \phi_i) \qquad \text{for all } i = 1, \dots, n-1.$$

Now $V \leq Q$ and so V also normalises N, whence $V \cap N$ is a normal subgroup of V. Applying Corollary 4.11(iii) we deduce that

$$V = (V \cap N)\left(\bigcap_{i=1}^{n-1} \ker \phi_i\right).$$

Recall that $\operatorname{Core}_L Q \lneqq N \leq (\operatorname{Core}_L Q)(L \cap T)$. Since by (b), $\operatorname{Core}_L Q \leq V$ we see that $V \cap N \leq (\operatorname{Core}_L Q)(V \cap T)$; again using (b) it follows that

$$V = (V \cap T)\left(\bigcap_{i=1}^{n-1} \ker \phi_i\right).$$

But this gives a contradiction for $n \geq 3$ as by Lemma 4.10

$V \big/ \bigcap_{i=1}^{n-1} \ker \phi_i \cong F^{n-1}$ while

$$(V \cap T) \bigcap_{i=1}^{n-1} \ker \phi_i \Big/ \bigcap_{i=1}^{n-1} \ker \phi_i \cong F.$$

That (ii) holds is immediate. We turn to (iii). Since $H = QT$ and $Q \leq L$ by (a), we have

$$L = L \cap (QT) = Q(L \cap T)$$

whence $\overline{L} = \overline{Q}(\overline{L \cap T})$; as $\overline{Q}$ is, by the definition of $\operatorname{Core}_Q L$, a core-free subgroup of $\overline{L}$ we deduce that (iii) holds. We turn to (iv). We claim that

$$\overline{L \cap T} \cap \overline{Q} = \overline{Q \cap T}.$$

Certainly $\overline{Q \cap T} \leq \overline{L \cap T} \cap \overline{Q}$ since $Q \leq L$. Suppose that $x \in L \cap T$ and $q \in Q$ are such that $\bar{x} = \bar{q}$. Then $x = qu$ for some $u \in \operatorname{Core}_L Q$ and consequently, $u \in Q$ whence so is $x = qu$ and $\bar{x} \in \overline{Q \cap T}$ as required. We deduce that

$$\eta\left(\overline{L \cap T} \cap \overline{Q}\right) = \eta(\overline{Q \cap T}) = \phi(Q \cap T) \geq \operatorname{Inn} F$$

and that the restriction $\eta|_{\overline{Q \cap T}}$ possesses a strict extension in $\overline{L \cap T}$ if and only if the restriction $\phi|_{Q \cap T}$ possesses a strict extension in $L \cap T$. As the latter does not possess a strict extension even in T we see that (iv) holds.

It remains to show that the (4.1)-tuple $(\overline{L}, \overline{L \cap T}, F, \overline{Q}, \eta)$ has rank n. Define a map α from the set of homomorphisms $Q \to \operatorname{Aut} F$ with kernel containing $\operatorname{Core}_L Q$ to the set of homomorphisms $\overline{Q} \to \operatorname{Aut} F$ given by

$$\alpha(\xi)(\bar{q}) = \xi(q) \qquad \text{for all } q \in Q.$$

It is straightforward, and we leave it to the reader, to show that α restricts to give a bijection between the set of homomorphisms $\xi : Q \to \operatorname{Aut} F$ satisfying

$$\xi|_{Q \cap T} = \phi|_{Q \cap T} \qquad \text{and} \qquad \tilde{\xi} = \tilde{\phi},$$

and the set of homomorphisms $\xi : \overline{Q} \to \operatorname{Aut} F$ satisfying

$$\xi|_{\overline{Q \cap T}} = \eta|_{\overline{Q \cap T}} \qquad \text{and} \qquad \tilde{\xi} = \tilde{\eta},$$

where as usual $\tilde{}$ denotes composition with the natural map $\operatorname{Aut} F \to \operatorname{Out} F$. Hence the rank of $(\overline{L}, \overline{L \cap T}, F, \overline{Q}, \eta)$ is equal to the rank of (H, T, F, Q, ϕ) and we are finished.

Proof of Proposition 4.9. In Remark 4.6 we noted that $F \operatorname{twr}_\phi H$ in its action on the coset space $(F \operatorname{twr}_\phi H : H)$ is a primitive permutation group of twisted wreath type. The results of [3] give a great deal of information about H, Q, and ϕ. In particular, Theorem 5.4 of [3] implies that:

(i) $\operatorname{Soc} H$ is a minimal normal subgroup of H and is non-abelian;

(ii) if K is the largest normal subgroup of $\operatorname{Soc} H$ that is contained in Q, then

$$Q \le N_H(K) = N_H(C_{\operatorname{Soc} H}(K)) \qquad \text{and} \qquad \ker \phi \ge C_H(C_{\operatorname{Soc} H}(K));$$

(iii) if Q does not contain a maximal normal subgroup of $\operatorname{Soc} H$, then $\ker \phi$ is determined by knowledge of H and $V = \phi^{-1}(\operatorname{Inn} F)$ (cf. [3, 6.4]);

(iv) if Q does contain a maximal normal subgroup of $\operatorname{Soc} H$, then $E \not\cong F$ where E is any minimal normal subgroup of $\operatorname{Soc} H$; in fact, F is isomorphic to a proper section of E (cf. [3, 9.13(1)]).

Let E be a minimal normal subgroup of $\operatorname{Soc} H$ not contained in Q: this is possible as Q is a core-free subgroup of H. Note that E is a non-abelian simple group as by (i), $\operatorname{Soc} H$ is a non-abelian minimal normal subgroup of H. Set $L = N_H(E)$. It is our intention to apply Lemma 4.12 and to do this we must show that conditions (a)–(c) of that lemma all hold.

Recall that by supposition $n \ge 3$, whence ϕ_1, ϕ_2 both exist and may both play the role of ϕ. Thus Corollary 4.11 implies that $\ker \phi$ is not determined by knowledge of H and V, whence by (iii), Q contains some maximal normal subgroup K of $\operatorname{Soc} H$. As $E \not\leq Q$ we must have $K = C_{\operatorname{Soc} H}(E) \leq Q$. By (ii), $Q \leq L = N_H(E)$, whence (a) holds, and moreover, $C_H(E) \leq \ker \phi \leq Q$. Now $C_H(E)$ is a normal subgroup of $L = N_H(E)$ and so $C_H(E) \leq \operatorname{Core}_L Q$; furthermore, the conjugation action of L on E induces an isomorphism between the quotient $L/C_H(E)$ and some subgroup of $\operatorname{Aut} E$. Since $E \leq L$ induces all inner automorphisms of E, we see that $L/C_H(E)$ is almost simple with socle $EC_H(E)/E \cong E$. Thus, either $C_H(E) = \operatorname{Core}_L Q$, or $EC_H(E) \leq \operatorname{Core}_L Q$. The latter is impossible as $E \not\leq Q$. Condition (b) follows since again by (ii) we have

$$C_H(E) \leq \ker \phi_i \qquad \text{for all } i = 1, \ldots, n-1.$$

We turn to condition (c). Now T is a core-free maximal subgroup of H. We deduce that $H = (\operatorname{Soc} H)T$ and that $(\operatorname{Soc} H) \cap T$ is a maximal T-invariant proper subgroup of $\operatorname{Soc} H$. As in Lemma 4.12 we let $\bar{\ }$ denote reduction modulo $\operatorname{Core}_L Q = C_H(E)$. By arguing as in the proof of the Aschbacher–O'Nan–Scott theorem as given in [14], we deduce that either $\overline{N_T(E)} = \overline{N_H(E)}$ or that $\overline{N_T(E)} <_{\max} \overline{N_H(E)}$. If the latter then (c) holds as required, and so we assume that the former holds, whence

$$N_H(E) = C_H(E) N_T(E).$$

As already noted by (ii) we have $C_H(E) \leq \ker \phi_i$ for each $i = 1, \ldots, n-1$, and so

$$\ker \phi_i = C_H(E)(\ker \phi_i \cap T) = C_H(E) \ker \sigma$$

is independent of i; given that $n \geq 3$ this contradicts Corollary 4.11.

We now finish by applying Lemma 4.12 to see that $(\overline{L}, \overline{L \cap T}, F, \overline{Q}, \eta)$ is again a (4.1)-tuple of rank n. As $\overline{L}$ is almost simple and $|\overline{L}| \leq |H|$ the minimality assumption implies that equality holds and that $H \cong L$ is also almost simple as required.

5. THE T-COMPLEMENT CASE: AN INITIAL REDUCTION

In this and the following two sections, we are concerned with the determination of $\Omega(2.\mathrm{E})$. The purpose of the present section is to translate the problem into the language of twisted wreath products. In so doing we identify two natural subcases which are then dealt with separately in Sections 6 and 7.

Suppose that T is a subgroup of a group G such that for some $n \in \mathbb{N}$ the lattice $[G/T]$ is isomorphic to M_n, and such that the socle of G is non-abelian, is a minimal normal subgroup of G, and is complemented by T. Set $M = \operatorname{Soc} G$. As M is a non-abelian minimal normal subgroup of G, M is a non-abelian characteristically simple group and so is the direct product of its minimal normal subgroups, each of which is non-abelian, simple, and G-conjugate to any other. Let F be a minimal normal subgroup of M. Set $S = N_T(F)$ and let $\phi : S \to \operatorname{Aut} F$ be induced by the conjugation action of S on F. Applying Lemma 3.5 we see that there is an isomorphism $G \to F \operatorname{twr}_\phi T$ mapping M to the base group $B^F_{\phi,T}$ and T to T. In particular, it follows that $B^F_{\phi,T}$ is the socle of $F \operatorname{twr}_\phi T$ and so by Lemma 3.9 the core of $\phi^{-1}(\operatorname{Inn} F)$ in T is trivial. Furthermore, the above isomorphism $G \to F \operatorname{twr}_\phi T$ also implies that

$$\left[B^F_{\phi,T}\big/\{\mathrm{id}\}\right]_T \cong \left[F \operatorname{twr}_\phi T\big/T\right] \cong \left[G/T\right] \cong \mathsf{M}_n. \tag{5.A}$$

Let L be a $\phi(S)$-invariant non-trivial proper subgroup of F. Then, as discussed in Subsection 3.4, we may view $B^L_{\phi,T}$ as a T-invariant non-trivial proper subgroup of $B^F_{\phi,T}$. Moreover, if K is also a $\phi(S)$-invariant non-trivial proper subgroup of F and $K < L$, then $B^K_{\phi,T} < B^L_{\phi,T}$ are a strictly comparable pair of T-invariant non-trivial proper subgroups of $B^F_{\phi,T}$. As this contradicts (5.A) it follows that

$$\left[F/\{\mathrm{id}\}\right]_{\phi(S)} \cong \mathsf{M}_m \tag{5.B}$$

for some non-negative integer $m \leq n$. (Recall that M_0 is the lattice of length 1.) Corollary 3.25 applies and we see that either $m = 0$ and $\phi(S) \geq \operatorname{Inn} F$, or $m = 1$ and $\phi(S) \cap \operatorname{Inn} F$ is a non-trivial proper subgroup of $\operatorname{Inn} F$. We consider these two cases separately, but before so doing we briefly pause to note that in both cases $\phi^{-1}(\operatorname{Inn} F)$ is non-trivial, whence S is a proper subgroup of T as $\operatorname{Core}_T(\phi^{-1}(\operatorname{Inn} F))$ is trivial.

Firstly we suppose that $\phi(S) \geq \operatorname{Inn} F$. Lemma 3.6 implies that the T-invariant non-trivial proper subgroups of $B^F_{\phi,T}$ are precisely the subgroups of the form $B^F_{\rho,T}$ where ρ varies over all the strict extensions of ϕ in T. Noting that $B^F_{\sigma,T} < B^F_{\rho,T}$ if and only if σ is in turn a strict extension of ρ we see that the tuple (T, F, S, ϕ) is a (5.1)-tuple of rank n in the meaning of the following definition.

DEFINITION 5.1. We say that the tuple (T, F, S, ϕ) *satisfies* (5.1), or *is a* (5.1)-*tuple*, if the following all hold:

(i) ϕ is a homomorphism $S \to \operatorname{Aut} F$ such that $\phi(S) \geq \operatorname{Inn} F$, where S is a proper subgroup of T and where F is a non-abelian simple group;

(ii) $\phi^{-1}(\mathrm{Inn}\,F)$ is a core-free subgroup of T;

(iii) $S <_{\max} R$ whenever $\rho : R \to \mathrm{Aut}\,F$ is a strict extension of ϕ in T.

Moreover, we say that the tuple (T, F, S, ϕ) is a (5.1)-*tuple of rank* n, if it satisfies (5.1) and there exist precisely n strict extensions of ϕ in T.

The subset $\Omega(5.1)$ of $\mathbb{N}$ is defined by

$$\Omega(5.1) = \{n \geq 16\text{: there exists a } (5.1)\text{-tuple of rank } n\}.$$

Secondly we suppose that $\phi(S) \not\geq \mathrm{Inn}\,F$. As noted above (5.B) together with Corollary 3.25 implies that

$$[F/\mathrm{id}]_{\phi(S)} \cong M_1$$

and that $\phi(S) \cap \mathrm{Inn}\,F$ is a non-trivial proper subgroup of F, and moreover, that $\phi(S) \cap \mathrm{Inn}\,F$ is the unique $\phi(S)$-invariant such subgroup, whence $\phi(S)$ is a maximal subgroup of $\phi(S)(\mathrm{Inn}\,F)$ and $\phi(S)$ normalises no non-trivial proper subgroup of $\phi(S) \cap \mathrm{Inn}\,F$. Observe that $B_{\phi,T}^{\phi(S)\cap \mathrm{Inn}\,F}$ is a T-invariant non-trivial proper subgroup of $B_{\phi,T}^{F}$. From (5.B) we deduce that T is a maximal subgroup of $(B_{\phi,T}^{\phi(S)\cap \mathrm{Inn}\,F})T$, or equivalently, that T is a maximal subgroup of $(\phi(S) \cap \mathrm{Inn}\,F)\mathrm{twr}_{\iota\circ\phi}T$ where $\iota \circ \phi$ is the composition of ϕ with the map $\iota : N_{\mathrm{Aut}\,F}(\phi(S) \cap \mathrm{Inn}\,F) \to \mathrm{Aut}(\phi(S) \cap \mathrm{Inn}\,F)$ induced by conjugation. On noting that Corollary 3.25 also implies that $\phi(S) \cap \mathrm{Inn}\,F$ is non-abelian and characteristically simple, we see that the ideas used to prove Corollary 3.7 can be adapted to show that T is maximal in $(\phi(S) \cap \mathrm{Inn}\,F)\mathrm{twr}_{\iota\circ\phi}T$ only if $\phi(S)$ normalises no non-trivial proper subgroup of $\phi(S) \cap \mathrm{Inn}\,F$ and there exists no strict extension of $\iota \circ \phi$ in T (with image in $\mathrm{Aut}(\phi(S) \cap \mathrm{Inn}\,F)$). (The converse however only holds if $\phi(S) \cap \mathrm{Inn}\,F$ is non-abelian and simple.)

Suppose now that X is a T-invariant non-trivial proper subgroup of $B_{\phi,T}^{F}$, and suppose further that there does not exist an extension ρ of ϕ in T with $X = B_{\rho,T}^{F}$. Since $[B_{\phi,T}^{F}/\{\mathrm{id}\}]_T \cong M_n$, Lemma 3.6 implies that $X = B_{\phi,T}^{\phi(S)\cap \mathrm{Inn}\,F}$. It follows that the T-invariant non-trivial proper subgroups of $B_{\phi,T}^{F}$ distinct from $B_{\phi,T}^{\phi(S)\cap \mathrm{Inn}\,F}$ are all of the form $B_{\rho,T}^{F}$ for some strict extension ρ of ϕ in T. Moreover, given a strict extension $\rho : R \to \mathrm{Aut}\,T$ of ϕ in T, we have that

$$T <_{\max} F\,\mathrm{twr}_{\rho}T <_{\max} F\,\mathrm{twr}_{\phi}T.$$

This, together with Corollary 3.7, implies that $\rho(R) \geq \operatorname{Inn} F$, whence

$$S \leq (\ker \rho)S < \rho^{-1}(\operatorname{Inn} F)S \leq R,$$

and that $S <_{\max} R$. We deduce that $S = (\ker \rho)S$ whence $\ker \rho = \ker \phi$, and that $R = \rho^{-1}(\operatorname{Inn} F)S$ whence $\rho(R) = (\operatorname{Inn} F)\phi(S)$ since as noted above $\rho(R) \geq \operatorname{Inn} F$. We have now shown that (T, F, S, ϕ) is a (5.2)-tuple of rank n in the meaning of the following definition.

DEFINITION 5.2. We say that the tuple (T, F, S, ϕ) *satisfies* (5.2), or *is a* (5.2)-*tuple*, if the following all hold:

(i) ϕ is a homomorphism $S \to \operatorname{Aut} F$ such that $\phi(S) \cap \operatorname{Inn} F$ is a non-trivial proper subgroup of $\operatorname{Inn} F$, where S is a proper subgroup of T and where F is a non-abelian simple group;

(ii) $\phi^{-1}(\operatorname{Inn} F)$ is a core-free subgroup of T;

(iii) $\phi(S) <_{\max} (\operatorname{Inn} F)\phi(S)$;

(iv) $T <_{\max} (\phi(S) \cap \operatorname{Inn} F)\operatorname{twr}_{\iota \circ \phi} T$ where $\iota \circ \phi$ is the composition of ϕ with the map $\iota : N_{\operatorname{Aut} F}(\phi(S) \cap \operatorname{Inn} F) \to \operatorname{Aut}(\phi(S) \cap \operatorname{Inn} F)$ induced by conjugation;

(v) if $\rho : R \to \operatorname{Aut} F$ is a strict extension of ϕ in T, then $\ker \phi = \ker \rho$ and $\rho(R) = (\operatorname{Inn} F)\phi(S)$.

Moreover, we say that the tuple (T, F, S, ϕ) is a (5.2)-*tuple of rank* n, if it satisfies (5.2) and there exist precisely $n - 1$ strict extensions of ϕ in T.

The subset $\Omega(5.2)$ of $\mathbb{N}$ is defined by

$$\Omega(5.2) = \{n \geq 16\colon \text{there exists a } (5.2)\text{-tuple of rank } n\}.$$

THEOREM 5.3. $\Omega(2.E) = \Omega(5.1) \cup \Omega(5.2)$.

Proof. We have already seen that

$$\Omega(2.E) \subseteq \Omega(5.1) \cup \Omega(5.2).$$

To show the containment in the reverse direction it is sufficient to show that if (T, F, S, ϕ) is either a (5.1)-tuple of rank n or a (5.2)-tuple of rank n, then

$$\left[F \operatorname{twr}_\phi T \middle/ T\right] \cong M_n$$

and $\operatorname{Soc}(F \operatorname{twr}_\phi T) = B^F_{\phi, T}$. The verification of this is left to the reader as it requires little more than reversing the arguments used above.

Remark 5.4. We note that an integral part of the above proof is to show that Definition 5.2(iii) and (iv) together imply that

$$\left[F/\{\mathrm{id}\}\right]_{\phi(S)} \cong M_1.$$

We close this section with a result that applies both to (5.1)-tuples and to (5.2)-tuples.

LEMMA 5.5. *Let (T, F, S, ϕ) be either a* (5.1)*-tuple of rank n or a* (5.2)*-tuple of rank $n+1$; in either case let $\rho_i : R_i \to \operatorname{Aut} F$ for $i = 1, \ldots, n$ be the n strict extensions of ϕ in T. Suppose that X is an overgroup of S in T such that*

$$\operatorname{Core}_X\bigl(\phi^{-1}(\operatorname{Inn} F)\bigr) \le \ker \phi.$$

Let $\bar{\ }$ denote reduction modulo $\operatorname{Core}_X(\phi^{-1}(\operatorname{Inn} F))$ and set $m = |\{i : R_i \le X\}|$. Then the condition

$$\eta(\bar{s}) = \phi(s) \qquad \textit{for all } s \in S$$

uniquely defines a homomorphism $\eta : \bar{S} \to \operatorname{Aut} F$, and moreover, if (T, F, S, ϕ) is a (5.1)*-tuple then $(\bar{X}, F, \bar{S}, \eta)$ is a* (5.1)*-tuple of rank m, while if (T, F, S, ϕ) is a* (5.2)*-tuple then $(\bar{X}, F, \bar{S}, \eta)$ is a* (5.2)*-tuple of rank $m+1$.*

Proof. This is straightforward and, other than to say that Lemma 3.8 is useful in verifying condition Definition 5.2(iv) in the relevant case, we leave it to the reader.

6. THE T-COMPLEMENT CASE: SUBCASE $\phi(S) \ge \operatorname{Inn} F$

In this section we study the problem of determining $\Omega(5.1)$. We start with a series of results leading to Corollary 6.3 which is an application of Lemma 5.5.

LEMMA 6.1. *Let (T, F, S, ϕ) be a* (5.1)*-tuple. Suppose that the homomorphisms $\rho_1, \ldots, \rho_m : R \to \operatorname{Aut} F$ are distinct strict extensions of ϕ in T with $m \ge 2$ and with a common domain. Then $m = 2$ and* $\ker \phi = \operatorname{Core}_R(\phi^{-1}(\operatorname{Inn} F))$.

Proof. For each $i = 1, \ldots, m$ we have

$$\ker \phi = \ker \rho_i \cap S_i$$

and so either $\ker\phi = \ker\rho_1 \cap \ker\rho_2$, or $S \not\geq \ker\rho_1 \cap \ker\rho_2$. If the latter holds then

$$S < (\ker\rho_1 \cap \ker\rho_2) \leq R,$$

and the maximality of S in R forces $R = (\ker\rho_1 \cap \ker\rho_2)S$. However, this is impossible as the distinct maps ρ_1, ρ_2 agree both on S and on $\ker\rho_1 \cap \ker\rho_2$, and so also on their common domain R.

Hence $\ker\phi = \ker\rho_1 \cap \ker\rho_2$. This implies that $\ker\phi$ is a normal subgroup of R. Let $^-$ denote reduction modulo $\ker\phi$. Observe that the homomorphisms $R \to \operatorname{Aut} F$ extending ϕ are in a one-to-one correspondence with the homomorphisms $\overline{R} \to \operatorname{Aut} F$ extending the map $\overline{\phi}: \overline{S} \to \operatorname{Aut} F$ defined by

$$\overline{\phi}: \bar{q} \mapsto \phi(q) \qquad \text{for all } q \in S.$$

Suppose that $\ker\phi = \ker\rho_1$. Then $\overline{R} \cong \rho_1(R)$ is almost simple with socle $\overline{\phi^{-1}(\operatorname{Inn} F)} \cong F$. The "Schreier conjecture" implies that any homomorphism $\overline{R} \to \operatorname{Aut} F$ with image containing $\operatorname{Inn} F$ is a monomorphism. Furthermore, it is clear that any monomorphism $\overline{R} \to \operatorname{Aut} F$ is determined uniquely by knowledge of its restriction to $\operatorname{Soc} \overline{R}$. Thus there exists a unique homomorphism $\overline{R} \to \operatorname{Aut} F$ extending $\overline{\phi}$, contradicting the hypothesis that $m \geq 2$.

So $\ker\rho_i > \ker\phi$ for all $i = 1, \ldots, m$. Recalling that $\ker\phi = \ker\rho_1 \cap \ker\rho_2$ it follows that $\rho_1(\ker\rho_2)$ is a non-trivial normal subgroup of $\rho_1(R)$, whence $\rho_1(\ker\rho_2)$ contains $\operatorname{Inn} F$; similarly, $\rho_2(\ker\rho_1) \geq \operatorname{Inn} F$. We conclude that the map $\overline{R} \to (\operatorname{Aut} F)^2$ given by

$$\bar{r} \mapsto \big(\rho_1(r), \rho_2(r)\big) \qquad \text{for all } r \in R$$

is a well-defined monomorphism with image containing $(\operatorname{Inn} F)^2$. Note also that the image of $\overline{\phi^{-1}(\operatorname{Inn} F)}$ is equal to the diagonal subgroup $\{(x, x): x \in \operatorname{Inn} F\}$ of $(\operatorname{Inn} F)^2$. It is now straightforward to see that there are precisely two homomorphisms $\overline{R} \to \operatorname{Aut} F$ extending $\overline{\phi}$, namely those corresponding to ρ_1 and to ρ_2, and that $\operatorname{Core}_{\overline{R}}(\overline{\phi^{-1}(\operatorname{Inn} F)})$ is trivial; equivalently, that $m = 2$ and that $\ker\phi = \operatorname{Core}_R(\phi^{-1}(\operatorname{Inn} F))$ as required.

Lemma 6.2. *Let (T, F, S, ϕ) be a* (5.1)*-tuple of rank $n \geq 2$, and for $i = 1, \ldots, m$ with $2 \leq m \leq n$ let $\rho_i : R_i \to \operatorname{Aut} F$ be distinct strict extensions of ϕ in T. Set $X = \langle R_1, \ldots, R_m \rangle$. Then*

$$\operatorname{Core}_X\big(\phi^{-1}(\operatorname{Inn} F)\big) \leq \ker\phi.$$

Proof. For convenience set $C = \operatorname{Core}_X(\phi^{-1}(\operatorname{Inn} F))$. Now C is a subgroup of T and is normalised by each R_i. Since $\rho_i(R_i) \geq \phi(S) \geq \operatorname{Inn} F$

and since no strict extensions of ρ_i in T exist, we have by Lemma 3.12 and Remark 3.14 either that $C \le \ker \rho_i$, or that

$$R_i = N_T\big(C \cap \ker \rho_i, C \cap \rho_i^{-1}(\operatorname{Inn} F)\big). \tag{6.A}$$

Note that since $C \le \phi^{-1}(\operatorname{Inn} F) \le S$, we have

$$\begin{aligned} C \cap \ker \rho_i &= C \cap \ker \phi \qquad \text{and} \\ C \cap \rho_i^{-1}(\operatorname{Inn} F) &= C \cap \phi^{-1}(\operatorname{Inn} F) = C. \end{aligned}$$

Thus if $C \le \ker \rho_i$, then $C \le \ker \phi$ as required, and we may therefore suppose that (6.A) holds for all $i = 1, \dots, m$. It follows that

$$R_i = N_T(C \cap \ker \phi, C) \qquad \text{for all } i = 1, \dots, m.$$

Thus $X = R_1$, and we finish by applying Lemma 5.5 to deduce that the X-core of $\phi^{-1}(\operatorname{Inn} F)$ is $\ker \phi$ which is certainly contained in $\ker \phi$ as required.

COROLLARY 6.3. *Let* (T, F, S, ϕ) *be a* (5.1)*-tuple of rank* $n \ge 2$, *and for* $i = 1, \dots, n$ *let* $\rho_i : R_i \to \operatorname{Aut} F$ *be the* n *strict extensions of* ϕ *in* T. *Suppose that* $|T|$ *is minimal among all such* (5.1)*-tuples. Then* $T = \langle R_1, \dots, R_n \rangle$.

Proof. This is an immediate corollary of Lemmas 5.5 and 6.2.

The next result is highly significant in that it says that if (T, F, S, ϕ) is a (5.1)-tuple, then the socle of T is non-abelian and minimal normal in T, and moreover, that each strict extension ρ of ϕ in T is uniquely determined by the subgroup $(\operatorname{Soc} T) \cap \ker \rho$.

THEOREM 6.4. *Let* (T, F, S, ϕ) *be a* (5.1)*-tuple of rank* $n \ge 3$. *Let* N *be any minimal normal subgroup of* T. *Then the following all hold*:

(i) $\phi(N \cap S) \ge \operatorname{Inn} F$;

(ii) *if* $P = N_T(N \cap \ker \phi, N \cap \phi^{-1}(\operatorname{Inn} F))$ *and* $\sigma : P \to \operatorname{Aut} F$ *is defined by requiring that*

$$\phi(x^y) = \phi(x)^{\sigma(y)} \qquad \textit{for all } x \in N \cap \phi^{-1}(\operatorname{Inn} F) \textit{ and } y \in P,$$

then σ *is a well-defined extension of* ϕ *in* T;

(iii) *N is non-abelian and equals the socle of T* (*whence N is the unique minimal normal subgroup of T*);

(iv) *any strict extension* $\rho : R \to \operatorname{Aut} F$ *of* ϕ *in* T *is uniquely determined by knowledge of* $N \cap \ker \rho$: *more precisely, if* $N \cap \ker \rho = N \cap \ker \phi$ *then* $\rho = \sigma$ (*where* σ *is as in* (ii)), *and if* $N \cap \ker \rho > N \cap \ker \phi$ *then*

$R = (N \cap \ker \rho)S$ *and* ρ *is given by*

$$\rho(xy) = \phi(y) \qquad \textit{for all } x \in N \cap \ker \phi \textit{ and } y \in S.$$

Proof. Let N be a minimal normal subgroup of T, and let $\rho : R \to \operatorname{Aut} F$ be any strict extension of ϕ in T. Suppose that $N \cap \ker \rho > N \cap \ker \phi$. As ρ extends ϕ we have $S \cap \ker \rho = \ker \phi$, whence $N \cap \ker \rho \nleq S$. Thus $(N \cap \ker \rho)S > S$ and the maximality of S in R forces $R = (N \cap \ker \rho)S$. Noting that $\rho(x) = \mathrm{id}$ if $x \in \ker \rho$ and that $\rho(y) = \phi(y)$ if $y \in S$ we see that (iv) holds for such a strict extension.

Suppose now that $N \leq \ker \rho$, or equivalently that $N = N \cap \ker \rho$. As ρ extends ϕ we have $N \leq S$ if and only if $N \leq \ker \phi$. But by the definition of (5.1)-tuples, $\ker \phi$ is a core-free subgroup of T and so $N \nleq S$, whence

$$N \cap \ker \rho = N > N \cap S = N \cap \ker \rho \cap S = N \cap \ker \phi.$$

The preceding paragraph implies that $R = NS$ and that ρ is the unique strict extension of ϕ with N in its kernel. Let $\xi : Q \to \operatorname{Aut} F$ be a strict extension of ϕ distinct from ρ. Thus $N \nleq \ker \xi$, whence by Lemma 3.12 the image $\xi(N \cap Q)$ is a non-trivial normal subgroup of $\xi(Q)$. Recall that $\xi(Q) \geq \phi(S) \geq \operatorname{Inn} F$, which means that $\xi(Q)$ is almost simple with a unique minimal normal subgroup, namely $\operatorname{Inn} F$. We deduce that $\xi(N \cap Q) \geq \operatorname{Inn} F$. Now $N \leq \ker \rho$ whence

$$\phi(N \cap S) = \rho(N \cap S) \leq \rho(N) = \{\mathrm{id}\}.$$

Hence $N \cap Q \nleq S$ and $S < (N \cap Q)S \leq Q$. The maximality of S in Q implies that

$$Q = (N \cap Q)S.$$

However, the latter is contained in $NS = R$ and the maximality of S in R forces $Q = R$. As by assumption $n \geq 3$ and ξ was any strict extension of ϕ distinct from ρ this contradicts Lemma 6.1. We conclude that $N \nleq \ker \rho$, and moreover, that $\ker \rho$ is a core-free subgroup of T.

On the other hand, suppose that $N \cap S \leq \ker \phi$. Then it is clear that the homomorphism $\rho : NS \to \operatorname{Aut} F$ given by

$$\rho(xs) = \phi(s) \qquad \text{for all } x \in N \text{ and } s \in S$$

is a well-defined strict extension of ϕ in T with $N \leq \ker \rho$. This contradicts the above conclusion and so $\phi(N \cap S)$ is a non-trivial normal subgroup of $\phi(S)$. It follows that $\phi(N \cap S) \geq \operatorname{Inn} F$ since $\phi(S)$ is almost simple with socle $\operatorname{Inn} F$. We conclude that (i) holds.

We turn to part (ii). Given part (i) this is immediate from Remark 3.14. From now on we assume that P and σ are as defined in (ii). Note that either $\sigma = \phi$ and $P = S$, or σ is a strict extension of ϕ and $S <_{\max} P$. Now the definition of σ forces $C_T(N) \leq \ker \sigma$. If $P = S$ then the fact that S is a core-free subgroup of T implies that $C_T(N)$ is trivial; on the other hand, if $P > S$ then, as we saw above, $\ker \sigma$ is a core-free subgroup of T, which again implies that $C_T(N)$ is trivial. Hence N is the unique minimal normal subgroup of T and so $N = \operatorname{Soc} T$. As part (i) implies that N is non-abelian, we see that (iii) holds.

We turn to part (iv). Once again let $\rho : R \to \operatorname{Aut} F$ be an arbitrary strict extension of ϕ in T. In the first paragraph we saw that (iv) holds if $N \cap \ker \rho$ strictly contains $N \cap \ker \phi$, and so it remains only to show that if $N \cap \ker \rho = N \cap \ker \phi$, then $\rho = \sigma$. By (i) we have that $\phi(N \cap S) \geq \operatorname{Inn} F$; it follows that

$$N \cap \rho^{-1}(\operatorname{Inn} F) = N \cap \phi^{-1}(\operatorname{Inn} F)$$

whence $R \leq N_T(N \cap \ker \phi, N \cap \phi^{-1}(\operatorname{Inn} F)) = P$. The maximality of S in the domain of any strict extension implies that $R = P$. Choose $x \in R$ and $y \in N \cap \phi^{-1}(\operatorname{Inn} F)$. Then $y^x \in N \cap \phi^{-1}(\operatorname{Inn} F)$ and

$$\phi(y^x) = \rho(y^x) = \rho(y)^{\rho(x)} = \phi(y)^{\rho(x)}.$$

As this is precisely the requirement defining σ we see that $\rho = \sigma$ as required.

We remark that given an (5.1)-tuple (T, F, S, ϕ) of rank $n \geq 3$, then the above theorem implies that T has a unique minimal normal subgroup, whence the minimal normal subgroup N, the subgroup P, and the extension σ of ϕ in T defined in the statement of the theorem are uniquely defined. Hence the statement of the following result is sensible.

COROLLARY 6.5. *Let* (T, F, S, ϕ) *be a* (5.1)*-tuple of rank* $n \geq 3$. *Let* P *and* σ *be as defined in Theorem* 6.4, *and for* $i = 1, \ldots, n$ *let* $\rho_i : R_i \to \operatorname{Aut} F$ *be the* n *strict extensions of* ϕ *in* T. *Suppose that* $T = \langle R_1, \ldots, R_n \rangle$. *Then* $T = NP$.

Proof. Let N be the unique minimal normal subgroup of T. Theorem 6.4(iv) implies that for each $i = 1, \ldots, n$ we have either $R_i = P$ or $R_i = (N \cap \ker \rho_i)S$. In either case we have $R_i \leq NP$ whence

$$\langle R_1, \ldots, R_n \rangle \leq NP$$

whence the result follows.

In what follows it proves more useful to assume that a given (5.1)-tuple satisfies the conclusion of the above corollary, rather than to make the stronger assumption that the hypothesis holds. For convenience we make the following definition.

DEFINITION 6.6. We say that the tuple (T, F, S, ϕ) is a *small-*(5.1)*-tuple* of rank n if $n \geq 3$, the tuple is a (5.1)-tuple of rank n, and

$$T = (\operatorname{Soc} T) N_T\big((\operatorname{Soc} T) \cap \ker \phi, (\operatorname{Soc} T) \cap \phi^{-1}(\operatorname{Inn} F)\big).$$

Note that Corollaries 6.3 and 6.5 imply that

$$\Omega(5.1) = \{n \geq 16\colon \text{there exists a small-}(5.1)\text{-tuple of rank } n\}. \quad (6.\mathrm{B})$$

Thus from now on we need only consider small-(5.1)-tuples.

Suppose that (T, F, S, ϕ) is a small-(5.1)-tuple of rank n. Set $N = \operatorname{Soc} T$, let E be a minimal normal subgroup of N, and let σ, P be as defined in Theorem 6.4(ii). In the following we see that P is transitive on the minimal normal subgroups of N, and furthermore, that (E, N, T, F, P, σ) is a (3.18)-tuple. This is then used to show that if $n \geq 4$ then S, as well as P, is transitive on the minimal normal subgroups of N, whence (E, N, T, F, S, ϕ) is also a (3.18)-tuple.

LEMMA 6.7. *Let (T, F, S, ϕ) be a small-*(5.1)*-tuple of rank $n \geq 4$. Then S is transitive on the minimal normal subgroups of* $\operatorname{Soc} T$.

Proof. As usual set $N = \operatorname{Soc} T$, and let σ and P be defined as in Theorem 6.4(ii). Also let E be a minimal normal subgroup of N. We assume that S is intransitive on the minimal normal subgroups of N and argue for a contradiction. Note that this assumption means that N has at least two minimal normal subgroups and so $E \neq N$.

As $T = NP$ we see that P is transitive on the minimal normal subgroups of N, or equivalently that $T = N_T(E)P$. It follows that (E, N, T, F, P, σ) is a (3.18)-tuple with $T = NP$. If $P \leq NS$ then $T = NS$ and the same argument shows that S is transitive on the minimal normal subgroups of N, contradicting our assumption. We deduce therefore that $P \nleq NS$. Thus

$$S \leq P \cap (NS) = (N \cap P)S < P,$$

and the maximality of S in P implies that $S = (N \cap P)S$ whence $N \cap S = N \cap P$. It immediately follows that

$$N \cap \ker \phi = N \cap \ker \sigma \qquad \text{and} \qquad N \cap \phi^{-1}(\operatorname{Inn} F) = N \cap \sigma^{-1}(\operatorname{Inn} F).$$

Let κ, C, D, and η be as defined after Definition 3.18 in terms of the (3.18)-tuple (E, N, T, F, P, σ), that is, $\kappa : N_T(E) \to \operatorname{Aut} E$ is induced by conjugation,

$$C = \kappa(N \cap \ker \phi), \qquad D = \kappa\big(N \cap \phi^{-1}(\operatorname{Inn} F)\big),$$

and $\eta : N_{\operatorname{Aut} E}(C, D) \to \operatorname{Aut} F$ is induced by the conjugation action on the quotient D/C together with the appropriate identification of D/C with F. As $P > S$, the map σ is a strict extension of ϕ in T, and so given that (T, F, S, ϕ) is a (5.1)-tuple σ is equal to any map that extends σ in T. Let $x_1, \dots, x_l$ be a right transversal for $N_P(E)$ in P. Corollary 3.22 implies that

$$N \cap \ker \sigma = \prod_{i=1}^{l} C^{x_i} \qquad \text{and} \qquad Y \cap P = Y \cap \ker \sigma = \prod_{\{i \,:\, E^{x_i} \leq Y\}} C^{x_i} \tag{6.C}$$

where Y is any proper normal subgroup of N. Let Y be any proper normal non-trivial subgroup of N that is normalised by S. As $N \cap \ker \sigma = N \cap \ker \phi$ we have $Y \cap P = Y \cap S \leq \ker \phi$. Thus the map $\rho_Y : YS \to \operatorname{Aut} F$ given by

$$\rho_Y : ys \mapsto \phi(s) \qquad \text{for all } y \in Y \text{ and } s \in S$$

is well-defined and extends ϕ. Given (6.C) an easy calculation shows that

$$N \cap \ker \rho_Y = \Bigg(\prod_{\{i \,:\, E^{x_i} \not\leq Y\}} C^{x_i} \Bigg) \times Y. \tag{6.D}$$

It is then clear that if Y_0 is also a proper normal non-trivial subgroup of N normalised by S with $Y_0 < Y$, then ρ_Y strictly extends ρ_{Y_0}, which in turn strictly extends ϕ. However, as (T, F, S, ϕ) is a (5.1)-tuple, any strict extension of ϕ in T has itself no strict extensions in T, and we deduce that the proper normal non-trivial subgroups of N that are normalised by S are pairwise incomparable by inclusion. Hence S has precisely two orbits on minimal normal subgroups of N.

On the other hand, if $\rho : R \to \operatorname{Aut} F$ is a strict extension of ϕ in T distinct from σ, then by Theorem 6.4 we have $R = (N \cap \ker \rho)S$ and $N \neq N \cap \ker \rho$. Set $Y = \operatorname{Core}_N(N \cap \ker \rho)$ which is clearly a proper normal subgroup of N normalised by R. Given that R is equal to $(N \cap \ker \rho)S$ we see that the normal subgroups of N normalised by R are precisely the same as those normalised by S. As S has two orbits on minimal normal subgroups we see that R normalises precisely two non-trivial proper normal subgroups of N, and moreover, that given any minimal normal subgroup E_0 of N there exists a non-trivial normal subgroup of N

normalised by R and not containing E_0. Now ρ is a strict extension of ϕ and so ρ has no strict extensions in T. Corollary 3.15(i) together with Lemma 3.16(ii) implies that $Y = \mathrm{Core}_N(N \cap \ker \rho)$ is a non-trivial subgroup of N. Thus $\rho_Y : YS \to \mathrm{Aut}\, F$ as defined earlier is a strict extension of ϕ in T. Also ρ is an extension of ρ_Y as ρ and ρ_Y agree on both S and on Y. We conclude that $\rho = \rho_Y$ as S is maximal in the domain of any strict extension of ϕ in T. As there are only two possible choices for Y (as a non-trivial normal proper subgroup of N normalised by R), there are only two possible choices for ρ (as a strict extension of ϕ in T distinct from σ). This contradicts the hypothesis that (T, F, S, ϕ) has rank $n \geq 4$, and we are finished.

Construction 6.8. The input for this construction is a small-(5.1)-tuple (T, F, S, ϕ) of rank n with either $T = (\mathrm{Soc}\, T)S$ or $n \geq 4$. The output is a tuple of objects which will be seen to be of key significance in the study of (T, F, S, ϕ).

As usual let N be the unique minimal normal subgroup of T; let E be a minimal normal subgroup of N and let $\kappa : N_T(E) \to \mathrm{Aut}\, E$ be induced by conjugation. Observe that (E, N, T, F, S, ϕ) is a (3.18)-tuple—this is immediate if $T = NS$ and follows from the above theorem if $n \geq 4$. Let C, D, P, and σ be as defined immediately after Definition 3.18. (Note that we do not yet know that σ is the same map as defined in the proof of Theorem 6.4; however, this will become clear in the course of proving Theorem 6.10 below.) The output of the construction is then the tuple

$$\big(C, D, E, \kappa\big(N_S(E)\big), \kappa\big(N_P(E)\big), \phi(S)\big).$$

For convenience, we shall refer to this tuple and its entries as being *obtained* from (T, F, S, ϕ) via Construction 6.8.

The main thrust of the following is to obtain necessary conditions on an arbitrary (ordered) tuple (C, D, E, K, L, A) for it to be the tuple obtained from some small-(5.1)-tuple via the above construction. The key conditions pertain just to C, D, E, and K and are given in Definition 6.9, while the remainder are given in Definition 6.17. Obviously we would also like such conditions to be also sufficient, but we are able only to achieve this in some special cases. The chief obstacle to ensuring sufficiency in all cases lies in attempting to construct a suitable small-(5.1)-tuple given a tuple obtained from a small-(5.1)-tuple (T, F, S, ϕ) satisfying $T \neq (\mathrm{Soc}\, T)S$. (Note that we do not necessarily want to reconstruct the original tuple, but rather any small-(5.1)-tuple of the same rank.) However, we do not view this to be a big problem as it is easy to see that if (T, F, S, ϕ) is a small-(5.1)-tuple of rank $n \geq 4$ with $T \neq (\mathrm{Soc}\, T)S$, then $((\mathrm{Soc}\, T)S, F, S, \phi)$ is also a small-(5.1)-tuple but of rank $n - 1$ instead of rank n.

DEFINITION 6.9. We say that the tuple (C, D, E, K) *satisfies* (6.9), or *is a* (6.9)-*tuple*, if the following conditions both hold:

(i) (C, D) is a non-abelian simple proper section of the non-abelian simple group E with $K \leq N_{\mathrm{Aut}\,E}(C, D)$;

(ii) if (C, D) is strictly K-contained in the section (C_0, D_0) of E, then (C_0, D_0) is a K-maximal section of E.

Moreover, we say that (C, D, E, K) is a (6.9)-*tuple of degree* d, if it satisfies (6.9) and there exist precisely d sections of E strictly K-containing (C, D).

The subset $\Delta(6.9)$ of $\mathbb{N}$ is defined by

$$\Delta(6.9) = \{d \in \mathbb{N} : \text{there exists a } (6.9)\text{-tuple of degree } d\}.$$

Observe that we use the terminology degree and the notation Δ, rather than rank and Ω, as we do not have an exact correspondence between (5.1)-tuples and (6.9)-tuples. Instead, we have the following.

THEOREM 6.10. *Let (T, F, S, ϕ) be a small-*(5.1)*-tuple of rank $n \geq 4$ and assume the notation of Construction* 6.8. *Then $(C, D, E, \kappa(N_S(E)))$ is a* (6.9)*-tuple of degree d with D contained in $\kappa(N_S(E))$ where*

$$d = \begin{cases} n & \textit{if } P = (\ker \sigma \cap N)S; \\ n-1 & \textit{otherwise}. \end{cases}$$

Moreover, if ρ is an extension of ϕ in T then

$$N \cap \ker \rho = \prod_{i=1}^{l} \kappa(N \cap \ker \rho)^{x_i},$$

where $x_1, \ldots, x_l$ is any right transversal for $N_S(E)$ in S; also, the map

$$\rho \mapsto (\kappa(N \cap \ker \rho), \kappa(N \cap \ker \rho)D)$$

is a bijection between the extensions ρ of ϕ in T satisfying $N \cap \ker \rho > N \cap \ker \phi$ and the sections of E strictly K-containing (C, D).

COROLLARY 6.11. $\Omega(5.1) \subseteq \Delta(6.9) \cup \{n \in \mathbb{N} : n - 1 \in \Delta(6.9)\}$

Proof. This is straightforward.

Proof of Theorem 6.10. Set $K = \kappa(N_S(E))$. It is clear that E is a non-abelian simple group, that (C, D) is a section of E isomorphic to the non-abelian simple group F, and that $D \leq K \leq N_{\mathrm{Aut}\,E}(C, D)$. Thus Definition 6.9(i) holds provided only that (C, D) is a proper section of E, or equivalently that $D/C \not\cong E$. However, if there exists at least one section of

E strictly K-containing (C, D), then (C, D) is not K-maximal whence D/C is certainly not isomorphic to E. Thus given that $n \geq 4$ and that, as already noted, $D \leq K$, to prove the first assertion it is sufficient to show that Definition 6.9(ii) holds and that there are precisely d sections of E strictly K-containing (C, D) where d is as defined in the statement of the theorem.

As usual for $i = 1, \dots, n$ let $\rho_i : R_i \to \operatorname{Aut} F$ be the strict extensions of ϕ in T. Notice that if $P = (N \cap \ker \sigma)S$ then either $P = S$ and $\sigma = \phi$ or $P \neq S$ and $N \cap \ker \sigma > N \cap \ker \phi$. On the other hand $P \neq (N \cap \ker \sigma)S$, equivalently $d = n - 1$, if and only if σ is a strict extension of ϕ with $N \cap \ker \sigma = N \cap \ker \phi$. We relabel so that if $d = n - 1$, then $\rho_n = \sigma$. For $i = 1, \dots, n$ we define subgroups C_i, D_i of E by

$$C_i = \kappa(N \cap \ker \rho_i) \qquad \text{and} \qquad D_i = \kappa\big(N \cap \rho_i^{-1}(\operatorname{Inn} F)\big).$$

We claim that $(C_1, D_1), \dots, (C_d, D_d)$ are distinct K-maximal sections of E, all strictly K-containing (C, D).

By the first part of Lemma 3.16 we see that the sections (C_i, D_i) all K-contain (C, D). Recall that $d \geq n - 1 \geq 3$. If (C_i, D_i) for $i = 1, \dots, d$ are distinct K-maximal sections, then they must all strictly K-contain (C, D) since if one is equal to (C, D), then (C, D) is K-maximal and so equal to any section K-containing it. Now by Theorem 6.7 we have $T = N_T(E)S$, whence $T = N_T(E)R_i$ for each $i = 1, \dots, n$. It follows that $(E, N, T, F, R_i, \rho_i)$ is a (3.18)-tuple for each $i = 1, \dots, n$. Let $x_1, \dots, x_l$ be a right transversal for $N_S(E)$ in S, and so also for $N_{R_i}(E)$ in R_i. As each ρ_i has no strict extensions in T we deduce from Proposition 3.20 that (C_i, D_i) is a $\kappa(N_{R_i}(E))$-maximal section of E, and from Corollary 3.22 that

$$N \cap \ker \rho_i = \prod_{j=1}^{l} C_i^{x_j} \qquad \text{for all } i = 1, \dots, n. \tag{6.E}$$

By Theorem 6.4 the map ρ_i is uniquely determined by the subgroup $N \cap \ker \rho_i$, whence the subgroups $C_1, \dots, C_n$ are distinct. Suppose that $N \cap \ker \rho_i > N \cap \ker \phi$. Then by Theorem 6.4 we have $R_i = (N \cap \ker \rho_i)S$, whence $\kappa(N_{R_i}(E)) = C_i K$. By Proposition 3.11, the section (C_i, D_i) is K-maximal if and only if it is $C_i K$-maximal, and so to verify the claim it is enough to show that $N \cap \ker \rho_i > N \cap \ker \phi$ for $i = 1, \dots, d$, or equivalently, that if ρ is a strict extension of ϕ in T with $N \cap \ker \rho = N \cap \ker \phi$ then $d = n - 1$ and $\rho = \sigma$. Suppose that ρ is such an extension. Note that the deduction $\rho = \sigma$ follows from Theorem 6.4 provided only that the subgroup P and the map σ are precisely the same as those defined by Theorem 6.4(ii). On comparing the two alternative definitions of P and σ

we see that they are the same if

$$N \cap \ker \phi = \prod_{j=1}^{l} C^{x_j}. \tag{6.F}$$

Given that $d \geq 3$, Theorem 6.4 implies that we may certainly relabel so that $R_i = (N \cap \ker \rho_i)S$ for at least $i = 1, 2$. It follows that

$$S \leq (N \cap \ker \rho_1 \cap \ker \rho_2)S \leq R_1 \cap R_2.$$

As ρ_1, ρ_2 clearly agree on $(N \cap \ker \rho_1 \cap \ker \rho_2)S$ we have

$$(N \cap \ker \rho_1 \cap \ker \rho_2)S < R_i \qquad \text{for all } i = 1, 2,$$

whence the maximality of S in R_i implies that $S = (N \cap \ker \rho_1 \cap \ker \rho_2)S$. We deduce that $N \cap \ker \phi = N \cap \ker \rho_1 \cap \ker \rho_2$, whence by (6.E) we have

$$N \cap \ker \phi = \prod_{j=1}^{l} (C_1 \cap C_2)^{x_j} \qquad \text{and} \qquad \kappa(N \cap \ker \phi) = C_1 \cap C_2.$$

As $\kappa(N \cap \ker \phi) = C$ we deduce that (6.F) holds, and that ρ does indeed equal σ. Hence $P \neq S$ and

$$N \cap \ker \rho = N \cap \ker \sigma = N \cap \ker \phi,$$

whence $P \neq (N \cap \ker \sigma)S$ and $d = n - 1$ as required. We conclude that the claim holds.

To prove the first assertion of the theorem it now remains only to show that if (C_0, D_0) is any section of E strictly K-containing (C, D), then (C_0, D_0) is equal to (C_i, D_i) for some $i = 1, \ldots, d$. To see this suppose that (C_0, D_0) strictly K-contains (C, D). Thus C_0 is a strict overgroup of C in E, is normalised by K, and meets D in precisely C. It follows that

$$\prod_{j=1}^{l} C_0^{x_j}$$

is a subgroup of N normalised by S and meeting $N \cap \phi^{-1}(\mathrm{Inn}\, F)$ in $N \cap \ker \phi$. Define a map $\xi : (\prod_{j=1}^{l} C_0^{x_j})S \to \mathrm{Aut}\, F$ by

$$\xi : xy \mapsto \phi(y) \qquad \text{for all } x \in \prod_{j=1}^{l} C_0^{x_j} \text{ and } y \in S.$$

Clearly ξ extends ϕ and so either $\xi = \phi$ or $\xi = \rho_i$ for some $i = 1, \dots, n$. Observe that $N \cap \ker \xi = \prod_{j=1}^{l} C_0^{x_j}$ and that

$$\kappa(N \cap \ker \xi) = C_0 \neq C = \kappa(N \cap \ker \phi),$$

whence $N \cap \ker \xi \neq N \cap \ker \phi$. Recall that if $n > d = n - 1$ then σ is a strict extension of ϕ, $N \cap \ker \sigma = N \cap \ker \phi$, and we have labelled so that $\rho_n = \sigma$. Thus $\xi = \rho_i$ for some $i = 1, \dots, d$ and by (6.E) it follows that $C_0 = C_i$ for some $i = 1, \dots, d$ as required.

Finally we turn to the "Moreover" statement. The first part of this holds by (6.E) and (6.F), while the second part follows by noting that in the course of the above we have seen that if ρ is any extension of ϕ in T satisfying $N \cap \ker \rho > N \cap \ker \phi$, then ρ is one of $\rho_1, \dots, \rho_d$ and that $(C_1, D_1), \dots, (C_d, D_d)$ are distinct and are the only sections of E strictly K-containing (C, D).

Given a (6.9)-tuple (C, D, E, K) of degree d, we fix some notation which will apply for the rest of this section. Let F be the quotient D/C; note that by definition F is a non-abelian simple group not isomorphic to E. As usual we identify F with $\operatorname{Inn} F$ and E with $\operatorname{Inn} E$. Observe that there is a natural action induced by conjugation of $N_{\operatorname{Aut} E}(C, D)$ on $F = D/C$; let $\eta : N_{\operatorname{Aut} E}(C, D) \to \operatorname{Aut} F$ be the associated map. (In the situation where (C, D, E, K) is obtained via Construction 6.8 applied with input (T, F, S, ϕ) (so that (E, N, T, F, S, ϕ) is a (3.18)-tuple), then we note that the definition of η just given agrees with that given immediately after Definition 3.18 in terms of the (3.18)-tuple (E, N, T, F, S, ϕ) subject to using ϕ to identify D/C with F.) Let $(C_1, D_1), \dots, (C_d, D_d)$ be the d sections of E that strictly K-contain (C, D). Now for each i we have $D_i = C_i D$ and so we can identify the quotient D_i/C_i with F via the map $F = D/C \to D_i/C_i$ given by

$$Cx \mapsto C_i x \qquad \text{for all } x \in D;$$

let $\eta_i : N_{\operatorname{Aut} E}(C_i, D_i) \to \operatorname{Aut} F$ be defined in an analogous fashion to η. Note that if $x \in N_{\operatorname{Aut} E}(C, D) \cap N_{\operatorname{Aut} E}(C_i, D_i)$ then

$$\eta(x) = \eta_i(x).$$

LEMMA 6.12. *Let (C, D, E, K) be a (6.9)-tuple of degree d. Then the following hold*:

(1) *The tuple (C, D, E, DK) is a (6.9)-tuple of degree d.*

(2) *If $d \geq 2$ and $D \leq K$ then*

$$D = K \cap \eta^{-1}(\operatorname{Inn} F) \cap E \qquad \textit{and} \qquad C = K \cap \ker \eta \cap E.$$

Proof. We start by noting that DK is, as implicitly assumed in the above statement, a subgroup of $N_{\mathrm{Aut}\,E}(C, D)$ since both D and K are subgroups of $N_{\mathrm{Aut}\,E}(C, D)$ and since D is normalised by K. Now the definition of section containment is such that a section K-contains (C, D) if and only if it DK-contains (C, D); part (1) follows from this observation.

To see (2) suppose that $d \geq 2$ and $D \leq K$. As $\eta^{-1}(\mathrm{Inn}\, F) = D(\ker \eta)$ it is enough to show that $C = K \cap \ker \eta \cap E$. For convenience set $C_0 = K \cap \ker \eta \cap E$. Clearly $C \leq C_0$. Suppose that $C < C_0$. As $\eta(C_0)$ is trivial, we have $C_0 \cap D = C$. Now K, and so also D, normalises C_0 and so $(C_0, C_0 D)$ is a section of E strictly K-containing (C, D); by the definition of (6.9)-tuples (C_0, D_0) is a K-maximal section of E. Let (C_i, D_i) be a section of E strictly K-containing (C, D) that is distinct from (C_0, D_0); this is possible as $d \geq 2$. However,

$$C_0 = K \cap \ker \eta \cap E \leq (C_i K) \cap \ker \eta_i \cap E$$

since η_i and η agree on K. The K-maximality of (C_i, D_i) implies that C_i equals $(C_i K) \cap \ker \eta_i \cap E$, whence $C_0 \leq C_i$ whereupon the K-maximality of (C_0, D_0) implies $C_0 = C_i$. But the section (C_i, D_i) was chosen to be distinct from (C_0, D_0), a contradiction.

Construction 6.13. The input is a (6.9)-tuple (C, D, E, K) of degree $d \geq 2$ with $D \leq K$, together with a subgroup A of $\mathrm{Aut}\, F$ that contains $\eta(K)$ (where we assume the notation set out immediately prior to Lemma 6.12). The construction attempts to output a (5.1)-tuple defined in terms of this input. More precisely, the construction outputs a tuple $(\hat{T}, F, \hat{S}, \hat{\phi})$ and in Proposition 6.16 below, we give necessary and sufficient conditions for this to be a (5.1)-tuple.

Set $a = |A : \eta(K)|$. We construct $\hat{S}$ as a subgroup of $(\mathrm{Aut}\, E) \wr S_a$. In fact, we construct $\hat{S}$ as a subgroup of $K \wr S_a \leq N_{\mathrm{Aut}\,E}(C, D) \wr S_a$. Let $\eta^{(a)}$ be the epimorphism $K \wr S_a \to \eta(K) \wr S_a$ with kernel $(\ker \eta \cap K)^a$ induced by η. Choose (and fix) a right transversal $\alpha_1 (= \mathrm{id}_A), \alpha_2, \ldots, \alpha_a$ for $\eta(K)$ in A, and let $\iota : A \to \eta(K) \wr S_a$ be the map defined by

$$\iota(\alpha) = \left(\alpha_1 \alpha \alpha_{1\pi}^{-1}, \ldots, \alpha_a \alpha \alpha_{a\pi}^{-1}\right)\pi,$$

where $\pi \in S_a$ is defined by the condition

$$\eta(K)\alpha_i \alpha = \eta(K)\alpha_{i\pi} \qquad \text{for all } i = 1, \ldots, a.$$

Theorem 3.3 shows that ι is a well-defined monomorphism. We now define $\hat{S}$ to be the full inverse image in $K \wr S_a$ under $\eta^{(a)}$ of $\iota(A)$. Thus $\hat{S}$ is a (not necessarily split) extension of $(K \cap \ker \eta)^a$ by A. Let $\hat{\phi} : \hat{S} \to$

$\operatorname{Aut} F$ be defined by

$$\hat{\phi}(x) = \alpha \qquad \text{for all } x \in \hat{S},$$

where $\alpha \in A \leq \operatorname{Aut} F$ is such that $\iota(\alpha) = \eta^{(a)}(x)$. Given our usual identification of E and $\operatorname{Inn} E$, we see that E^a is a normal subgroup of $(\operatorname{Aut} E) \wr S_a$. We finish by setting $\hat{T} = E^a\hat{S}$.

LEMMA 6.14. *With the notation of Construction* 6.13, *the following all hold*:

(1) *E^a is the unique minimal normal subgroup of $\hat{T}$*;

(2) $E^a \cap \ker \hat{\phi} = C^a$;

(3) *if we identify E with a subgroup of E^a via the map*

$$x \mapsto (x, \mathrm{id}_E, \dots, \mathrm{id}_E) \qquad \textit{for all } x \in E,$$

and let $\kappa : N_{(\operatorname{Aut} E) \wr S_a}(E) \to \operatorname{Aut} E$ *be induced by conjugation, then* $\kappa(N_{\hat{S}}(E)) = K$ *and*

$$\hat{\phi}(x) = \eta(\kappa(x)) \qquad \textit{for all } x \in N_{\hat{S}}(E);$$

(4) $\hat{\phi}(E^a \cap \hat{S}) = \operatorname{Core}_A(\eta(K \cap E)) \geq \operatorname{Inn} F$.

Proof. It is clear that $\hat{S} \leq \hat{T}$ is transitive on the simple direct factors of $E^a \leq \hat{T}$, whence E^a is indeed a minimal normal subgroup of $\hat{T}$. Moreover, as the centraliser in $(\operatorname{Aut} E) \wr S_a$ of E^a is trivial, we see that E^a is the unique such subgroup, and so (1) holds.

By construction we have

$$\ker \hat{\phi} = \ker \eta^{(a)} = (K \cap \ker \eta)^a.$$

Thus $E^a \cap \ker \hat{\phi} = (K \cap \ker \eta \cap E)^a$ and Lemma 6.12(2) implies that (2) holds.

We turn to (3). Now $\hat{S}$ is defined as the inverse image under the epimorphism $\eta^{(a)} : K \wr S_a \to \eta(K) \wr S_a$ of the group $\iota(A)$. Observe that $N_{K \wr S_a}(E)$ contains $\ker \eta^{(a)}$ and that

$$\eta^{(a)}\big(N_{K \wr S_a}(E)\big) = \{(x_1, \dots, x_a)\pi \in \eta(K) \wr S_a : 1\pi = 1\}.$$

Therefore on inspecting the definition of ι we see that $N_{\hat{S}}(E)$ is precisely the inverse image under $\eta^{(a)}$ of $\iota(\eta(K))$, whence the definition of $\hat{\phi}$ implies that $\hat{\phi}(N_{\hat{S}}(E))$ is equal to $\eta(K)$. Furthermore, if $x \in N_{K \wr S_a}(E)$

then $\eta^{(a)}(x)$ is of the form

$$(\eta(\kappa(x)), \dots)\pi$$

for some $\pi \in S_a$, while if $y \in K$ then $\iota(\eta(y))$ is of the form

$$(\eta(y), \dots)\tau$$

for some $\tau \in S_a$; thus if $x \in N_{\hat{S}}(E)$ and $y \in K$ are such that $\eta^{(a)}(x) = \iota(\eta(y))$, then we must have equality between $\eta(\kappa(x))$ and $\eta(y)$. It follows by the definition of $\hat{\phi}$ that

$$\eta(\kappa(x)) = \hat{\phi}(x) \qquad \text{for all } x \in N_{\hat{S}}(E).$$

To see the remainder of (3), namely that $\kappa(N_{\hat{S}}(E)) = K$, we note that $\kappa(N_{\hat{S}}(E))$ is certainly a subgroup of K as $\hat{S} \leq K \wr S_a$. Now by the above

$$\eta\big(\kappa(N_{\hat{S}}(E))\big) = \hat{\phi}(N_{\hat{S}}(E)) = \eta(K),$$

and so $\kappa(N_{\hat{S}}(E)) = K$ if and only if $\kappa(N_{\hat{S}}(E)) \geq \ker \eta \cap K$. However, the latter holds as $N_{\hat{S}}(E)$ clearly contains $(\ker \eta^{(a)}) \cap (K \wr S_a) = (\ker \eta \cap K)^a$.

Finally we consider (4). As $\hat{\phi}(\hat{S}) = A$ and as E^a is a normal subgroup of $\hat{T} \geq \hat{S}$, we see that $\hat{\phi}(E^a \cap \hat{S})$ is a normal subgroup of A. By (3) we have

$$\hat{\phi}(E^a \cap \hat{S}) = \eta\big(\kappa(E^a \cap \hat{S})\big) \leq \eta\big(\kappa(E^a) \cap \kappa(N_{\hat{S}}(E))\big) = \eta(E \cap K).$$

Hence $\hat{\phi}(E^a \cap \hat{S})$ is contained in the A-core of $\eta(K \cap E)$. On the other hand, the definition of ι implies that

$$\iota\big(\mathrm{Core}_A(\eta(K \cap E))\big) \leq (\eta(K \cap E))^a = \eta^{(a)}\big((K \cap E)^a\big),$$

and this together with the definition of $\hat{\phi}$ ensures that $\mathrm{Core}_A(\eta(K \cap E))$ is contained in $\hat{\phi}(E^a \cap \hat{S})$. Thus $\hat{\phi}(E^a \cap \hat{S}) = \mathrm{Core}_A(\eta(K \cap E))$, and on noting that $\eta(K \cap E)$ contains $\eta(D) = \mathrm{Inn}\, F$, part (4) follows.

Remark 6.15. There is a strong sense in which Construction 6.13 followed by Construction 6.8 has the effect of doing nothing. We leave the precise formulation of what we mean by this statement, together with its verification (of which part (3) of the above lemma is an essential part) to the reader. We stress however that the composition in the other direction, namely Construction 6.8 followed by Construction 6.13, does not have a trivial effect.

PROPOSITION 6.16. *Suppose that (C, D, E, K) is a* (6.9)*-tuple of degree $d \geq 2$ with $D \leq K$, and that A is a subgroup of $\mathrm{Aut}\, F$ containing $\eta(K)$* (*where we assume our usual notation in relation to this* (6.9)*-tuple.*) *Let*

$(\hat{T}, F, \hat{S}, \hat{\phi})$ *be the corresponding output of Construction* 6.13. *Then* $(\hat{T}, F, \hat{S}, \hat{\phi})$ *is a* (5.1)-*tuple if and only if the following conditions all hold*:

(i) *for all* $i = 1, \dots, d$

$$\left[C_i^a / C^a\right]_{\hat{S}} \cong \mathsf{M}_0;$$

(ii) *for all* $i = 1, \dots, d$

$$\mathrm{Core}_A(\eta(K \cap E)) = \mathrm{Core}_A\big(\eta_i\big(N_E(C_i, D_i)\big)\big);$$

(iii) *either*

$$\mathrm{Core}_A\big(\eta\big(N_E(C, D)\big)\big) = \mathrm{Core}_A(\eta(K \cap E)) \tag{6.G}$$

or

$$\left[\mathrm{Core}_A\big(\eta\big(N_E(C, D)\big)\big) \big/ \mathrm{Core}_A(\eta(K \cap E))\right]_A \cong \mathsf{M}_0. \tag{6.H}$$

Moreover, if $(\hat{T}, F, \hat{S}, \hat{\phi})$ *is a* (5.1)-*tuple, then it has rank n where*

$$n = \begin{cases} d & \textit{if } (6.G) \textit{ holds}; \\ d + 1 & \textit{otherwise}. \end{cases}$$

Proof. We start by noting that it is easy to see that the tuple $(\hat{T}, F, \hat{S}, \hat{\phi})$ satisfies the conditions of Definition 5.1(i)–(ii). So to prove the first assertion it is enough to show that the three conditions of the statement hold if and only if Definition 5.1(iii) holds.

As in Lemma 6.14(3) we identify E with a subgroup of E^a via the map

$$x \mapsto (x, \mathrm{id}_E, \dots, \mathrm{id}_E) \qquad \text{for all } x \in E.$$

Using Lemma 6.14(1) and Lemma 6.14(4) we see that $(E, E^a, \hat{T}, F, R, \rho)$ is a (3.18)-tuple with $\hat{T}$ equal to $E^a R$ whenever $\rho: R \to \mathrm{Aut}\, F$ is an extension of $\hat{\phi}$ in $\hat{T}$.

In particular $(E, E^a, \hat{T}, F, \hat{S}, \hat{\phi})$ is a (3.18)-tuple. Let the overgroup P of S and the extension $\sigma: P \to \mathrm{Aut}\, F$ of $\hat{\phi}$ in $\hat{T}$ be as defined immediately prior to Lemma 3.19 in terms of this (3.18)-tuple. Note that since $E^a \cap \ker \hat{\phi} = C^a$ by Lemma 6.14(2), we in fact have $P = N_{\hat{T}}(E^a \cap \ker \hat{\phi}, E^a \cap \hat{\phi}^{-1}(\mathrm{Inn}\, F))$ and that σ is defined by requiring that

$$\hat{\phi}(x)^{\sigma(y)} = \hat{\phi}(x^y)$$

whenever $x \in E^a \cap \hat{\phi}^{-1}(\mathrm{Inn}\, F)$ and $y \in P$. Note also that since $\hat{T} = E^a \hat{S}$ we have $P = (E^a \cap P)\hat{S}$, whence σ is a strict extension of $\hat{\phi}$ if and only if $E^a \cap P > E^a \cap \hat{S}$.

Recall that for $i = 1, \ldots, d$ the section (C, D) of E is strictly K-contained in the section (C_i, D_i). For each i we claim that the subgroup C_i^a of E^a is normalised by $\hat{S}$ and that the intersection $C_i^a \cap \hat{S}$ is contained in $\ker \hat{\phi}$. Certainly C_i^a is normalised by $\hat{S}$ as C_i is normalised by K and since $\hat{S} \leq K \wr S_a$. Now if $x \in C_i^a \cap \hat{S}$ then Lemma 6.14(3) implies that $\hat{\phi}(x) = \eta(\kappa(x)) \in \eta(C_i)$; the latter is trivial and so the claim holds. Thus for $i = 1, \ldots, d$ the map $\rho_i : C_i^a\hat{S} \to \operatorname{Aut} F$ given by

$$\rho_i : xs \mapsto \hat{\phi}(s) \qquad \text{for all } x \in C_i^a \text{ and } x \in \hat{S},$$

is a well-defined extension of $\hat{\phi}$ in $\hat{T} = E^a\hat{S}$. Moreover, by Lemma 6.14(2)

$$E^a \cap \ker \hat{\phi} = C^a < C_i^a \leq E^a \cap \ker \rho_i$$

and so each such map is a strict extension of $\hat{\phi}$. For convenience we use R_i to denote the domain $C_i^a\hat{S}$ of ρ_i.

Now let $\rho : R \to \operatorname{Aut} F$ be any extension of $\hat{\phi}$ in T and assume that ρ itself has no strict extensions in $\hat{T}$. Let $x_1 (= \mathrm{id}), \ldots, x_a$ be a right transversal for $N_{\hat{S}}(E)$ in $\hat{S}$. By applying Lemma 3.19 to the (3.18)-tuple $(E, E^a, \hat{T}, F, R, \rho)$ we deduce that

$$E^a \cap \ker \rho = \prod_{i=1}^{a} X^{x_i}$$

for some $X \leq E$. By Lemma 6.14(4) we have $\hat{\phi}(E^a \cap \hat{S}) \geq \operatorname{Inn} F$ and so

$$E^a \cap \rho^{-1}(\operatorname{Inn} F) = (E^a \cap \ker \rho)\big(E^a \cap \hat{\phi}^{-1}(\operatorname{Inn} F)\big).$$

Thus if $X = C$ then $E^a \cap \ker \rho = E^a \cap \ker \hat{\phi}$ and $E^a \cap \rho^{-1}(\operatorname{Inn} F) = E^a \cap \hat{\phi}^{-1}(\operatorname{Inn} F)$, whence it follows that σ, as defined above, is an extension of ρ. On the other hand, if $X > C$ then it is straightforward to see that X is a subgroup of E normalised by K and satisfying $X \cap D = C$, whence the definition of (6.9)-tuples ensures that $X = C_i$ for some $i = 1, \ldots, d$ and it now follows that ρ is an extension of ρ_i (for the same i). Hence the tuple $(\hat{T}, F, \hat{S}, \hat{\phi})$ satisfies Definition 5.1(iii) if and only if all of the following conditions hold:

(a) $\hat{S} <_{\max} R_i$ for all $i = 1, \ldots, d$;

(b) ρ_i has no strict extension in $\hat{T}$ for all $i = 1, \ldots, d$;

(c) either $P = \hat{S}$ or $\hat{S} <_{\max} P$.

Condition (a) is easily seen to be equivalent to condition (i) of the statement. By applying Proposition 3.20 to the (3.18)-tuple $(E, E^a, \hat{T}, F, R_i, \rho_i)$ we see that (b) is equivalent to (ii). Recall that $P = (E^a \cap P)\hat{S}$. In the following we apply Lemma 3.19 to the (3.18)-tuple

$(E, E^a, \hat{T}, F, \hat{S}, \hat{\phi})$. By Lemma 3.19(i) we have

$$E^a \cap \ker \sigma = \prod_{i=1}^{a} (E \cap \ker \eta)^{x_i},$$

where as above $x_1(= \mathrm{id}), \ldots, x_a$ is a right transversal for $N_{\hat{S}}(E)$ in $\hat{S}$. We consider two cases, namely:

(1) $E \cap \ker \eta > C$;

(2) $E \cap \ker \eta = C$.

If (1) holds then $E^a \cap \ker \sigma > E^a \cap \ker \hat{\phi}$ and arguing as above for the arbitrary extension ρ we see that σ is an extension of ρ_i for some i. In particular, $P \geq R_i > \hat{S}$. Hence if (a)–(c), and consequently (i)–(ii) of the statement, all hold, then for some $i = 1, \ldots, d$ we have $P = R_i > \hat{S}$, $\sigma = \rho_i$, and (iii) follows from (ii). Conversely, if conditions (i)–(ii) of the statement hold, then we have already seen that both (a) and (b) hold, whence it follows that $\hat{S} <_{max} R_i = P$, and so (c) holds. On the other hand, if (2) holds, then $E^a \cap \ker \sigma = E^a \cap \ker \hat{\phi}$ and so (c) is equivalent to the condition:

(c)′ either $\hat{\phi}(E^a \cap \hat{S}) = \sigma(E^a \cap P)$ or $\hat{\phi}(E^a \cap \hat{S})$ is a maximal $\hat{\phi}(S)$-invariant subgroup of $\sigma(E^a \cap P)$.

But by Lemmas 3.19(ii) and 6.14(iv) the latter is equivalent to (iii) of the statement. We have now verified the first assertion of the proposition.

We turn to the "Moreover" statement. So we assume that $(\hat{T}, F, \hat{S}, \hat{\phi})$ is a (5.1)-tuple of rank n. In the notation of the above we see that the set of strict extensions of $\hat{\phi}$ in $\hat{T}$ is contained in the set $\{\rho_1, \ldots, \rho_d, \sigma\}$. Conversely, $\rho_1, \ldots, \rho_d$ are certainly distinct strict extensions of $\hat{\phi}$; by inspecting the above arguments, we see that σ is strict extension of $\hat{\phi}$ distinct from $\rho_1, \ldots, \rho_d$ if and only if condition (2) above holds and $\hat{S} \neq P$, and that in turn this conjunction of conditions holds if and only if $\hat{\phi}(E^a \cap \hat{S})$ is a maximal $\hat{\phi}(S)$-invariant subgroup of $\sigma(E^a \cap P)$, or equivalently if (6.H) holds. The "Moreover" statement now follows.

DEFINITION 6.17. We say that the tuple (C, D, E, K, L, A) *satisfies* (6.17), or *is a* (6.17)-*tuple*, if the following conditions (i)–(ix) all hold:

(i) (C, D, E, K) is a (6.9)-tuple of degree $d \geq 2$ with $D \leq K$ (in the following we assume that F, η, $C_1, \ldots, C_d$ and $\eta_1, \ldots, \eta_d$ are defined in terms of the (6.9)-tuple (C, D, E, K) as immediately prior to Lemma 6.12);

(ii) $\eta(K) \leq A \leq \operatorname{Aut} F$ (note that conditions (i) and (ii) are sufficient for the implementation of Construction 6.13: the output tuple $(\hat{T}, F, \hat{S}, \hat{\phi})$ is referred to in condition (vii));

(iii) $K \leq L \leq N_{\operatorname{Aut} E}(C, D)$;

(iv) $\eta(L)A$ is a subgroup of $\operatorname{Aut} F$;

(v) either $\eta(L) \leq A$ or

$$A <_{\max} \eta(L)A, \qquad \eta(K) = A \cap \eta(L), \qquad \text{and}$$
$$K \cap \ker \eta = L \cap \ker \eta;$$

(vi) set $X = E \cap \ker \eta$ and $Y = E \cap \eta^{-1}(\operatorname{Core}_A(\eta(N_E(C, D))))$; then one of the following mutually exclusive conditions holds:

(a) $K = XK = YK = L$;

(b) $K < XK = YK = L$;

(c) $K = XK < YK = L$;

(d) $K = XK = YK < L$;

(vii) for all $i = 1, \ldots, d$

$$\left[C_i^a / C^a\right]_{\hat{S}} \cong M_0;$$

(viii) for all $i = 1, \ldots, d$

$$\operatorname{Core}_A(\eta(K \cap E)) = \operatorname{Core}_A\bigl(\eta_i\bigl(N_E(C_i, D_i)\bigr)\bigr);$$

(ix) if (vi)(c) above holds, then

$$\left[\operatorname{Core}_A\bigl(\eta\bigl(N_E(C, D)\bigr)\bigr) \big/ \operatorname{Core}_A(\eta(K \cap E))\right]_A \cong M_0.$$

Moreover, we say that (C, D, E, K, L, A) is a (6.17)-*tuple of rank* n, if it satisfies (6.17) and

$$n = \begin{cases} d & \text{if (vii)(a) or (vii)(b) holds;} \\ d + 1 & \text{otherwise,} \end{cases}$$

where d is the degree of the (6.9)-tuple (C, D, E, K).

The subset $\Delta(6.17(\text{a}))$ of $\mathbb{N}$ is defined by

$$\Delta(6.17(\text{a})) = \left\{ n \in \mathbb{N} : \begin{array}{c} \text{there exists a (6.17)-tuple of rank } n \\ \text{such that (vi)(a) holds} \end{array} \right\}.$$

The subsets $\Delta(6.17(b))$, $\Delta(6.17(c))$, and $\Delta(6.17(d))$ are defined analogously; the subset $\Delta(6.17)$ is defined by

$$\Delta(6.17) = \Delta(6.17(a)) \cup \Delta(6.17(b)) \cup \Delta(6.17(c)) \cup \Delta(6.17(d)).$$

Remark 6.18. It is immediate from the definition that if (C, D, E, K, L, A) is a (6.17)-tuple of rank n, then (C, D, E, K) is a (6.9)-tuple of degree n if Definition 6.17(vi)(a) or (vi)(b) holds, and of degree $n-1$ otherwise; also if Definition 6.17(vi)(d) holds, then (C, D, E, K, K, A) is a (6.17)-tuple of rank $n-1$ satisfying Definition 6.17(vi)(a). Hence

$$\Delta(6.17(a)) \cup \Delta(6.17(b)) \subseteq \Delta(6.9),$$
$$\Delta(6.17(c)) \cup \Delta(6.17(d)) \subseteq \{n : n-1 \in \Delta(6.9)\}$$

and

$$\Delta(6.17(d)) \subseteq \{n : n-1 \in \Delta(6.17(a))\}.$$

Observe that we use the notation Δ, rather than Ω, as we do not have an exact correspondence between (5.1)-tuples and (6.17)-tuples. Instead, we have the following.

THEOREM 6.19. *Let (T, F, S, ϕ) be a small-*(5.1)*-tuple of rank n with either $T = (\operatorname{Soc} T)S$ or $n \geq 4$. As usual set $N = \operatorname{Soc} T$ and let E be any minimal normal subgroup of N. Let κ, C, D, P, and σ be as defined immediately after Definition* 3.18 *in terms of the* (3.18)*-tuple (E, N, T, F, S, ϕ). Then the tuple*

$$(C, D, E, \kappa(N_S(E)), \kappa(N_T(E)), \phi(S))$$

obtained from (T, F, S, ϕ) via Construction 6.8 *is a* (6.17)*-tuple of rank n. Moreover, the cases* (a)–(d) *of Definition* 6.17(vi) *are respectively equivalent to cases* (1)–(4) *below*:

(1) *$T = NS$ and $S = (N \cap \ker \sigma)S = P$*;

(2) *$T = NS$ and $S < (N \cap \ker \sigma)S = P$*;

(3) *$T = NS$ and $S = (N \cap \ker \sigma)S < P$*;

(4) *$T \neq NS$.*

COROLLARY 6.20. *The following all hold*:

(1) $\Omega(5.1) \subseteq \Delta(6.17)$.

(2) $\Omega(5.1) \subseteq \Delta(6.17(\mathrm{a})) \cup \Delta(6.17(\mathrm{b})) \cup \Delta(6.17(\mathrm{c})) \cup \{n : n-1 \in \Delta(6.17(\mathrm{a}))\}$.

(3) $\{n \geq 16 : n \in \Delta(6.17(\mathrm{a})) \cup \Delta(6.17(\mathrm{b})) \cup \Delta(6.17(\mathrm{c}))\} \subseteq \Omega(5.1)$.

Proof. Recall from (6.B) that

$$\Omega(5.1) = \{n \geq 16 : \text{there exists a small-}(5.1)\text{-tuple of rank } n\}.$$

Part (1) is now immediate from Theorem 6.19, and part (2) follows from part (1) and Remark 6.18.

To see part (3) suppose that (C, D, E, K, L, A) is a (6.17)-tuple of rank n satisfying one of Definition 6.17(vi)(a)–(c). We wish to apply Proposition 6.16 to demonstrate the existence of a (5.1)-tuple of rank n. To achieve this it is enough to show, with respect to (C, D, E, K) and A, that conditions (i) and (ii) of Proposition 6.16 both hold, and that condition (6.G) holds if one of Definition 6.17(vi)(a)–(b) applies, while condition (6.H) holds if Definition 6.17(vi)(c) applies. Now (i) and (ii) of Proposition 6.16 are identical to (vii) and (viii) of Definition 6.17, respectively, and if Definition 6.17(vi)(c) applies, then (6.H) is identical to Definition 6.17(ix). So it remains to show that (6.G) holds if one of Definition 6.17(vi)(a)–(b) applies. In either case we have

$$(E \cap \ker \eta)K = \big(E \cap \eta^{-1}\big(\mathrm{Core}_A\big(\eta\big(N_E(C, D)\big)\big)\big)\big)K;$$

by intersecting both sides with E and then applying η we deduce that

$$\eta(K \cap E) = \mathrm{Core}_A\big(\eta\big(N_E(C, D)\big)\big)\eta(K \cap E).$$

Thus $\eta(K \cap E) \geq \mathrm{Core}_A(\eta(N_E(C, D)))$ and so

$$\mathrm{Core}_A\big(\eta(K \cap E)\big) \geq \mathrm{Core}_A\big(\eta\big(N_E(C, D)\big)\big).$$

As the containment in the reverse direction follows easily from the observation that $K \leq N_{\mathrm{Aut}\,E}(C, D)$, we see that equality, and hence (6.G), holds as required.

The proof of Theorem 6.19 requires the following lemma.

LEMMA 6.21. *Let l be a positive integer and let A, B, H be groups such that*

$$A < B \qquad \textit{and} \qquad H \leq \mathrm{Aut}(B^l)$$

with H transitive on the l direct factors of B^l. Identify B with the first direct factor of B^l and let $L \leq H$ be the normaliser in H of B. Suppose that A^l is *invariant under H and that B does not normalise A. Then*

$$\left[B^l/A^l\right]_H \cong \mathsf{M}_0 \text{ if and only if } \left[B/A\right]_L \cong \mathsf{M}_0.$$

Proof. Firstly note that if A^l is invariant under H, then A is invariant under L and the assertion is sensible.

Now the result in one direction is easy: if A_0 is invariant under L with $A < A_0 < B$ then it is straightforward to see that there is a group $\hat{A}_0$ invariant under H such that $A^l < \hat{A}_0 < B^l$ and $\hat{A}_0 \cong A_0^l$.

We consider the reverse direction. We assume that there are no proper L-invariant subgroups of B that strictly contain A and suppose that X is a strict H-invariant overgroup of A^l in B^l. We must show that $X \geq B^l$ as equality then follows. Let κ be the projection $B^l \to B$ which restricts to give the identity map between the direct factor identified with B and B. As $A^l < X$ we have $A < \kappa(X)$. Now $\kappa(X)$ is certainly invariant under L as X is invariant under H, and so by assumption $\kappa(X) = B$. Recall that B has been identified with a direct factor of B^l; consider $X \cap B$. This is normalised by X, and so also by $\kappa(X) = B$, is invariant under L, and contains A. As B does not normalise A, we have $A < X \cap B$; our assumption then forces $X \cap B = B$. It follows that X contains the conjugates of B under H, and so $X \geq B^l$ as required.

Proof of Theorem 6.19. We suppose that (T, F, S, ϕ) is a small-(5.1)-tuple of rank n with $T = (\operatorname{Soc} T)S$ or $n \geq 4$. As in the statement of the theorem we set $N = \operatorname{Soc} T$ and let E be a minimal normal subgroup of N. Observe that S is transitive on the minimal normal subgroups of T: this is immediate if $T = NS$ and is implied by Theorem 6.7 if $n \geq 4$. (Our reason for not proving a weaker theorem, namely one covering only those small-(5.1)-tuples of rank $n \geq 4$, is that we shall have cause later in the proof to replace (T, F, S, ϕ) by (NS, F, S, ϕ): our hypotheses are thus chosen so that they are still satisfied after such a replacement.) It follows that (E, N, T, F, S, ϕ) is a (3.18)-tuple, as is implicitly assumed in the statement of the theorem. Let κ, C, D, η, and P be as in the statement of the theorem, i.e., as defined in terms of the (3.18)-tuple (E, N, T, F, S, ϕ) as immediately after Definition 3.18. Note that in the proof of Theorem 6.10 we saw that this definition of P and σ is equivalent to that given by Theorem 6.4(ii), that is, $P = N_T(N \cap \ker \phi, N \cap \phi^{-1}(\operatorname{Inn} F))$ and $\sigma\colon P \to \operatorname{Aut} F$ is defined by requiring that

$$\phi(x)^{\sigma(y)} = \phi(x^y) \qquad \text{for all } x \in N \cap \phi^{-1}(\operatorname{Inn} F) \text{ and } y \in P.$$

We recall from (3.A) that $\kappa(N_P(E)) \leq N_{\operatorname{Aut} E}(C, D)$ and that

$$\eta(\kappa(x)) = \sigma(x) \qquad \text{for all } x \in N_P(E).$$

In the following this will often be used without explicit mention. For convenience we set $K = \kappa(N_S(E))$, $L = \kappa(N_P(E))$, and $A = \phi(S)$. Let $x_1, \dots, x_l$ be a right transversal for $N_S(E)$ in S with $x_1 = \mathrm{id}_S$. We consider conditions (i)–(ix) of Definition 6.17.

Definition 6.17(i) follows from Theorema 6.10, while conditions (ii)–(iii) of Definition 6.17 follow by the definition of C, D, K, L, and η. As noted in the first paragraph we have $T = N_T(E)S$, whence $P = N_P(E)S$ and

$$\sigma(P) = \sigma\big(N_P(E)\big)\sigma(S) = \eta\big(\kappa\big(N_P(E)\big)\big)\phi(S) = \eta(L)A.$$

As $\sigma(P)$ is certainly a subgroup of Aut F, Definition 6.17(iv) follows.

We turn to Definition 6.17(v); we suppose that $\eta(L) \nleq A$ whence A is a proper subgroup of $\eta(L)A$. Recall that $\sigma(P) = \eta(L)A$ and that $A = \phi(S) = \sigma(S)$. Using the fact that S is either equal to P or maximal in P we deduce that $A <_{\max} \eta(L)A$ as required, and also that $\ker\sigma = \ker\phi$. Now $\eta(K)$ is clearly contained in both A and $\eta(L)$. To see the reverse containment we choose $w \in \eta(L) \cap A$. Then there exist $x \in N_P(E)$ and $y \in S$ with

$$w = \eta(\kappa(x)) = \phi(y).$$

As σ extends ϕ and as $\eta(\kappa(x)) = \sigma(x)$ it follows that $xy^{-1} \in \ker\sigma$. We have just seen that $\ker\phi = \ker\sigma \le S$, whence $x \in S$ as $y \in S$. We deduce that $x \in N_S(E)$, that $\kappa(x) \in K$, and consequently that

$$w = \eta(\kappa(x)) \in \eta(K)$$

as required. Finally to see the remainder of (v) observe that since $\eta(\kappa(x)) = \sigma(x)$ for all $x \in N_P(E)$ we have

$$K \cap \ker\eta = \kappa\big(N_{\ker\phi}(E)\big) \qquad \text{and} \qquad L \cap \ker\eta = \kappa\big(N_{\ker\sigma}(E)\big).$$

As $\ker\phi = \ker\sigma$ we have equality between the above and Definition 6.17(v) holds.

Before verifying Definition 6.17(vi) we claim that $P = S$ if and only if $K = L$. In one direction this is obvious. To verify the other direction, we suppose that $S < P$: we must show that $K < L$. Let $\rho : R \to \mathrm{Aut}\, F$ be a strict extension of ϕ distinct from σ. (This is possible as the small-(5.1)-tuple (T, F, S, ϕ) has rank $n \ge 3$ by Definition 6.6.) Note that S normalises

$$N_{N \cap \ker\rho}\big(N \cap \ker\phi, N \cap \phi^{-1}(\mathrm{Inn}\, F)\big) = N \cap \ker\rho \cap P.$$

Given that $\rho \neq \sigma$, Theorem 6.4 implies that $R = (N \cap \ker\rho)S$, whence the maximality of S in R forces

$$\big[N \cap \ker\rho / N \cap \ker\phi\big]_S \cong M_0.$$

Thus either $N \cap \ker \rho \leq P$ or

$$N \cap \ker \rho \cap P = N \cap \ker \phi. \tag{6.I}$$

If the former holds, then $R = (N \cap \ker \rho)S \leq P$ and the maximality of S in P forces $R = P$. Choose $x \in N \cap \phi^{-1}(\mathrm{Inn}\, F)$ and $y \in R = P$. Then

$$\phi(x)^{\rho(y)} = \rho(x)^{\rho(y)} = \rho(x^y) = \phi(x^y)$$

as $x^y \in N \cap \phi^{-1}(\mathrm{Inn}\, F)$ and as ρ extends ϕ. On comparing this with the definition of σ we see that $\rho = \sigma$, a contradiction. Hence (6.I) holds.

As σ is a strict extension of ϕ in T, σ itself has no strict extensions in T and by Corollary 3.15, $(N \cap \ker \sigma, N \cap \sigma^{-1}(\mathrm{Inn}\, F))$ is a P-maximal section of N. Given (6.I) this implies that $N \cap \ker \rho$ is not normalised by P. By Theorem 6.10, $N \cap \ker \rho$ is equal to $\prod_{i=1}^{l} \kappa(N \cap \ker \rho)^{x_i}$. As P does not normalise $N \cap \ker \rho$ it is clear that there exist an integer j with $1 \leq j \leq l$ and elements $x \in \kappa(N \cap \ker \rho)$ and $y \in P$ such that

$$x^{x_j y} \notin N \cap \ker \rho.$$

Let $1 \leq k \leq l$ be such that $x_j y x_k^{-1} \in N_T(E)$. Then as $x_j, x_k \in S \leq P$, we have $x_j y x_k^{-1} \in N_P(E)$, $x^{x_j y x_k^{-1}} \in E$, and

$$x^{x_j y x_k^{-1}} \notin E \cap (N \cap \ker \rho)^{x_k^{-1}} = E \cap (N \cap \ker \rho) = \kappa(N \cap \ker \rho).$$

By the definition of κ we have $x^{x_j y x_k^{-1}} = x^{\kappa(x_j y x_k^{-1})}$ whence

$$\kappa\left(x_j y x_k^{-1}\right) \notin N_{\mathrm{Aut}\, E}(\kappa(N \cap \ker \rho)) \qquad \text{and}$$
$$L = \kappa(N_P(E)) \nleq N_{\mathrm{Aut}\, E}(\kappa(N \cap \ker \rho)).$$

As the latter contains K we have $L > K$ as required.

We now consider Definition 6.17(vi). As S is either equal to P or maximal in P it is clear that one, and only one, of the following holds:

(A) $S = (N \cap \ker \sigma)S = (N \cap P)S = P$;

(B) $S < (N \cap \ker \sigma)S = (N \cap P)S = P$;

(C) $S = (N \cap \ker \sigma)S < (N \cap P)S = P$;

(D) $S = (N \cap \ker \sigma)S = (N \cap P)S < P$.

Furthermore, as $T = N_T(E)S$ and as $N \leq N_T(E)$ we see that the above cases can be equivalently described as:

(A)′ $N_S(E) = (N \cap \ker \sigma)N_S(E) = (N \cap P)N_S(E) = N_P(E)$;

(B)′ $N_S(E) < (N \cap \ker \sigma)N_S(E) = (N \cap P)N_S(E) = N_P(E)$;

(C)′ $N_S(E) = (N \cap \ker \sigma) N_S(E) < (N \cap P) N_S(E) = N_P(E)$;

(D)′ $N_S(E) = (N \cap \ker \sigma) N_S(E) = (N \cap P) N_S(E) < N_P(E)$.

Now $K = \kappa(N_S(E))$, $L = \kappa(N_P(E))$, and by Lemma 3.19(i) and (iii) we have

$$X = \kappa(N \cap \ker \sigma) \qquad \text{and} \qquad Y = \kappa(N \cap P),$$

where X and Y are as defined in Definition 6.17(vi). Thus by applying κ to each condition, and noting that by the above claim $P = S$ if and only if $K = L$, we deduce that the conditions (A)′–(D)′ are respectively equivalent to the cases (a)–(d) of Definition 6.17(vi).

Before proceeding with Definition 6.17(vii)–(ix) we assume that they do hold and consider the consequences. Thus (C, D, E, K, L, A) is a (6.17)-tuple. Using the information that either $S = P$ or $S <_{\max} P$ and that $T = NP$ (since (T, F, S, ϕ) is a small-(5.1)-tuple), it is easy to see that the cases (1)–(4) of the theorem are respectively equivalent to cases (A)–(D) above. The "Moreover" statement follows.

Now observe that one of (1) and (2) holds if and only if $P = (N \cap \ker \sigma)S$. From this together with Theorem 6.10 it follows that (C, D, E, K, L, A) is indeed a (6.17)-tuple of rank n. We conclude that the theorem holds provided only that the remaining conditions (vii)–(ix) of Definition 6.17 all hold.

Note that Definition 6.17(vii)–(ix) depend only on the (6.9)-tuple (C, D, E, K) and the subgroup $A = \phi(S)$ of $\operatorname{Aut} F$, and not on the subgroup L of $\operatorname{Aut} E$. Note also that if we replace the small-(5.1)-tuple (T, F, S, ϕ) by the small-(5.1)-tuple (NS, F, S, ϕ) then the (6.9)-tuple (C, D, E, K) and the subgroup A remain unchanged. Hence we may assume that $T = NS$.

By Theorem 3.3 the map $T \to (\operatorname{Aut} E) \wr S_l$ given by

$$x \mapsto \left(\kappa\left(x_1 x x_{1\pi}^{-1}\right), \ldots, \kappa\left(x_l x x_{l\pi}^{-1}\right)\right)\pi,$$

where $\pi \in S_l$ is such that $x_i x x_{i\pi}^{-1} \in N_T(E)$, is a monomorphism. We use this to identify T with a subgroup of $(\operatorname{Aut} E) \wr S_l$. Note that under this identification we have

$$\begin{aligned} N &= \{(y_1, \ldots, y_l) : y_i \in E\} \cong E^l, \\ E &= \{(y, \mathrm{id}, \ldots, \mathrm{id}) : y \in E\} \cong E, \\ N_T(E) &= T \cap \{(y_1, \ldots, y_l)\pi \in (\operatorname{Aut} E) \wr S_l : 1\pi = 1\}, \\ S &\le K \wr S_l \le (\operatorname{Aut} E) \wr S_l, \end{aligned}$$

and the map $\kappa : N_T(E) \to \operatorname{Aut} E$ is given by

$$\kappa : x \mapsto y_1$$

where $x = (y_1, \dots, y_l)\pi \in N_T(E)$. (These statements depend on our assumption that $x_1, \dots, x_l$ is a right transversal for $N_S(E)$ in S with $x_1 = \mathrm{id}_S$.) Also note that by Theorem 6.10 we have

$$N \cap \ker \phi = \prod_{i=1}^{l} C^{x_i},$$

whence given the above identification

$$N \cap \ker \phi = \{(y_1, \dots, y_l) : y_i \in C\} = C^l.$$

It is clear that $K \wr S_l$, and so also S, normalises the subgroup $(K \cap \ker \eta)^l$ of $(\operatorname{Aut} E) \wr S_l$. Set $S_0 = (K \cap \ker \eta)^l S$ and observe that NS_0 is a well-defined subgroup of $(\operatorname{Aut} E) \wr S_l$ as N is a normal subgroup of the latter. If $x \in (K \cap \ker \eta)^l \cap S$, then $x \in N_S(E)$ and

$$\phi(x) = \eta(\kappa(x)) = \mathrm{id}.$$

Hence $(K \cap \ker \eta)^l \cap S \leq \ker \phi$ and the map $\phi_0 : S_0 \to \operatorname{Aut} F$ given by

$$\phi_0 : xs \mapsto \phi(s) \qquad \text{for all } x \in (K \cap \ker \eta)^l \text{ and } s \in S$$

is well-defined. We claim that (NS_0, F, S_0, ϕ_0) is small-(5.1)-tuple of the same rank as (T, F, S, ϕ), and moreover, that if we replace (T, F, S, ϕ) by (NS_0, F, S_0, ϕ_0) then the (6.9)-tuple (C, D, E, K) and the subgroup A of $\operatorname{Aut} F$ remain unchanged. This is straightforward and left to the reader. Hence we may assume that $S \geq (K \cap \ker \eta)^l$.

As in the proof of Theorem 6.10 for $i = 1, \dots, n$ we let $\rho_i : R_i \to \operatorname{Aut} F$ be the n strict extensions of ϕ in T labelled so that $\rho_n = \sigma$ if $n = d + 1$, where d is the degree of the (6.9)-tuple (C, D, E, K), and set $C_i = \kappa(N \cap \ker \rho_i)$. Given the current identification of T with a subgroup of $(\operatorname{Aut} E) \wr S_l$ we deduce from Theorems 6.10 and 6.4 that

$$N \cap \ker \rho_i = C_i^l \qquad \text{for all } i = 1, \dots, n$$

and that $R_i = C_i^l S$ for all $i = 1, \dots, d$.

We turn to Definition 6.17(vii). The maximality of S in each R_i implies that

$$\left[C_i^l S / S\right] \cong M_0 \qquad \text{for all } i = 1, \dots, d. \tag{6.J}$$

Given that for $i = 1, \dots, d$ we have $C_i^l S = (C_i(K \cap \ker \eta))^l S$ while $S \cap \ker \rho_i = \ker \phi$ implies that

$$S \cap C_i^l = C^l \qquad \text{and} \qquad S \cap \left(C_i(K \cap \ker \eta)\right)^l = (K \cap \ker \eta)^l,$$

we deduce that (6.J) is equivalent to either of

$$\left[C_i^l / C^l\right]_S \cong M_0 \qquad \text{for all } i = 1, \dots, d, \tag{6.K}$$

$$\left[\left(C_i(K \cap \ker \eta)\right)^l / (K \cap \ker \eta)^l\right]_S \cong M_0 \qquad \text{for all } i = 1, \dots, d. \tag{6.L}$$

On the other hand, if $a = |A : \eta(K)|$ and $\hat{S}$ is defined in terms of (C, D, E, K) and A as in Construction 6.13, then given that $(K \cap \ker \eta)^a \leq \hat{S}$ we see that Definition 6.17(vii) is equivalent to either (6.K) or (6.L), but with l replaced by a and with S replaced by $\hat{S}$. If for all $i = 1, \dots, d$ the subgroup $C_i(K \cap \ker \eta)$ does not normalise $K \cap \ker \eta$, then Definition 6.17(vii) follows via two applications of Lemma 6.21—the first deducing from (6.L) that $[C_i(K \cap \ker \eta)/(K \cap \ker \eta)]_K \cong M_0$ for $i = 1, \dots, d$, and the second deducing from this that (6.L) holds, but with l replaced by a and with S replaced by $\hat{S}$. Hence we may assume that for some $i = 1, \dots, d$ that $K \cap \ker \eta$ is normalised by $C_i(K \cap \ker \eta)$. With this assumption we claim that $\ker \phi \leq N_S(E)$.

The claim essentially follows from the observation that, as $\ker \phi$ is normal in S, the $\ker \phi$-orbits on minimal normal subgroups of N form a system of imprimitivity for the action of S on such subgroups—this system of imprimitivity, if non-trivial, can be used to construct a strict S-invariant overgroup of $(K \cap \ker \eta)^l$ that is strictly contained in $(C_i(K \cap \ker \eta))^l$, contrary to (6.L). More formally, to see the claim we proceed as follows. We assume that the transversal $x_1, \dots, x_l$ for $N_S(E)$ in S is chosen so that for all $j, k = 1, \dots, l$

$$x_j x_k^{-1} \in N_S(E)(\ker \phi) \qquad \text{if and only if} \qquad x_j x_k^{-1} \in \ker \phi.$$

(This can be achieved by firstly choosing a right transversal for $N_{\ker \phi}(E)$ in $\ker \phi$ and a right transversal for $N_S(E)(\ker \phi)$ in S, and then combining these to give a right transversal for $N_S(E)$ in S.) Define the subgroup W of $(C_i(K \cap \ker \eta))^l$ by

$$W = \left\{(y_1, \dots, y_l) \in \left(C_i(K \cap \ker \eta)\right)^l : y_j y_k^{-1} \in K \cap \ker \eta \right.$$
$$\left. \text{whenever } x_j x_k^{-1} \in \ker \phi \right\}.$$

Observe that W strictly contains $(K \cap \ker \eta)^l$, and moreover, that W is a proper subgroup of $(C_i(K \cap \ker \eta))^l$ if and only if $\ker \phi \nleq N_S(E)$. Hence, given that (6.L) holds, to verify the claim it suffices to show that W is normalised by S. Choose $x \in S$ and $y = (y_1, \ldots, y_l) \in W$. Let $\pi \in S_l$ be such that $x_j x x_{j\pi}^{-1} \in N_S(E)$ for all $j = 1, \ldots, l$ so that x is identified with the element

$$\left(\kappa\left(x_1 x x_{1\pi}^{-1}\right), \ldots, \kappa\left(x_l x x_{l\pi}^{-1}\right)\right)\pi$$

of $(\mathrm{Aut}\, E) \wr S_l$. Thus

$$y^x = \left(y_{1\pi^{-1}}^{\kappa(x_{1\pi^{-1}} x x_1^{-1})}, \ldots, y_{l\pi^{-1}}^{\kappa(x_{l\pi^{-1}} x x_l^{-1})}\right).$$

Now

$$x_{j\pi} x_{k\pi}^{-1} = x_{j\pi} x^{-1} x_j^{-1} x_j x_k^{-1} x_k x x_{k\pi}^{-1} \in N_S(E) x_j x_k^{-1} N_S(E).$$

Recalling that $\ker \phi$ is a normal subgroup of $S \geq N_S(E)$, we see that $x_j x_k^{-1}$ is in $N_S(E)(\ker \phi)$ if and only if $x_{j\pi} x_{k\pi}^{-1} \in N_S(E)(\ker \phi)$, and so by the choice of transversal, $x_j x_k^{-1} \in \ker \phi$ if and only if $x_{j\pi} x_{k\pi}^{-1} \in \ker \phi$. Suppose $x_j x_k^{-1} \in \ker \phi$; to show that $y^x \in W$ we must show that

$$y_{j\pi^{-1}}^{\kappa(x_{j\pi^{-1}} x x_j^{-1})} \left(y_{k\pi^{-1}}^{\kappa(x_{k\pi^{-1}} x x_k^{-1})}\right)^{-1} \in K \cap \ker \eta.$$

Now

$$y_{j\pi^{-1}}^{\kappa(x_{j\pi^{-1}} x x_j^{-1})} \left(y_{k\pi^{-1}}^{\kappa(x_{k\pi^{-1}} x x_k^{-1})}\right)^{-1} = \left(y_{j\pi^{-1}}^{\kappa(x_{j\pi^{-1}} x x_j^{-1} x_k x^{-1} x_{k\pi^{-1}}^{-1})} y_{k\pi^{-1}}^{-1}\right)^{\kappa(x_{k\pi^{-1}} x x_k^{-1})}.$$

But

$$\begin{aligned}
\eta\left(\kappa\left(x_{j\pi^{-1}} x x_j^{-1} x_k x^{-1} x_{k\pi^{-1}}^{-1}\right)\right) &= \phi\left(x_{j\pi^{-1}} x x_j^{-1} x_k x^{-1} x_{k\pi^{-1}}^{-1}\right) \\
&= \phi\left(x_{j\pi^{-1}} x x^{-1} x_{k\pi^{-1}}^{-1}\right), \qquad \text{as } x_j x_k^{-1} \in \ker \phi, \\
&= \phi\left(x_{j\pi^{-1}} x_{k\pi^{-1}}^{-1}\right) \\
&= \mathrm{id}_{\mathrm{Aut}\, F}, \qquad \text{as } x_{j\pi^{-1}} x_{k\pi^{-1}}^{-1} \in \ker \phi,
\end{aligned}$$

whence $\kappa(x_{j\pi^{-1}} x x_j^{-1} x_k x^{-1} x_{k\pi^{-1}}^{-1}) \in K \cap \ker \eta$. Hence

$$\begin{aligned}
&y_{j\pi^{-1}}^{\kappa(x_{j\pi^{-1}} x x_j^{-1})} \left(y_{k\pi^{-1}}^{\kappa(x_{k\pi^{-1}} x x_k^{-1})}\right)^{-1} \\
&\quad \in \left((K \cap \ker \eta) y_{j\pi^{-1}} y_{k\pi^{-1}}^{-1}\right)^{\kappa(x_{k\pi^{-1}} x x_k^{-1})}, \quad \text{since } C_i \text{ normalises } K \cap \ker \eta, \\
&\quad \in (K \cap \ker \eta)^{\kappa(x_{k\pi^{-1}} x x_k^{-1})}, \qquad \text{since } x_{j\pi^{-1}} x_{k\pi^{-1}}^{-1} \in \ker \phi, \\
&\quad \in K \cap \ker \eta, \qquad \text{since } K \cap \ker \eta \text{ is normal in } K = \kappa\left(N_S(E)\right).
\end{aligned}$$

We have now shown that S normalises W and so have verified the claim that $\ker \phi \leq N_S(E)$. We deduce that $\phi(x_1), \ldots, \phi(x_l)$ is a right transversal for $\eta(K)$, which equals $\phi(N_S(E))$, in $A = \phi(S)$. It follows that $l = a$, and more significantly, that if $\hat{S}$ is constructed using the transversal $\phi(x_1), \ldots, \phi(x_l)$, then $S = \hat{S}$. Definition 6.17(vii) holds as it is now identical to (6.K) and as it does not depend on the transversal chosen in Construction 6.13.

We turn to Definition 6.17(viii). Note that for each $i = 1, \ldots, d$ the tuple $(E, N, T, F, R_i, \rho_i)$ is a (3.18)-tuple such that ρ_i has no strict extensions in T. Applying Proposition 3.20 we see that

$$\rho_i(N \cap R_i) = \mathrm{Core}_{\rho_i(R_i)}\big(\eta_i(N_E(C_i, D_i))\big) \qquad \text{for all } i = 1, \ldots, d,$$

where η_i is as defined immediately prior to Lemma 6.12. Now for each $i = 1, \ldots, d$ we have $R_i = C_i^l S$ and $C_i^l = N \cap \ker \rho_i$, whence $\rho_i(R_i) = \rho_i(S) = \phi(S) = A$ and

$$\rho_i(N \cap R_i) = \rho_i\big(C_i^l(N \cap S)\big) = \rho_i(N \cap S) = \phi(N \cap S).$$

Clearly $\kappa(N \cap S) \leq K \cap E$, while $\phi(N \cap S)$ is a normal subgroup of $\phi(S) = A$. Hence $\phi(N \cap S) = \eta(\kappa(N \cap S))$ is certainly contained in the A-core of $\eta(K \cap E)$. Summarising we have

$$\mathrm{Core}_A\big(\eta_i(N_E(C_i, D_i))\big) = \rho_i(N \cap R_i) = \phi(N \cap S) \leq \mathrm{Core}_A(\eta(K \cap E))$$

for all $i = 1, \ldots, d$. Recalling that η and η_i agree on $K \leq N_{\mathrm{Aut}\, E}(C_i, D_i)$, it is immediate that $\eta(K \cap E)$ is contained in $\eta_i(N_E(C_i, D_i))$, whence the A-core of the former is contained in the A-core of the latter. Definition 6.17(viii) follows. We also deduce that $\phi(N \cap S)$ is equal to $\mathrm{Core}_A(\eta(K \cap E))$.

Finally we turn to Definition 6.17(ix). We assume that Definition 6.17(vi)(c) holds, or equivalently that (C) above holds, i.e., that

$$S = (N \cap \ker \sigma)S < (N \cap P)S = P.$$

The maximality of S in P implies that

$$\big[N \cap P / N \cap S\big]_S \cong M_0. \tag{6.M}$$

Now $S = (N \cap \ker \sigma)S$ forces $N \cap \ker \phi = N \cap \ker \sigma$, whence (6.M) holds if and only if

$$\big[\sigma(N \cap P)/\phi(N \cap S)\big]_{\phi(S)} \cong M_0.$$

At the end of the previous paragraph we saw that $\phi(N \cap S) = \mathrm{Core}_A(\eta(K \cap E))$, while by applying Lemma 3.19 to the (3.18)-tuple (E, N, T, F, S, ϕ)

we have that

$$\sigma(N \cap P) = \mathrm{Core}_A\big(\eta\big((N_E(C, D))\big)\big).$$

Definition 6.17(ix) follows and we are finished.

7. THE T-COMPLEMENT CASE: SUBCASE $\phi(S) \not\geq \mathrm{Inn}\, F$

In this section we study the problem of determining $\Omega(5.2)$. We start with some easy consequences of the definition of (5.2)-tuples.

LEMMA 7.1. *Let (T, F, S, ϕ) be a* (5.2)*-tuple, and let $\rho : R \to \mathrm{Aut}\, F$ be a strict extension of ϕ in T. Then the following all hold*:

(i) *T is a maximal subgroup of the twisted wreath product $F\, \mathrm{twr}_\rho T$*;

(ii) *the twisted wreath product $F\, \mathrm{twr}_\rho T$ in its action on the coset space $(F\, \mathrm{twr}_\rho T : T)$ is a primitive permutation group with a non-abelian regular normal subgroup*;

(iii) *the socle $\mathrm{Soc}\, T$ of T is non-abelian and is the unique minimal normal subgroup of T*;

(iv) $\rho(\mathrm{Soc}\, T \cap R) \geq \mathrm{Inn}\, F$;

(v) $R = (\mathrm{Soc}\, T \cap \rho^{-1}(\mathrm{Inn}\, F))S$;

(vi) $R = (\mathrm{Soc}\, T \cap R)S$.

Proof. By Definition 5.2(v) and Corollary 3.7, the top group T is maximal in $F\, \mathrm{twr}_\rho T$ and (i) follows. The action of $F\, \mathrm{twr}_\rho T$ on the (right) cosets of T is thus primitive. As T complements B_ρ in $F\, \mathrm{twr}_\rho T$ we see that B_ρ acts both regularly and faithfully in this action (that is, B_ρ is transitive and meets every point-stabilizer trivially). We can therefore identify the cosets of T with elements of B_ρ so that the action of T becomes that of conjugation on B_ρ. Hence the kernel of the action is equal to $C_T(B_\rho)$ which by Lemma 3.9 equals $\mathrm{Core}_T(\ker \rho)$. The latter is trivial as by Definition 5.2(ii) and (v), the kernel $\ker \rho$ is a core-free subgroup of T. Given that F is a non-abelian simple group, whence B_ρ is indeed a non-abelian regular normal subgroup, we see that (ii) holds. That (iii) and (iv) hold follows from (ii) together with [3, 5.4]. From (iv) we see that

$$\rho\big(\mathrm{Soc}\, T \cap \rho^{-1}(\mathrm{Inn}\, F)\big) = \mathrm{Inn}\, F,$$

whence by Definition 5.2(i), $\mathrm{Soc}\, T \cap \rho^{-1}(\mathrm{Inn}\, F) \not\leq S$. Now S normalises $\mathrm{Soc}\, T \cap \rho^{-1}(\mathrm{Inn}\, F)$ and so $(\mathrm{Soc}\, T \cap \rho^{-1}(\mathrm{Inn}\, F))S$ is a strict overgroup of S in R. Part (v) follows as by Definition 5.2(v) $S <_{\max} R$. Finally (vi) is an immediate consequence of (v).

Recall that a (5.2)-tuple has rank n if and only if there exist $n-1$ strict extensions of ϕ in T. In the following corollary to Lemma 5.5 we consider (5.2)-tuples of rank $n \geq 2$ so that such strict extensions exist.

COROLLARY 7.2. *Let* (T, F, S, ϕ) *be a* (5.2)-*tuple of rank* $n \geq 2$, *and for* $i = 1, \dots, n-1$ *let* $\rho_i : R_i \to \operatorname{Aut} F$ *be the strict extensions of* ϕ *in* T. *Suppose that* $|T|$ *is minimal among all such* (5.2)-*tuples*. *Then*

$$T = \langle R_1, \dots, R_{n-1} \rangle.$$

Proof. Set $X = \langle R_1, \dots, R_{n-1} \rangle$. The result follows from Lemma 5.5 provided that

$$\operatorname{Core}_X\big(\phi^{-1}(\operatorname{Inn} F)\big) \leq \ker \phi.$$

In fact, since $R_1 \leq X$ it is enough to show that

$$\operatorname{Core}_{R_1}\big(\phi^{-1}(\operatorname{Inn} F)\big) \leq \ker \phi.$$

By Definition 5.2(v), $\ker \phi = \ker \rho_1$ and on applying the homomorphism ρ_1 we see that the above holds if and only if $\phi(S) \cap \operatorname{Inn} F$ is a core-free subgroup of $\rho_1(R_1)$. But the latter is true, since by Definition 5.2(v) the image $\rho_1(R_1)$ is almost simple with socle $\operatorname{Inn} F$, while by Definition 5.2(i) the intersection $\phi(S) \cap \operatorname{Inn} F$ is a proper subgroup of $\operatorname{Inn} F$.

The conclusion of the above result turns out to be fundamental in what follows and for convenience we give the following definition.

DEFINITION 7.3. We say that the tuple (T, F, S, ϕ) is a *small*-(5.2)-*tuple of rank* n if $n \geq 2$, the tuple is a (5.2)-tuple of rank n, and the conclusion of Corollary 7.2 holds, i.e.,

$$T = \langle R_1, \dots, R_{n-1} \rangle,$$

where for $i = 1, \dots, n-1$ the maps $\rho_i : R_i \to \operatorname{Aut} F$ are the strict extensions of ϕ in T.

PROPOSITION 7.4. *Let* (T, F, S, ϕ) *be a small*-(5.2)-*tuple of rank* n, *and for* $i = 1, \dots, n-1$ *let* $\rho_i : R_i \to \operatorname{Aut} F$ *be the strict extensions of* ϕ *in* T. *Then the following all hold*:

(i) $T = (\operatorname{Soc} T)S$;

(ii) $\ker \phi$ *is trivial*;

(iii) *for each* $i = 1, \dots, n-1$ *the inverse image* $\rho_i^{-1}(\operatorname{Inn} F)$ *is a subgroup of* $\operatorname{Soc} T$ *isomorphic to* $\operatorname{Inn} F$ *via* ρ_i, *and moreover*,

$$\operatorname{Soc} T = \left\langle \rho_1^{-1}(\operatorname{Inn} F), \dots, \rho_{n-1}^{-1}(\operatorname{Inn} F) \right\rangle.$$

Proof. By Lemma 7.1(vi) we have $R_i \leq (\operatorname{Soc} T)S$ for all $i = 1, \ldots, n-1$, whence

$$\langle R_1, \ldots, R_{n-1} \rangle \leq (\operatorname{Soc} T)S \leq T.$$

Equality follows by the definition of small-(5.2)-tuples, and so (i) holds.

We turn to (ii); by Definition 5.2(v), $\ker \phi = \ker \rho_i$ for all $i = 1, \ldots, n-1$ and so $\ker \phi$ is normalised by $\langle R_1, \ldots, R_{n-1} \rangle$. As the latter is equal to T by assumption we see that $\ker \phi$ is a normal subgroup of T whence by Definition 5.2(ii), $\ker \phi$ is trivial as required.

To see (iii) note that by (ii) and Definition 5.2(v) the kernel $\ker \rho_i$ is trivial for all $i = 1, \ldots, n-1$, whence the inverse image $\rho_i^{-1}(\operatorname{Inn} F)$ is isomorphic to a subgroup of $\operatorname{Inn} F$ via ρ_i. However, by Lemma 7.1(iv), $\operatorname{Soc} T \cap \rho_i^{-1}(\operatorname{Inn} F)$ is a subgroup of $\rho_i^{-1}(\operatorname{Inn} F)$ with $\operatorname{Inn} F$ as a homomorphic image, whence equality holds and

$$\rho_i^{-1}(\operatorname{Inn} F) \leq \operatorname{Soc} T \qquad \text{for all } i = 1, \ldots, n-1.$$

Finally by repeating the argument used to prove (i), but using Lemma 7.1(v) in place of Lemma 7.1(vi), we see that

$$T = \langle \rho_1^{-1}(\operatorname{Inn} F), \ldots, \rho_{n-1}^{-1}(\operatorname{Inn} F) \rangle S.$$

As $\langle \rho_1^{-1}(\operatorname{Inn} F), \ldots, \rho_{n-1}^{-1}(\operatorname{Inn} F) \rangle$ is normalised by both itself and by S, we see that it is a normal subgroup of T. Noting that $\langle \rho_1^{-1}(\operatorname{Inn} F), \ldots, \rho_{n-1}^{-1}(\operatorname{Inn} F) \rangle$ is non-trivial and is contained in $\operatorname{Soc} T$, which by Lemma 7.1(iii) is a minimal normal subgroup of T, we deduce that

$$\operatorname{Soc} T = \left\langle \rho_1^{-1}(\operatorname{Inn} F), \ldots, \rho_{n-1}^{-1}(\operatorname{Inn} F) \right\rangle$$

and are finished.

Suppose that (T, F, S, ϕ) is a small-(5.2)-tuple of rank n so that the conclusions of Lemma 7.1 and Proposition 7.4 all hold. Let E be a minimal normal subgroup of $\operatorname{Soc} T$. As $\operatorname{Soc} T$ is non-abelian and minimal normal in T the group E is non-abelian and simple, and $\operatorname{Soc} T$ is the direct product of the T-conjugates of E. Let $\kappa : N_T(E) \to \operatorname{Aut} E$ be induced by conjugation; we identify E with $\operatorname{Inn} E$ in the usual way so that κ restricts to give the identity on E. As usual for $i = 1, \ldots, n-1$ let $\rho_i : R_i \to \operatorname{Aut} F$ be the strict extensions of ϕ in T. Now by Proposition 7.4(iii) the inverse image $\rho_i^{-1}(\operatorname{Inn} F)$ is a subgroup of $\operatorname{Soc} T$ isomorphic to $\operatorname{Inn} F$ via ρ_i. Note that by Proposition 7.4(i) the group S, and so also each R_i, is transitive on the minimal normal subgroups of $\operatorname{Soc} T$. This means that the projections of $\rho_i^{-1}(\operatorname{Inn} F)$ onto each simple direct factor of $\operatorname{Soc} T$

are isomorphic to each other. As $\rho_i^{-1}(\operatorname{Inn} F)$ is non-trivial we see that $\kappa(\rho_i^{-1}(\operatorname{Inn} F))$ is non-trivial. Furthermore, as $\rho_i^{-1}(\operatorname{Inn} F)$ is simple we see that κ restricts to give an isomorphism between $\rho_i^{-1}(\operatorname{Inn} F)$ and its image under κ. (For later reference we note also that $\rho_i^{-1}(\operatorname{Inn} F)$ meets every proper normal subgroup of $\operatorname{Soc} T$ trivially.) For each $i = 1, \dots, n-1$ we define the monomorphism $\alpha_i : \operatorname{Inn} F \to E$ by

$$\alpha_i(\rho_i(x)) = \kappa(x) \qquad \text{for all } x \in \rho_i^{-1}(\operatorname{Inn} F).$$

We also set $L = \phi(S)$.

DEFINITION 7.5. The tuple $(E, F, \alpha_1, \dots, \alpha_{n-1}, L)$ as defined in the above discussion is referred to as the tuple *obtained* from the small-(5.2)-tuple (T, F, S, ϕ).

In Theorem 7.10 we see that the small-(5.2)-tuple (T, F, S, ϕ) is recoverable from the tuple $(E, F, \alpha_1, \dots, \alpha_{n-1}, L)$ obtained from it. However, before defining exactly what we mean by recoverable we wish to make explicit the more important properties of the tuple $(E, F, \alpha_1, \dots, \alpha_{n-1}, L)$.

DEFINITION 7.6. Let m be a positive integer. We say that $(E, F, \alpha_1, \dots, \alpha_m, L)$ is a (7.6)-*tuple* if the following all hold:

(i) E and F are non-abelian simple groups;

(ii) $\alpha_1, \dots, \alpha_m$ are distinct monomorphisms $F \to E$ such that their images generate E, that is, $E = \langle \alpha_1(F), \dots, \alpha_m(F) \rangle$;

(iii) for each $i = 1, \dots, m$ the section $(\{\mathrm{id}\}, \alpha_i(F))$ is a maximal section of E;

(iv) L is a subgroup of $\operatorname{Aut} F$ such that

$$\left[F/\{\mathrm{id}_F\}\right]_L \cong M_1;$$

(v) for all $i, j = 1, \dots, m$ we have

$$\alpha_i|_{L \cap \operatorname{Inn} F} = \alpha_j|_{L \cap \operatorname{Inn} F}.$$

(Note that in (v) we have implicitly assumed our usual identification between F and $\operatorname{Inn} F$.)

Furthermore, a (7.6)-tuple $(E, F, \alpha_1, \dots, \alpha_m, L)$ is said to be either a (7.6(a))-*tuple*, or a (7.6(b))-tuple, depending on which of the following holds:

(a) $E \ncong F$;

(b) $E \cong F$.

We refer to the integer m as the degree of the (7.6)-tuple, and define the subsets $\Delta(7.6(\mathrm{a}))$, $\Delta(7.6(\mathrm{b}))$, and $\Delta(7.6)$ of $\mathbb{N}$ by

$$\Delta(7.6(\mathrm{a})) = \{n \geq 16 : \text{there exists a } (7.6(\mathrm{a}))\text{-tuple of degree } n-1\},$$
$$\Delta(7.6(\mathrm{b})) = \{n \geq 16 : \text{there exists a } (7.6(\mathrm{b}))\text{-tuple of degree } n-1\},$$

and

$$\Delta(7.6) = \Delta(7.6(\mathrm{a})) \cup \Delta(7.6(\mathrm{b})).$$

We remark that the notation and terminology, Δ and degree, have been used to emphasize the fact that we do not have a complete correspondence between (5.2)-tuples and (7.6)-tuples. We do however have the following.

THEOREM 7.7. $\Omega(5.2) \subseteq \Delta(7.6)$.

The theorem is proved by supposing that (T, F, S, ϕ) is a small-(5.2)-tuple of degree $n \geq 2$, and then showing that the tuple $(E, F, \alpha_1, \dots, \alpha_{n-1}, L)$ obtained from (T, F, S, ϕ) is a (7.6)-tuple. However, we note that the tuple $(E, F, \alpha_1, \dots, \alpha_{n-1}, L)$ is not uniquely defined by the small-(5.2)-tuple (T, F, S, ϕ): indeed, it is defined only up to the choice of the minimal normal subgroup E of $\operatorname{Soc} T$ and up to ordering of the maps $\alpha_1, \dots, \alpha_{n-1}$.

DEFINITION 7.8. Let the tuple $(E, F, \alpha_1, \dots, \alpha_m, L)$ be either a (7.6)-tuple or be a tuple obtained from some small-(5.2)-tuple; let $(D, F, \beta_1, \dots, \beta_l, L)$ be another such tuple. We say that the two tuples are *equivalent* if there exists an isomorphism $\chi : D \to E$ such that

$$\{\alpha_1, \dots, \alpha_m\} = \{\chi \circ \beta_1, \dots, \chi \circ \beta_l\},$$

where $\chi \circ \beta_i$ denotes the composition of β_i followed by χ.

Observe that two (7.6)-tuples are equivalent only if they have the same degree; observe also that if two tuples are equivalent and one is a (7.6)-tuple, then so is the other.

Proof of Theorem 7.7. As noted above we prove the theorem by supposing that (T, F, S, ϕ) is a small-(5.2)-tuple of degree $n \geq 2$, and then showing that the tuple $(E, F, \alpha_1, \dots, \alpha_{n-1}, L)$ obtained from (T, F, S, ϕ) is a (7.6)-tuple, i.e., that conditions (i)–(v) of Definition 7.6 all hold. (Note that by the above observations it does not matter which tuple obtained from (T, F, S, ϕ) is considered as they are clearly equivalent to each other.)

It is clear that Definition 7.6(i) holds. We consider Definition 7.6(ii). Given our usual identification between F and $\operatorname{Inn} F$, we certainly have that $\alpha_1, \dots, \alpha_{n-1}$ are monomorphisms $F \to E$. Suppose that $\alpha_i = \alpha_j$. We claim that this forces $\rho_i = \rho_j$. To see this we start by considering $\rho_i^{-1}(\mathrm{Inn}\, F)$ which we recall is a subgroup of $\mathrm{Soc}\, T$ isomorphic to $\mathrm{Inn}\, F$ and such that the projection κ restricts to give an isomorphism between $\rho_i^{-1}(\mathrm{Inn}\, F)$ and $\alpha_i(F)$. Now as noted above S is transitive on the simple direct factors of $\mathrm{Soc}\, T$. Set $l = |S : N_S(E)|$; the transitivity of S implies that there exist $s_1(= \mathrm{id}_S), \dots, s_l \in S$ with

$$\mathrm{Soc}\, T = E^{s_1} \times \cdots \times E^{s_l},$$

whence for all $x \in \mathrm{Soc}\, T$ we have

$$x = \left(\kappa\left(x^{s_1^{-1}}\right)\right)^{s_1} \cdots \left(\kappa\left(x^{s_l^{-1}}\right)\right)^{s_l}. \tag{7.A}$$

Choose $x \in \rho_i^{-1}(\mathrm{Inn}\, F) \le \mathrm{Soc}\, T$. For each $s \in S$ we have $x^s \in \rho_i^{-1}(\mathrm{Inn}\, F)$ since S normalises $\rho_i^{-1}(\mathrm{Inn}\, F)$. Now $\rho_i(x^s) = \rho_i(x)^{\phi(s)}$ since ρ_i is a homomorphism extending ϕ, while by the definition of α_i we have

$$\alpha_i(\rho_i(x^s)) = \kappa(x^s).$$

Hence for each $k = 1, \dots, l$ we have

$$\kappa\left(x^{s_k^{-1}}\right) = \alpha_i\left(\rho_i(x)^{\phi(s_k^{-1})}\right).$$

This, together with (7.A), implies that

$$x = \left(\alpha_i\left(\rho_i(x)^{\phi(s_1^{-1})}\right)\right)^{s_1} \cdots \left(\alpha_i\left(\rho_i(x)^{\phi(s_l^{-1})}\right)\right)^{s_l}. \tag{7.B}$$

Suppose now that $\tilde{x} \in \rho_j^{-1}(\mathrm{Inn}\, F)$ is such that $\rho_j(\tilde{x}) = \rho_i(x)$. As $\alpha_i = \alpha_j$, inspection of (7.B) shows that $x = \tilde{x}$. We conclude that not only are the subgroups $\rho_i^{-1}(\mathrm{Inn}\, F)$ and $\rho_j^{-1}(\mathrm{Inn}\, F)$ identical, but also that ρ_i and ρ_j agree on these subgroups. Since they also agree on S we deduce from Lemma 7.1(v) that $\rho_i = \rho_j$. Since the ρ_i are distinct, it follows that $\alpha_1, \dots, \alpha_{n-1}$ are also all distinct.

To see that E is generated by the images $\alpha_1(F), \dots, \alpha_{n-1}(F)$ note that by Proposition 7.4(iii)

$$\mathrm{Soc}\, T = \left\langle \rho_i^{-1}(\mathrm{Inn}\, F), \dots, \rho_{n-1}^{-1}(\mathrm{Inn}\, F)\right\rangle;$$

the required result follows by applying κ to both sides and recalling that $\alpha_i(F)$ is equal to $\kappa(\rho_i^{-1}(\mathrm{Inn}\, F))$ by the definition of α_i. We have now shown that Definition 7.6(ii) holds.

To see Definition 7.6(iii) we note that for each $i = 1, \dots, n-1$ the tuple $(E, \mathrm{Soc}\, T, T, F, R_i, \rho_i)$ is a (3.18)-tuple with $T = (\mathrm{Soc}\, T)R_i$, and such that ρ_i has no strict extensions in T; Definition 7.6(iii) then follows from

Proposition 3.20(i). Definition 7.6(iv) is immediate from Remark 5.4. Finally Definition 7.6(v) is an easy consequence of the fact that the restrictions of each ρ_i to $\rho_i^{-1}(\mathrm{Inn}\, F) \cap S$ are all equal.

We must stress here that the concept of a (7.6(a))-tuple is an extremely restricted one: certainly Definition 7.6(iv) imposes a great restriction on L and F, and if $E \not\cong F$, then Definition 7.6(iii) severely limits the possibilities for E and the monomorphisms $\alpha_1, \dots, \alpha_m$. Our intuitive feeling is that the set $\Delta(7.6(\mathrm{a}))$ is likely to be empty. On the other hand, if $E \cong F$ then Definition 7.6(iii) is trivially satisfied, and the concept of a (7.6(b))-tuple is not a useful one. Indeed, we note in Remark 8.5 that $\Delta(7.6(\mathrm{b})) = \{n \in \mathbb{N} \colon n \geq 16\}$! Thus we must establish what extra conditions must be satisfied by a (7.6)-tuple so that it is a (7.6)-tuple obtained from some small-(5.2)-tuple. The key to doing this is the already advertised result, Theorem 7.10, which says (among other things) that if $(E, F, \alpha_1, \dots, \alpha_{n-1}, L)$ is the (7.6)-tuple of degree $n-1$ obtained from the small-(5.2)-tuple (T, F, S, ϕ) of rank n, then (T, F, S, ϕ) is recoverable from $(E, F, \alpha_1, \dots, \alpha_{n-1}, L)$. To be precise about what is meant by "recoverable" we need the following construction.

Construction 7.9. The input to this construction is a (7.6)-tuple

$$(E, F, \alpha_1, \dots, \alpha_m, L)$$

of degree m, and the output is a tuple (T, F, S, ϕ). En route to constructing the output tuple we shall have cause to construct various objects: in their order of definition these are denoted

$$\eta_1, \dots, \eta_m, \eta, K, \alpha, l, \psi.$$

In addition to these objects which are an integral part of the construction we shall also define maps χ and ι that will be useful later. For convenience, we will often subsequently refer to the input tuple as x, to the output tuple as $\Gamma(x)$, and to the components of the output tuple and associated objects as T_x, F_x etc.

So we start with a (7.6)-tuple $(E, F, \alpha_1, \dots, \alpha_m, L)$ of degree m. For each $i = 1, \dots, m$ we define a homomorphism $\eta_i : N_{\mathrm{Aut}\, E}(\alpha_i(F)) \to \mathrm{Aut}\, F$ by requiring that for all $x \in N_{\mathrm{Aut}\, E}(\alpha_i(F))$ the automorphism $\eta_i(x)$ of F is such that the following diagram commutes:

$$\begin{array}{ccc} F & \xrightarrow{\eta_i(x)} & F \\ \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \alpha_i} \\ E & \xrightarrow{\text{conjugation by } x} & E \end{array} \tag{7.C}$$

We define a homomorphism $\eta : \bigcap_{i=1}^{m} N_{\mathrm{Aut}\, E}(\alpha_i(F)) \to (\mathrm{Aut}\, F)^m$ by

$$\eta(x) = \big(\eta_1(x), \dots, \eta_m(x)\big) \qquad \text{for all } x \in \bigcap_{i=1}^{m} N_{\mathrm{Aut}\, E}\big(\alpha_i(F)\big). \quad (7.\mathrm{D})$$

Observe that $x \in \mathrm{Aut}\, E$ is in the kernel of η if and only if x centralizes $\alpha_i(F)$ for all $i = 1, \dots, m$; by Definition 7.6(ii) this holds if and only if x centralizes E. We conclude that η is a monomorphism.

Let K be the subgroup of L given by

$$K = \{x \in L : (x, \dots, x) \in \mathrm{Im}\, \eta\}, \qquad (7.\mathrm{E})$$

and define a map $\alpha : K \to \bigcap_{i=1}^{m} N_{\mathrm{Aut}\, E}(\alpha_i(F))$ by requiring that

$$\eta(\alpha(x)) = (x, \dots, x) \qquad \text{for all } x \in K; \qquad (7.\mathrm{F})$$

observe that α is a well-defined monomorphism, and moreover that for all $i = 1, \dots, m$

$$\alpha_i(y^x) = \alpha_i(y)^{\alpha(x)} \qquad \text{for all } y \in F \text{ and } x \in K. \qquad (7.\mathrm{G})$$

We deduce from Definition 7.6(v) that $L \cap \mathrm{Inn}\, F \le K$ and moreover that

$$\alpha_i|_{L \cap \mathrm{Inn}\, F} = \alpha|_{L \cap \mathrm{Inn}\, F} \qquad \text{for all } i = 1, \dots, m. \qquad (7.\mathrm{H})$$

Set $l = |L : K|$ and choose a right transversal $x_1, \dots, x_l$ for K in L. Define a map $\psi : L \to \mathrm{Aut}\, E \wr S_l$ by

$$\psi : x \mapsto \Big(\alpha\big(x_1 x x_{1\pi}^{-1}\big), \dots, \alpha\big(x_l x x_{l\pi}^{-1}\big)\Big)\pi \qquad \text{for all } x \in L, \qquad (7.\mathrm{I})$$

where $\pi \in S_l$ is such that $x_i x x_{i\pi}^{-1} \in K$ for all $i = 1, \dots, l$. Theorem 3.3 shows that ψ is a monomorphism. Let $\phi : \psi(L) \to \mathrm{Aut}\, F$ be inverse to ψ. Note that $\psi(L)$ normalises $(\mathrm{Inn}\, E)^l \le \mathrm{Aut}\, E \wr S_l$; let T be the subgroup of $\mathrm{Aut}\, E \wr S_l$ given by $T = (\mathrm{Inn}\, E)^l \psi(L)$.

Finally we set $S = \psi(L)$ so that the output tuple (T, F, S, ϕ) of the construction has now been defined.

Before moving on we define in terms of the (7.6)-tuple x two more objects, namely χ and ι which play a role later.

Define the homomorphism $\chi : N_{\mathrm{Aut}\, E}(\alpha(L \cap \mathrm{Inn}\, F)) \to \mathrm{Aut}(L \cap \mathrm{Inn}\, F)$ by requiring that for each $x \in N_{\mathrm{Aut}\, E}(\alpha(L \cap \mathrm{Inn}\, F))$ the automorphism $\chi(x)$ of $L \cap \mathrm{Inn}\, F$ is such that the following diagram commutes:

$$\begin{tikzcd} L \cap \mathrm{Inn}\, F \arrow[r, "\chi(x)"] \arrow[d, "\alpha"] & L \cap \mathrm{Inn}\, F \arrow[d, "\alpha"] \\ E \arrow[r, "\text{conjugation by } x"] & E \end{tikzcd} \qquad (7.\mathrm{J})$$

Also define the homomorphism $\iota : N_{\mathrm{Aut}\, F}(L \cap \mathrm{Inn}\, F) \to \mathrm{Aut}(L \cap \mathrm{Inn}\, F)$ to be that induced by the conjugation action on $L \cap \mathrm{Inn}\, F$. We observe that $\alpha(K)$ is contained in $N_{\mathrm{Aut}\, E}(\alpha(L \cap \mathrm{Inn}\, F))$, and moreover that

$$y^{\chi(\alpha(x))} = y^x = y^{\iota(x)} \qquad \text{for all } x \in K \text{ and } y \in L \cap \mathrm{Inn}\, F.$$

THEOREM 7.10. *Let $x = (E, F, \alpha_1, \ldots, \alpha_m, L)$ and y be equivalent* (7.6)-*tuples of degree m, and let $\Gamma(x)$, $\Gamma(y)$ be the outputs of Construction* 7.9 *as applied to x, y, respectively. Then the following all hold.*

(i) *if $\Gamma(x)$ is a* (5.2)-*tuple of rank $m + 1$, then both $\Gamma(x)$ and $\Gamma(y)$ are small-*(5.2)-*tuples and moreover, the* (7.6)-*tuple obtained from $\Gamma(x)$ is equivalent to x*;

(ii) *if x is obtained from some small-*(5.2)-*tuple of rank $m + 1$, then $\Gamma(x)$ is a small-*(5.2)-*tuple of rank $m + 1$*;

(iii) *the tuple $\Gamma(x)$ is a* (5.2)-*tuple of rank $m + 1$ if and only if x is obtained from some small-*(5.2)-*tuple of rank $m + 1$.*

The proof of Theorem 7.10 uses the following two lemmas.

LEMMA 7.11. *Let $x = (E, F, \alpha_1, \ldots, \alpha_m, L)$ be a* (7.6)-*tuple of degree m. Let $\Gamma(x) = (T, F, S, \phi)$ be the output of Construction* 7.9 *as applied to x. In the following we use the notation of Construction* 7.9, *in particular $x_1(= \mathrm{id}), \ldots, x_l$ is the right transversal for K in L chosen in the process of constructing $\Gamma(x)$.*

For each $i = 1, \ldots, m$ let $\beta_i : F \to (\mathrm{Inn}\, E)^l \leq T$ be given by

$$\beta_i(x) = \left(\alpha_i(x^{x_1^{-1}}), \ldots, \alpha_i(x^{x_l^{-1}})\right) \qquad \textit{for all } x \in F.$$

(*Note that β_i is a well-defined monomorphism as for $j = 1, \ldots, l$ each α_j is a monomorphism $F \to E$ and each x_j is an automorphism of F.*) *Then the images $\beta_1(F), \ldots, \beta_m(F)$ are all normalised by S, and moreover, for $i = 1, \ldots, m$ the maps $\rho_i : \beta_i(F)S \to \mathrm{Aut}\, F$ given by*

$$\rho_i : \beta_i(x)y \mapsto x\phi(y) \qquad \textit{for all } x \in F \textit{ and } y \in S, \tag{7.K}$$

are well-defined distinct monomorphisms strictly extending ϕ in T.

Proof. We fix i and choose $x \in F$ and $y \in L$ so that $\psi(y) \in \psi(L) = S$. Let $\pi \in S_l$ be such that $x_j y x_{j\pi}^{-1} \in K$ for all $j = 1, \ldots, l$. By the definition of ψ, (7.I),

$$\psi(y) = \left(\alpha\left(x_1 y x_{1\pi}^{-1}\right), \ldots, \alpha\left(x_l y x_{l\pi}^{-1}\right)\right)\pi$$

whence

$$\beta_i(x)^{\psi(y)} = \Big(\alpha_i\big(x^{x_{1\pi^{-1}}^{-1}}\big)^{\alpha(x_{1\pi^{-1}}yx_1^{-1})}, \dots, \alpha_i\big(x^{x_{l\pi^{-1}}^{-1}}\big)^{\alpha(x_{l\pi^{-1}}yx_l^{-1})}\Big).$$

Now $\alpha(K)$ is contained in $N_{\mathrm{Aut}\,E}(\alpha_i(F))$ and by the definition of η_i, (7.C), we have

$$\alpha_i\big(x^{x_{j\pi^{-1}}^{-1}}\big)^{\alpha(x_{j\pi^{-1}}yx_j^{-1})} = \alpha_i\Big(\big(x^{x_{j\pi^{-1}}^{-1}}\big)^{\eta_i(\alpha(x_{j\pi^{-1}}yx_j^{-1}))}\Big)$$

for all $j = 1, \dots, l$. But α is defined so that $\eta_i(\alpha(t)) = t$ for all $t \in K$ and so

$$\beta_i(x)^{\psi(y)} = \big(\alpha_i(x^{yx_1^{-1}}), \dots, \alpha_i(x^{yx_l^{-1}})\big) = \beta_i(x^y). \qquad \text{(7.L)}$$

Thus $\beta_i(F)$ is normalised by $\psi(L) = S$.

To see that ρ_i is a well-defined homomorphism we must show, firstly that

$$\rho_i\big(\beta_i(x)^y\big) = x^{\phi(y)} \qquad \text{for all } x \in F \text{ and } y \in S,$$

and secondly that if $y = \beta_i(x) \in \beta_i(F) \cap S$ then $x = \phi(y)$, or equivalently given the definition of ϕ, that if $x \in F$ and $z \in L$ are such that $\beta_i(x) = \psi(z)$ then $x = z$. The former follows immediately from the definitions of ϕ and ρ_i and from (7.L). To see the latter suppose $x \in F$ and $z \in L$ are such that $\beta_i(x) = \psi(z)$. Now

$$\psi(z) = \big(\alpha\big(x_1zx_{1\pi}^{-1}\big), \dots, \alpha\big(x_lzx_{l\pi}^{-1}\big)\big)\pi,$$

where $\pi \in S_l$ satisfies $x_jzx_{j\pi}^{-1} \in K$ for all $j = 1, \dots, l$. By comparing this with the expression given for $\beta_i(x)$ we deduce that $\pi = \mathrm{id}$, whence $x_jzx_j^{-1} \in K$ for all $j = 1, \dots, l$, and that $\alpha(x_jzx_j^{-1}) \in \alpha_i(F)$ for all $j = 1, \dots, l$. In particular, and on recalling that $x_1 = \mathrm{id}$, we have $\alpha(z) \in \alpha_i(F)$. From (7.G) we deduce that conjugation by $\alpha(z)$ is an inner automorphism of $\alpha_i(F)$ if and only if conjugation by z is an inner automorphism of F. Hence $\alpha(z) \in \alpha_i(F)$ implies that $z \in \mathrm{Inn}\, F$. On the other hand, if $z \in L \cap \mathrm{Inn}\, F \leq F$, then by using (7.H) it is easy to see that $\psi(z) = \beta_i(z)$, and so $\beta_i(x) = \psi(z)$ forces $x = z$ as β_i is a monomorphism.

To see that ρ_i is a monomorphism, suppose that $x \in F$ and $z \in L$ are such that $\beta_i(x)\psi(z) \in \ker \rho_i$. Then

$$\mathrm{id} = \rho_i\big(\beta_i(x)\psi(z)\big) = xz,$$

whence $z = x^{-1} \in L \cap \mathrm{Inn}\, F$. As noted above, this means that $\psi(z) = \beta_i(z)$, whence $\beta_i(x)\psi(z) = \mathrm{id}$ and the kernel of ρ_i is trivial as required.

It is clear from its definition that ρ_i extends ϕ in T. Moreover ρ_i strictly extends ϕ as $\rho_i(\beta_i(F)S) \geq \rho_i(\beta_i(F)) = \operatorname{Inn} F$ while $\phi(S) = L$ which by Definition 7.6(iv) and Corollary 3.25 does not contain $\operatorname{Inn} F$.

Finally to see that $\rho_1, \dots, \rho_m$ are distinct we let κ be the projection map $(\operatorname{Inn} E)^l \to \operatorname{Inn} E$ given by

$$(y_1, \dots, y_l) \mapsto y_1 \qquad \text{for all } y_1, \dots, y_l \in \operatorname{Inn} E.$$

Observe that the map $F \to E$ given by

$$x \mapsto \kappa(\beta_i(x)) \qquad \text{for all } x \in F$$

is equal both to α_i (given our usual identification between E and $\operatorname{Inn} E$) and to the composition of ρ_i^{-1} followed by κ. As $\alpha_1, \dots, \alpha_m$ are distinct we see that $\rho_1, \dots, \rho_m$ are also distinct.

LEMMA 7.12. *Let E be a non-abelian simple group with subgroups $K_1, \dots, K_m$, and let l be a positive integer. For each $i = 1, \dots, m$ let $\beta_{i1}, \dots, \beta_{il}$ be automorphisms of K_i. Define subgroups $V_1, \dots, V_m$ of E^l by*

$$V_i = \left\{ (k^{\beta_{i1}}, \dots, k^{\beta_{il}}) : k \in K_i \right\} \cong K_i.$$

Then $E^l = \langle V_1, \dots, V_m \rangle$ if and only if both of the following hold:

(i) $E = \langle K_1, \dots, K_m \rangle$;

(ii) *there do not exist integers $1 \leq j < k \leq l$ and an automorphism $\beta \in \operatorname{Aut} E$ such that $\beta \in \bigcap_{i=1}^m N_{\operatorname{Aut} E}(K_i)$ and such that for each $i = 1, \dots, m$*

$$x^{\beta_{ij}\beta} = x^{\beta_{ik}} \qquad \textit{for all } x \in K_i.$$

Proof. The necessity of the two conditions is easy to see. To see that they are also sufficient we assume that they both hold and set $H = \langle V_1, \dots, V_m \rangle$. For each $i = 1, \dots, l$, let $\pi_i : E^l \to E$ be the projection map

$$(e_1, \dots, e_l) \mapsto e_i.$$

Now for each $i = 1, \dots, l$

$$\pi_i(H) = \langle \pi_i(V_1), \dots, \pi_i(V_m) \rangle = \langle K_1, \dots, K_m \rangle$$

whence by condition (i), H is a subgroup of E^l projecting onto each simple direct factor. A standard argument (see for instance the lemma on p. 328 of [20]) shows that H is the direct product of full diagonal subgroups, and we deduce that $H < E^l$ if and only if there exist integers $1 \leq j < k \leq l$

and an automorphism $\beta \in \operatorname{Aut} E$ such that $H \leq X$ where X is given by

$$X = \left\{ (x_1, \dots, x_l) \in E^l : x_j^{\beta} = x_k \right\}.$$

Now for $i = 1, \dots, m$, the subgroup V_i is contained in X if and only if

$$x^{\beta_{ij}\beta} = x^{\beta_{ik}} \qquad \text{for all } x \in K_i.$$

If the latter holds for a given i, then

$$K_i^{\beta} = \left\{ x^{\beta_{ij}} : x \in K_i \right\}^{\beta} = \left\{ x^{\beta_{ik}} : x \in K_i \right\} = K_i,$$

that is, β normalises K_i. Hence condition (ii) implies that one of $V_1, \dots, V_m$ is not contained in X, whence $H = E^l$ as required.

Proof of Theorem 7.10. We assume the notation of Construction 7.9; in particular, we assume that $x_1 (= \mathrm{id}), \dots, x_l$ is the right transversal for K in L chosen in the process of constructing the tuple $\Gamma(x)$. For $i = 1, \dots, m$ we let β_i and ρ_i be as in Lemma 7.11.

We start with part (i) and assume that $\Gamma(x) = (T, F, S, \phi)$ is a (5.2)-tuple of rank $m + 1$. By Lemma 7.11 the maps $\rho_1, \dots, \rho_m$ are distinct strict extensions of ϕ in T, and so they are the only strict extensions of ϕ in T. Thus to show that $\Gamma(x)$ is a small-(5.2)-tuple we must show that

$$T = \langle \beta_1(F)S, \dots, \beta_m(F)S \rangle.$$

By construction $T = (\operatorname{Inn} E)^l S$ and so it is enough to show that

$$(\operatorname{Inn} E)^l = \langle \beta_1(F), \dots, \beta_m(F) \rangle.$$

To do this we aim to apply Lemma 7.12. For $i = 1, \dots, m$ set $K_i = \alpha_i(F)$. Recall that for $x \in F$

$$\beta_i(x) = \left(\alpha_i(x^{x_1^{-1}}), \dots, \alpha_i(x^{x_l^{-1}}) \right).$$

For $i = 1, \dots, m$ and $j = 1, \dots, l$ let β_{ij} be the automorphism of $K_i = \alpha_i(F)$ given by

$$\beta_{ij}(\alpha_i(x)) = \alpha_i(x^{x_j^{-1}}) \qquad \text{for all } x \in F. \tag{7.M}$$

By Lemma 7.12 we have $(\operatorname{Inn} E)^l = \langle \beta_1(F), \dots, \beta_m(F) \rangle$ if and only if conditions (i) and (ii) of Lemma 7.12 both hold. Lemma 7.12(i) is precisely Definition 7.6(ii) and so holds as x is a (7.6)-tuple. Suppose that Lemma 7.12(ii) does not hold, i.e., that there exist integers $1 \leq j < k \leq l$ and an automorphism $\beta \in \bigcap_{i=1}^{m} N_{\operatorname{Aut} E}(\alpha_i(F))$ such that for each $i = 1, \dots, m$

$$x^{\beta_{ij}\beta} = x^{\beta_{ik}} \qquad \text{for all } x \in \alpha_i(F).$$

As $\beta \in \bigcap_{i=1}^{m} N_{\mathrm{Aut}\, E}(\alpha_i(F))$ we can consider the image of β under $\eta_1, \dots, \eta_m$ and η. Recalling the definition of η_i we see that

$$\alpha_i(x^{\eta_i(\beta)}) = \alpha_i(x)^{\beta} = \alpha_i(x)^{\beta_{ij}^{-1}\beta_{ik}} \qquad \text{for all } x \in F.$$

But (7.M) implies that for $x \in F$

$$\alpha_i(x)^{\beta_{ij}^{-1}\beta_{ik}} = \beta_{ik}\left(\beta_{ij}^{-1}(\alpha_i(x))\right) = \beta_{ik}(\alpha_i(x^{x_j})) = \alpha_i(x^{x_j x_k^{-1}}).$$

It follows that $\eta_i(\beta) = x_j x_k^{-1}$ and is independent of i. Hence

$$\eta(\beta) = \left(x_j x_k^{-1}, \dots, x_j x_k^{-1}\right)$$

and the definition of K forces $x_j x_k^{-1} \in K$. This is impossible as x_j, x_k are distinct elements in a transversal for K in L. We conclude that Lemma 7.12(ii) holds as required.

Having shown that $\Gamma(x)$ is a small-(5.2)-tuple we must show that the (7.6)-tuple obtained from $\Gamma(x)$ is equivalent to x. By construction $S = \psi(L)$ is a subgroup of $(\mathrm{Aut}\, E) \wr S_l$ acting transitively on the l components of $(\mathrm{Aut}\, E)^l$ and so $T = (\mathrm{Inn}\, E)^l \psi(L)$ contains $(\mathrm{Inn}\, E)^l$ as a minimal normal subgroup. As by assumption $\Gamma(x)$ is a (5.2)-tuple and so has a unique minimal normal subgroup by Lemma 7.1(iii), we have $\mathrm{Soc}\, T = (\mathrm{Inn}\, E)^l$. To construct the (7.6)-tuple obtained from $\Gamma(x)$ we start by choosing a minimal normal subgroup of $\mathrm{Soc}\, T$. We choose this to be E where we identify E with the subgroup

$$\{(x, \mathrm{id}, \dots, \mathrm{id}) : x \in \mathrm{Inn}\, E\}$$

of $(\mathrm{Inn}\, E)^l$ in the obvious way. With this choice of minimal normal subgroup of $\mathrm{Soc}\, T$ and recalling that the maps $\rho_1, \dots, \rho_m$ given in Lemma 7.11 are the strict extensions of ϕ in T, it is now straightforward to see that the (7.6)-tuple obtained from $\Gamma(x)$ is, up to possible reordering of $\alpha_1, \dots, \alpha_m$, precisely $x = (E, F, \alpha_1, \dots, \alpha_m, L)$. To see that (i) holds it remains only to show that $\Gamma(y)$ is also a small-(5.2)-tuple, given that y is a (7.6)-tuple equivalent to x. By the definition of equivalence (7.8) we have $y = (D, F, \chi \circ \alpha_1, \dots, \chi \circ \alpha_m, L)$ for some isomorphism $\chi : E \to D$. It is straightforward to construct, using the isomorphism χ, an isomorphism $\tilde{\chi} : T_x \to T_y$ such that $\tilde{\chi}(S_x) = S_y$ and such that

$$\phi_x(w) = \phi_y(\tilde{\chi}(w)) \qquad \text{for all } w \in S_x.$$

Having already shown that $\Gamma(x)$ is a small-(5.2)-tuple, it is now immediate that $\Gamma(y)$ is also a small-(5.2)-tuple. Hence (i) holds.

We turn to (ii). We assume that x is obtained from a small-(5.2)-tuple (T, F, S, ϕ), and to distinguish this tuple from the tuple $\Gamma(x)$ we attach a subscript x to the components of $\Gamma(x)$, that is, we write $\Gamma(x) = (T_x, F_x, S_x, \phi_x)$. To show that (ii) holds, it is clearly enough, given that $F = F_x$, to show that there exists an isomorphism $\xi : T \to T_x$ with $\xi(S) = S_x$ and

$$\phi_x(\xi(y)) = \phi(y) \qquad \text{for all } y \in S.$$

Now as $x = (E, F, \alpha_1, \dots, \alpha_m, L)$ is obtained from (T, F, S, ϕ) we have that E is a minimal normal subgroup of $\operatorname{Soc} T$. Let $\kappa : N_T(E) \to \operatorname{Aut} E$ be induced by the conjugation action of $N_T(E)$ on E. Since $\operatorname{Soc} T \le N_T(E)$, Proposition 7.4(i) implies that $T = N_T(E)S$ and so a right transversal for $N_S(E)$ in S is also a right transversal for $N_T(E)$ in T. Set $k = |S : N_S(E)|$ and let $y_1, \dots, y_k$ be a right transversal for $N_S(E)$ in S. Define a map $\xi : T \to (\operatorname{Aut} E) \wr S_k$ by

$$\xi : x \mapsto \left(\kappa\left(y_1 x y_{1\pi}^{-1}\right), \dots, \kappa\left(y_k x y_{k\pi}^{-1}\right)\right)\pi \qquad \text{for all } x \in T, \tag{7.N}$$

where $\pi \in S_k$ is such that $y_i x y_{i\pi}^{-1} \in N_T(E)$ for all $i = 1, \dots, k$. Theorem 3.3 shows that ξ is a homomorphism and that $\ker \xi = \operatorname{Core}_T(C_T(E))$. Now $\operatorname{Core}_T(C_T(E))$ is equal to $C_T(\operatorname{Soc} T)$ since a normal subgroup of T centralizing E must also centralize the T-conjugates of E and since $\operatorname{Soc} T$ is the direct product of the T-conjugates of E. From Lemma 7.1(iii) we deduce that $\ker \xi$ is trivial, whence ξ is a monomorphism. Observe also that $\xi(\operatorname{Soc} T) = (\operatorname{Inn} E)^k$ whence by Proposition 7.4(i), $\xi(T) = (\operatorname{Inn} E)^k\, \xi(S)$. We claim that $k = l$ and that $y_1, \dots, y_l$ can be chosen so that $\xi(S) = S_x$ and

$$\phi_x(\xi(y)) = \phi(y) \qquad \text{for all } y \in S.$$

Given the observation that $\xi(T) = (\operatorname{Inn} E)^k\, \xi(S)$, verification of this claim is enough to prove that (ii) holds.

To see the claim we start by showing that $N_S(E) = \phi^{-1}(K)$ where K is as defined by (7.E). The containment $N_S(E) \subseteq \phi^{-1}(K)$ is straightforward. To see the reverse containment we suppose that there exists $x \in S \setminus N_S(E)$ with $\phi(x) \in K$, and argue for a contradiction. Let $\rho_1, \dots, \rho_m$ be the strict extensions of ϕ in T, and let $V_1, \dots, V_m$ be the subgroups of $\rho_1^{-1}(\operatorname{Inn} F), \dots, \rho_m^{-1}(\operatorname{Inn} F)$, which by Proposition 7.4(iii) are subgroups of $\operatorname{Soc} T$ isomorphic to $\operatorname{Inn} F$ and that generate $\operatorname{Soc} T$. Note that x normalises each of the V_i and that $\alpha(\phi(x))$ is an automorphism of E normalising each of the images $\kappa(V_i) = \alpha_i(F)$, where α is given by (7.F). In fact the definition of α is such that

$$\kappa(y^x) = \kappa(y)^{\alpha(\phi(x))} \qquad \text{for all } y \in V_i,\ i = 1, \dots, m.$$

From this we deduce that $\operatorname{Soc} T \neq \langle V_1, \ldots, V_m \rangle$, which gives the required contradiction, either by choosing any identification between $\operatorname{Soc} T$ and E^k and then applying Lemma 7.12, or by adapting the argument given in Lemma 7.12 to the present notation.

On recalling that ϕ is a monomorphism we see that

$$k = |S : N_S(E)| = |\phi(S) : K| = |L : K| = l$$

as required. Moreover we see that if $y_1, \ldots, y_l$ are chosen so that

$$\phi(y_i) = x_i \qquad \text{for all } i = 1, \ldots, l,$$

where we recall that $x_1, \ldots, x_l$ is the given right transversal for K in L, then $y_1, \ldots, y_l$ is a right transversal for $N_S(E)$ in S. It is now a routine calculation to verify that the claim holds given this choice of transversal.

Finally we note that (iii) is an immediate consequence of (i) and (ii).

Theorem 7.10 means that instead of studying (5.2)-tuples of rank n, we can study (7.6)-tuples x of degree $n - 1$ such that the tuple $\Gamma(x)$ obtained via Construction 7.9 is a (5.2)-tuple of rank n. Our next task in this section is to find necessary and sufficient conditions on the (7.6)-tuple x for $\Gamma(x)$ to be a (5.2)-tuple.

DEFINITION 7.13. We say that the tuple $x = (E, F, \alpha_1, \ldots, \alpha_m, L)$ *satisfies* (7.13), or *is a* (7.13)-*tuple*, if x is a (7.6)-tuple and if the following conditions all hold (in which K, α, $\eta_1, \ldots, \eta_m$, ι, χ are as defined in Construction 7.9):

(i) if β is a monomorphism $F \to E$ such that $\beta(F)$ is normalised by $\alpha(K)$ and such that for all $x \in F$ and $y \in K$

$$\beta(x^y) = \beta(x)^{\alpha(y)},$$

then $\beta = \alpha_i$ for some $i = 1, \ldots, m$;

(ii) for each $i = 1, \ldots, m$ the section $(\{\mathrm{id}_E\}, \alpha_i(F))$ of E is $\alpha(K)$-maximal;

(iii) for each $i = 1, \ldots, m$

$$\operatorname{Core}_{(\operatorname{Inn} F)L}\Big(\eta_i\big(N_E\big(\alpha_i(F)\big)\big)\Big) = (\operatorname{Inn} F)\operatorname{Core}_L\big(\alpha^{-1}(E)\big)$$

(where $\alpha^{-1}(E) = \{x \in K : \alpha(x) \in E\}$);

(iv) one of the following holds:

(a) $L \cap \operatorname{Inn} F$ is a non-abelian simple group, the section $(\{\mathrm{id}\}, \alpha(L \cap \operatorname{Inn} F))$ of E is $\alpha(K)$-maximal, and

$$\operatorname{Core}_{\iota(L)}\Big(\chi\big(N_E\big(\alpha(L \cap \operatorname{Inn} F)\big)\big)\Big) = \iota\Big(\operatorname{Core}_L\big(\alpha^{-1}(E)\big)\Big);$$

(b) $K = L$, $N_E(\alpha(L \cap \operatorname{Inn} F)) \leq \alpha(K)$, and if D is any minimal normal subgroup of $L \cap \operatorname{Inn} F$, then

$$C_E(\alpha(D)) \leq \alpha(K)$$

and the section $(C_E(\alpha(D)), \alpha(D)C_E(\alpha(D)))$ of E is $\alpha(K)$-maximal.

Moreover, we say that x is a (7.13)-*tuple of rank* n, if it satisfies (7.13) and $n = m + 1$, where m is the degree of x as a (7.6)-tuple.

The subset $\Omega(7.13)$ of $\mathbb{N}$ is defined by

$$\Omega(7.13) = \{n \geq 16 : \text{there exists a } (7.13)\text{-tuple of rank } n\}.$$

Remark 7.14. In the course of proving Theorem 7.15 below we see that Definition 7.13(ii) is implied by Definition 7.13(iv). Thus Definition 7.13(ii) could be omitted in the above definition. However, it has not been so because we feel that Definition 7.13(ii) provides a starting point in determining $\Omega(7.13)$ by directing attention towards maximal non-abelian simple sections of non-abelian simple groups.

Before giving the results that justify the above definition we pause to consider Definition 7.13(iv). Let $x = (E, F, \alpha_1, \dots, \alpha_m, L)$ be a (7.6)-tuple, and assume the notation of Construction 7.9. Clearly the following cases are exhaustive and mutually exclusive:

(A) $K = L$ and $L \cap \operatorname{Inn} F$ is non-abelian and simple;

(B) $K \neq L$ and $L \cap \operatorname{Inn} F$ is non-abelian and simple;

(C) $K = L$ and $L \cap \operatorname{Inn} F$ is either abelian or not simple;

(D) $K \neq L$ and $L \cap \operatorname{Inn} F$ is either abelian or not simple.

Obviously if (D) applies then neither Definition 7.13(iv)(a) nor (iv)(b) can hold; if (C) applies then only Definition 7.13(iv)(b) can hold; if (B) applies then only Definition 7.13(iv)(a) can hold. However, if (A) applies then it appears that either can hold. In fact the situation is simpler than it may first appear as if (A) holds, then Definition 7.13(iv)(b) is implied by Definition 7.13(iv)(a).

To see this suppose that (A) does hold. Further suppose that Definition 7.13(iv)(a) holds. Let D be a minimal normal subgroup of $L \cap \operatorname{Inn} F$. As the latter is simple we have $D = L \cap \operatorname{Inn} F$. Now $\alpha(L \cap \operatorname{Inn} F) \cong L \cap \operatorname{Inn} F$ is non-abelian and simple, and so has a trivial centre. Hence $C_E(\alpha(L \cap \operatorname{Inn} F))$ meets $\alpha(L \cap \operatorname{Inn} F)$ trivially. As $C_E(\alpha(L \cap \operatorname{Inn} F))$ is normalised by $\alpha(K)$ we see that the section

$$\big(C_E(\alpha(L \cap \operatorname{Inn} F)), C_E(\alpha(L \cap \operatorname{Inn} F))\,\alpha(L \cap \operatorname{Inn} F)\big)$$

$\alpha(K)$-contains the section $(\{\mathrm{id}\}, \alpha(L \cap (\mathrm{Inn}\, F))$ of E. But the latter is $\alpha(K)$-maximal, whence both sections are $\alpha(K)$-maximal and in fact are equal. It follows that $C_E(\alpha(L \cap \mathrm{Inn}\, F))$ is trivial and is certainly contained in $\alpha(K)$. Now by assumption $K = L$. Also the composition $\chi \circ \alpha : K \to \mathrm{Aut}(L \cap \mathrm{Inn}\, F)$ is identical to the map ι. As $\alpha(K) = \alpha(L)$ normalises $N_E(\alpha(L \cap \mathrm{Inn}\, F))$ we deduce that

$$\mathrm{Core}_{\iota(L)}\Big(\chi\big(N_E\big(\alpha(L \cap \mathrm{Inn}\, F)\big)\big)\Big) = \chi\big(N_E\big(\alpha(L \cap \mathrm{Inn}\, F)\big)\big).$$

By Definition 7.13(iv)(a) the latter is contained in $\iota(\mathrm{Core}_L(\alpha^{-1}(E)))$ which in turn is clearly contained in $\iota(L)$. Thus

$$\chi\big(N_E\big(\alpha(L \cap \mathrm{Inn}\, F)\big)\big) \le \iota(L)$$

and on recalling that $C_E(\alpha(L \cap \mathrm{Inn}\, F)) = (\ker \chi) \cap N_E(\alpha(L \cap \mathrm{Inn}\, F))$ is trivial and by applying χ^{-1} to both sides we deduce that $N_E(\alpha(L \cap \mathrm{Inn}\, F)) \le \alpha(L) = \alpha(K)$ as required. Definition 7.13(iv)(b) follows.

THEOREM 7.15. *Let* $x = (E, F, \alpha_1, \dots, \alpha_m, L)$ *be a* (7.6)-*tuple of degree* $m \ge 1$ *and let* $\Gamma(x) = (T, F, S, \phi)$ *be the output of Construction* 7.9 *as applied to* x. *Then the following all hold*:

(i) $\Gamma(x)$ *satisfies Definition* 5.2(i)–(iii);

(ii) $\Gamma(x)$ *satisfies Definition* 5.2(iv) *if and only if* x *satisfies either Definition* 7.13(iv)(a) *or* (iv)(b);

(iii) $\Gamma(x)$ *is a* (5.2)-*tuple of rank* $m + 1$ *if and only if* x *is a* (7.13)-*tuple*.

COROLLARY 7.16. $\Omega(5.2) = \Omega(7.13)$.

Proof. This is immediate from Theorem 7.15(iii).

Proof of Theorem 7.15. Let x and $\Gamma(x)$ be as in the statement of the theorem. We assume the notation of Construction 7.9 and of Lemma 7.11. In particular, $x_1 (= \mathrm{id}), \dots, x_m$ is the right transversal for K in L used to define $\Gamma(x)$, and $\rho_1, \dots, \rho_m$ are the distinct strict extensions of ϕ in T as defined by (7.K).

Recall that T is constructed as the subgroup $(\mathrm{Inn}\, E)^l S$ of $(\mathrm{Aut}\, E) \wr S_l$. We claim that $(\mathrm{Inn}\, E)^l$ is the unique minimal normal subgroup of T, and so equal to the socle $\mathrm{Soc}\, T$ of T. Certainly its centralizer in $(\mathrm{Aut}\, E) \wr S_l$, and so also its centralizer in T, is trivial. Furthermore, $(\mathrm{Inn}\, E)^l$ is minimal normal in T as S is transitive on the l simple direct factors of $(\mathrm{Inn}\, E)^l$. The claim now follows.

Throughout the proof we identify E with a minimal normal subgroup of $\operatorname{Soc} T = (\operatorname{Inn} E)^l$ via the map

$$x \mapsto (x, \mathrm{id}_E, \ldots, \mathrm{id}_E) \qquad \text{for all } x \in E, \tag{7.O}$$

and let $\kappa : N_T(E) \to \operatorname{Aut} E$ be the map induced by conjugation. Note that $N_S(E) = \psi(K)$ and that for all $x \in K$

$$\kappa(\psi(x)) = \alpha(x). \tag{7.P}$$

On several occasions in the proof we shall have cause to apply earlier results on (3.18)-tuples. To facilitate this we now determine the subgroup $\psi(L) \cap \operatorname{Soc} T$. Recall that for $x \in L$ we have

$$\psi(x) = \left(\alpha\left(x_1 x x_{1\pi}^{-1}\right), \ldots, \alpha\left(x_l x x_{l\pi}^{-1}\right)\right)\pi,$$

where $x_1, \ldots, x_l$ is the chosen right transversal for K in L and where $\pi \in S_l$ is such that $x_i x x_{i\pi}^{-1} \in K$ for all $i = 1, \ldots, l$. Thus $\psi(x) \in (\operatorname{Aut} E)^l$ if and only if

$$x_i x x_i^{-1} \in K \qquad \text{for all } i = 1, \ldots, l,$$

and moreover, $\psi(x) \in (\operatorname{Inn} E)^l$ if and only if

$$x_i x x_i^{-1} \in \alpha^{-1}(E) \qquad \text{for all } i = 1, \ldots, l.$$

Hence $\psi(L) \cap (\operatorname{Aut} E)^l = \psi(\operatorname{Core}_L K)$ and

$$\psi(L) \cap (\operatorname{Inn} E)^l = \psi\left(\operatorname{Core}_L \alpha^{-1}(E)\right).$$

We now consider part (i) of the theorem. We must show that Definition 5.2(i)–(iii) all hold with respect to the tuple $\Gamma(x) = (T, F, S, \phi)$. Certainly ϕ is a homomorphism $S \to \operatorname{Aut} F$ and F is a non-abelian simple group. By Definition 7.6(iv) and Corollary 3.25, the intersection $\phi(S) \cap \operatorname{Inn} F = L \cap \operatorname{Inn} F$ is a non-trivial proper subgroup of $\operatorname{Inn} F$. As strict extensions of ϕ in T exist, namely $\rho_1, \ldots, \rho_m$, we see that S is a proper subgroup of T. Thus Definition 5.2(i) holds.

Now we have already seen that $(\operatorname{Inn} E)^l$ is the socle of T and is the unique minimal normal subgroup of T. Note that $(\operatorname{Inn} E)^l$ is not contained in S as $T = (\operatorname{Inn} E)^l S$ and as S is a proper subgroup of T (as proved in the preceding paragraph). We deduce that S, and so also $\phi^{-1}(\operatorname{Inn} F)$, is a core-free subgroup of T, whence Definition 5.2(ii) holds. Finally Definition 5.2(iii) is an immediate consequence of Definition 7.6(iv).

We turn to part (ii). By Definition 7.6(iv) and Corollary 3.25, the intersection $L \cap \operatorname{Inn} F$ is non-abelian and is a minimal normal subgroup of L. Let D be a minimal normal subgroup of $L \cap \operatorname{Inn} F$. Then D is a non-abelian simple group and $L \cap \operatorname{Inn} F$ is the direct product of the L-conjugates of D. As ψ is a monomorphism the same statement holds with D, $L \cap \operatorname{Inn} F$, and L replaced respectively by $\psi(D)$, $\psi(L \cap \operatorname{Inn} F)$, and $\psi(L) = S$. Let $\sigma : N_S(\psi(D)) = \psi(N_L(D)) \to \operatorname{Aut} D$ be induced by conjugation and the isomorphism $D \to \psi(D)$ obtained by restricting ψ. It is straightforward, in fact it is an application of Lemma 3.5, to see that Definition 5.2(iv) holds if and only if T is a maximal subgroup of the twisted wreath product $D \operatorname{twr}_\sigma T$.

Observe that as $\alpha(L \cap \operatorname{Inn} F) \leq E$ we have $\psi(L \cap \operatorname{Inn} F) \leq (\operatorname{Inn} E)^l$, which by the proof of part (i) is the socle of T. Thus $\psi(D) \leq \operatorname{Soc} T$. Note also that $\psi(D)$ is contained in the domain of σ and that $\sigma(\psi(D)) = \operatorname{Inn} D$. Hence

$$\operatorname{Soc} T \cap \sigma^{-1}(\operatorname{Inn} D) = \psi(D)(\operatorname{Soc} T \cap \ker \sigma)$$

and $\sigma(\operatorname{Soc} T \cap \sigma^{-1}(\operatorname{Inn} D)) = \operatorname{Inn} D$. By definition $\ker \sigma = \psi((C_L(D))$ and so by Corollaries 3.7 and 3.15 the following statements are equivalent:

(a) Definition 5.2(iv) holds;

(b) there exist no strict extensions of σ in T;

(c) the section

$$\big(\operatorname{Soc} T \cap \psi(C_L(D)), \psi(D)\big(\operatorname{Soc} T \cap \psi(C_L(D))\big)\big)$$

is a $\psi(N_L(D))$-maximal section of $\operatorname{Soc} T$ with normaliser in T equal to $\psi(N_L(D))$.

We split into two cases: if $K = L$ then we show that Definition 5.2(iv) holds if and only if 7.13(iv)(b) holds, while if $K \neq L$ then we show that Definition 5.2(iv) holds if and only if Definition 7.13(iv)(a) holds. Given the discussion immediately prior to the statement of the theorem this is sufficient to prove part (ii).

Suppose that $K = L$. Note that in such circumstances Construction 7.9 is much simplified, indeed $l = 1$, $\alpha = \psi$, $T \leq \operatorname{Aut} E$ and we can identify E with $\operatorname{Soc} T = \operatorname{Inn} E$. Recall that $\alpha : K = L \to \operatorname{Aut} E$ is a monomorphism; for convenience we use α to identify L with $S = \alpha(L) \leq T$. With these conventions condition (c) above becomes

(c)′ the section $(E \cap C_L(D), D(E \cap C_L(D))$ is a $N_L(D)$-maximal section of E with normaliser in T equal to $N_L(D)$.

Assume now that Definition 5.2(iv) does indeed hold, or equivalently that (c)′ holds. Observe that $C_E(D)$ is normalised by $N_L(D)$ and meets $D(E \cap C_L(D))$ in $E \cap C_L(D)$. Thus the $N_L(D)$-maximality of section given in (c)′ implies that $C_E(D) = E \cap C_L(D)$ is a subgroup of L and that the section given in (c)′ is equal to the section

$$(C_E(D), DC_E(D))$$

of E. As a section is certainly L-maximal if it is $N_L(D)$-maximal (since $N_L(D) \leq L$), to verify Definition 7.13(iv)(b) it remains only to show that $N_E(L \cap \operatorname{Inn} F) \leq L$; in fact we show that $N_T(L \cap \operatorname{Inn} F) \leq L$. Now $N_T(L \cap \operatorname{Inn} F)$ clearly contains L and acts on the minimal normal subgroups of $L \cap \operatorname{Inn} F$. As L acts transitively on such subgroups, one of which is D, we have

$$N_T(L \cap \operatorname{Inn} F) = N_T(D, L \cap \operatorname{Inn} F)L.$$

But $N_T(D)$ certainly normalises the section $(C_E(D), DC_E(D))$; as by (c)′ the normaliser in T of this section is contained in L we have $N_T(D) \leq L$, whence $N_T(L \cap \operatorname{Inn} F) \leq L$ as required.

Conversely, we assume that Definition 7.13(iv)(b) holds. It is enough to show that (c)′ holds. By Definition 7.13(iv)(b) we have, in particular, $C_E(D) \leq L$ whence $E \cap C_L(D)$ is equal to $C_E(D)$. Thus the section given in (c)′ is equal to

$$(C_E(D), DC_E(D)).$$

We claim that the normaliser in T of this section is contained in $N_T(D, L \cap \operatorname{Inn} F)$. For any group H let $H^{(\infty)}$ be the normal subgroup of H that is minimal subject to $H/H^{(\infty)}$ being soluble; note that $H^{(\infty)}$ is a characteristic subgroup of H. Now $L \cap \operatorname{Inn} F$ is contained in E and so

$$C_{L \cap \operatorname{Inn} F}(D) \leq C_E(D) \leq L.$$

As $L \cap \operatorname{Inn} F$ is non-abelian and characteristically simple and as by the "Schreier conjecture" $L/L \cap \operatorname{Inn} F$ is soluble we see that

$$C_{L \cap \operatorname{Inn} F}(D) = C_E(D)^{(\infty)} \qquad \text{and} \qquad L \cap \operatorname{Inn} F = \big(DC_E(D)\big)^{(\infty)}.$$

Hence the normaliser of the section $(C_E(D), DC_E(D))$ also normalises $C_{L \cap \operatorname{Inn} F}(D)$, $L \cap \operatorname{Inn} F$, and

$$C_{L \cap \operatorname{Inn} F}\big(C_{L \cap \operatorname{Inn} F}(D)\big) = D,$$

and so is contained in $N_T(D, L \cap \operatorname{Inn} F)$. The claim follows.

By the claim $N_T(C_E(D), DC_E(D))$ is contained in $N_T(L \cap \operatorname{Inn} F)$. As $T = EL$ we have

$$N_T(L \cap \operatorname{Inn} F) = N_E(L \cap \operatorname{Inn} F)L$$

which is contained in L as by Definition 7.13(iv)(b), $N_E(L \cap \operatorname{Inn} F) \leq L$. We deduce that

$$N_T\big(C_E(D), DC_E(D)\big) \leq N_T(D, L \cap \operatorname{Inn} F) \cap L = N_L(D).$$

On the other hand, $N_L(D)$ certainly normalises the section $(C_E(D), DC_E(D))$ and so $N_L(D)$ is indeed equal to the normaliser in T of this section. It remains only to show that the section given in (c)$'$ is $N_L(D)$-maximal. This follows from the assumption that it is L-maximal together with the observation that its normaliser in T, and so also in L, is contained in $N_L(D)$.

We now consider the second case referred to above, namely that in which $K \neq L$. Let D and σ be as above. We initially show that if either Definition 5.2(iv) or 7.13(iv)(a) holds, then $\operatorname{Soc} T \cap \psi(C_L(D))$ is trivial.

Assume that Definition 5.2(iv) holds, whence condition (c) above holds. In particular, the section

$$\big(\operatorname{Soc} T \cap \psi(C_L(D)), \psi(D)\big(\operatorname{Soc} T \cap \psi(C_L(D))\big)\big)$$

is a maximal section of $\operatorname{Soc} T$. Lemma 3.16 together with Remark 3.17 implies that

$$\operatorname{Soc} T \cap \psi\big(C_L(D)\big) = \prod_{W \lhd_{\min} \operatorname{Soc} T} W \cap \psi\big(C_L(D)\big), \tag{7.R}$$

where the direct product is taken over all minimal normal subgroups W of $\operatorname{Soc} T$. Now $\psi(C_L(D)) \leq \psi(N_L(D)) \leq \psi(L) = S$. Also $S \leq \beta_1(F)S$ where β_1 is as defined by Lemma 7.11. In Lemma 7.11 it was shown that ρ_1 is a monomorphism mapping $\beta_1(F)S$ to a subgroup of $\operatorname{Aut} F$ with $\rho_1(\beta_1(F)) = \operatorname{Inn} F$. Hence $\beta_1(F)$ is the unique minimal normal subgroup of $\beta_1(F)S$. Recall that $l > 1$ since $K \neq L$; observe that the map β_1 is defined so that $\beta_1(F)$ meets every non-trivial normal subgroup of $\operatorname{Soc} T = (\operatorname{Inn} E)^l$ trivially. As

$$\prod_{W \lhd_{\min} \operatorname{Soc} T} W \cap \big(\beta_1(F)S\big)$$

is a normal subgroup of $\beta_1(F)S$, it is non-trivial if and only if

$$\beta_1(F) \leq \prod_{W \lhd_{\min} \operatorname{Soc} T} W \cap \big(\beta_1(F)S\big).$$

As above, for any group H let $H^{(\infty)}$ be the normal subgroup of H that is minimal subject to $H/H^{(\infty)}$ being soluble. As $\beta_1(F)^{(\infty)} = \beta_1(F)$, while

$$\left(\prod_{W \lhd_{\min} \operatorname{Soc} T} W \cap \big(\beta_1(F)S\big)\right)^{(\infty)} \leq \prod_{W \lhd_{\min} \operatorname{Soc} T} W \cap \beta_1(F) = \prod \{\mathrm{id}\} = \{\mathrm{id}\},$$

we deduce that $\prod_{W \lhd_{\min} \operatorname{Soc} T} W \cap (\beta_1(F)S)$ is trivial, whence by (7.R) and the containment $\psi(C_L(D)) \leq \beta_1(F)S$ we see that $\operatorname{Soc} T \cap \psi(C_L(D))$ is also trivial.

On the other hand, assume that Definition 7.13(iv)(a) holds. Then $L \cap \operatorname{Inn} F$ is a non-abelian simple group, whence $D = L \cap \operatorname{Inn} F$. As $(\{\mathrm{id}\}, \alpha(L \cap \operatorname{Inn} F))$ is a maximal section of E, we have $C_E(\alpha(D)) = \{\mathrm{id}\}$. An easy calculation shows that $C_{\operatorname{Soc} T}(\psi(D))$ is also trivial, whence $\operatorname{Soc} T \cap \psi(C_L(D))$ is trivial as required.

Given the above two paragraphs it is enough to make the extra assumption that $\operatorname{Soc} T \cap \psi(C_L(D))$ is trivial and to show that Definition 5.2(iv) holds if and only if Definition 7.13(iv)(a) holds. As $\psi(C_{L \cap \operatorname{Inn} F}(D)) \leq \operatorname{Soc} T \cap \psi(C_L(D))$, it follows that $D = L \cap \operatorname{Inn} F$ is a non-abelian simple group, that $N_S(\psi(D)) = S$, and that the restriction to L of the homomorphism $\iota : N_{\operatorname{Aut} F}(L \cap \operatorname{Inn} F) \to \operatorname{Aut}(L \cap \operatorname{Inn} F)$ defined at the end of Construction 7.9 is identical to the composition of ψ followed by σ. Recall that we have identified E with a minimal normal subgroup of $\operatorname{Soc} T = (\operatorname{Inn} E)^l$ via (7.O) and that the map $\kappa : N_T(E) \to \operatorname{Aut} E$ is induced by conjugation. We claim that

$$(E, \operatorname{Soc} T, T, L \cap \operatorname{Inn} F, S, \sigma)$$

is a (3.18)-tuple with $T = (\operatorname{Soc} T)S$. Certainly E is a minimal normal subgroup of $\operatorname{Soc} T$; also at the beginning of this proof we saw the $\operatorname{Soc} T = (\operatorname{Inn} E)^l$ is the unique minimal normal subgroup of T, whence Definition 3.18(i) holds. That Definition 3.18(ii) holds is immediate. By construction $T = (\operatorname{Soc} T)S$ and so Definition 3.18(iii) holds since $\operatorname{Soc} T$ is contained in $N_T(E)$. By (7.Q) we have $S \cap (\operatorname{Soc} T) = \psi(\operatorname{Core}_L(\alpha^{-1}(E)))$ and so

$$\sigma(S \cap (\operatorname{Soc} T)) = \sigma\Big(\psi\big(\operatorname{Core}_L(\alpha^{-1}(E))\big)\Big) = \iota\big(\operatorname{Core}_L(\alpha^{-1}(E))\big).$$

Now $\alpha^{-1}(E) \geq L \cap \operatorname{Inn} F$ which is a normal subgroup of L, whence

$$\sigma(S \cap (\operatorname{Soc} T)) \geq \iota(L \cap \operatorname{Inn} F) = \operatorname{Inn}(L \cap \operatorname{Inn} F)$$

and Definition 3.18(iv) holds. The claim follows.

From above $\operatorname{Soc} T \cap \psi(C_L(L \cap \operatorname{Inn} F))$ is trivial, whence $\kappa(\operatorname{Soc} T \cap \ker \sigma) = \{\mathrm{id}\}$ and

$$\begin{aligned} &\kappa\Big(\psi(L \cap \operatorname{Inn} F)\big(\operatorname{Soc} T \cap \psi(C_L(L \cap \operatorname{Inn} F))\big)\Big) \\ &\quad = \kappa(\psi(L \cap \operatorname{Inn} F)) = \alpha(L \cap \operatorname{Inn} F). \end{aligned}$$

From Proposition 3.20 we deduce that condition (b) above holds if and only if the section $(\{\mathrm{id}\}, \alpha(L \cap \operatorname{Inn} F))$ is a $\alpha(K)$-maximal section of E

and if

$$\mathrm{Core}_{\iota(L)}\big(\chi\big(N_E(\alpha(L \cap \mathrm{Inn}\, F))\big)\big) = \sigma(S \cap \mathrm{Soc}\, T).$$

We have just seen that

$$\sigma(S \cap \mathrm{Soc}\, T) = \iota\big(\mathrm{Core}_L\, \alpha^{-1}(E)\big)$$

and so we have shown that condition (b) holds if and only if Definition 7.13(iv)(a) holds: part (ii) follows.

Finally we turn to part (iii). By parts (i) and (ii) it is enough to assume that $\Gamma(x)$ satisfies Definition 5.2(iv), and then to show that x satisfies Definition 7.13(i)–(iii) if and only if $\Gamma(x)$ satisfies Definition 5.2(v) and there are precisely m strict extensions of ϕ in T. Recall that by Lemma 7.11, $\rho_1, \dots, \rho_m$ are distinct strict extensions of ϕ in T with

$$\ker \rho_i = \ker \phi = \{\mathrm{id}\} \qquad \text{and} \qquad \mathrm{Im}\, \rho_i = (\mathrm{Inn}\, F)L = (\mathrm{Inn}\, F)\phi(S)$$

for all $i = 1, \dots, m$. Thus it is in fact enough to show that Definition 7.13(i)–(iii) hold if and only if $\rho = \rho_i$ for some $i = 1, \dots, m$ whenever ρ is a strict extension of ϕ in T.

Suppose that $\rho : R \to \mathrm{Aut}\, F$ is a strict extension of ϕ in T. We claim that

$$S = \rho^{-1}\big(N_{\mathrm{Aut}\, F}(L \cap \mathrm{Inn}\, F)\big). \tag{7.S}$$

Suppose not. Set $X = \rho^{-1}(N_{\mathrm{Aut}\, F}(L \cap \mathrm{Inn}\, F))$. Note that the restriction $\rho|_X$ of ρ to X is a map $X \to \mathrm{Aut}\, F$ strictly extending ϕ. Thus the composition $\iota \circ (\rho|_X)$ is a strict extension of the composition $\iota \circ \phi$ contradicting the assumption that Definition 5.2(iv) holds. Hence (7.S) holds. As $\ker \phi = \ker \rho \cap S$ we deduce that $\ker \phi = \ker \rho$.

(Recall that in Remark 7.14 we claimed that Definition 7.13(ii) is a consequence of Definition 7.13(iv), or equivalently given part (ii), that Definition 7.13(ii) is a consequence of Definition 5.2(iv). The purpose of the present paragraph is merely to justify this and does not form part of the proof of Theorem 7.15. Fix $i = 1, \dots, m$ and consider the strict extension ρ_i of ϕ in T. Note that any extension of ρ_i in T is necessarily a strict extension of ϕ in T. Thus the previous paragraph shows that $\ker \rho = \ker \rho_i = \ker \phi = \{\mathrm{id}\}$ whenever ρ is an extension of ρ_i. By the definition of β_i we have $\beta_i(F) \le \mathrm{Soc}\, T = (\mathrm{Inn}\, E)^l$ and $\rho_i(\beta_i(F)) = \mathrm{Inn}\, F$. From Lemma 3.13 we deduce that the section

$$\big((\mathrm{Soc}\, T) \cap \ker \rho_i, \mathrm{Soc}\, T \cap \rho_i^{-1}(\mathrm{Inn}\, F)\big) = \big(\{\mathrm{id}\}, \beta_i(F)\big)$$

is a $\beta_i(F)S$-maximal section of $\mathrm{Soc}\, T$. As in part (ii) we identify E with a minimal normal subgroup of $\mathrm{Soc}\, T$ via the map given by (7.O), and let

$\kappa : N_T(E) \to \operatorname{Aut} E$ be induced by conjugation. Observe that $\kappa(\beta_i(F)) = \alpha_i(F)$, that $N_S(E) = \psi(K)$, and that $\kappa(\psi(K)) = \alpha(K)$. By Lemma 3.16 the section $(\kappa(\{\mathrm{id}\}), \kappa(\beta_i(F)))$, which is equal to the section $(\{\mathrm{id}\}, \alpha_i(F))$, is a $\alpha(K)$-maximal section of E and Definition 7.13(ii) holds.)

We return to the proof proper and continue with the assumptions in place before the bracketed paragraph. Thus $\rho : R \to \operatorname{Aut} F$ is assumed to be a strict extension of ϕ in T. We have already seen that $\ker \rho = \ker \phi$ and that $\rho(R) \nleq N_{\operatorname{Aut} F}(L \cap \operatorname{Inn} F)$. We now claim that $\rho(R) \geq \operatorname{Inn} F$. To see this observe that $\rho(R) \cap \operatorname{Inn} F$ contains $L \cap \operatorname{Inn} F$ and is normalised by both $L = \phi(S)$ and $\rho(R)$. As $\rho(R)$ does not normalise $L \cap \operatorname{Inn} F$ we see that $\rho(R) \cap \operatorname{Inn} F$ is a strict overgroup of $L \cap \operatorname{Inn} F$ in $\operatorname{Inn} F$. From Definition 7.6(iv), $\rho(R) \cap \operatorname{Inn} F$ must equal $\operatorname{Inn} F$ and the claim holds.

We conclude that R is almost simple with socle $\rho^{-1}(\operatorname{Inn} F)$. Since $R \cap \operatorname{Soc} T$ is a normal subgroup of R and since $\{\mathrm{id}\} \neq \psi(L \cap \operatorname{Inn} F) \leq \operatorname{Soc} T \cap R$ we deduce that $\rho^{-1}(\operatorname{Inn} F) \leq \operatorname{Soc} T$. As above let $\kappa : N_T(E) \to \operatorname{Aut} E$ be as defined following the identification of E with a subgroup of $\operatorname{Soc} T$ given by (7.O). Define $\beta : F \to E$ by

$$\rho(x) \mapsto \kappa(x) \qquad \text{for all } x \in \rho^{-1}(\operatorname{Inn} F).$$

It is straightforward to see that β is a monomorphism $F \to E$ such that $\beta(F)$ is normalised by $\alpha(K)$ and such that for all $x \in F$ and $y \in K$

$$\beta(x^y) = \beta(x)^{\alpha(y)},$$

and moreover, that

$$\rho^{-1}(\operatorname{Inn} F) = \left\{\left(\beta(x^{x_1^{-1}}), \dots, \beta(x^{x_l^{-1}})\right) : x \in F\right\}.$$

On comparing this with the definition of $\beta_i(F)$ for $i = 1, \dots, m$ we see that every strict extension of ϕ in T is a strict extension of ρ_i for some $i = 1, \dots, m$ if and only if Definition 7.13(i) holds. We have thus reduced to showing that for $i = 1, \dots, m$ no strict extensions of ρ_i exist if and only if Definition 7.13(ii)–(iii) both hold. Observe that for each $i = 1, \dots, m$ the tuple $(E, \operatorname{Soc} T, T, F, \beta_i(F)S, \rho_i)$ is a (3.18)-tuple with $T = (\operatorname{Soc} T)S$. By Proposition 3.20, ρ_i has no strict extensions for $i = 1, \dots, m$ if and only if Definition 7.13(ii) holds and

$$\operatorname{Core}_{(\operatorname{Inn} F)L}\left(\eta_i\left(N_E\left(\alpha_i(F)\right)\right)\right) = \rho_i\left(\left(\beta_i(F)S\right) \cap \operatorname{Soc} T\right) \qquad \text{for all } i = 1, \dots, m. \tag{7.T}$$

On noting that $\beta_i(F) \leq \operatorname{Soc} T$ and that ρ_i extends ϕ, and by using (7.Q), we see that

$$\begin{aligned}\rho_i\left(\left(\beta_i(F)S\right) \cap \operatorname{Soc} T\right) &= \rho_i\left(\beta_i(F)\right)\phi(S \cap \operatorname{Soc} T)\\ &= (\operatorname{Inn} F)\operatorname{Core}_L\left(\alpha^{-1}(E)\right).\end{aligned}$$

Hence (7.T) is equivalent to Definition 7.13(iii) and the proof of Theorem 7.15 is finished.

Let $x = (E, F, \alpha_1, \ldots, \alpha_m, L)$ be a (7.6)-tuple. Recall that Definition 7.6 identified x as a (7.6(a))-tuple, or as a (7.6(b))-tuple, depending on whether: (a) $E \not\cong F$; or (b) $E \cong F$. If the latter then the conditions of Definition 7.13 can be greatly simplified; our final task in this section is to make use of this simplification to replace the concept of a (7.13)-tuple satisfying (b) above with an equivalent concept that is much more amenable to analysis.

We start by supposing that $(E, F, \alpha_1, \ldots, \alpha_m, L)$ is a (7.6(b))-tuple. Thus the monomorphism $\alpha_1 : F \to E$ is in fact an isomorphism and so the tuple $(E, F, \ldots)$ is equivalent to the tuple $(F, F, \alpha_1^{-1} \circ \alpha_1, \ldots, \alpha_1^{-1} \circ \alpha_m, L)$. Note that $\alpha_1^{-1} \circ \alpha_1$ is the identity map $F \to F$, and moreover, that for $i = 1, \ldots, m$ the monomorphisms $\alpha_1^{-1} \circ \alpha_i$ are automorphisms of F. Conversely we have the following result.

LEMMA 7.17. *Let F be a non-abelian simple group.* (*As usual we identify F with* $\mathrm{Inn}\, F$.) *Let $\alpha_1, \ldots, \alpha_m$ be distinct automorphisms of F with $\alpha_1 = \mathrm{id}_{\mathrm{Aut}\, F}$, let L be a subgroup of* $\mathrm{Aut}\, F$, *and set $x = (F, F, \alpha_1, \ldots, \alpha_m, L)$. Then x is a* (7.13)-*tuple if and only if the following conditions all hold*:

(1) $[F/\{\mathrm{id}\}]_L \cong M_1$;

(2) $\alpha_i \in C_{\mathrm{Aut}\, F}(L \cap \mathrm{Inn}\, F)$ *for all* $i = 1, \ldots, m$;

(3) $\{\alpha_1, \ldots, \alpha_m\} = C_{\mathrm{Aut}\, F}(C_L(\alpha_1) \cap \cdots \cap C_L(\alpha_m))$;

(4) *one of the following holds*:

(a) *$L \cap F$ is non-abelian and simple, and* $(\{\mathrm{id}\}, L \cap F)$ *is a* $\bigcap_{i=1}^{m} C_L(\alpha_i)$-*maximal section of F*;

(b) *$\alpha_i \in C_{\mathrm{Aut}\, F}(L)$ for all $i = 1, \ldots, m$, and if D is any minimal normal subgroup of $L \cap$* $\mathrm{Inn}\, F$, *then $C_F(D)D = L \cap F$.*

Proof. Now x is a (7.6)-tuple if and only if Definition 7.6(i)–(v) all hold. Definition 7.6(i)–(iii) are trivially satisfied, while Definition 7.6(iv) and (v) are equivalent to conditions (1) and (2), respectively. Hence x is a (7.6)-tuple if and only if (1) and (2) both hold. It is thus enough to assume that x is a (7.6)-tuple, and then to show that x is a (7.13)-tuple if and only if conditions (3) and (4) both hold. To do this we must first determine the objects $\eta_1, \ldots, \eta_m, K, \alpha, \iota, \chi$ defined in the course of applying Construction 7.9 to x.

For each $i = 1, \ldots, m$ the image $\alpha_i(F)$ is equal to F, and so the maps $\eta_1, \ldots, \eta_m$ are automorphisms of $\mathrm{Aut}\, F$; in fact we have

$$\eta_i(x) = \alpha_i x \alpha_i^{-1} \qquad \text{for all } x \in \mathrm{Aut}\, F,\ i = 1, \ldots, m.$$

In particular, η_1 is the identity map on Aut F. It follows that

$$K = \bigcap_{i=1}^{m} C_L(\alpha_i^{-1}) = \bigcap_{i=1}^{m} C_L(\alpha_i),$$

and that the map $\alpha : K \to \operatorname{Aut} F$ is the identity map on K. The homomorphisms ι and χ are equal to the map $N_{\operatorname{Aut} F}(L \cap F) \to \operatorname{Aut}(L \cap F)$ induced by the conjugation action on Aut F.

Given the above information it is straightforward to verify that Definition 7.13(i)–(iv) can be rewritten respectively as:

(i) if β is an automorphism of F such that F is normalised by $\bigcap_{i=1}^{m} C_L(\alpha_i)$ (which is a subgroup of Aut F) and such that β centralises $\bigcap_{i=1}^{m} C_L(\alpha_i)$, then $\beta = \alpha_i$ for some $i = 1, \ldots, m$;

(ii) for each $i = 1, \ldots, m$ the section $(\{\mathrm{id}\}, F)$ of F is $\bigcap_{i=1}^{m} C_L(\alpha_i)$-maximal;

(iii) for each $i = 1, \ldots, m$,

$$F = F;$$

(iv) one of the following holds:

(a) $L \cap \operatorname{Inn} F$ is a non-abelian simple group, the section $(\{\mathrm{id}\}, L \cap \operatorname{Inn} F)$ of F is $\bigcap_{i=1}^{m} C_L(\alpha_i)$-maximal, and

$$L \cap \operatorname{Inn} F = L \cap \operatorname{Inn} F;$$

(b) $L = \bigcap_{i=1}^{m} C_L(\alpha_i)$, $L \cap \operatorname{Inn} F \leq K$, and if D is any minimal normal subgroup of $L \cap \operatorname{Inn} F$, then

$$C_F(D) \leq K$$

and the section $(C_F(D), DC_F(D))$ of F is L-maximal.

Clearly Definition 7.13(ii) and (iii) are trivial, while given that (1) and (2) hold, it follows that Definition 7.13(i) and (iv) are equivalent to (3) and (4), respectively, as required.

Definition 7.18. We say that the tuple (F, K, L) *satisfies* (7.18), or *is a* (7.18)-*tuple*, if the following conditions all hold:

(i) F is a non-abelian simple group;

(ii) K and L are subgroups of Aut F with $L \cap \operatorname{Inn} F \leq K \leq L$;

(iii) $[F/\{\mathrm{id}\}]_L \cong M_1$;

(iv) $K = C_L(C_{\mathrm{Aut}\,F}(K))$;

(v) one of the following holds:

(a) $L \cap \mathrm{Inn}\,F$ is a non-abelian simple group, and $(\{\mathrm{id}\}, L \cap \mathrm{Inn}\,F)$ is a K-maximal section of F;

(b) $K = L$ and if D is any minimal normal subgroup of $L \cap \mathrm{Inn}\,F$, then

$$DC_{\mathrm{Inn}\,F}(D) = L \cap \mathrm{Inn}\,F.$$

Moreover we say that (F, K, L) *is a* (7.18)-*tuple of rank* n if it satisfies (7.18) and

$$n = \left|C_{\mathrm{Aut}\,F}(K)\right| + 1.$$

The subset $\Omega(7.18)$ of $\mathbb{N}$ is defined by

$$\Omega(7.18) = \{n \geq 16 : \text{there exists a } (7.18)\text{-tuple of rank } n\}.$$

COROLLARY 7.19.

$$\Omega(7.18) = \left\{ n \geq 16 : \begin{array}{c} \textit{there exists a } (7.13)\textit{-tuple} \\ (E, F, \alpha_1, \ldots, \alpha_m, L) \\ \textit{of rank } n \textit{ with } E \cong F \end{array} \right\}.$$

Proof. Given the remarks preceding Lemma 7.17 together with the implication of Theorem 7.10 that a (7.6)-tuple is a (7.13)-tuple if and only if any of its equivalents are, the corollary is a straightforward application of Lemma 7.17.

8. FINAL COMMENTS AND EXAMPLES

From the Results Diagram (Fig. 1) we see that

$$\begin{aligned} \Omega \subseteq \Omega(4.7) &\cup \{n \leq 50 : n \in \Omega\} \cup \Delta(6.9) \\ &\cup \{n \in \mathbb{N} : n - 1 \in \Delta(6.9)\} \cup \Delta(7.6(\mathrm{a})) \\ &\cup \Omega(7.18) \cup \mathcal{K} \cup \mathcal{S}. \end{aligned}$$

It is our hope that further investigation of the sets $\Omega(4.7)$, $\Delta(6.9)$, $\Delta(7.6(\mathrm{a}))$, $\Omega(7.18)$, and $\mathcal{S}$ will be sufficient to show that $\Omega \neq \mathbb{N}$. We comment on the difficulties involved in the determination of the first four of these sets.

Problem 1. Determine $\Omega(4.7)$.

It seems likely that $\Omega(4.7)$ is the empty set; we offer three reasons for this.

Firstly, we have no examples of a (4.1)-tuple (H, T, F, Q, ϕ) of rank n with H almost simple and with $n \geq 3$. We do however have an example with $n = 2$. Let $H = A_8$, let $F = L_3(2)$, and let T, Q be maximal subgroups of H with $T \cong A_7$ and with $Q \cong 2^3{:}L_3(2)$. Then there exist obvious homomorphisms $\phi : Q \to \operatorname{Aut} F$ with kernel isomorphic to 2^3. Moreover, if ϕ is any such homomorphism, then it is easy to see that (H, T, F, Q, ϕ) is a (4.1)-tuple of rank 2 with H almost simple.

Secondly, suppose that $n \in \Omega(4.7)$: then $n \geq 16$ and there exists a (4.1)-tuple (H, T, F, Q, ϕ) of rank n with H almost simple. In what follows we assume the notation of Lemma 4.10. Thus $\bigcap_{i=1}^{n-1} \ker \phi_i$ and V are normal subgroups of Q such that the quotient

$$\bar{V} = V \Big/ \bigcap\nolimits_{i=1}^{n-1} \ker \phi_i$$

is isomorphic to F^{n-1}. Let $X = N_H(\bigcap_{i=1}^{n-1} \ker \phi_i, V)$ so that X acts on $\bar{V}$ by conjugation. Further, let $\tilde{X} \leq S_{n-1}$ be the permutation group induced by the action of X on the $n - 1$ maximal proper normal subgroups of $\bar{V}$. It is an easy consequence of Definition 4.1 that $\phi_i(Q) \geq \operatorname{Inn} F$ and that ϕ_i has no strict extensions in H. By Corollary 3.15 we have

$$Q = N_H(\ker \phi_i, V),$$

whence in particular $Q = N_X(\ker \phi_i)$ for all $i = 1, \ldots, n - 1$. Thus the action of X on the $n - 1$ maximal proper normal subgroups of $\bar{V} \cong F^{n-1}$ is such that the stabilizer of any one stabilizes all others; equivalently, the stabilizer in $\tilde{X} \leq S_{n-1}$ of any point is trivial. It is our intuitive feeling that, for H almost simple, the permutation group $\tilde{X}$ is likely to be "close" to S_{n-1}. This intuition, together with the above regularity condition, suggests that n must be small, perhaps even that $n \leq 4$.

Thirdly, and more significantly, our investigation to date of $\Omega(4.7)$ has produced the following result:

> *If* (H, T, F, Q, ϕ) *is a* (4.1)*-tuple with* $n \geq 16$ *and H almost simple, then H is not alternating, sporadic, or exceptional of Lie type.*

The major tools used in proving this result are the classification of non-trivial maximal factorisations of almost simple groups due to Liebeck, Praeger, and Saxl [15], and, in the alternating case, a description of the maximal non-abelian simple sections of the alternating groups [2]. (The former is relevant to Definition 4.1(iii), and the latter to Definition 4.1(iv).) Resolution of the one remaining case depends upon developing a useful description of the maximal non-abelian simple sections of the classical groups, but we are hopeful that this can be achieved.

Problem 2. Determine $\Delta(6.9)$.

Progress on this problem, in view of Definition 6.9(i)–(ii), depends on the existence of an adequate theory of non-abelian simple sections of non-abelian simple groups. As mentioned above, we already have a description of the maximal non-abelian simple sections of the alternating groups [2], and it is hoped that analogous descriptions can be obtained in the remaining cases.

We remark that if the concept of a proper non-abelian simple section (C, D) is replaced in Definition 6.9 by that of a proper subgroup, then we obtain the definition of, instead the set $\Delta(6.9)$, the set $\mathcal{S}$. Hence we can expect the degree of difficulty involved in the determination of $\Delta(6.9)$ to be comparable to that involved in the determination of $\mathcal{S}$. Furthermore, given that our intuition suggests that $\mathcal{S}$ is a highly restricted set, possibly even bounded, then it seems reasonable to expect $\Delta(6.9)$ to be similarly restricted.

Finally we note that we have examples of (6.9)-tuples of degree d only for $d \leq 2$. Of these examples, some give rise to (6.17)-tuples of ranks d and $d+1$, while others do not. Below we give an example of the latter behaviour as this seems to best illustrate the delicacy of the conditions involved in Definition 6.17.

EXAMPLE 8.1. For $n > 2$, let $E = GL_{2n}(2)$, the group of all $2n \times 2n$ matrices over the field $\mathbb{F}_2$ of two elements, and define the subgroups (C, D) of E as

$$C = \left\{ \begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix} : B \in M_n(2) \right\},$$

$$D = \left\{ \begin{pmatrix} A & B \\ 0 & A \end{pmatrix} : A \in GL_n(2), B \in M_n(2) \right\},$$

where $M_n(2)$ is the set of all $n \times n$ matrices over $\mathbb{F}_2$, and where I_n is the identity matrix. Note that (C, D) is a section of E isomorphic to $GL_n(2)$: let F be the quotient D/C. We claim that:

(1) (C, D, E, D) is a (6.9)-tuple of degree 2;

(2) (C, D, E, D, D, F) is a (6.17)-tuple of rank 2.

To see the claim we note that in Example 4.10 of [3] it is shown that

$$\hat{D} = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} : A, C \in GL_n(2), B \in M_n(2) \right\}$$

is the unique maximal subgroup of E containing D. As $\hat{D}$ normalises C with quotient $\hat{D}/C \cong F \times F$, it follows that there are precisely two sections of E strictly D-containing (C, D). Given this the verification of the claim is straightforward.

Further let σ be the involutory automorphism of E given by first taking the inverse transpose, and then conjugating by $\left(\begin{smallmatrix} 0 & I_n \\ I_n & 0 \end{smallmatrix}\right)$: direct calculation shows that C, D, and $\hat{D}$ are all σ-invariant, and we also use σ to denote the involutory automorphism of $F = D/C$ induced by σ. In Example 4.10 of [3] it is also shown that (C, D) is a $\langle D, \sigma \rangle$-maximal section of E, and it is tempting to hope that $(C, D, E, D, \langle D, \sigma \rangle, \langle F, \sigma \rangle)$ is a (6.17)-tuple of rank 3. However, this is not the case—explicit calculation shows that Definition 6.17(vii) fails. Moreover, such a calculation can be generalised to give a proof of the following result:

> *Let (C, D, E, K, L, A) be a* (6.17)*-tuple. Suppose that $(C_1, D_1), (C_2, D_2)$ are sections of E strictly K-containing (C, D) with $C_1 \leq N_E(D_2)$. Then $D_1 = D_2$ and $A = \eta(K)$, where η is the homomorphism induced by the conjugation action of $N_{\mathrm{Aut}\, E}(C, D)$ on the quotient $F = D/C$ (as in Construction* 6.13).

Problem 3. Determine $\Delta(7.6(\mathrm{a}))$.

As in the previous problem, progress here depends on being able to count maximal non-abelian simple sections of non-abelian simple groups subject to various conditions. We must stress that the set $\Delta(7.6(\mathrm{a}))$ could prove to be bounded above, which is the desirable outcome, or if not, then it would be likely to contain all but finitely many positive integers. The reason for this is the following observation.

If $(E, F, \alpha_1, \ldots, \alpha_m, L)$ is a (7.6)-tuple with

$$C_{\mathrm{Aut}\, F}(L \cap \mathrm{Inn}\, F) = \{\beta_1, \ldots, \beta_r\},$$

then $(E, F, \gamma_1, \ldots, \gamma_s, L)$ is a (7.6)-tuple, where $\gamma_1, \ldots, \gamma_s$ are any distinct elements of the set

$$\left\{\alpha_i \circ \beta_j : i = 1, \ldots, m, j = 1, \ldots, r\right\}$$

chosen so that $E = \langle \gamma_1(F), \ldots, \gamma_s(F) \rangle$.

If the latter unbounded scenario occurs, then we would have to incorporate some version of the extra conditions as satisfied by (7.13)-tuples; probably the most helpful being some version of Definition 7.13(i).

We note that we have no examples of (7.6)-tuples of degree d with $d \geq 4$.

Example 8.2. Let $E = O_8^+(q)$ with $q > 2$, and let F be a maximal subgroup of E with $F \cong O_7(q)$. Let τ be a triality automorphism of E, that is, $\tau^3 = \mathrm{id}_{\mathrm{Aut}\, E}$ and τ induces a non-trivial symmetry of the Dynkin diagram of E. Set $\alpha_1 = \mathrm{id}$, $\alpha_2 = \tau|_F$, $\alpha_3 = \tau^2|_F$, and $L = C_F(\tau)$. By [9, 3.1.1(vi)] we have

$$L = F^{\alpha_1} \cap F^{\alpha_2} = F^{\alpha_2} \cap F^{\alpha_3} = F^{\alpha_1} \cap F^{\alpha_3} \cong G_2(q).$$

Also L is non-abelian and simple and L is a maximal subgroup of F, and moreover, it is straightforward to verify that $(E, F, \alpha_1, \alpha_2, \alpha_3, L)$ is both a (7.6)-tuple of degree 3 and a (7.13)-tuple of rank 4.

Problem 4. Determine $\Omega(7.18)$.

This problem contrasts with the earlier ones in that the fundamental information required is not that of maximal non-abelian simple sections of non-abelian simple groups. Instead it seems sensible to focus on the following necessary condition.

If (F, K, L) is a (7.18)-tuple of rank n, then F is a non-abelian simple group with $L \leq \operatorname{Aut} F$ such that

$$\left[F/\{iid\}\right]_L \cong M_1 \qquad \text{and} \qquad (n-1)\Big|\left|C_{\operatorname{Aut} F}(L \cap \operatorname{Inn} F)\right|.$$

Note that this necessary condition does not involve K; in fact we are hopeful that it may be enough for our purposes to only find those F, L, and n satisfying this condition. Note also that $C_{\operatorname{Aut} F}(L \cap \operatorname{Inn} F)$ is isomorphic to a subgroup of Out F since $C_{\operatorname{Inn} F}(L \cap \operatorname{Inn} F)$ is trivial: thus to find all (7.18)-tuples (F, K, L) of degree n with $n \geq 16$, we need only consider F of Lie type, and not alternating or sporadic.

We stress that $\Omega(7.18)$ is unbounded as the following examples show.

EXAMPLE 8.3. Let $F = PSL_n(q^2)$ with $n = q^2 - 1$ be the quotient of $SL_n(q^2)$ by its centre Z, and define the subgroup D of F by

$$D = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} Z : A \in SL_{n-1}(q^2) \right\} \cong PSL_{n-1}(q^2).$$

Let $\sigma \in \operatorname{Aut} F$ be given by

$$\sigma : (a_{ij})Z \mapsto \left(a_{ji}^q\right)^{-1} Z,$$

and set $K = L = \langle D, \sigma \rangle$. Then $|C_{\operatorname{Aut} F}(L)| = q + 1$ and (F, K, L) is a (7.18)-tuple of rank $q + 2$.

EXAMPLE 8.4. Let $F = PSL_2(p^f)$ with both p and f prime, and let $K = L = PSL_2(p)$, which we view as a subgroup of F in the natural way. Then $|C_{\operatorname{Aut} E}(L)| = f$ and (F, K, L) is a (7.18)-tuple of rank $f + 1$.

Remark 8.5. Either of the above two examples can be used to show that $\Delta(7.6(\mathrm{b})) = \{n \in \mathbb{N} : n \geq 16\}$, thus justifying the dichotomy introduced in Definition 7.6.

Our final comment on Problem 4 is that the conditions defining a (7.18)-tuple are similar, but not identical, to those investigated by the second author in the "Crucial Case" of [16]. Given that the latter was resolved successfully, it seems reasonable to hope that the present problem can also be resolved.

We finish by stressing the two general problems on which the successful resolution of Problems 1–4 depend:

- Describe the maximal non-abelian simple sections of the non-abelian simple groups (cf. [2]).
- Describe all pairs (F, L) with F non-abelian and simple, $L \leq \mathrm{Aut}\, F$, and $[F/\{\mathrm{id}\}]_L \cong M_1$.

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