Characters, Fields and Degrees

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Abstract

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I have relied mainly on my notes from the lectures. So, any errors are the product of the note-taking and are not to be attributed to the content of the lectures

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Chapter 1

Day One

We first set some notation to be used throughout the lectures.

Let G be a finite group and p be a prime. We denote by $\mathrm{Syl}_p(G)$ the set of Sylow p-subgroups of G. The symbol $\mathrm{Irr}(G)$ denotes the set of irreducible characters of G. Let F be a subfield of $\mathbb{C}$, we denote by $\mathrm{Irr}_F(G)$ the set of irreducible characters χ of G such that $\chi(g) \in F$ for any $g \in G$. Similarly, if $\chi \in \mathrm{Irr}(G)$, then $\mathbb{Q}(\chi)$ denotes the field $\mathbb{Q}[\chi(g) \mid g \in G]$. In particular, $\mathrm{Irr}_{\mathbb{R}}(G)$ denotes the set of real valued irreducible characters of G. We use the symbol $\mathrm{cd}_F(G)$ for set $\{\chi(1) \mid \chi \in \mathrm{Irr}_F(G)\}$ and $\mathrm{cl}(G)$ for the set of conjugacy classes of G. The element x of G is said to be real if $x^g = x^{-1}$ for some $g \in G$. In particular $\mathrm{cl}_r(G)$ denotes the set of conjugacy classes of real elements of G.

The main aim of this short course is proving some the following results.

Theorem 1 (Ito-Michler (CFSGs)) *Let G be a finite group and p a prime number. If $p \not| \chi(1)$ for every $\chi \in \mathrm{Irr}(G)$, then $P \lhd G$, where $P \in \mathrm{Syl}_p(G)$.*

Theorem 2 (Dolfi-Navarro-Tiep) *If $2 \not| \chi(1)$ for every $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$, then $P \lhd G$, where $P \in \mathrm{Syl}_2(G)$.*

Theorem 3 (Navarro-Tiep) *If $2 \not| \chi(1)$ for any $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ non-linear, then G has a normal 2-complement.*

We quote the following two results relating the structure of a finite group G with $|\mathrm{cd}_F(G)|$.

Theorem 4 *If $|\mathrm{cd}_{\mathbb{C}}(G)| \leq 4$, then G is solvable.*

Clearly, Theorem 4 is the best possible, in fact $\mathrm{cd}_{\mathbb{C}}(\mathrm{Alt}(5)) = \{1, 3, 4, 5\}$.

Theorem 5 *If $|\mathrm{cd}_{\mathbb{R}}(G)| \leq 3$, then G is solvable.*

Clearly, Theorem 5 is the best possible, in fact $\mathrm{cd}_{\mathrm{R}}(\mathrm{Alt}(5)) = \mathrm{cd}_{\mathbb{C}}(\mathrm{Alt}(5)) = \{1, 3, 4, 5\}$.

Let n be in $\mathbb{N}$. Set $\mathbb{Q}_n = \mathbb{Q}[\xi]$, where $\xi \in \mathbb{C}$ is a primitive nth root of unity. It is a classical (easy) result that if $\mathbb{Q} \subseteq F \subset \mathbb{Q}_n$ and $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$, then σ extends to some $\hat{\sigma}$ in $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$.

Furthermore, if G is a finite group and $|G| \mid n$, then $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ acts naturally on $\mathrm{Irr}(G)$, i.e. $(\chi, \sigma) \mapsto \chi^\sigma \in \mathrm{Irr}(G)$, where $\chi^\sigma(g) = (\chi(g))^\sigma$. In other words, if $\rho : G \to \mathrm{GL}(m, \mathbb{Q}_n)$ is the representation affording the irreducible character χ and $\sigma \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$, then χ^σ is the character afforded by the irreducible representation $\rho^\sigma : G \to \mathrm{GL}(m, \mathbb{Q}_n)$ defined by $(\rho^\sigma)(g) = (a_{ij}^\sigma)_{ij}$, where $\rho(g) = (a_{ij})_{ij}$.

Lemma 1 *Let H and G be finite groups, $\psi \in \mathrm{Irr}(H)$ and $\chi \in \mathrm{Irr}(G)$. Assume $\mathbb{Q}(\psi), \mathbb{Q}(\chi) \subseteq \mathbb{Q}_m$, for some $m \in \mathbb{N}$. We have $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\psi)$ if and only if whenever $\psi^\sigma = \psi$, for some $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$, then $\chi^\sigma = \chi$.*

Proof. Clear from the definitions and straightforward Galois Theory. □

Lemma 2 *Let A be a group acting, as a group of automorphisms, on a finite group G of order dividing n. Set $\mathcal{G} = Gal(\mathbb{Q}_n/\mathbb{Q})$. The group A acts on $\mathrm{Irr}(G)$ by $\chi^a(g) = \chi(g^{a^{-1}})$, for $a \in A, g \in G, \chi \in \mathrm{Irr}(G)$. Furthermore, the action of A on $\mathrm{Irr}(G)$ commutes with the action of $\mathcal{G}$ on $\mathrm{Irr}(G)$, i.e. $(\chi^a)^\sigma = (\chi^\sigma)^a$, for any $\chi \in \mathrm{Irr}(G), a \in A, g \in \mathcal{G}$.*

Proof. Exercise! □

If χ is a character of G and H is a subgroup of G, then χ_H denotes the restriction of the character χ to the subgroup H. Furthermore, if N is a normal subgroup of G and $\theta \in \mathrm{Irr}(N)$, then $\mathrm{Irr}(G \mid \theta)$ denotes the set of irreducible characters of G that restricted to N have constituent θ.

Lemma 3 (Isaacs) *Let N be a normal subgroup of a group G. Let H be a subgroup of G, $M = N \cap H$ and θ be a G-invariant irreducible character of N. Assume $\theta_M = \varphi \in \mathrm{Irr}(M)$. Then the map $_H : \mathrm{Irr}(G \mid \theta) \to \mathrm{Irr}(H \mid \varphi)$ defined by $\chi \mapsto \chi_H$ is a well-defined bijection.*

Proof. Let χ be an element of $\mathrm{Irr}(G \mid \theta)$. So, by Clifford's Theorem, $\chi_N = e\theta = \frac{\chi(1)}{\theta(1)}\theta = \frac{\chi(1)}{\varphi(1)}\theta$. Furthermore, $\chi_H = e_1\xi_1 + \cdots + e_2\xi_s$, for some $\xi_i \in \mathrm{Irr}(H \mid \varphi)$. In particular,

$$(\dagger) \qquad \frac{\chi(1)}{\varphi(1)} = e_1\frac{\xi_1(1)}{\varphi(1)} + \cdots + e_s\frac{\xi_s(1)}{\varphi(1)}.$$

Similarly, by the hypothesis on φ, we have $\frac{\chi(1)}{\varphi(1)}\varphi$.

By Frobenius Reciprocity Law, $\xi_i^G = e_i\chi + \Delta_i$. So,

$$((\xi_i)^G)_N \quad = \quad e_i\chi_N + (\Delta_i)_N = e_i\frac{\chi(1)}{\varphi(1)}\theta + (\Delta_i)_N. \tag{1.1}$$

But, using MacKey, we get

$$((\xi_i)^G)_N \quad = \quad ((\xi_i)_M)^N = \frac{\xi_i(1)}{\varphi(1)}\varphi^N = \frac{\xi_i(1)}{\varphi(1)}\theta + \Lambda, \tag{1.2}$$

where Λ is a character of N not containing θ.

Summing up Equation (1.1), (1.2), we get $e_i \frac{\chi(1)}{\varphi(1)} \leq \frac{\xi_i(1)}{\varphi(1)}$. Therefore, $e_i^2 \frac{\chi(1)}{\varphi(1)} \leq e_i \frac{\xi_i(1)}{\varphi(1)}$. Now, using (†), we have $e_i^2 \frac{\chi(1)}{\varphi(1)} \leq \frac{\chi(1)}{\chi(1)}$. Thus, $e_i^2 = 1$. So, $\chi(1) = \xi_i(1)$. Therefore, $\chi_H = \xi_i$. □

Lemma 4 (Brauer) *Let A be a group acting on $\mathrm{Irr}(G)$ and on $\mathrm{cl}(G)$, such that $\chi^a(x^a) = \chi(x)$ for any $a \in A$. Then $|\{\chi \in \mathrm{Irr}(G) \mid \chi^a = \chi$ for any $a \in A\}| = |\{c \in \mathrm{cl}(G) \mid c^a = c$ for any $a \in A\}|$.*

Lemma 4 has the following well-known application.

Corollary 1 $|\,\mathrm{Irr}_{\mathbb{R}}(G)| = |\,\mathrm{cl}_r(G)|$.

Proof. Let σ be an element of order 2 and define $\chi^\sigma = \overline{\chi}$ and $(x^G)^\sigma = (x^{-1})^G$. Set $A = \langle\sigma\rangle$. Clearly, $\chi^\sigma(x^\sigma) = \overline{\chi(x^\sigma)} = \overline{\chi(x^{-1})} = \chi(x)$. Therefore, this proposition follows from Lemma 4. □

Corollary 2 $|\,\mathrm{Irr}_{\mathbb{R}}(G)| = 1$ *if and only if $|G|$ has odd order.*

Proof. If 2 divides the order of G, then G has an involution x. Clearly, x is a real element in $G\backslash\{1\}$, therefore, by Corollary 1, $|\,\mathrm{Irr}_{\mathbb{R}}(G)| \geq 2$, a contradiction.

Conversely, if $|\,\mathrm{Irr}_{\mathbb{R}}(G)| \geq 2$, then, by Corollary 1, there exists $x \in G\backslash\{1\}$ real. In particular, $x^g = x^{-1}$, for some $g \in G$. Thus $x^{g^2} = x$. If G has odd order, we have $\langle g^2\rangle = \langle g\rangle$. This yields $x^{-1} = x^g = x$. Thence x is an element of order 2, a contradiction. □

Now, we turn to a subtle problem. Suppose N is a normal subgroup of G, θ is a character of N and χ is a character of G such that $\chi_N = \theta$. *Is there any control on $\mathbb{Q}(\chi)$ if $\mathbb{Q}(\theta)$ is "under control"?*

In general, there isn't much to say. For instance, if $G = \langle x\rangle \cong C_4$ and $N = \langle x^2\rangle$, then any character of N is rational but the characters of G are not rational (they are not even real!).

Shortly we are gonna have to use the following result.

Gallagher's correspondence

Let N be a normal subgroup of the finite group G, let $\theta \in \mathrm{Irr}(N)$ and $\chi \in \mathrm{Irr}(G)$ extending θ to G. Then $\mathrm{Irr}(G \mid \theta) = \{\beta\chi \mid \beta \in \mathrm{Irr}(G/N)\}$. Furthermore,

$$\begin{array}{ccc} \mathrm{Irr}(G/N) & \longrightarrow & \mathrm{Irr}(G \mid \theta) \\ \beta & \longmapsto & \beta\chi \end{array}$$

is a well-defined bijection.

Theorem 6 *Assume G/N is a group of odd order and θ is a G-invariant element of $\mathrm{Irr}_{\mathbb{R}}(N)$. Then there exists a unique real valued $\chi \in \mathrm{Irr}(G \mid \theta)$. In fact, $\chi_N = \theta$. Furthermore, $\mathbb{Q}(\chi) = \mathbb{Q}(\theta)$.*

Proof. We prove that θ has a unique real valued extension by induction on $|G : N|$.

Suppose $N < M \lhd G$. Then $|M : N| < |G : N|$, so, by induction, θ has a unique real valued extension η. Note that, by uniqueness, η is G-invariant: since M is a normal subgroup of G, if $g \in G$, then η, η^g are two real valued extensions of θ living in M, so, by uniqueness, $\eta^g = \eta$.

Now $|G : M| < |G : N|$, so η has a unique real valued extension χ, i.e. $\chi_M = \eta$. Note that $\chi_N = (\chi_M)_N = \eta_N = \theta$, therefore χ is an extension of θ.

Assume, by a way of contradiction, that ψ is another real valued extension of θ to G. So, by Gallagher's correspondence, $\psi = \beta\chi$ for some $\beta \in \mathrm{Irr}(G \mid N)$. Now, $\psi = \overline{\psi} = \overline{\beta\chi} = \overline{\beta}\overline{\chi} = \overline{\beta}\chi$. Gallagher's correspondence yields that $\beta = \overline{\beta}$ (Gallagher's correspondence is a bijection!). So, $\beta \in \mathrm{Irr}_{\mathbb{R}}(G/N)$, but G/N has odd order, thence $\beta = 1$. Thus $\psi = \chi$.

Since, by the Odd Order Theorem, G/N is solvable, it remains to prove the result when G/N is cyclic of prime order p. In this case, it is well-known (and easy) that θ extends to G. Let ξ be an extension. The map $\lambda \mapsto \lambda\xi$ is a bijection from $\mathrm{Irr}(G/N)$ to $\mathrm{Irr}(G \mid \theta)$. Since $|\mathrm{Irr}(G/N)| = p$, we have $|\mathrm{Irr}(G \mid \theta)| = p$. Now, θ is real, so χ^G is real. Therefore, complex conjugaction acts on $\mathrm{Irr}(G \mid \theta)$ as a group of order 2 on a odd set. (Another way to see this is the following. If $\psi \in \mathrm{Irr}(G)$ and $\psi_N = \theta$, then, complex conjugaction and restriction commute, so, $(\overline{\psi})_N = \overline{\psi_N} = \overline{\theta} = \theta$) Therefore, there exists a fixed point χ, i.e. $\chi \in \mathrm{Irr}_{\mathbb{R}}(G \mid \theta)$.

Like in the previous case, Gallagher's correspondence yields that χ is unique (If ψ is another real valued extension of θ, then $\psi = \lambda\chi$ for some $\lambda \in \mathrm{Irr}(G/N)$. So, $\psi = \overline{\psi} = \overline{\lambda\chi} = \overline{\lambda}\chi$. Thus, we have $\lambda = \overline{\lambda}$. So, $\lambda \in \mathrm{Irr}_{\mathbb{R}}(G/N) = \{1\}$. Thence $\psi = \chi$ and χ is unique).

It remains to prove that $\mathbb{Q}(\chi) = \mathbb{Q}(\theta)$. Let n be the order of G. We have $\mathbb{Q}(\chi), \mathbb{Q}(\theta) \subseteq \mathbb{Q}_n$. If $\sigma \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ fixes χ, then $\theta^\sigma = (\chi_N)^\sigma = (\chi^\sigma)_N = \chi_N = \theta$. So, σ fixes θ. Conversely, assume $\sigma \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ fixes θ. Now, $(\chi^\sigma)_N = (\chi_N)^\sigma = \theta^\sigma = \theta$. Furthermore, since $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ is an abelian group, we have $\overline{\chi^\sigma} = (\overline{\chi})^\sigma = \chi^\sigma$. This says that χ^σ is a real valued extension of θ to G. By uniqueness, $\chi^\sigma = \chi$. Lemma 1 yields that $\mathbb{Q}(\chi) = \mathbb{Q}(\theta)$. □

Theorem 7 (CFSGs) *If G has even order, then G has a non-trivial irreducible rational character*

Proof. By induction on $|G|$. If $1 < N \lhd G$ and G/N has even order, then, by induction, we are done. Assume G/N is odd and without loss of generality we may as well assume that N is a minimal normal subgroup of G. If N is abelian, then N is an elementary abelian 2-group. So, $\mathrm{Irr}(N) = \mathrm{Irr}_{\mathbb{Q}}(N)$. Pick $1 \neq \lambda \in \mathrm{Irr}(N)$. Consider the inertia group $T = I_G(\lambda)$. Now $N \leq T \leq G$. By Theorem 6, there exists a unique real valued extension χ of λ to T and

$\mathbb{Q}(\chi) = \mathbb{Q}(\lambda) = \mathbb{Q}$. Now, by Clifford's theorem (T is the inertia group of λ!) $\chi^G \in \mathrm{Irr}(G)$. Furthermore, since χ is rational, we have that χ^G is rational.

It remains to prove the result when N is direct product of isomorphic non-abelian finite simple groups. Now, get your hands dirty and prove that every finite non-abelian simple group has a non-linear rational character λ. Namely, if S is a sporadic group, then $|\mathrm{Irr}_{\mathbb{Q}}(S)| \geq 6$. If S is an alternating group, then $|\mathrm{Irr}_{\mathbb{Q}}(S)| = n-1$. If S is a group of Lie type, then $|\mathrm{Irr}_{\mathbb{Q}}(S)| \geq 2$. With such a λ the rest of the proof is like in the soluble case. □

Definition. Let χ be the irreducible character afforded by the representation $\rho : G \to \mathrm{GL}(m, \mathbb{C})$. Note, that $\det(\rho) : g \mapsto \det(\rho(g))$ is a linear character of G (in fact $\det(\rho)$ depends only on χ and not on the representation ρ affording the character χ). Define $o(\chi)$ to be the order of the linear character $\det(\chi)$ as element of $\mathrm{Hom}(G, \mathbb{C})$.

We note that if $\rho : G \to \mathrm{GL}(m, \mathbb{C})$ is a representation of G affording the character χ of G and $\sigma \in \mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$ (where $|G| \mid n$), then $o(\chi) = o(\chi^\sigma)$. Indeed, $\rho^\sigma : G \to \mathrm{GL}(m, \mathbb{C})$ is the representation affording the character χ^σ. So, if $\det(\rho) = \lambda$, then $\det(\rho^\sigma) = \lambda^\sigma$. In particular, $\lambda^s = 1$, iff, $(\lambda^s)^\sigma = 1$, iff, $(\lambda^\sigma)^s = 1$. Thus $o(\chi) = o(\chi^\sigma)$. In particular, $o(\chi) = o(\overline{\chi})$.

Theorem 8 *Let N be a normal subgroup of G, θ be a G-invariant element in* $\mathrm{Irr}(N)$. *If* $\mathrm{lcd}\{\theta(1)o(\theta), |G : N|\} = 1$, *then θ extends to G. In fact, there exists a unique extension χ such that $o(\chi) = o(\theta)$. In particular, $\mathbb{Q}(\chi) = \mathbb{Q}(\theta)$.*

Lemma 5 *Assume P acts on K as a group of automorphisms. If $2 \mid |P/C_P(K)|$, then there exists $1 \neq \theta \in \mathrm{Irr}(K)$ and $x \in P$ such that $\theta^x = \overline{\theta}$.*

Proof. Let $xC_P(K) \in P/C_P(K)$ be an involution. There exists $k \in K$ such that $k^x \neq k$, otherwise $x \in C_P(K)$. Let $1 \neq y = k^{-1}k^x$. Now, $y^x = (k^{-1})^x k^{x^2} = (k^{-1})xk = y^{-1}$.

Consider $\langle\sigma\rangle$ a group of order 2. Define an action of σ on $\mathrm{Irr}(K)$ by $\chi^\sigma = \overline{\chi}^x$. This is a well-defined action, indeed, $\chi^{\sigma^2}(g) = \overline{\chi^\sigma(g^{x^{-1}})} = \overline{\overline{\chi(g^{x^{-2}})}} = \chi(g)$, so, $\chi^{\sigma^2} = \chi$. The element σ acts on the classes $\mathrm{cl}(K)$, indeed, $(g^K)^\sigma = ((g^{-1})^x)^K$. These two actions are compatible in the sense of Brauer's lemma. Indeed, $\chi^\sigma(g^\sigma) = \chi^\sigma((g^{-1})^x) = \overline{\chi(g^{-1})} = \chi(g)$. Thus, by Lemma 4, we have that the number of σ-invariant conjugacy classes of K is equal to the number of σ-invariant irreducible characters of K. Since y is a σ-invariant non-trivial element of K, we have that there exists $1 \neq \theta \in \mathrm{Irr}(K)$ such that $\theta^\sigma = \theta$, so $\overline{\theta}^x = \theta$. In other words, $\overline{\theta} = \theta^x$. □

Lemma 6 *Let G/N be a group of odd order and $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$. Then every irreducible constituent of χ_N is real.*

Proof. Let θ be an irreducible constituent of χ_N, so $[\theta, \chi_N]_N \neq 0$. In particular, $\overline{\theta}$ is an irreducible constituent of χ_N, indeed, $0 \neq \overline{[\theta, \chi_N]_N} = [\overline{\theta}, \chi_N]_N$.

So, by Clifford's Theorem, $\overline{\theta} = \theta^g$, for some $g \in G$. So, $\theta^{g^2} = \theta$. Therefore, $g^2 \in T = I_G(\theta)$. Now, $N \leq T \leq G$ and G/N is odd, thus $\langle gN\rangle = \langle g^2N\rangle$. This yields $gN \in T/N$. So, $g \in T$. So, $\overline{\theta} = \theta^g = \theta$. So θ is real. □

We say that a finite group G is of Chillag-Mann type if every element in $\mathrm{Irr}_{\mathbb{R}}(G)$ is linear.

Theorem 9 (Chillag-Mann) *If G is of Chillag-Mann type then $G = K \times P$, where $P \in \mathrm{Syl}_2(G)$ is of Chillag-Mann type.*

Theorem 10 (Tiep) *Let S be a non-abelian simple group, $S \lhd G$, $C_G(S) = 1$ and G/S a 2-group. Then there exists a character $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ of even degree such that $[\chi_S, 1_S] = 0$.*

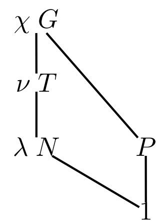
Theorem 11 *All elements of $\mathrm{Irr}_{\mathbb{R}}(G)$ have odd degree if and only if G has a normal Sylow 2-subgroup P of Chillag-Mann type.*

Proof. First we assume that G has a normal Sylow 2-subgroup of Chillag-Mann type. Let χ be an element in $\mathrm{Irr}_{\mathbb{R}}(G)$. We have to prove that $\chi(1)$ is odd. Let θ be an irreducible constituent of χ_P. By Lemma 6, θ lies in $\mathrm{Irr}_{\mathbb{R}}(P)$. Thus, by hypothesis, θ is a linear character. Now, $\chi(1) = \frac{\chi(1)}{\theta(1)} \mid |G_P|$. Therefore, $\chi(1)$ is odd.

Conveserly. We argue by induction on $|G|$. Let P be a Sylow 2-subgroup of G. If N is a nontrivial normal subgroup of G, then every element in $\mathrm{Irr}_{\mathbb{R}}(G/N)$ has odd degree, so, by induction, PN/N is a normal subgroup of G/N of Chillag-Mann type. Let θ be in $\mathrm{Irr}_{\mathbb{R}}(PN)$. The character θ has a unique T-invariant extension ψ to the inertia subgroup $T = I_G(\theta)$. The uniqueness of ψ yields that ψ is real valued. Now, $\chi = \psi^G$ is a real valued irreducible character of G. Thus $\chi(1)$ is odd. Hence $\theta(1)$ is odd. This says that all elements in $\mathrm{Irr}_{\mathbb{R}}(PN)$ have odd degree. If $PN < G$, then, by induction, P is a normal subgroup of PN of Chillag-Mann type. Therefore, P is a normal subgroup of G of Chillag-Mann type (PN is normal in G!).

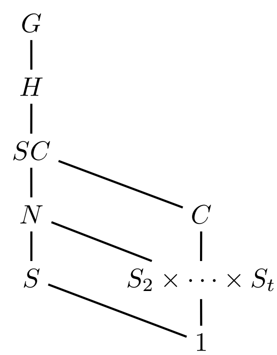
This shows that we may as well assume that G has a unique minimal normal subgroup N and G/N is a 2-group.

Assume N is soluble. If $2 \mid |N|$, then $G = P$ and we are done. So, $|N|$ is odd. Now, $C_P(N)$ is a normal of G. If $C_P(N) = P$, then we are done. Therefore assume $C_P(N) < P$. Now, P is an even group acting (non-trivially) on N, thus, by Lemma 5, P inverts some irreducible character of N. In other words, there exists $\lambda \in \mathrm{Irr}(N) \backslash \{1\}$ and $x \in P$, such that $\lambda^x = \overline{\lambda}$. Set $T = I_G(\lambda)$ the inertia group of λ. The character λ has a canonical extension ν to T. Furthermore, ν is the only extension such that $o(\nu) = o(\lambda)$. Moreover, 'cause ν is it canonical, we have $\overline{\nu}^x = \nu$ (indeed, $\overline{\lambda}^x = \lambda$, so uniqueness of ν yields that the same equation holds for ν). By Clifford's theorem, $\chi = \nu^G$ is an irreducible character of G.



We have $\overline{\chi} = \overline{\nu^G} = (\overline{\nu})^G = (\nu^x)^G = \nu^G = \chi$ (inducing ν or a conjugate of ν is the same! so, $\nu^G = (\nu^x)^G$!). So, $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$. Thence, $\chi(1)$ is odd. Therefore, $T = G$. This proves that λ is G-invariant. Therefore, $\overline{\lambda} = \lambda^x = \lambda$. The group N has odd order, so, $\lambda = 1$, a contradiction.

Assume N is not soluble. So, $N = S_1 \times \cdots \times S_t$, where the S_is are isomorphic non-abelian simple groups. Note that the group G acts transitively on $\{S_1, \ldots, S_t\}$. Set $S = S_1$, $H = N_G(S)$, $C_G(S) = C$, $H/C = \overline{H}$, $SC/C = \overline{S}$ and $D/C = C_{\overline{H}}(\overline{S})$ for a suitable subgroup D of G. Since $\overline{S}$ is a non-abelian simple group, we have $\overline{S} \cap C_{\overline{H}}(\overline{S}) = 1$. Therefore, $D \cap SC = C$. Hence, $D \cap S = D \cap SC \cap S = C \cap S = 1$. This proves that D and S are normal subgroups of H such that $D \cap S = 1$, thus $[D, S] = 1$. So $D = 1$ and $C_{\overline{H}}(\overline{S}) = 1$.



By Theorem 10, there exists $\chi \in \mathrm{Irr}_{\mathbb{R}}(\overline{H})$ such that $\chi(1)$ is even and $[\chi_{\overline{S}}, 1_{\overline{S}}] = 1$. In particular, $\chi \in \mathrm{Irr}_{\mathbb{R}}(H)$ with $C \subseteq \mathrm{Ker}\,\chi$ and $[\chi_S, 1_S] = 0$. Let $\delta \neq 1$ be an irreducible character of S lying under χ. Define $\psi = \delta \times 1_{S_2} \times \cdots \times 1_{S_t}$. Note that χ lies over ψ too.

Set $T = I_G(\psi)$. We claim $T \subseteq H$. Let g be in G such that $\psi^g = \psi$. In particular, $\mathrm{Ker}\,\psi$ is g-invariant. But, $\mathrm{Ker}\,\psi = S_2 \times \cdots \times S_t$, because $\delta \neq 1$. So, g normalizes $S_2 \times \cdots \times S_t$. Thus g normalizes S_1. Hence $g \in H$.

Let ξ be in $\mathrm{Irr}(T \mid \psi)$. Now, $\xi^H = \chi$. Moreover, by definition of T, we get $\chi^G = \xi^G \in \mathrm{Irr}(G)$. Now, χ has even degree because it lies over ψ and it is real valued. This proves that χ^G is an irreducible real valued character of even degree, a contradiction. □

Chapter 2

Day Two

Some recall on our aims.

Theorem 12 *If 2 does not divide $\chi(1)$ for any $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$, then G has a normal Sylow 2-subgroup.*

This theorem is false for rational characters: if $G = \mathrm{PSL}(2, 27)$, then $\mathrm{Irr}_{\mathbb{Q}}(G) = \{1, \theta\}$ where $\theta(1) = 27$.

Theorem 13 *If 2 divides $\chi(1)$ for any $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ non-linear, then G has a normal 2-complement.*

This theorem is true even for rational characters, the proof is a very deep!

In the following theorem the symbol $E^p(G)$ denotes the smallest normal subgroup E such that G/E is an elementary abelian p-group.

Theorem 14 (J.Thompson) *Let N be a subgroup of G and p a prime such that p does not divide $|G : N|$. Suppose $E^p(G) \cap N = E^p(N)$. Then $O^p(G) \cap N = O^p(N)$.*

We'll soon need the previous theorem.

Lemma 7 *Let λ be a linear character of G and $x \in G$. If $(o(\lambda), |x|) = 1$, then $\lambda(x) = 1$.*

Proof. Now, $\lambda^{o(\lambda)} = 1$, so, $\lambda^{o(\lambda)}(x) = 1$. Thence $\lambda(x^{o(\lambda)}) = 1$. Therefore, $x^{o(\lambda)} \in \mathrm{Ker}\,\lambda$. Since $\langle x \rangle = \langle x^{o(\lambda)} \rangle$, we have $x \in \mathrm{Ker}\,\lambda$. □

Lemma 8 *If λ is a linear character of G, $o(\lambda) = p^f$ and $\lambda_P = 1$ for some $P \in \mathrm{Syl}_p(G)$, then $\lambda = 1$.*

Proof. Let x be an element of G. Write x as $x_p x_{p'}$, where x_p is a p-element and $x_{p'}$ is a p'-element. By Lemma 7, we have $\lambda(x) = \lambda(x_p x_{p'}) = \lambda(x_p) = 1$. □

Lemma 9 *Let P be a Sylow p-subgroup of a finite group G. If λ is a linear character of a p-group P that extends to G, then there exists a unique (up to a canonical choice) δ extending λ with the same order, i.e. $o(\lambda) = o(\delta)$.*

Proof. Let say that ψ extends λ to G. So, $\psi_P = \lambda$. Since ψ is linear, we can write $\psi = \psi_p \psi_{p'}$, where ψ_p, respectively $\psi_{p'}$, is the p-part, respectively p'-part, of ψ. Since $(\psi_{p'})_P = 1$, we may as well assume that $\psi = \psi_p$ (this is the canonical choice to make!). In particular $o(\psi) = p^a$. Since $\psi_P = \lambda$, we have $(\psi^{o(\lambda)})_P = 1$. Therefore, by Lemma 8, we get $\psi^{o(\lambda)} = 1$. So, $o(\psi) \leq o(\lambda)$. Trivially, we have $o(\lambda) \leq o(\psi)$. Thus the lemma is proved. $\square$

Theorem 15 *Let G be a finite group and $P \in \mathrm{Syl}_p(G)$, then the followings are equivalent.*

(a) *G has a normal p-complement;*

(b) *every character $\theta \in \mathrm{Irr}(P)$ extends to G;*

(c) *every $\lambda \in \mathrm{Irr}(P/\Phi(P))$ extends to G.*

Proof. Clearly, (a) yields (b), and, (b) yields (c). It remains to prove that (c) yields (a).

Let λ be a character of $P/\Phi(P)$. Note that $o(\lambda) = 1$. By hypothesis, λ extends to a linear character $\chi_\lambda \in \mathrm{Irr}(G)$. By Lemma 9, we can take χ_λ so that $o(\chi_\lambda) = p$. This yields $|G/\operatorname{Ker}\chi_\lambda| = p$, therefore $E^p(G) \subseteq \operatorname{Ker}\chi_\lambda$. So, $E^p(G) \cap P \subseteq \operatorname{Ker}((\chi_\lambda)_P) = \ker\lambda$. This argument holds for any λ. Thus $E^p(G) \cap P \subseteq \Phi(P)$. Clearly, $E^p(G) \cap G$ is a normal subgroup of P and $P/(E^p(G) \cap P)$ is elementary abelian. Therefore $\Phi(P) \subseteq E^p(G) \cap P$. This proves that $E^p(G) \cap P = \Phi(G) = E^p(P)$.

Now, by Theorem 14, we have $O^p(G) \cap P = O^p(P) = 1$. This says that P is a complement of $O^p(G)$ in G, i.e. G has a normal p-complement. $\square$

Theorem 16 *If 2 divides $\chi(1)$ for any $\chi \in \mathrm{Irr}_{\mathbb{R}}(G)$ non-linear, then G has a normal 2-complement.*

Proof. Let P be a Sylow 2-subgroup of G. By Theorem 15, it is enough to prove that any character $\lambda \in \mathrm{Irr}(P/\Phi(P))$ extends to G. Note that the character λ is rational. Consider $\lambda^g = \Delta_1 + \Delta_2 + \Delta_3$, where Δ_1 is the sum of the constituents of even degree of λ^G, Δ_2 is the sum of the real valued constituents of odd degree and Δ_3 is the sum of the non-real valued constituents of odd degree.

Let χ be a non-real constituent of odd degree. We have $[\lambda^G, \chi] = [\overline{(\lambda^G)}, \overline{\chi}] = [\lambda^G, \overline{\chi}]$. This proves that if χ is a non-real constituent of odd degree of λ^G, then $\overline{\chi}$ is also a constituent. This says that Δ_3 has even degree. In particular, $\chi(1) \equiv \Delta_2(1) \mod 2$. Now, $|G : P|$ is odd and $\lambda(1) = 1$, therefore $1 \equiv \Delta_2(1) \mod 2$. So, $\Delta_2 \neq 0$. This proves that there exists some character χ real valued of odd degree over λ. By hypothesis, χ is linear. Hence $\chi(1) = 1$ and $\chi_P = \lambda$. The proof is complete. $\square$

Theorem 17 *If* $|\operatorname{cd}_{\mathbb{R}}(G)| = 2$*, then* G *is soluble.*

Proof. We have $\operatorname{cd}_{\mathbb{R}}(G) = \{1, m\}$. If m is even then, by Theorem 16, G has a normal 2-complement, and so G is soluble by the Odd Order Theorem. If m is odd, then, by Theorem 11, G has a normal Sylow 2-subgroup, and so G is soluble. □

We point out that Theorem 17 is also true if $|\operatorname{cd}_{\mathbb{R}}(G)| = 3$, but the proof requires the CFSGs.

Theorem 18 (Ito) *If* $\{|G : C_G(x)| \mid x \in G\} = \{1, m\}$*, then* G *is nilpotent.*

Conjecture 1 (Navarro) *If* $\{|G : C_G(x)| \mid x \in G, x \text{ real}\} = \{1, m\}$*, then* G *is solvable.*

Note that we cannot replace "solvable" with "nilpotent". In fact, if $G = \operatorname{Alt}(4)$, then $\{|G : C_G(x)| \mid x \text{ real}\} = \{1, 3\}$.

The following is going to be useful later on.

Theorem 19 (Gow) *If* G *is soluble,* $\chi \in \operatorname{Irr}_{\mathbb{R}}(G)$ *of odd degree, then* χ *is rational. In fact,* $\chi = \lambda^G$ *where* $o(\lambda) = 2$.

Proof. Let N be a normal subgroup of G and $\theta \in \operatorname{Irr}(N)$ a constituent of χ_N. We claim that θ is real. In fact, if $\theta \subseteq \chi_N$, then $\overline{\theta} \subseteq \overline{\chi_N} = \chi_N$. So, by Clifford's correspondence, θ and $\overline{\theta}$ are conjugate, $\theta^g = \overline{\theta}$. Take $T = I_G(\theta) = I_G(\overline{\theta}) = I_G(\theta^g) = I_G(\theta)^g = T^g$, so g normalizes T and $g^2 \in T$. So the order of gT/T divides 2. The character θ extends to a character ψ in T and $\psi^G = \chi$. We have $\chi(1) = |G : T|\psi(1)$ odd. So, $|N_G(T) : T|$ is odd. Therefore $gT = T$ and $g \in T$. Hence $\overline{\theta} = \theta$.

Now, we prove that $\chi = \lambda^G$ and $o(\lambda) = 2$ by induction on G.

Step 1. We can assume that $\operatorname{Ker}\chi = 1$.

Step 2. We can assume that $O_{2'}(G) = 1$. If $\theta \in \operatorname{Irr}(O_{2'}(G))$ is under χ, then θ is real, furthermore $|O_{2'}(G)|$ is odd. Thus $\theta = 1$.

Step 3. We can assume that χ is quasiprimitive, i.e. $\chi_N = e\theta$ for any N normal subgroup G where $\theta \in \operatorname{Irr}(N)$. In fact, let N be a normal subgroup of G, ψ the corresponding character in the inertia subgroup T. Since θ is real, we have $\psi, \overline{\psi} \in \operatorname{Irr}(T \mid \theta)$. So, $\chi = \psi^G = (\overline{\psi})^G = \overline{(\psi^G)} = \overline{\chi} = \chi$. So, by the uniqueness in the Clifford's correspondence, $\psi = \overline{\psi}$. So, ψ is odd and real valued. If $T < G$, then $\psi = \lambda^T$ and $o(\lambda) = 2$ for some λ. Then, $\chi = \psi^G = (\lambda^T)^G = \lambda^G$.

Step 4. Take $N = O_2(G)$. Let θ be an irreducible constituent of χ_N (so $\chi_N = e\theta$). By the previous step we have θ is real and G-invariant, so, by Step 1, $\operatorname{Ker}\theta = 1$. Now, $\chi(1)$ is odd, so $\theta(1)$ is odd and N is a 2-group. Thence $\theta(1) = 1$, i.e. θ is a linear character, in particular $o(\theta) = 2$ and $|N : \operatorname{Ker}\theta| = 2$. But $\operatorname{Ker}\theta = 1$ and so we have $|N| = 2$. Therefore $N \subseteq \xi(G)$, but $C_G(O_2(G)) \subseteq O_2(G)$. This yields $G = N$ and now the theorem is trivially proved. □

Theorem 20 (Tiep) *Let S be a non-abelian simple group. Then there exists an* $\mathrm{Aut}(S)$*-orbit Y of characters of S, $Y \subseteq \mathrm{Irr}(S)\backslash\{1\}$ such that*

(i) Y is odd;

(ii) if $\alpha \in Y$, then α is rational of odd degree.

In fact, Y can be chosen so that $|Y| = 1$, except for $S = \mathrm{PSL}(2, 2^f), \mathrm{PSU}(3, 2^f)$ where $|Y| = 3$.

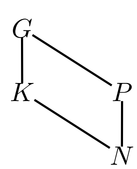
Theorem 21 *2 divides $\chi(1)$ for any $\chi \in \mathrm{Irr}_{\mathbb{Q}}(G)$ non-linear if and only if G has a normal 2-complement.*

Proof. Assume G has a normal 2-complement K. Let χ be an element in $\mathrm{Irr}_{\mathbb{R}}(G)$ non-linear. We want to prove that 2 divides $\chi(1)$. Deny it. Let θ be a constituent of χ. Now, $\chi(1)/\theta(1)$ divides $|G : K|$ (a power of 2). So, if 2 does not divide $\chi(1)$, then $\chi_K = \theta$. The character θ is real valued and K has odd order, so, $\theta = 1$. So, $\chi \in \mathrm{Irr}(G/K)$. Now, G/K is a 2-group and $\chi(1)$ is odd, therefore χ is linear, a contradiction.

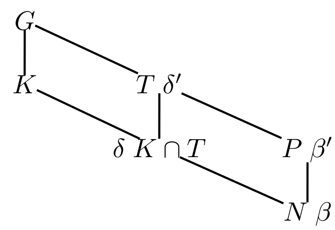
Vice versa. We argue by induction on $|G|$. Let N be a minimal normal subgroup of G. The group G/N has a normal 2-complement by induction. If N is abelian, then G is soluble. We claim that any real valued non linear character of G has even degree and so this theorem would follow from Theorem 16. Let χ be an irreducible real valued non-linear character of G of odd degree. Then, by Theorem 19, χ is rational.

We may assume that $N = S_1 \times \cdots \times S_t$, where the S_is are non-abelian simple groups. Fix $S = S_1$. We have $S_i = S^{g_i}$, for some $g_i \in G$. So, by Theorem 20, there exists Y a $\mathrm{Aut}(S)$-orbit of $\mathrm{Irr}(S)\backslash\{1\}$ of odd size such that any element in Y is rational of odd degree. Clearly, this set Y is $N_G(S)$-invariant, in fact $N_G(S)/C_G(S) \subseteq \mathrm{Aut}(S)$. Take $Y_i = Y^{g_i} \subseteq \mathrm{Irr}(S^{g_i}) = \mathrm{Irr}(S_i)$. Set $Z = \{\alpha_1 \cdots \alpha_t \mid \alpha_i \in Y_i\} \subseteq \mathrm{Irr}(N)$. Note that if β lies in Z, then β has odd degree and is rational.

Let P be a Sylow 2-subgroup of G and K a normal 2-complement $\mod N$.



Now, P/N is a 2-group acting on the odd set Z. So, P/N fixes some $\beta \in \mathrm{Irr}(N)$ rational of odd degree. Set $T = I_G(\beta)$. Now, $\det \beta$ is a linear character of N, thus $\det \beta = 1$, so, $o(\beta) = 1$. Moreover, $\beta(1)$ is odd, therefore, $(\beta(1)o(\beta), |G : N|) = 1$, so, by Theorem 8, there exists $\beta' \in \mathrm{Irr}(P)$ rational that extends β. Furthermore, β is real and $|T \cap N : N|$ is odd so, by Theorem 6, there exists a unique real valued δ extension of β to $T \cap K$, in fact $\mathbb{Q}(\delta) = \mathbb{Q}(\beta) = \mathbb{Q}$ and so δ is rational.



The reader might check that the uniqueness of δ yields that δ is P-invariant. Now, using Lemma 3, we have a bijection

$$\begin{array}{ccc}\mathrm{Irr}(T \mid \delta) & \longrightarrow & \mathrm{Irr}(P \mid \beta) \\ \chi & \longmapsto & \chi_P\end{array}$$

Let δ' be the δ-corresponding character in T, so δ' is the unique character in $\mathrm{Irr}(T \mid \delta)$ such that $(\delta')_P = \beta'$.

Let σ be in $\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})$, where $n = |G|$. Now, $(\delta')^\sigma$ is a character of T over δ ($(\delta')^\sigma \in \mathrm{Irr}(T \mid \delta^\sigma) = \mathrm{Irr}(T \mid \delta)$). Moreover, $((\delta')^\sigma)_P = (\beta')^\sigma = \beta'$ (β' is rational). So, by uniqueness, $(\delta')^\sigma = \delta'$. This proves that δ' is rational.

Now, by Clifford's correspondence, $(\delta')^G = \chi$ is a rational irreducible character of G of odd degree (in fact $|G : T|$ is odd and $(\delta')(1) = \delta(1) = \beta(1)$ is odd). This proves that χ is an irreducible rational character of G of odd degree. Thus, χ is linear! So, β is a linear character of N. Hence $\beta = 1$, a contradiction. The theorem is proved. $\square$

Theorem 21 has the following natural generalization.

Theorem 22 *If p divides $\chi(1)$ for any $\chi \in \mathrm{Irr}_{\mathbb{Q}_p}(G)$ non-linear, then G has a normal p-complement.*

From now on G is a soluble group. The rest of this course is devoted in proving that $|\{\chi \in \mathrm{Irr}_{\mathbb{Q}}(G) \mid \chi(1) \text{ is odd}\}|$ is locally group theoretically determined.

Lemma 10 (MacKey) *If $\nu \in \mathrm{Irr}(H)$ and $H \leq G$, then ν^G is irreducible if and only if $[\nu_{H\cap H^g}, (\nu^g)_{H\cap H^g}] = 0$ for any $g \in G \backslash H$.*

Proof. This is a trivial application of MacKey's formula. Recall that if $\mathcal{T}$ is a set of representatives of (H, H)-double cosets of G, i.e. $G = \coprod_{t\in\mathcal{T}} HtH$, then

$$(\nu^G)_H = \sum_{t\in\mathcal{T}} ((\nu^t)_{H\cap H^t})^H.$$

Now,

$$\begin{aligned}[\nu^G, \nu^G] &= [(\nu^G)_H, \nu]_H = \sum_{t\in\mathcal{T}} [(\nu^t)_{H\cap H^g}, \nu_{H\cap H^t}]_{H\cap H^t} \\ &= [\nu, \nu] + \sum_{t\in\mathcal{T}, t\notin H} [(\nu^t)_{H\cap H^t}, \nu_{H\cap H^t}].\end{aligned}$$

This proves that $[\nu^G, \nu^G] = 1$ if and only if $[(\nu^t)_{H\cap H^t}, \nu_{H\cap H^t}] = 0$ for any $t \in G\backslash H$. □

Lemma 11 *Let ν, λ be linear characters of G and $P \in \mathrm{Syl}_p(G)$. If $\nu_{N_G(P)} = \lambda_{N_G(P)}$, then $\nu = \lambda$.*

Proof. Set $\delta = \lambda\overline{\nu}$. By hypothesis, $\delta_{N_G(P)} = 1$. So, $N_G(P) \subseteq \mathrm{Ker}\,\delta \lhd G$. Using the Frattini argument, we get $\mathrm{Ker}\,\delta = G$. Thence, $\delta = 1$ and $\nu = \lambda$. □

MacKey Conjecture

$$|\operatorname*{Irr}_{p'}(G)| = |\operatorname*{Irr}_{p'}(N_G(P))|, \qquad P \in \operatorname*{Syl}_p(G).$$

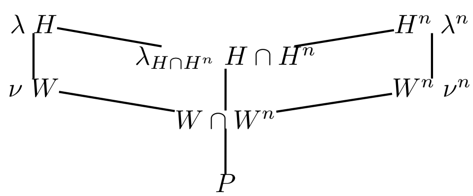
It is fairly well-known that there cannot be any natural-canonical bijection between $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(N_G(P))$.

Let χ be a real valued character of G of odd degree. By Theorem 19, the character χ is actually rational and $\chi = \lambda^G$, for some linear character $\lambda \in \mathrm{Irr}(H)$ and $o(\lambda) = 2$. In particular, it is easy to notice that χ has odd degree if and only if H contains a Sylow 2-subgroup P of G.

Theorem 23 *Using the previous notation. $(\lambda_{N_G(P)})^{N_G(P)} \in \mathrm{Irr}_{\mathbb{Q},odd}(N_G(P))$.*

Proof. It is enough to prove that $(\lambda_{N_G(P)})^{N_G(P)}$ is irreducible. Note that this theorem sets a "natural" correspondence between rational irreducible characters of odd degree of G and rational irreducible characters of odd degree of $N_G(P)$.

Set $W = N_H(P)$, $N = N_G(P)$ and $\nu = \lambda_W$. We want to prove that ν^N is irreducible. Take n in $N\backslash W$, by Lemma 10, we have to prove that $[\nu_{W\cap W^n}, (\nu^n)_{W\cap W^n}] = 0$. Deny it. Since, $\nu_{W\cap W^n}$ and $(\nu^n)_{W\cap W^n}$ are both linear, we have to prove that they coincide. Consider the following picture ($W^n = N_{H^w}(P)$, $W \cap W^n = N_{H\cap H^n}(P)$).



The characters $\lambda_{H\cap H^n}$ and $(\lambda^n)_{H\cap H^n}$ restricted to $W \cap W^n$ are equal:

$$(\lambda_{H\cap H^n})_{W\cap W^n} = \nu_{W\cap W^n} = (\nu^n)_{W\cap W^n} = (\lambda^n)_{W\cap W^n} = ((\lambda^n)_{H\cap H^n})_{W\cap W^n}.$$

By Lemma 11, we have $\lambda_{H\cap H^n} = (\lambda^n)_{H\cap H^n}$. So, $[\lambda_{H\cap H^n}, (\lambda^n)_{H\cap H^n}] \neq 0$. but, $\lambda^G \in \mathrm{Irr}(G)$. So, by Lemma 10, n lies H. This yields $n \in H \cap N = N_H(P) = W$, a contradiction. $\square$

Using all the previous results one might check that there exists a well-defined natural bijection from $\mathrm{Irr}_{\mathbb{Q},odd}(G)$ into $\mathrm{Irr}_{\mathbb{Q},odd}(N_G(P))$ (one has for example to check that the character constructed before does not depend on the subgroup H of G). This result is clearly false if G is not soluble (take $G = \mathrm{Alt}(6)$).

Now it is pretty easy to compute the size of $\mathrm{Irr}_{\mathbb{Q},odd}(G)$. Indeed, the size of $\mathrm{Irr}_{\mathbb{Q},odd}(N_G(P))$ is easy to get. We leave it to the reader to check that the number of elements in $\mathrm{Irr}_{\mathbb{Q},odd}(N_G(P))$ is equal to the number of $N_G(P)$-orbits on $P/\Phi(P)$. Thus, we have:

$$|\mathrm{Irr}_{\mathbb{Q},odd}(G)| = \#N_G(P)\text{-orbits on } P/\Phi(P).$$

This result holds only for the prime 2, in the sense $|\,\mathrm{Irr}_{\mathbb{Q},p'}(G)| \neq |\,\mathrm{Irr}_{\mathbb{Q},p'}(N_G(P))|$ (use $G = \mathrm{GL}(2,3)$ and $p = 3$).