The Monster

The Monster M is the largest, the most famous and still the most mysterious of the 26 sporadic finite simple groups. Its order (i.e. the number of elements) is indeed substantial:

$$|M| = 2^{46}.3^{20}.5^9.7^6.11^2.13^3.17.19.23.29.31.41.47.59.71 =$$

$$808,017,424,794,512,875,886,459,904,961,710,575,005,754,368,000,000,000.$$

The existence of the Monster was predicted in 1973 independently by B. Fischer and R. Griess. This was a very active period in the classification of finite simple groups and by that time mathematicians have already got ideas which groups can arise as centralizers of involutions in finite simple groups. An involution is an element of order 2 and by the Feit–Thompson theorem such an element can be found in every non-abelian (i.e. non-commutative) finite simple group. The Monster was predicted as a finite simple group M containing an involution τ such that the elements in M which commute with τ form a subgroup C (called the centralizer of τ in M and denote by $C_M(\tau)$) of the shape

$$C \sim 2_+^{1+24}.Co_1.$$

This means that C contains a normal subgroup Q of order 2^{25} which is extraspecial of plus type. By definition Q is generated by 24 involutions x_i, $1 \leq i \leq 24$ such that $(x_i \cdot x_j)^2$ equals to τ if $|i-j| = 12$ and to the identity element otherwise. Thus the quotient of Q over the subgroup $\langle \tau \rangle$ generated by τ can be considered as 24-dimensional vectors space over the field of 2 elements. The quotient C/Q is another sporadic simple group, the first Conway group Co_1. This group is associated with the so-called the Leech lattice. This is the unique (up to equivalence) set Λ of vectors in 24-dimensional real vector space, which is closed under addition and such that (a) the length of every vector from Λ is an even integer; (b) there are no vectors outside Λ having integral inner product with every vector in Λ; (c) Λ does not contain vectors of length 2. In other terms Λ is a 24-dimensional lattice which is (a) even; (b) unimodular; (c) without roots. The automorphism group of Λ is a (non-split) extension of Co_1 by a subgroup of order 2 generated by the element which negates every vector in Λ. The vectors in Λ whose halves are also in Λ form a sublattice 2Λ in Λ of index 2^{24}. The action of Co_1 on $\Lambda/2\Lambda$ and on $Q/\langle \tau \rangle$ are isomorphic. These conditions leave us with exactly two possible isomorphism types of C. The difference between these two types are quite delicate but only one of the groups can be the involution centralizer in a finite simple group.

If α is an element in M commuting with τ, i.e. if $\alpha \in C$ then the centralizer of α in C is the centralizer of an involution in $C_M(\alpha)$. From this is was deduced that the centralizers in M of various elements involve many other sporadic simple groups and for some of these groups this was the first evidence for existence. For instance there is an involution $\sigma \in Q$ whose centralizer turned out to be a new sporadic simple group, the Fischer Baby Monster group B, extended (in a non-split way) by $\langle \sigma \rangle$. In fact every involution in M is either a conjugate of τ (Conway involutions) or it is a conjugate of σ (Fischer involutions). Besides τ the subgroup C contains many other Conway involutions both inside and outside Q. In the restriction to C of any representation of M by linear transformations these involutions must have the same character (i.e. trace). It turns out that $d_1 = 196,883$ is dimension of the smallest faithful linear representation of C over complex numbers such that all Conway involutions in C have the same character. This means that d_1 is dimension of the smallest complex vector space V in which M might have a faithful representation. It was conjectured (and lately proved) that M indeed has a representation in V. Based on this conjecture the whole table of complex characters of M was computed. Under a number of assumption including the structure of the involution centralizer C and existence of complex representation of dimension d_1, J. Thompson proved in [Th79] the uniqueness of the Monster. The strategy was the following. The Monster is known to be generated by C and another subgroup N of the shape

$$N \sim 2^{2+11+22}.(M_{24} \times S_3),$$

which is the normalizer in M of a subgroup of order 4, all whose non-trivial elements are Conway involutions and one of them is τ. Consider the amalgam $\{C, N\}$ by which we understand the set-theoretical union of C and N with group operations coinciding on the intersection

$$C \cap N \sim 2_+^{1+24}.2^{11}.M_{24} \sim 2^{2+11+22}.(M_{24} \times 2).$$

It was shown in [Th79] that up to conjugation there is at most one embedding of this amalgam in $GL(V)$.

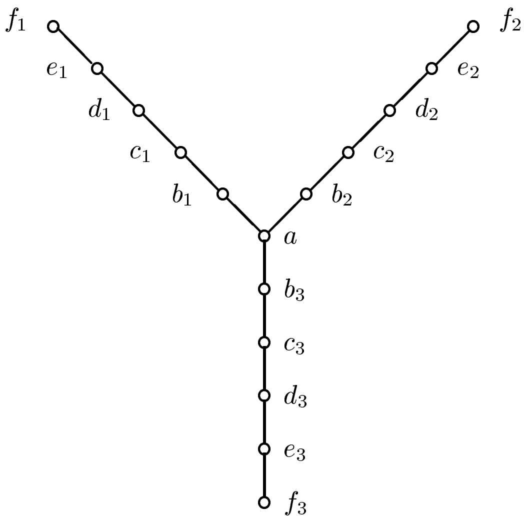
In [Gri82] the Monster was proved to exist in a certain sense by making constructive the uniqueness proof in [Th79]. Namely the embedding in $GL(V)$ of the amalgam $\{C, N\}$ was explicitly constructed in [Gri82] and the group generated by the image was identified with the Monster. Namely it was shown that C is the full centralizer of τ in the group generated by the image. Both on construction and identification stages a certain bilinear form and an algebra structure preserved by the Monster have played a very important role. The invariant bilinear form exists since V is selfdual as M-module and it was notices by S. Norton that the invariant algebra structure exists since V appears with non-zero multiplicity in decomposition into M-irreducibles of the tensor square of V. The M-invariant algebra structure on V is now known as the Griess algebra.

The Fischer involutions in M are sometimes called transpositions. The term come from classical notion of transposition in a symmetric group. The product of any two distinct transpositions in a symmetric group has order 2 or 3. Moreover the symmetric group of the set $\{1, 2, ..., n\}$ is generated by the transpositions $e_i = (i, i+1)$ for $1 \leq i \leq i-1$ subject to the relations that e_i and e_j commute if $|i - j| > 1$ and $(e_i \cdot e_{i+1})^3 = 1$ (that is why they are called 3-transpositions). The product of any two distinct Monster transpositions (Fischer involutions) has order 2, 3, 4, 5 or 6 and these products are in eight different conjugacy classes of M. Moreover, the orbit of M containing given pair of transpositions is uniquely determined by the conjugacy class of their product. This means that M when acting by conjugation of the set X of its transpositions produces a permutation group of rank 9. It is actually the smallest faithful permutation representation of M and its degree (i.e. the number of transpositions in the Monster) is

$$|X| = 97, 239, 461, 142, 990, 186, 000.$$

In [Nor85] the numbers of triples (x, y, z) of transpositions were computed depending on conjugacy classes containing $x \cdot y$, $x \cdot z$ and $y \cdot z$. This gives the intersection numbers of the centralizer algebra of M on X. The primitive idempotents of this algebra were determined and one of them turned out to be of rank $d_1 = 196, 883$. This means that under the action of M the space U of functions from X into complex numbers slips into a direct sum of irreducible M-modules and one of them is V. The Griess algebra can be realized as the projection to V of the pointwise multiplication in U. Thus the existence of d_1-dimensional faithful representation of M was deduced from its local properties (that is from the structure of centralizers certain elements). However proofs for some of the local properties needed was never published and a different self-contained uniqueness proof for the Monster appeared in [GMS89].

It was observed by B. Fischer on early stages of study of the Monster that it is generated by a small number of transpositions with pairwise products being of order 2 or 3, extending the analogy between the symmetric and Monster transpositions. A particular nice set of generators can be found



in the Bimonster BM which is the the wreath product $(M \wr 2)$ of M and a group of order 2. In other terms Bimonster is the direct product of two copies of M extended by an involutory automorphism permuting the factors.

It was shown in [CNS88] that BM is generated by 16 involutions corresponding to the nodes on the diagram above such that the involutions commute (i.e. their product is of order 2) unless the nodes are joined by an edge and if they are joined then the product is of order 3. Besides these relations the following additional equality holds:

$$(a.b_1.c_1.a.b_2.c_2.a.b_3.c_3)^{10} = 1.$$

Because the above diagram looks alike the letter "Y" the group defined by the above presentation was denoted by Y_{555}, where the indices show the lengths of the three arms of the diagram. Similar notation are used for groups with shorter arms satisfying the conditions on the order of products and the above additional equality. It was conjectured by J. Conway that Y_{555} is the Bimonster. In favour of this conjecture was the identification of some smaller Y-groups which turned out to be isomorphic to the corresponding subgroups in the Bimonster:

$$Y_{222} \cong 3^5.O_5(3); \quad Y_{322} \cong O_7(3) \times 2; \quad Y_{332} \cong 2^2 \cdot Fi_{22};$$

$$Y_{333} \cong 2^3 \cdot{}^2 E_6(2); \quad Y_{432} \cong Fi_{23} \times 2, \quad Y_{442} \cong 3 \cdot Fi_{24}.$$

Here Fi_{22}, Fi_{23} and Fi_{24} are sporadic 3-transposition groups discovered by B. Fischer. Eventually the Conway conjecture was proved by joining the results in [Nor92] and [Ivn92]. The Monster contains another important subgroup L which is the normalizer in M of a Conway-pure elementary abelian subgroup of order 8, such that

$$L \sim 2^{3+6+12+18}.(3 \cdot S_6 \times L_3(2))$$

and both $L \cap C$ and $L \cap N$ have index 7 in L. Within the classification of certain geometries (so-called tilde geometries) mostly associated with sporadic simple groups and analogous to the Tits buildings for Lie type groups, it it was proved in [Ivn92] that M is the only group which contains the amalgam $\{C, N, L\}$ which generates it. It was known from [Soi89] that $Y_{555} \cong (Z \wr 2)$ where Z is the index 2 subgroup in Y_{443} which contains even products of generators only. In [Nor92] the amalgam $\{C, N, L\}$

was reconstructed inside Z and was shown to generate Z. The available proof of the Y-presentation for the Monster relies on the existence of the Monster and on a number of concrete properties of M. On the other hand an independent proof would provide a completely different existence proof for the Monster. In such a proof one would probably start with showing that Y_{555} is non-trivial in that sense that it contains at least three distinct elements. At present nobody knows how to do this not assuming that the Monster exists.

A very excited area associated with the Monster was originated by an observation by J. McKay that the dimension d_1 of the smallest possible representation of M is one less the linear coefficient in in modular function

$$J(z) = \sum_{n=-1}^{\infty} a_n \, q^n = q^{-1} + 0 + 196,884q + 21,493,760q^2 + ...$$

This is unique up to normalization function invariant under the natural action of $SL_2(\mathbf{Z})$ on the upper half of the complex plane, and $q = e^{2\pi i z}$. Soon after that J. Thompson notices than a few further coefficients in the q-expansion of $J(z)$ are simple linear combinations of degrees of irreducible characters of M. Further investigations led to discovery of deeper correlations between M and $J(z)$, which were summarized in a unique paper [CN79] and now known under the name of *monstrous moonshine*. The main conjecture posed in [CN79] is that there exists a graded M-nodule, say $V^\natural$ whose graded dimensions are exactly the coefficients in the q-expansion of $J(z)$, i.e.

$$V^\natural = V_{-1} \oplus V_1 \oplus \; ... \; \oplus V_n \oplus \; ... \; ,$$

where dim $(V_n) = a_n$, in particular V_1 is the direct sum of V and the trivial 1-dimension M-module. For an element $g \in M$ one defines what is called the Thompson series as

$$T_g(z) = \sum_{n=-1}^{\infty} \mathrm{tr}(V_n \mid g) \; q^n,$$

where $\mathrm{tr}(V_n \mid g)$ is the trace of g when acting on the homogeneous component V_n. It was conjectured in [CN79] that $T_g(q)$ is a genus zero function when defined on the upper half-plane.

The module $V^\natural$ was constructed in [FLM88] so that the action of M on it preserves a structure of vertex operator algebra (VOA) of which the Griess algebra is one of the homogeneous components. The genus zero conjecture was proved in [Bor92]. Vertex operator algebras, also known as conformal field theories, play a role in various areas of theoretical physics including dual resonance models and string theory. An important characteristic of a VOA is its central charge, which in case of $V^\natural$ equals to 24. Construction of $V^\natural$ was based on the Leech lattice and performed in two steps. First the so-called non-twisted construction was applied to Λ to produce a VOA $\mathcal{H}(\Lambda)$ and then the latter was twisted to give $\widetilde{\mathcal{H}}(\Lambda) = V^\natural$. Besides Λ there are 23 even unimodular 24-dimensional lattices, known as the Niemeier lattices. To each of these lattices one can also apply both non-twisted and twisted constructions to produce a pair of VOA's with central charge 24. These constructions are analogous to non-twisted and twisted constructions of lattices from binary codes. It is known that Λ can be obtained by applying the twisted construction to the Golay code. The VOA's coming from even unimodular lattices via non-twisted construction were characterized in [DGM90] as the ones containing a full rank Heisenberg algebra. It would be extremely important to have a similar characterization for VOA's coming from lattices via twisted construction. The Leech lattice is better understood along with the Niemeier lattices. Similarly one would expect that the moonshine module would be nicer to treat having the whole family of VOA's with $c = 24$. It was conjectured in [Sch93] that the family contains exactly 71 relatives of the Monster module. We believe that this conjucture provides a very attractive perspective for future study of the Monster and related objects.

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