$l \geqslant r(p)$ we have

$$G \in \mathfrak{A}_{p^k}\mathfrak{X}_{p'} \cap \mathfrak{W}^{(r(p))} = \mathfrak{A}_{p^k}\mathfrak{X}_{p'} \cap \mathfrak{W}.$$

Thus $G \in \mathfrak{W}$ is not $\mathfrak{W}$-critical, and we have a contradiction. This completes the proof of Theorem 1.2.

REFERENCES

1. J. Cossey, 'Critical groups and the lattice of varieties', *Proc. Amer. Math. Soc.* 20 (1969) 217–21.
2. G. Higman, 'Some remarks on varieties of groups', *Quart. J. Math. Oxford* (2) 10 (1959) 165–78.
3. —— 'The orders of relatively free groups', *Proc. Internat. Conf. Theory of Groups, Austral. Nat. Univ., Canberra*, 1965 (Gordon and Breach, 1967).
4. L. G. Kovács and M. F. Newman, 'Cross varieties of groups', *Proc. Royal Soc. Ser. A* 292 (1966) 530–36.
5. —— 'On non-Cross varieties of groups', *J. Austral. Math. Soc.* 12 (1971) 129–44.
6. H. Neumann, *Varieties of groups* (Springer-Verlag, Berlin, 1967).
7. S. Oates and M. B. Powell, 'Identical relations in finite groups', *J. Algebra* 1 (1964) 11–39.
8. A. Ju. Ol'šanskiĭ, 'On the problem of a finite basis for the identities of groups', *Izv. Akad. Nauk S.S.S.R.* 34 (1970) 376–84 (in Russian).
9. W. R. Scott, *Group theory* (Prentice-Hall, New Jersey, 1964).

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BASES FOR POLYNILPOTENT GROUPS

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1. Introduction

The word 'basis' will be used here in the following sense: a basis for a group G is a well-ordered subset B of G with the property that every member of G can be written in the 'canonic' form 1 or $b_1^{\beta_1}b_2^{\beta_2}\dots b_k^{\beta_k}$, where each b_i is a member of B, each β_i is a non-zero integer and $b_1 < b_2 < \dots < b_k$. This is an obvious generalization of the notion of a basis for an abelian group and also of the role of conventional basic commutators in free nilpotent groups.

The objects of this paper are, firstly, to find a free generating set for each of the terms

$$P_{k_1,k_2,\dots,k_d}(F) = \gamma_{k_d}(\gamma_{k_{d-1}}(\dots\gamma_{k_1}(F))\dots)$$

of a polycentral series of an absolutely free group F of arbitrary rank; secondly, to find bases for the free polynilpotent groups $F/P_{k_1,k_2,\dots,k_d}(F)$ and, finally, to describe a 'collecting process'—an effective and reasonably efficient process for converting any word in the free generating set for F into the product of a canonic form modulo the subgroup $P_{k_1,k_2,\dots,k_d}(F)$ and a word in the free generators for that subgroup.

One such subgroup is the derived group, δF. Gruenberg ([**1**] Theorem 5.2) has already found a free generating set for this subgroup which is similar to, but not quite the same as, the one given in this paper.

This paper will use the techniques of an earlier paper of mine ([**2**]). In particular the language developed there, especially that introduced in §14, will be used here without redefinition.

In [**2**] Theorem 9.1, bases are given for factors of the form $\hat{W}_\Phi(F)/\hat{W}_\Psi(F)$, where F is an absolutely free group, provided that Ψ does not dominate Φ—a rather restrictive condition. In the corollaries to Theorem 3 of this paper, bases are given for factors of the form $P_{K_{d'}}(F)/P_{K_d}(F)$ whenever $d' \leqslant d$, which removes the non-domination restriction in the former theorem for some important choices of Φ and Ψ.

Throughout this paper, except where otherwise explicitly stated, it will be assumed that we are working with a fixed polyweight range $Q = Q^K$ defined by a sequence $K = (k_1, k_2, \dots)$ in terms of which all definitions will be made. For any expression $\mathrm{x} \in \mathbf{A}$, $\pi(\mathrm{x})$ is a member of Q and hence a function $\omega \to \omega$; thus, for any $j \in \omega$, $\pi(\mathbf{x})(j)$ is defined and is itself a member of ω. For any non-negative integer d the *d-weight* of $\mathbf{x}$ is the integer $\pi(\mathbf{x})(d)$. Note that 0-weight is ordinary weight. The *depth* $d(\mathbf{x})$ of $\mathbf{x}$ is defined by $d(\mathbf{x}) = d_{\pi(\mathbf{x})}$ (see [2] Definition 14.2(A)). The depth of $\mathbf{x}$ may alternatively be defined by

$$\pi(\mathbf{x})(j) \neq 0 \Leftrightarrow j \leqslant d(\mathbf{x})$$

or, using [2] Definition 14.3, by

$$\delta_{d(\mathbf{x})} \leqslant \pi(\mathbf{x}) < \delta_{d(\mathbf{x})+1}.$$

Observe that, if $\mathbf{x}$ is of $(d-1)$-weight k_d then it is of d-weight 1 and that if $\mathbf{x}$ is of d-weight 1 then $\delta_d \leqslant \pi(\mathbf{x}) < 2\delta_d$.

The expressions in $\mathbf{A}$ are normally used to describe elements in groups of some variety $\mathfrak{B}$ (see [2] Definition 1.6). This gives rise to a natural congruence on $\mathbf{A}$: write

$$\mathbf{x} \equiv \mathbf{y} \pmod{\mathfrak{B}}$$

if $\mathbf{x}\rho = \mathbf{y}\rho$ for any description ρ of any group in $\mathfrak{B}$. If $\mathfrak{B}$ is not explicitly specified, it is assumed to be the variety $\mathfrak{O}$ of all groups.

2. Invertators

A particular type of expression, which may be roughly described as a 'commutator sprinkled with inverses' is now defined recursively: $\mathbf{x}$ is an *invertator* if either $\mathbf{x} \in \mathbf{G}$ or $\mathbf{x} = [\mathbf{b}^\beta, \mathbf{a}^\alpha] \in \mathbf{A}$, where $\mathbf{a}$ and $\mathbf{b}$ are invertators, $\alpha = \pm 1$ and $\beta = \pm 1$. The set of all invertators will be denoted by $\mathbf{I}$ and the set of all invertators and their inverses by $\mathbf{I}^{\pm 1}$. Observe that all commutators are invertators.

DEFINITION 1. The relation $<$ on the set $\mathbf{I}^{\pm 1}$ is defined as follows.

(i) If $\mathbf{a}^\alpha, \mathbf{b}^\beta \in \mathbf{I}^{\pm 1}$ and $\pi(\mathbf{a}) = \pi(\mathbf{b}) = 1$ so that $\mathbf{a}, \mathbf{b} \in \mathbf{G}$, then $\mathbf{a}^\alpha < \mathbf{b}^\beta$ if and only if
  (a) $\mathbf{a} < \mathbf{b}$ under the well-ordering of $\mathbf{G}$ induced by the indexing as given in [2] Definition 1.1 or
  (b) $\mathbf{a} = \mathbf{b}$, $\alpha = +1$, and $\beta = -1$.

(ii) If $\mathbf{a}^\alpha, \mathbf{b}^\beta \in \mathbf{I}^{\pm 1}$ and $\pi(\mathbf{a}) \neq \pi(\mathbf{b})$ then $\mathbf{a}^\alpha < \mathbf{b}^\beta$ if and only if $\pi(\mathbf{a}) < \pi(\mathbf{b})$.

(iii) If $\mathbf{a}^\alpha, \mathbf{b}^\beta \in \mathbf{I}^{\pm 1}$ and $\pi(\mathbf{a}) = \pi(\mathbf{b}) > 1$ then '$\mathbf{a}^\alpha < \mathbf{b}^\beta$' is defined recursively over $\pi(\mathbf{a})$. Write $\mathbf{a} = [\mathbf{a}_1^{\alpha_1}, \mathbf{a}_2^{\alpha_2}]$ and define

$$\mathrm{ld}(\mathbf{a}) = \mathbf{a}_1^{\alpha_1} \quad \text{and} \quad \mathrm{tr}(\mathbf{a}) = \mathbf{a}_2^{\alpha_2} \quad \text{if} \quad \mathbf{a}_2^{\alpha_2} < \mathbf{a}_1^{\alpha_1} \quad \text{or} \quad \mathbf{a}_2^{\alpha_2} = \mathbf{a}_1^{\alpha_1},$$

$$\mathrm{ld}(\mathbf{a}) = \mathbf{a}_2^{\alpha_2} \quad \text{and} \quad \mathrm{tr}(\mathbf{a}) = \mathbf{a}_1^{\alpha_1} \quad \text{otherwise.}$$

Write $\mathbf{b} = [\mathbf{b}_1^{\beta_1}, \mathbf{b}_2^{\beta_2}]$ and define $\mathrm{ld}(\mathbf{b})$ and $\mathrm{tr}(\mathbf{b})$ similarly. Then $\mathbf{a}^\alpha < \mathbf{b}^\beta$ if and only if

  (a) $\mathrm{ld}(\mathbf{a}) < \mathrm{ld}(\mathbf{b})$,
  (b) $\mathrm{ld}(\mathbf{a}) = \mathrm{ld}(\mathbf{b})$ and $\mathrm{tr}(\mathbf{a}) < \mathrm{tr}(\mathbf{b})$,
  (c) $\mathrm{ld}(\mathbf{a}) = \mathrm{ld}(\mathbf{b})$, $\mathrm{tr}(\mathbf{a}) = \mathrm{tr}(\mathbf{b})$, and $\mathbf{b}_1^{\beta_1} < \mathbf{a}_1^{\alpha_1}$, or
  (d) $\mathbf{a} = \mathbf{b}$, $\alpha = +1$, and $\beta = -1$.

(iv) Finally, write $\mathbf{a}^\alpha \leqslant \mathbf{b}^\beta$ if $\mathbf{a}^\alpha < \mathbf{b}^\beta$ or $\mathbf{a}^\alpha = \mathbf{b}^\beta$.

It will be observed that this definition is an obvious slight generalization of [2] Definition 6.1. A minor modification to the proof of [2] Lemma 6.1 yields:

LEMMA 1. *With the notation of Definition* 1, $\leqslant$ *is a (full) well ordering of* $\mathbf{I}^{\pm 1}$ *which extends the ordering of the set of commutators given in* [2] *Definition* 6.1.

DEFINITION 2. (A) A *Q-basic invertator* is defined recursively. $\mathbf{x}$ is a Q-basic invertator if and only if it is of one of the forms

(i) $\mathbf{x} \in \mathbf{G}$ or
(ii) $\mathbf{x} = [\mathbf{b}, \mathbf{a}^\alpha] \in \mathbf{I}$ where
  (a) $\mathbf{a}$ and $\mathbf{b}$ are both Q-basic invertators,
  (b) $\mathbf{a} < \mathbf{b}$,
  (c) if $\alpha = -1$ then $d(\mathbf{a}) < d(\mathbf{x})$,
  (d) if $\mathbf{b} = [\mathbf{b}_1, \mathbf{b}_2^\beta]$ then $\mathbf{b}_2 \leqslant \mathbf{a}$, and
  (e) if $\mathbf{b} = [\mathbf{b}_1, \mathbf{a}^\beta]$ and $d(\mathbf{a}) < d(\mathbf{b}_1)$ then $\beta = \alpha$.

(B) A *Q-basic invertator product* is an expression of the form $\mathbf{1}$ or $\mathbf{b}_1^{\beta_1}\mathbf{b}_2^{\beta_2}\ldots\mathbf{b}_r^{\beta_r}$ $(r \geqslant 1)$, where each $\mathbf{b}_i$ is a Q-basic invertator, each β_i is a non-zero integer, and $\mathbf{b}_1 < \mathbf{b}_2 < \ldots < \mathbf{b}_r$.

Whenever the polyweight range Q may be understood from the context, the prefix 'Q-' may be omitted. An easy inductive argument yields the following alternative description of basic invertators.

LEMMA 2. (i) *An expression of depth* 0 *(that is, of weight* $< k_1$*) is a basic invertator if and only if it is a basic commutator.*

(ii) *An expression of d-weight* 1, *where* $d \geqslant 1$, *is a basic invertator if and only if it is of the form*

$$[\mathbf{b}, n_1\mathbf{a}_1^{\alpha_1}, n_2\mathbf{a}_2^{\alpha_2}, \ldots, n_t\mathbf{a}_t^{\alpha_t}]$$

(using [2] *Definition* 7.3*), where* $t \geqslant 1$, *each* $n_i \geqslant 1$, $\mathbf{b}$ *and each* $\mathbf{a}_i$ *is a basic invertator of depth* $d-1$ *but* $[\mathbf{b}, \mathbf{a}_1^{\alpha_1}]$ *is of depth* d, $\mathbf{b} > \mathbf{a}_1 < \mathbf{a}_2 < \ldots < \mathbf{a}_t$ *each* $\alpha_i = \pm 1$ *and, if* $\mathbf{b} = [\mathbf{b}_1, \mathbf{b}_2]$, *then* $\mathbf{b}_2 \leqslant \mathbf{a}_1$.

(iii) *An expression of depth d but d-weight* > 1 *(and hence of d-weight* $< k_{d+1}$*) is a basic invertator if and only if it is of the form* $[\mathbf{b}, \mathbf{a}]$, *where* $\mathbf{a}$ *and* $\mathbf{b}$ *are basic invertators of depth d*, $\mathbf{a} < \mathbf{b}$ *and, if* $\mathbf{b} = [\mathbf{b}_1, \mathbf{b}_2^{\beta_2}]$, $\mathbf{b}_2 \leqslant \mathbf{a}$.

Henceforth, for any non-negative integer d, $\mathbf{J}_d$ will denote the set of all basic invertators of d-weight 1. Following [**2**] Definition 8.2, a *product of invertators* and a *product of basic invertators* are defined in the obvious way.

3. The collecting process

DEFINITION 3. (i) Let $\mathbf{x}$ be a product of basic invertators and $\mathbf{a}$ be any basic invertator. The integer $\varepsilon_{\mathbf{a}}(\mathbf{x})$ is defined to be the sum of exponents on occurrences of $\mathbf{a}$ as a factor of $\mathbf{x}$: strictly,

$$\varepsilon_{\mathbf{a}}(\mathbf{1}) = 0,$$

for any basic invertator $\mathbf{b}$,

$$\varepsilon_{\mathbf{a}}(\mathbf{b}^\beta) = \begin{cases} 0 & \text{if } \mathbf{a} \neq \mathbf{b}, \\ \beta & \text{if } \mathbf{a} = \mathbf{b}, \end{cases}$$

and for any two products $\mathbf{x}$ and $\mathbf{y}$ of basic invertators

$$\varepsilon_{\mathbf{a}}(\mathbf{xy}) = \varepsilon_{\mathbf{a}}(\mathbf{x}) + \varepsilon_{\mathbf{a}}(\mathbf{y}).$$

(ii) A *working product* is a product $\mathbf{x}$ of basic invertators with the property that, if

$$\mathbf{x} = \mathbf{x}_1[\mathbf{b}_1, \mathbf{b}_2^{\beta_2}]^\beta \mathbf{x}_2 \mathbf{a}^\alpha \mathbf{x}_3,$$

where $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are (possibly empty) products, $\mathbf{a}$ and $[\mathbf{b}_1, \mathbf{b}_2^{\beta_2}]$ are basic invertators, $\alpha = \pm 1$ and $\beta = \pm 1$, then

(a) $\mathbf{a} \geqslant \mathbf{b}_2$ and

(b) if $\mathbf{a} = \mathbf{b}_2$ and $d(\mathbf{a}) < d(\mathbf{b}_1)$ then

$$\varepsilon_{\mathbf{a}}(\mathbf{x}_2\mathbf{a}^\alpha\mathbf{x}_3) > 0 \Rightarrow \beta_2 = +1$$

and

$$\varepsilon_{\mathbf{a}}(\mathbf{x}_2\mathbf{a}^\alpha\mathbf{x}_3) < 0 \Rightarrow \beta_2 = -1.$$

(iii) Let $\mathbf{a}$ and $\mathbf{b}$ be invertators, $\alpha = \pm 1$ and $\beta = \pm 1$. Then $\nu(\mathbf{b}, \mathbf{a})$ is the smallest non-negative integer ν such that

$$d([\mathbf{b}, (\nu+1)\mathbf{a}]) > d(\mathbf{a}).$$

(iv) The expression $\chi(\mathbf{b}^\beta, \mathbf{a}^\alpha)$ is defined as follows.

$$\chi(\mathbf{b}, \mathbf{a}) = \mathbf{b}[\mathbf{b}, \mathbf{a}],$$

$$\chi(\mathbf{b}^{-1}, \mathbf{a}) = [\mathbf{b}, \mathbf{a}]^{-1}\mathbf{b}^{-1},$$

$$\chi(\mathbf{b}, \mathbf{a}^{-1}) = \begin{cases} \mathbf{b}[\mathbf{b}, \mathbf{a}^{-1}] & \text{if } \nu(\mathbf{b}, \mathbf{a}) = 0, \\ \mathbf{b}\chi([\mathbf{b}, \mathbf{a}]^{-1}, \mathbf{a}^{-1}) & \text{if } \nu(\mathbf{b}, \mathbf{a}) > 0, \end{cases}$$

$$\chi(\mathbf{b}^{-1}, \mathbf{a}^{-1}) = \begin{cases} [\mathbf{b}, \mathbf{a}^{-1}]^{-1}\mathbf{b}^{-1} & \text{if } \nu(\mathbf{b}, \mathbf{a}) = 0, \\ \chi([\mathbf{b}, \mathbf{a}], \mathbf{a}^{-1})\mathbf{b}^{-1} & \text{if } \nu(\mathbf{b}, \mathbf{a}) > 0. \end{cases}$$

REMARK 1. In part (iii) of this definition, such integers ν exist, for instance $\nu = k_{d(\mathbf{a})+1}$, so $\nu(\mathbf{b}, \mathbf{a})$ is always defined. Further,

$$d(\mathbf{b}) > d(\mathbf{a}) \Rightarrow \nu(\mathbf{b}, \mathbf{a}) = 0$$

and thus

$$\nu(\mathbf{b}, \mathbf{a}) > 0 \Rightarrow d(\mathbf{b}) \leqslant d(\mathbf{a}) \Rightarrow \nu([\mathbf{b}, \mathbf{a}], \mathbf{a}) = \nu(\mathbf{b}, \mathbf{a}) - 1.$$

Hence Part (iv) of this definition makes sense as a recursive definition.

REMARK 2. Observe that $\chi(\mathbf{b}^\beta, \mathbf{a}^\alpha)$ is a product of invertators and that the invertators occurring as factors in the product $\chi(\mathbf{b}^\beta, \mathbf{a}^{-1})$ are, writing ν for $\nu(\mathbf{b}, \mathbf{a})$,

$$[\mathbf{b}, r\mathbf{a}] \quad (r = 0, 1, \ldots, \nu)$$

and

$$[\mathbf{b}, \nu\mathbf{a}, \mathbf{a}^{-1}].$$

From this it follows that, if $d(\mathbf{a}) = d$ and $\mathbf{b}$ is either of depth d or of $(d+1)$-weight 1, then the same is true of all factors of $\chi(\mathbf{b}^\beta, \mathbf{a}^\alpha)$.

LEMMA 3. (i) *With the notation of Definition* 3,

$$\chi(\mathbf{b}^\beta, \mathbf{a}^\alpha) \equiv \mathbf{a}^{-\alpha}\mathbf{b}^\beta\mathbf{a}^\alpha.$$

(ii) *Further, suppose that, with the same notation,*

(a) $\mathbf{a}$ *and* $\mathbf{b}$ *are both basic invertators,*

(b) $\mathbf{a} < \mathbf{b}$,

(c) *if* $\mathbf{b} = [\mathbf{b}_1, \mathbf{b}_2^{\beta_2}]$ *then* $\mathbf{b}_2 \leqslant \mathbf{a}$, *and*

(d) *if* $\mathbf{b} = [\mathbf{b}_1, \mathbf{a}^{\beta_2}]$ *and* $d(\mathbf{a}) < d(\mathbf{b}_1)$ *then* $\beta_2 = \alpha$.

Then $\chi(\mathbf{b}^\beta, \mathbf{a}^\alpha)$ *is a product of basic invertators.*

Proof. (i) is proved by induction over $\nu = \nu(\mathbf{b}, \mathbf{a})$. For $\chi(\mathbf{b}^\beta, \mathbf{a}^{-1})$ when $\nu = 0$ and for $\chi(\mathbf{b}^\beta, \mathbf{a})$ this assertion is obvious. For $\chi(\mathbf{b}, \mathbf{a}^{-1})$ when $\nu > 0$, $\nu([\mathbf{b}, \mathbf{a}], \mathbf{a}) < \nu$ so that, inductively,

$$\chi([\mathbf{b}, \mathbf{a}]^{-1}, \mathbf{a}^{-1}) \equiv (\mathbf{a}^{-1})^{-1}[\mathbf{b}, \mathbf{a}]^{-1}\mathbf{a}^{-1} \equiv [\mathbf{b}, \mathbf{a}^{-1}]$$

and then

$$\chi(\mathbf{b}, \mathbf{a}^{-1}) = \mathbf{b}\chi([\mathbf{b}, \mathbf{a}]^{-1}, \mathbf{a}^{-1}) \equiv \mathbf{b}[\mathbf{b}, \mathbf{a}^{-1}] \equiv (\mathbf{a}^{-1})^{-1}\mathbf{b}\mathbf{a}^{-1}.$$

The proof for $\chi(\mathbf{b}^{-1}, \mathbf{a}^{-1})$ when $\nu > 0$ is similar.

(ii) Suppose first that $\alpha = +1$. The invertators occurring as factors of $\chi(\mathbf{b}^\beta, \mathbf{a})$ are $\mathbf{b}$ and $[\mathbf{b}, \mathbf{a}]$. Now $\mathbf{b}$ is basic by assumption and, for $[\mathbf{b}, \mathbf{a}]$, all the conditions of Definition 2(A) are true by assumption also ((c) being vacuously fulfilled).

Now suppose that $\alpha = -1$. The invertators occurring as factors of $\chi(\mathbf{b}^\beta, \mathbf{a}^{-1})$ are $[\mathbf{b}, r\mathbf{a}]$ $(r = 0, 1, \ldots, \nu)$ and $[\mathbf{b}, \nu\mathbf{a}, \mathbf{a}^{-1}]$. Basicity of each $[\mathbf{b}, r\mathbf{a}]$ is confirmed by induction over r. For $r = 0$, $[\mathbf{b}, r\mathbf{a}] = \mathbf{b}$ is basic by assumption. Now suppose that $0 < r \leqslant \nu$ and $[\mathbf{b}, (r-1)\mathbf{a}]$ is basic. Condition (a) of Definition 2(A) is true by hypothesis. Also $\mathbf{a} < \mathbf{b}$ so $\mathbf{a} < [\mathbf{b}, (r-1)\mathbf{a}]$ and Condition (b) is true. Condition (c) is vacuously fulfilled. If $r = 1$, Conditions (d) and (e) are true by assumption, and if $r > 1$ they are automatic. Basicity of $[\mathbf{b}, \nu\mathbf{a}, \mathbf{a}^{-1}]$ is confirmed similarly:

for conditions (a), (b), and (d) the argument is the same. For condition (c),

$$d([\mathbf{b}, \nu\mathbf{a}, \mathbf{a}^{-1}]) = d([\mathbf{b}, (\nu+1)\mathbf{a}]) > d(\mathbf{a})$$

by choice of ν, and for condition (e), either $\nu = 0$, in which case the condition is true by assumption, or $\nu > 0$, in which case $d(\mathbf{a}) \nless d([\mathbf{b}, \nu\mathbf{a}])$ by choice of ν.

LEMMA 4. *Let* $\mathbf{x}$ *be a working product,* $\mathbf{a}$ *be the earliest invertator which occurs as a factor of* $\mathbf{x}$ *and suppose that* $\varepsilon_{\mathbf{a}}(\mathbf{x}) = 0$. *Write* $d = d(\mathbf{a})$. *Then there exists an expression* $\mathbf{y}$ *such that*

(i) $\mathbf{y} \equiv \mathbf{x}$,

(ii) $\mathbf{y}$ *is a working product,*

(iii) *all the factors of* $\mathbf{y}$ *are of d-weight not less than that of* $\mathbf{a}$, *and, if* $\mathbf{a}'$ *is an invertator other than but of the same d-weight as* $\mathbf{a}$, *then* $\varepsilon_{\mathbf{a}'}(\mathbf{y}) = \varepsilon_{\mathbf{a}'}(\mathbf{x})$,

(iv) $\mathbf{a}$ *occurs less often as a factor of* $\mathbf{y}$ *than it does as a factor of* $\mathbf{x}$,

(v) $\varepsilon_{\mathbf{a}}(\mathbf{y}) = 0$, *and*

(vi) *if every factor of* $\mathbf{x}$ *is either of depth* d *or of* $(d+1)$*-weight* 1 *then the same is true of* $\mathbf{y}$.

Proof. Suppose that the first (leftmost) occurrence of $\mathbf{a}$ as a factor of $\mathbf{x}$ has exponent $-\alpha = \pm 1$. Then, since $\varepsilon_{\mathbf{a}}(\mathbf{x}) = 0$, $\mathbf{a}$ must occur again to the right of this with exponent $+\alpha$. Thus $\mathbf{x}$ may be written in the form

$$\mathbf{x} = \mathbf{x}_1\mathbf{a}^{-\alpha}\mathbf{x}_2\mathbf{a}^{\alpha}\mathbf{x}_3,$$

where $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are (possibly empty) products, $\mathbf{a}$ does not occur as a factor of $\mathbf{x}_2$ at all and, if $\mathbf{a}$ occurs as a factor of $\mathbf{x}_1$, then it does so with exponent $-\alpha$. Write

$$\mathbf{x}_2 = \mathbf{b}_1^{\beta_1}\mathbf{b}_2^{\beta_2}\ldots\mathbf{b}_k^{\beta_k}$$

so that

$$\mathbf{x} = \mathbf{x}_1\mathbf{a}^{-\alpha}\mathbf{b}_1^{\beta_1}\mathbf{b}_2^{\beta_2}\ldots\mathbf{b}_k^{\beta_k}\mathbf{a}^{\alpha}\mathbf{x}_3$$

and none of the $\mathbf{b}_i$ are $\mathbf{a}$. Define

$$\mathbf{y} = \mathbf{x}_1\chi(\mathbf{b}_1^{\beta_1}, \mathbf{a}^{\alpha})\chi(\mathbf{b}_2^{\beta_2}, \mathbf{a}^{\alpha})\ldots\chi(\mathbf{b}_k^{\beta_k}, \mathbf{a}^{\alpha})\mathbf{x}_3;$$

the properties imputed to $\mathbf{y}$ above will now be proved. The factors of the products $\chi(\mathbf{b}_i^{\beta_i}, \mathbf{a}^{\alpha})$ other than the $\mathbf{b}_i$ themselves will be referred to as the *new* factors of $\mathbf{y}$. The others, namely the factors of $\mathbf{x}_1$ and $\mathbf{x}_3$ and the $\mathbf{b}_i$ in each $\chi(\mathbf{b}_i^{\beta_i}, \mathbf{a}^{\alpha})$, will be referred to as the *old* factors of y. Observe that the old factors of $\mathbf{y}$ are also factors of x and occur in the same order in $\mathbf{x}$ as they do in $\mathbf{y}$.

(i) $$\mathbf{y} \equiv \mathbf{x}_1\mathbf{a}^{-\alpha}\mathbf{b}_1^{\beta_1}\mathbf{a}^{\alpha}\mathbf{a}^{-\alpha}\mathbf{b}_2^{\beta_2}\mathbf{a}^{\alpha}\ldots\mathbf{a}^{-\alpha}\mathbf{b}_k^{\beta_k}\mathbf{a}^{\alpha}\mathbf{x}_3 \equiv \mathbf{x}$$

by Lemma 3(i).

(ii) Before proving that $\mathbf{y}$ is a working product it must be checked that $\mathbf{y}$ is a product of basic invertators. Any old factor of $\mathbf{y}$ is also a factor of $\mathbf{x}$ and hence is a basic invertator since $\mathbf{x}$ is a working product. It is now sufficient to show that each $\chi(\mathbf{b}_i^{\beta_i}, \mathbf{a}^{\alpha})$ is a product of basic invertators and this is done by establishing the conditions of Lemma 3(ii).

Condition (a): $\mathbf{a}$ and $\mathbf{b}_i$ are both basic invertators, being factors of $\mathbf{x}$.

Condition (b): $\mathbf{b}_i \neq \mathbf{a}$ by choice of $\mathbf{x}_2$ and then $\mathbf{a} < \mathbf{b}_i$ by choice of $\mathbf{a}$.

Condition (c): $\mathbf{x}$ is a working product and $\mathbf{a}^{\alpha}$ occurs to the right of $\mathbf{b}_i^{\beta_i}$ in $\mathbf{x}$, so

$$\mathbf{b}_i = [\mathbf{b}_{i,1}, \mathbf{b}_{i,2}^{\beta'}] \Rightarrow \mathbf{b}_{i,2} \leqslant \mathbf{a}$$

by Definition 3(ii) (a).

Condition (d): if $\mathbf{a}$ occurs as a factor of $\mathbf{x}_1$ then it does so with exponent $-\alpha$, so $\varepsilon_{\mathbf{a}}(\mathbf{x}_1\mathbf{a}^{-\alpha}\mathbf{b}_1^{\beta_1}\mathbf{b}_2^{\beta_2}\ldots\mathbf{b}_i^{\beta_i})$ is non-zero and has sign $-\alpha$. But $\varepsilon_{\mathbf{a}}(\mathbf{x}) = 0$ so $\varepsilon_{\mathbf{a}}(\mathbf{b}_{i+1}^{\beta_{i+1}}\mathbf{b}_{i+2}^{\beta_{i+2}}\ldots\mathbf{b}_k^{\beta_k}\mathbf{a}^{\alpha}\mathbf{x}_3)$ is non-zero and has sign $+\alpha$. Thus, by Definition 3(ii) (b), if $\mathbf{b}_i = [\mathbf{b}_{i,1}, \mathbf{a}^{\beta'}]$ and $d(\mathbf{a}) < d(\mathbf{b}_{i,1})$ then $\beta' = \alpha$.

It is now shown that $\mathbf{y}$ is indeed a working product by checking Parts (a) and (b) of Definition 3(ii). Suppose then that $\mathbf{y} = \mathbf{y}_1\mathbf{c}^{\gamma}\mathbf{y}_2\mathbf{d}^{\delta}\mathbf{y}_3$ where $\mathbf{c} = [\mathbf{c}_1, \mathbf{c}_2^{\gamma_2}]$.

Case 1: $\mathbf{c}$ *and* $\mathbf{d}$ *are both new factors of* $\mathbf{y}$. Then, by Remark 2 above, $\mathbf{c}_2 = \mathbf{a} < \mathbf{d}$ and both parts of Definition 3(ii) are true, the second vacuously.

Case 2: $\mathbf{c}$ *is an old and* $\mathbf{d}$ *a new factor of* $\mathbf{y}$. Since $\mathbf{d}$ is a new factor of $\mathbf{y}$, it is a factor of one of the products $\chi(\mathbf{b}_i^{\beta_i}, \mathbf{a}^{\alpha})$; but $\mathbf{c}$ lies to the left of $\mathbf{d}$ and thus to the left of a factor $\mathbf{a}^{\alpha}$ in $\mathbf{x}$. But $\mathbf{x}$ is a working product, so $\mathbf{c}_2 \leqslant \mathbf{a} < \mathbf{d}$ and again both parts of Definition 3(ii) are true.

Case 3: $\mathbf{c}$ *is a new and* $\mathbf{d}$ *and old factor of* $\mathbf{y}$. Then, as in Case 1, $\mathbf{c}_2 = \mathbf{a}$ and, by choice of $\mathbf{a}$, $\mathbf{c}_2 = \mathbf{a} \leqslant \mathbf{d}$. If $\mathbf{c}_2 = \mathbf{a} < \mathbf{d}$ then both parts of Definition 3(ii) are true as before; otherwise $\mathbf{c}_2 = \mathbf{a} = \mathbf{d}$ and Part (a) is satisfied. Write $\mathbf{y}'$ for that part of the product

$$\chi(\mathbf{b}_1^{\beta_1}, \mathbf{a}^{\alpha})\chi(\mathbf{b}_2^{\beta_2}, \mathbf{a}^{\alpha})\ldots\chi(\mathbf{b}_k^{\beta_k}, \mathbf{a}^{\alpha})$$

which lies to the right of $\mathbf{c}$. Now, by Remark 2, $\mathbf{a}$ does not occur as a factor of $\mathbf{y}'$, so $\varepsilon_{\mathbf{a}}(\mathbf{y}'\mathbf{x}_3) = \varepsilon_{\mathbf{a}}(\mathbf{x}_3)$. It has already been observed that $\varepsilon_{\mathbf{a}}(\mathbf{a}^{\alpha}\mathbf{x}_3)$ is non-zero and has sign $+\alpha$, so $\varepsilon_{\mathbf{a}}(\mathbf{x}_3)$ either is zero or has sign $+\alpha$. Thus, to establish Part (b) of Definition 3(ii) it is sufficient to show that

$$d(\mathbf{a}) < d(\mathbf{c}_1) \Rightarrow \alpha = \gamma_2.$$

But $\mathbf{c}$ is a new factor of $\mathbf{y}$, say a factor of $\chi(\mathbf{b}_i^{\beta_i}, \mathbf{a}^{\alpha})$ other than $\mathbf{b}_i$. Referring to Remark 2 and inspecting the various invertators of the form $\mathbf{c} = [\mathbf{c}_1, \mathbf{a}^{\gamma_2}]$ which can occur as factors of $\chi(\mathbf{b}_i^{\beta_i}, \mathbf{a}^{\alpha})$ it is seen that the only ones for which $\gamma_2 \neq \alpha$ are those of the form $\mathbf{c} = [\mathbf{b}_i, r\mathbf{a}]$ when $\alpha = -1$ and $r \leqslant \nu = \nu(\mathbf{b}_i, \mathbf{a})$; but then $\mathbf{c}_1 = [\mathbf{b}_i, (r-1)\mathbf{a}]$ and, by definition of ν, $d(\mathbf{c}_1) \ngtr d(\mathbf{a})$.

Case 4: **c** *and* **d** *are both old factors of* **y**. Since **c** and **d** both occur in **x** in the same order as they do in **y** then $\mathbf{c}_2 \leqslant \mathbf{d}$ and Part (a) of Definition 3(ii) is true. In order to verify Part (b) of this definition, three subcases are considered.

Case 4.1: $\mathbf{c}_2 < \mathbf{a}$. Then, by choice of **a**, $\mathbf{a} \leqslant \mathbf{d}$. Thus $\mathbf{c}_2 < \mathbf{d}$ and Part (b) is vacuously true as before.

Case 4.2: $\mathbf{c}_2 = \mathbf{a}$. Let $\mathbf{x}'$ be that part of **x** and $\mathbf{y}'$ that part of **y** which lie to the right of **c**. There are three possibilities. Firstly, if **c** is a factor of $\mathbf{x}_1$ then $\mathbf{y}'$ is formed from $\mathbf{x}'$ by deleting one occurrence of **a** with exponent $+\alpha$ and one occurrence with exponent $-\alpha$ and inserting some extra invertators (the new factors of **y**) which are all $> \mathbf{a}$. Hence $\varepsilon_{\mathbf{a}}(\mathbf{y}') = \varepsilon_{\mathbf{a}}(\mathbf{x}')$ and Part (b) is true since **x** is a working product. Secondly, if **c** is a factor of $\mathbf{x}_2$ then $\mathbf{y}'$ is formed from $\mathbf{x}'$ by deleting one occurrence of **a** with exponent $+\alpha$ and inserting some extra invertators (some of the new factors of **y**) which are all $> \mathbf{a}$. But, as already observed, $\varepsilon_{\mathbf{a}}(\mathbf{x}')$ is non-zero and has sign $+\alpha$. Thus $\varepsilon_{\mathbf{a}}(\mathbf{y}')$ is either zero or has sign $+\alpha$ and again Part (b) is true. Thirdly, if **c** is a factor of $\mathbf{x}_3$ then $\mathbf{x}' = \mathbf{y}'$ so that $\varepsilon_{\mathbf{a}}(\mathbf{y}') = \varepsilon_{\mathbf{a}}(\mathbf{x}')$ and and Part (b) is immediately true.

Case 4.3: $\mathbf{c}_2 > \mathbf{a}$. Since $\mathbf{c} = [\mathbf{c}_1, \mathbf{c}_2^{\gamma_2}]$ is a factor of the working product **x**, it must occur to the right of the rightmost occurrence of **a** as a factor of **x**. Thus **c** is a factor of $\mathbf{x}_3$ and Part (b) is again true.

Parts (iii), (iv), (v), and (vi) of this lemma are immediate consequences of the definition of **y**.

Observe that, by repeated application of this lemma, Part (iv) can be strengthened to

(IV) **a** *does not occur at all as a factor of* **y**.

LEMMA 5. *Let* **x** *be a working product. Suppose that the factors* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r$ *of* **x** *are all of depth d and d-weight c, that* $\mathbf{a}_1 < \mathbf{a}_2 < \ldots < \mathbf{a}_r$, *and that all other factors of* **x** *are of d-weight* $> c$. *Then*

$$\mathbf{x} \equiv \mathbf{a}_1^{\varepsilon_1}\mathbf{a}_2^{\varepsilon_2}\ldots\mathbf{a}_r^{\varepsilon_r}\mathbf{y},$$

where $\varepsilon_i = \varepsilon_{\mathbf{a}_i}(\mathbf{x})$ $(i = 1, 2, \ldots, r)$ *and* **y** *is a working product all of whose factors are of d-weight* $> c$.

Further, if all the factors of **x** *are either of depth d or* $(d+1)$*-weight* 1 *then the same is true of* **y**.

Proof. By induction over s it is shown that

$$\mathbf{x} \equiv \mathbf{a}_1^{\varepsilon_1}\mathbf{a}_2^{\varepsilon_2}\ldots\mathbf{a}_s^{\varepsilon_s}\mathbf{z}_s \quad (s = 0, 1, \ldots, r),$$

where $\mathbf{z}_s$ is a working product whose factors of depth d and d-weight c are $\mathbf{a}_{s+1}, \mathbf{a}_{s+2}, \ldots, \mathbf{a}_r$, all of whose other factors are of d-weight $> c$ and for

which $\varepsilon_{\mathbf{a}_i}(\mathbf{z}_s) = \varepsilon_i$ $(i = s+1, s+2, \ldots, r)$; further, if every factor of **x** is of depth d or $(d+1)$-weight 1 then the same is true of $\mathbf{z}_s$. Putting $\mathbf{z}_0 = \mathbf{x}$ takes care of the case $s = 0$. Now suppose that $0 \leqslant s < r$ and that $\mathbf{z}_s$ exists as claimed. Then $\mathbf{a}_{s+1}$ is the earliest invertator which occurs as a factor of $\mathbf{z}_s$ and hence also the earliest invertator which occurs as a factor of $\mathbf{a}_{s+1}^{-\varepsilon_{s+1}}\mathbf{z}_s$. But

$$\varepsilon_{\mathbf{a}_{s+1}}(\mathbf{a}_{s+1}^{-\varepsilon_{s+1}}\mathbf{z}_s) = 0,$$

so by the strengthened form of Lemma 4 there exists a working product $\mathbf{z}_{s+1}$ such that $\mathbf{z}_{s+1} \equiv \mathbf{a}_{s+1}^{-\varepsilon_{s+1}}\mathbf{z}_s$ (from which it follows that

$$\mathbf{x} \equiv \mathbf{a}_1^{\varepsilon_1}\mathbf{a}_2^{\varepsilon_2}\ldots\mathbf{a}_{s+1}^{\varepsilon_{s+1}}\mathbf{z}_{s+1})$$

and which has all the other required properties. This completes the induction; putting $\mathbf{y} = \mathbf{z}_r$ completes the proof of the lemma.

THEOREM 1. *Let* **x** *be any expression and d be a non-negative integer. Then there exists an expression* **yw** *satisfying*

(i) $\mathbf{x} \equiv \mathbf{yw}$,

(ii) **y** *is a basic invertator product consisting, if it is not* **1**, *of basic invertators of depth* $< d$, *and*

(iii) **w** *is a (possibly empty) working product whose factors are all of d-weight* 1.

Proof. This is by induction over d. Clearly there exists a product **w** of generators and their inverses such that $\mathbf{w} \equiv \mathbf{x}$ and then **w** is a working product all of whose factors are of 0-weight 1. Putting $\mathbf{y} = \mathbf{1}$ disposes of the case $d = 0$.

Now suppose that $d \geqslant 0$ and **yw** exists as claimed.

It is now proved, by a subsidiary induction over c, that when $1 \leqslant c \leqslant k_{d+1}$ there exists an expression $\mathbf{y}_c\mathbf{w}_c$ such that $\mathbf{x} \equiv \mathbf{y}_c\mathbf{w}_c$, $\mathbf{y}_c$ is a basic invertator product consisting, if it is not **1**, of basic invertators of d-weight $< c$ (and hence of depth $< d+1$) and $\mathbf{w}_c$ is a (possibly empty) working product consisting of basic invertators which are of d-weight $\geqslant c$ and $(d+1)$-weight 0 or 1. Putting $\mathbf{y}_1 = \mathbf{y}$ and $\mathbf{w}_1 = \mathbf{w}$ disposes of the case $c = 1$. Now suppose that $1 < c \leqslant k_{d+1}$ and $\mathbf{y}_{c-1}\mathbf{w}_{c-1}$ exists with the properties described above. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r$ be the basic invertators of d-weight $c-1$ which occur as factors of $\mathbf{w}_{c-1}$, written in order so that $\mathbf{a}_1 < \mathbf{a}_2 < \ldots < \mathbf{a}_r$. Then by Lemma 5,

$$\mathbf{w}_{c-1} \equiv \mathbf{a}_1^{\varepsilon_1}\mathbf{a}_2^{\varepsilon_2}\ldots\mathbf{a}_r^{\varepsilon_r}\mathbf{w}_c,$$

where $\varepsilon_i = \varepsilon_{\mathbf{a}_i}(\mathbf{w}_{c-1})$ $(i = 1, 2, \ldots, r)$ and $\mathbf{w}_c$ is a working product all of whose factors are of d-weight $\geqslant c$. Further, since all the factors of $\mathbf{w}_{c+1}$ are of $(d+1)$-weight 0 or 1, the same is true of $\mathbf{w}_c$. Writing $\mathbf{y}_c$ for the result of deleting any $\mathbf{a}_i^0 = \mathbf{1}$ from the product $\mathbf{y}_{c-1}\mathbf{a}_1^{\epsilon_1}\mathbf{a}_2^{\epsilon_2}\ldots\mathbf{a}_r^{\epsilon_r}$, the subsidiary induction is completed.

Putting $c = k_{d+1}$, $\mathbf{y}_c$ is a basic invertator product consisting, if it is not $\mathbf{1}$, of basic invertators of d-weight $< c = k_{d+1}$ and hence of depth $< d+1$ and $\mathbf{w}_c$ is a (possibly empty) working product all of whose factors are of d-weight $\geqslant c$ and $(d+1)$-weight 0 or 1. But since their d-weight is $\geqslant c = k_{d+1}$, their $(d+1)$-weight can only be 1. This completes the main induction.

4. A free generating set for $\gamma_c(F)$

DEFINITION 4. Let $\mathbf{D}$ be the set of all basic commutators of depth 0. Suppose that f and g are functions $D \to \omega$, where ω is the set of non-negative integers, and $\mathbf{A} \subseteq \mathbf{D}$. Then '$f \leqslant g$ on $\mathbf{A}$' will mean that

$$\mathbf{a} \in \mathbf{A} \Rightarrow f(\mathbf{a}) \leqslant g(\mathbf{a}).$$

For each $\mathbf{A} \subseteq \mathbf{D}$ this forms a pre-order of the functions $\mathbf{D} \to \omega$. '$f = g$ on $\mathbf{A}$' is defined similarly.

For any $\mathbf{x} = [\mathbf{b}, n_1\mathbf{a}_1^{\alpha_1}, n_2\mathbf{a}_2^{\alpha_2}, \ldots, n_t\mathbf{a}_t^{\alpha_t}] \in \mathbf{J}_1$, the element $\lambda(\mathbf{x})$ of $\mathbf{D}$, two subsets $\mathbf{A}^+(\mathbf{x})$ and $\mathbf{A}^-(\mathbf{x})$ of $\mathbf{D}$, and a function $\theta_{\mathbf{x}}\colon \mathbf{D} \to \omega$ are defined by

$$\lambda(\mathbf{x}) = \mathbf{b},$$

$$\mathbf{A}^+(\mathbf{x}) = \{\mathbf{a}_i\colon 1 \leqslant i \leqslant t,\ \alpha_i = +1\},$$

$$\mathbf{A}^-(\mathbf{x}) = \{\mathbf{a}_i\colon 1 \leqslant i \leqslant t,\ \alpha_i = -1\},$$

$$\theta_{\mathbf{x}}(\mathbf{u}) = \begin{cases} n_i & \text{if } \mathbf{u} = \mathbf{a}_i \text{ for any } 1 \leqslant i \leqslant t, \\ 0 & \text{otherwise.} \end{cases}$$

For any $\mathbf{x} \in \mathbf{J}_1$, $\mathbf{F}(\mathbf{x})$ is the set of all $\mathbf{y} \in \mathbf{J}_1$ such that $\lambda(\mathbf{y}) = \lambda(\mathbf{x})$, $\mathbf{A}^+(\mathbf{y}) = \mathbf{A}^+(\mathbf{x}) \cup \mathbf{A}^-(\mathbf{x})$, $\mathbf{A}^-(\mathbf{y}) = \varnothing$, $\theta_{\mathbf{y}} = \theta_{\mathbf{x}}$ on $\mathbf{A}^+(\mathbf{x})$, and $\theta_{\mathbf{y}} \geqslant \theta_{\mathbf{x}}$ on $\mathbf{A}^-(\mathbf{x})$. For any integer $c \geqslant 1$, $\mathbf{F}_c(\mathbf{x})$ is the set of all members of $\mathbf{F}(\mathbf{x})$ of weight $< c$.

Finally, for any basic commutator $\mathbf{y} \in \mathbf{J}_1$, the integer $E(\mathbf{x}, \mathbf{y})$ is defined by

$$E(\mathbf{x}, \mathbf{y}) = \begin{cases} \prod_{\mathbf{u} \in \mathbf{A}^-(\mathbf{x})} (-1)^{\theta_{\mathbf{y}}(\mathbf{u})} \binom{\theta_{\mathbf{y}}(\mathbf{u}) - 1}{\theta_{\mathbf{x}}(\mathbf{u}) - 1} & \text{if } \mathbf{y} \in \mathbf{F}(\mathbf{x}), \\ 0 & \text{otherwise;} \end{cases}$$

where $\binom{\theta_{\mathbf{y}}(\mathbf{u}) - 1}{\theta_{\mathbf{x}}(\mathbf{u}) - 1}$ denotes the usual binomial coefficient.

For any $\mathbf{x} \in \mathbf{J}_1$ the following observations are elementary.

$\mathbf{A}^+(\mathbf{x}) \cup \mathbf{A}^-(\mathbf{x})$ is the support of $\theta_{\mathbf{x}}$. The invertator $\mathbf{x}$ may be uniquely specified by giving $\lambda(\mathbf{x})$, $\mathbf{A}^+(\mathbf{x})$, $\mathbf{A}^-(\mathbf{x})$, and $\theta_{\mathbf{x}}$. $\mathbf{A}^+(\mathbf{x})$ and $\mathbf{A}^-(\mathbf{x})$ are

disjoint finite sets. $\mathbf{x}$ is a basic commutator if and only if $\mathbf{A}^-(\mathbf{x})$ is empty; consequently every member of $\mathbf{F}(\mathbf{x})$ is a basic commutator. $\mathbf{F}(\mathbf{x})$ may be defined alternatively as follows: writing

$$\mathbf{x} = [\mathbf{b}, n_1\mathbf{a}_1^{\alpha_1}, n_2\mathbf{a}_2^{\alpha_2}, \ldots, n_t\mathbf{a}_t^{\alpha_t}]$$

then $\mathbf{y} \in \mathbf{F}(\mathbf{x})$ if and only if it is of the form

$$\mathbf{y} = [\mathbf{b}, \nu_1\mathbf{a}_1^{\alpha_1}, \nu_2\mathbf{a}_2^{\alpha_2}, \ldots, \nu_t\mathbf{a}_t^{\alpha_t}],$$

where, for each i $(1 \leqslant i \leqslant t)$,

$$\alpha_i = +1 \Rightarrow \nu_i = n_i \quad \text{and} \quad \alpha_i = -1 \Rightarrow \nu_i \geqslant n_i.$$

Then

$$E(\mathbf{x}, \mathbf{y}) = \prod_{i=1}^{t} \alpha_i^{\nu_i} \binom{\nu_i - 1}{n_i - 1}.$$

Finally, for any function $f\colon \mathbf{D} \to \omega$ such that $f = \theta_{\mathbf{x}}$ on $\mathbf{A}^+(\mathbf{x})$, $f \geqslant \theta_{\mathbf{x}}$ on $\mathbf{A}^-(\mathbf{x})$ and $f = 0$ elsewhere, there exists a unique $\mathbf{y} \in \mathbf{F}(\mathbf{x})$ such that $\theta_{\mathbf{y}} = f$.

LEMMA 6. *Let $\mathbf{x} \in \mathbf{J}_1$ and c be a positive integer. Then*

$$x \equiv \prod_{\mathbf{y} \in \mathbf{F}_c(\mathbf{x})} \mathbf{y}^{E(\mathbf{x},\mathbf{y})} \quad (mod\, \mathfrak{N}_{c-1} \cap \mathfrak{A}.\mathfrak{N}_{k_1-1}),$$

the order of factors in the product being immaterial. If $\mathbf{F}_c(\mathbf{x})$ is empty, the product is to be interpreted as $\mathbf{1}$.

Proof. Write $\mathfrak{B} = \mathfrak{N}_{c-1} \cap \mathfrak{A}.\mathfrak{N}_{k_1-1}$. Since each $\mathbf{y} \in \mathbf{J}_1$, it is of depth 1 and thus of weight $\geqslant k_1$. Then all the $\mathbf{y}$ commute modulo $\mathfrak{A}.\mathfrak{N}_{k_1-1}$: hence it is true that the order of factors in the product above is immaterial.

Write $\mathbf{x} = [\mathbf{b}, n_1\mathbf{a}_1^{\alpha_1}, n_2\mathbf{a}_2^{\alpha_2}, \ldots, n_t\mathbf{a}_t^{\alpha_t}]$. It is required to prove that

$$\mathbf{x} \equiv \prod [\mathbf{b}, \nu_1\mathbf{a}_1, \nu_2\mathbf{a}_2, \ldots, \nu_t\mathbf{a}_t]^{\kappa} \quad (\text{mod } \mathfrak{B}),$$

where

$$\kappa = \alpha_1^{\nu_1}\alpha_2^{\nu_2}\ldots\alpha_t^{\nu_t} \binom{\nu_1 - 1}{n_1 - 1}\binom{\nu_2 - 1}{n_2 - 1}\cdots\binom{\nu_t - 1}{n_t - 1}$$

and the product is taken over those values of $\nu_1, \nu_2, \ldots, \nu_t$ satisfying

$$\alpha_i = +1 \Rightarrow \nu_i = n_i,$$

$$\alpha_i = -1 \Rightarrow \nu_i \geqslant n_i,$$

and

$$\text{wt}([\mathbf{b}, \nu_1\mathbf{a}_1, \nu_2\mathbf{a}_2, \ldots, \nu_t\mathbf{a}_t]) < c.$$

Suppose that $c \leqslant \text{wt}(\mathbf{x})$. Then $\mathbf{F}_c(\mathbf{x}) = \varnothing$ and $\mathbf{x} \equiv \mathbf{1}$ (mod $\mathfrak{N}_{c-1}$) so the formula is trivially true. Henceforth, assume that $c > \text{wt}(\mathbf{x})$, in which case $\mathbf{F}_c(\mathbf{x}) \neq \varnothing$ since it contains, for instance, the commutator

$$[\mathbf{b}, n_1\mathbf{a}_1, n_2\mathbf{a}_2, \ldots, n_t\mathbf{a}_t].$$

Consider first the case $t = 1$, so that $\mathbf{x} = [\mathbf{b}, n_1\mathbf{a}_1^{\alpha_1}]$. If $\alpha_1 = +1$ then $\mathbf{x} = [\mathbf{b}, n_1\mathbf{a}_1]$; but then $\mathbf{F}_c(\mathbf{x}) = \{\mathbf{x}\}$ and the formula is again trivially true.

On the other hand, if $\alpha_1 = -1$ then $\mathbf{x} = [\mathbf{b}, n_1\mathbf{a}_1^{-1}] = [\mathbf{b}, n\mathbf{a}^{-1}]$ say. But then

$$\mathbf{F}_c(\mathbf{x}) = \{[\mathbf{b}, n\mathbf{a}], [\mathbf{b}, (n+1)\mathbf{a}], \ldots, [\mathbf{b}, m\mathbf{a}]\},$$

where m is the largest integer for which $\mathrm{wt}([\mathbf{b}, m\mathbf{a}]) < c$, so the formula to be proved is

$$[\mathbf{b}, n\mathbf{a}^{-1}] = \prod_{\nu=n}^{m} [\mathbf{b}, \nu\mathbf{a}]^{\kappa} \pmod{\mathfrak{V}}, \tag{1}$$

where

$$\kappa = (-1)^{\nu}\binom{\nu-1}{n-1};$$

this is done by induction over n. The following group identity is easily checked:

$$[u, v^{-1}] \equiv [u, v, v^{-1}]^{-1}[u, v]^{-1}.$$

Applying this to $[\mathbf{b}, \mathbf{a}^{-1}]$ we have

$$\begin{aligned}[\mathbf{b}, \mathbf{a}^{-1}] &\equiv [\mathbf{b}, \mathbf{a}, \mathbf{a}^{-1}]^{-1}[\mathbf{b}, \mathbf{a}]^{-1} \\ &\equiv [\mathbf{b}, \mathbf{a}]^{-1}[\mathbf{b}, \mathbf{a}, \mathbf{a}^{-1}]^{-1} \pmod{\mathfrak{V}}\end{aligned}$$

since $\mathrm{wt}([\mathbf{b}, \mathbf{a}]) \geqslant k_1$. Applying the same trick to $[\mathbf{b}, \mathbf{a}, \mathbf{a}^{-1}]^{-1}$,

$$[\mathbf{b}, \mathbf{a}^{-1}] \equiv [\mathbf{b}, \mathbf{a}]^{-1}[\mathbf{b}, \mathbf{a}, \mathbf{a}][\mathbf{b}, \mathbf{a}, \mathbf{a}, \mathbf{a}^{-1}] \pmod{\mathfrak{V}}.$$

Continuing in this manner,

$$\begin{aligned}[\mathbf{b}, \mathbf{a}^{-1}] &\equiv \left(\prod_{\nu=1}^{m} [\mathbf{b}, \nu\mathbf{a}]^{(-1)^{\nu}}\right)[\mathbf{b}, m\mathbf{a}, \mathbf{a}^{-1}]^{(-1)^{m}} \pmod{\mathfrak{V}} \\ &\equiv \left(\prod_{\nu=1}^{m} [\mathbf{b}, \nu\mathbf{a}]^{(-1)^{\nu}}\right) \pmod{\mathfrak{V}}\end{aligned} \tag{2}$$

since, by the choice of m, $\mathrm{wt}([\mathbf{b}, m\mathbf{a}, \mathbf{a}^{-1}]) = \mathrm{wt}([\mathbf{b}, (m+1)\mathbf{a}]) \geqslant c$. This is formula (1) when $n = 1$. Now suppose that (1) is true for some $n \geqslant 1$; the corresponding formula for $n+1$, namely

$$[\mathbf{b}, (n+1)\mathbf{a}^{-1}] = \prod_{\nu=n+1}^{m} [\mathbf{b}, \nu\mathbf{a}]^{\kappa} \pmod{\mathfrak{V}}, \tag{3}$$

where

$$\kappa = (-1)^{\nu}\binom{\nu-1}{n},$$

is deduced. Now

$$\begin{aligned}[\mathbf{b}, (n+1)\mathbf{a}^{-1}] &= [\mathbf{b}, n\mathbf{a}^{-1}, \mathbf{a}^{-1}] \\ &\equiv \prod_{\nu=n}^{m} [\mathbf{b}, \nu\mathbf{a}, \mathbf{a}^{-1}]^{\kappa} \pmod{\mathfrak{V}},\end{aligned}$$

where

$$\kappa = (-1)^{\nu}\binom{\nu-1}{n-1},$$

by the inductive assumption, using the fact that, modulo $\mathfrak{A}.\mathfrak{N}_{k_1-1}$, commutation distributes over products of expressions of weight $\geqslant k_1$. But, by the choice of m, $\mathrm{wt}([\mathbf{b}, m\mathbf{a}, \mathbf{a}^{-1}]) \geqslant c$ and so

$$[\mathbf{b}, (n+1)\mathbf{a}^{-1}] \equiv \prod_{\nu=n}^{m-1} [\mathbf{b}, \nu\mathbf{a}, \mathbf{a}^{-1}]^{\kappa} \pmod{\mathfrak{V}},$$

where

$$\kappa = (-1)^{\nu}\binom{\nu-1}{n-1}. \tag{4}$$

Also, again by choice of m,

$$\mathrm{wt}([\mathbf{b}, \nu\mathbf{a}, (m-\nu)\mathbf{a}]) < c \leqslant \mathrm{wt}([\mathbf{b}, \nu\mathbf{a}, (m-\nu+1)\mathbf{a}])$$

for any ν $(n \leqslant \nu < m)$ and then (2) gives

$$\begin{aligned}[\mathbf{b}, \nu\mathbf{a}, \mathbf{a}^{-1}] &\equiv \prod_{r'=1}^{m-\nu} [\mathbf{b}, \nu\mathbf{a}, r'\mathbf{a}]^{(-1)^{r'}} \pmod{\mathfrak{V}} \\ &= \prod_{r=\nu+1}^{m} [\mathbf{b}, r\mathbf{a}]^{(-1)^{r-\nu}}.\end{aligned}$$

Substituting this in (4) gives

$$[\mathbf{b}, (n+1)\mathbf{a}^{-1}] \equiv \prod_{\nu=n}^{m-1} \prod_{r=\nu+1}^{m} [\mathbf{b}, r\mathbf{a}]^{\kappa} \pmod{\mathfrak{V}},$$

where

$$\kappa = (-1)^{r}\binom{\nu-1}{n-1}.$$

Changing the order of multiplication according to the rule

$$\prod_{\nu=n}^{m-1} \prod_{r=\nu+1}^{m} = \prod_{r=n+1}^{m} \prod_{\nu=n}^{r-1}$$

and using

$$\sum_{\nu=n}^{r-1} \binom{\nu-1}{n-1} = \binom{r-1}{n},$$

this gives formula (3) as required.

This completes the proof in the case $t = 1$; but the proof for $t > 1$ is now obvious.

LEMMA 7. *Let $\rho: \mathbf{A} \to F$ be a free description of the absolutely free group F. Then ρ maps $\mathbf{J}_1$ one-to-one into $\gamma_{k_1}(F)$ modulo $\delta(\gamma_{k_1}(F))$ and $\mathbf{J}_1\rho$ forms a basis for the free abelian group $\gamma_{k_1}(F)$ modulo $\delta(\gamma_{k_1}(F))$.*

Proof. Let $\mathbf{x} = \mathbf{x}_1^{\xi_1}\mathbf{x}_2^{\xi_2}\ldots\mathbf{x}_r^{\xi_r}$ be a non-trivial product of members of $\mathbf{J}_1$ where $\mathbf{x}_1 < \mathbf{x}_2 < \ldots < \mathbf{x}_r$ and each ξ_i is a non-zero integer: it is sufficient to show that $\mathbf{x}\rho \notin \delta(\gamma_{k_1}(F))$.

Choose $p\ (1 \leqslant p \leqslant r)$ so that $\mathbf{A}^-(\mathbf{x}_p)$ is maximal (under ordinary set inclusion) and so that, amongst the possible such choices of p, $\theta_{\mathbf{x}_p}$ is maximal on $\mathbf{A}^-(\mathbf{x}_p)$. In other words, choose p so that, when $i \neq p$, $\mathbf{A}^-(\mathbf{x}_p)$ is not a proper subset of $\mathbf{A}^-(\mathbf{x}_i)$ and, if $\mathbf{A}^-(\mathbf{x}_p) = \mathbf{A}^-(\mathbf{x}_i)$, then

$$\theta_{\mathbf{x}_p} \leqslant \theta_{\mathbf{x}_i} \text{ on } \mathbf{A}^-(\mathbf{x}_p) \Rightarrow \theta_{\mathbf{x}_p} = \theta_{\mathbf{x}_i} \text{ on } \mathbf{A}^-(\mathbf{x}_p).$$

Define an integer N by

$$N = \max\{\theta_{\mathbf{x}_i}(\mathbf{u}) : \mathbf{u} \in \mathbf{A}^-(\mathbf{x}_p),\ 1 \leqslant i \leqslant r\}.$$

Since $\mathbf{A}^-(\mathbf{x}_p)$ is a finite set, this definition is permissible. Write

$$\mathbf{A}^-(\mathbf{x}_p) = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_s\},$$

where the $\mathbf{u}_i$ are distinct, and write

$$n_i = \theta_{\mathbf{x}_p}(\mathbf{u}_i) \quad (i = 1, 2, \ldots, s).$$

Notice that, by the choice of N, each $n_i \leqslant N$.

Consider a set $\{\nu_1, \nu_2, \ldots, \nu_s\}$ of integers, all $> N$, and notice that $\nu_i > n_i\ (i = 1, 2, \ldots, s)$. It will be convenient to think of the ν_i as 'variables' temporarily; a specific choice will be made shortly. Define a commutator $\mathbf{y} = \mathbf{y}(\nu_1, \nu_2, \ldots, \nu_s) \in \mathbf{F}(\mathbf{x}_p)$ by

$$\theta_{\mathbf{y}}(\mathbf{u}_i) = \nu_i \quad (i = 1, 2, \ldots, s).$$

Note that, since $\nu_i \geqslant n_i\ (1 \leqslant i \leqslant s)$, $\mathbf{y}$ is uniquely defined by this, as was remarked following Definition 4.

Consider $E(\mathbf{x}_p, \mathbf{y})$. This is the integer

$$E(\mathbf{x}_p, \mathbf{y}) = \prod_{\mathbf{u} \in \mathbf{A}^-(\mathbf{x}_p)} (-1)^{\theta_{\mathbf{y}}(\mathbf{u})} \binom{\theta_{\mathbf{y}}(\mathbf{u}) - 1}{\theta_{\mathbf{x}}(\mathbf{u}) - 1}$$

$$= \prod_{j=1}^{s} (-1)^{\nu_j} \binom{\nu_j - 1}{n_j - 1},$$

which is $(-1)^{\nu_1+\nu_2+\ldots+\nu_s}$ times a polynomial in the variables $\nu_1, \nu_2, \ldots, \nu_s$ with leading term $\nu_1^{n_1-1}\nu_2^{n_2-1}\ldots\nu_s^{n_s-1}$: 'leading' in the sense that if $c\nu_1^{m_1}\nu_2^{m_2}\ldots\nu_s^{m_s}$ is any other term for which $c \neq 0$, then $m_1 \leqslant n_1 - 1$, $m_2 \leqslant n_2 - 1$, $\ldots$, $m_s \leqslant n_s - 1$.

Now consider $E(\mathbf{x}_i, \mathbf{y})$ for any $i \neq p$. If $\mathbf{y} \notin \mathbf{F}(\mathbf{x}_i)$ then $E(\mathbf{x}_i, \mathbf{y}) = 0$ by definition of $\mathbf{F}(\mathbf{x}_i)$. Otherwise $\mathbf{y} \in \mathbf{F}(\mathbf{x}_i)$ and then

$$\lambda(\mathbf{x}_p) = \lambda(\mathbf{y}) \quad \text{since } \mathbf{y} \in \mathbf{F}(\mathbf{x}_p)$$

$$= \lambda(\mathbf{x}_i) \quad \text{since } \mathbf{y} \in \mathbf{F}(\mathbf{x}_i).$$

Also

$$\mathbf{A}^+(\mathbf{x}_i) \cup \mathbf{A}^-(\mathbf{x}_i) = \mathbf{A}^+(\mathbf{y}) = \mathbf{A}^+(\mathbf{x}_p) \cup \mathbf{A}^-(\mathbf{x}_p)$$

for the same reason. Now if $\mathbf{A}^+(\mathbf{x}_i) \cap \mathbf{A}^-(\mathbf{x}_p) \neq \emptyset$ then this intersection contains some member, $\mathbf{u}_j$ say (being a member of $\mathbf{A}^-(\mathbf{x}_p)$). Then

$$\theta_{\mathbf{y}}(\mathbf{u}_j) = \theta_{\mathbf{x}_i}(\mathbf{u}_j) \quad \text{since } \mathbf{y} \in \mathbf{F}(\mathbf{x}_i) \text{ and } \mathbf{u}_j \in \mathbf{A}^+(\mathbf{x}_i)$$

$$\leqslant N \quad \text{by the choice of } N;$$

but also

$$\theta_{\mathbf{y}}(\mathbf{u}_j) = \nu_j \quad \text{by definition of } \mathbf{y}$$

$$> N \quad \text{by the choice of } \nu_j.$$

This contradiction proves that $\mathbf{A}^+(\mathbf{x}_i) \cap \mathbf{A}^-(\mathbf{x}_p) = \emptyset$. But $\mathbf{A}^+(\mathbf{x}_i) \cup \mathbf{A}^-(\mathbf{x}_i)$ and $\mathbf{A}^+(\mathbf{x}_p) \cup \mathbf{A}^-(\mathbf{x}_p)$ are partitions of the same set $\mathbf{A}^+(\mathbf{y})$. Thus $\mathbf{A}^-(\mathbf{x}_p) \subseteq \mathbf{A}^-(\mathbf{x}_i)$. Then, by the choice of $\mathbf{x}_p$, $\mathbf{A}^-(\mathbf{x}_p) = \mathbf{A}^-(\mathbf{x}_i)$ and thus $\mathbf{A}^+(\mathbf{x}_p) = \mathbf{A}^+(\mathbf{x}_i)$ also. Now, as remarked following Definition 4, $\mathbf{x}_i$ is uniquely defined by $\lambda(\mathbf{x}_i)$, $\mathbf{A}^+(\mathbf{x}_i)$, $\mathbf{A}^-(\mathbf{x}_i)$, and $\theta_{\mathbf{x}_i}$; it has been observed so far that $\mathbf{x}_i \neq \mathbf{x}_p$, $\lambda(\mathbf{x}_i) = \lambda(\mathbf{x}_p)$, $\mathbf{A}^+(\mathbf{x}_i) = \mathbf{A}^+(\mathbf{x}_p)$, and $\mathbf{A}^-(\mathbf{x}_i) = \mathbf{A}^-(\mathbf{x}_p)$, so it follows that $\theta_{\mathbf{x}_i} \neq \theta_{\mathbf{x}_p}$. Thus, again by the choice of $\mathbf{x}_p$, there exists $j_0\ (1 \leqslant j_0 \leqslant s)$ such that

$$\theta_{\mathbf{x}_i}(\mathbf{u}_{j_0}) < \theta_{\mathbf{x}_p}(\mathbf{u}_{j_0}) = n_j.$$

Now

$$E(\mathbf{x}_i, \mathbf{y}) = \prod_{j=1}^{s} (-1)^{\nu} \binom{\nu_j - 1}{\theta_{\mathbf{x}_i}(\mathbf{u}_j) - 1}$$

and which is, again, $(-1)^{\nu_1+\nu_2+\ldots+\nu_s}$ times a polynomial in the variables $\nu_1, \nu_2, \ldots, \nu_s$; but now, if

$$c\nu_1^{m_1}\nu_2^{m_2}\ldots\nu_s^{m_s}$$

is any term in this polynomial for which $c \neq 0$, then

$$m_{j_0} \leqslant \theta_{\mathbf{x}_i}(\mathbf{u}_j) - 1 < n_{j_0} - 1.$$

Consequently, this polynomial contains no term of the form

$$c\nu_1^{n_1-1}\nu_2^{n_2-1}\ldots\nu_s^{n_s-1} \quad (c \neq 0).$$

Now consider

$$\sum_{i=1}^{r} \xi_i E(\mathbf{x}_i, \mathbf{y}).$$

This is $(-1)^{\nu_1+\nu_2+\ldots+\nu_s}$ times a polynomial in $\nu_1, \nu_2, \ldots, \nu_s$. The summand $\xi_p E(\mathbf{x}_p, \mathbf{y})$ contributes a term $\xi_p \nu_1^{n_1-1}\nu_2^{n_2-1}\ldots\nu_s^{n_s-1}$ to this polynomial, but none of the other summands $\xi_i E(\mathbf{x}_i, \mathbf{y})$ for $i \neq p$ contain a term of the same order, so this polynomial is non-trivial. Thus it is possible to choose integer values for $\nu_1, \nu_2, \ldots, \nu_s$, all $> N$, such that this polynomial and hence $\sum_{i=1}^{r} \xi_i E(\mathbf{x}_i, \mathbf{y})$ also are non-zero. Consider this choice of values for $\nu_1, \nu_2, \ldots, \nu_s$ to be made from now on, let $\mathbf{y} = \mathbf{y}(\nu_1, \nu_2, \ldots, \nu_s)$ with this choice and let $c = \mathrm{wt}(\mathbf{y}) + 1$.

Write $\mathbf{F} = \bigcup_{i=1}^{r} \mathbf{F}_c(\mathbf{x}_i)$; this is the union of a finite number of finite sets. By Lemma 6, for $1, 2, \ldots, r$,

$$\mathbf{x}_i \equiv \prod_{\mathbf{z}\in\mathbf{F}} \mathbf{z}^{E(\mathbf{x}_i,\mathbf{z})} \pmod{\mathfrak{N}_{c-1} \cap \mathfrak{A}.\mathfrak{N}_{k_1-1}}.$$

The extension of the range of this product from $\mathbf{F}_c(\mathbf{x}_i)$ to $\mathbf{F}$ is permissible since, if $\mathbf{z} \notin \mathbf{F}_c(\mathbf{x}_i)$ then $E(\mathbf{x}_i, \mathbf{z}) = 0$ by definition. Thus

$$\mathbf{x} = \mathbf{x}_1^{\xi_1}\mathbf{x}_2^{\xi_2}\ldots\mathbf{x}_r^{\xi_r}$$
$$\equiv \prod_{\mathbf{z}\in\mathbf{F}} \mathbf{z}^{\Sigma\xi_i E(\mathbf{x}_i,\mathbf{z})} \pmod{\mathfrak{N}_{c-1} \cap \mathfrak{A}.\mathfrak{N}_{k_1-1}}.$$

Now at least one of the exponents $\sum \xi_i E(\mathbf{x}_i, \mathbf{z})$ is non-zero, namely the one for which $\mathbf{z} = \mathbf{y}$. Deleting all factors with zero exponent and ordering the remaining factors in order of increasing $\mathbf{z}$ results in a non-trivial basic expression consisting of basic commutators which are of weight $< c$ and of shape $< 2\delta_1$. Then, by the Basis Theorem ([**2**] Theorem 9.1D(iii)),

$$\mathbf{x}\rho \notin \gamma_c F.Q_{2\delta_1}(F) = \gamma_c(F).\delta(\gamma_{k_1}(F)).$$

LEMMA 8. *Let ρ: $\mathbf{A} \to F$ be a free description of the absolutely free group F. Then ρ maps the set $\mathbf{J}_1$ one-to-one onto a set $\mathbf{J}_1\rho$ of free generators for $\gamma_{k_1}(F)$.*

Proof. It must be shown that $\mathbf{J}_1\rho$ generates $\gamma_{k_1}(F)$: the lemma then follows from Lemma 7. Suppose then that $x \in \gamma_{k_1}(F)$. Then there exists an expression $\mathbf{x} \in \mathbf{A}$ such that $\mathbf{x}\rho = x$. Then, by Theorem 1, there exists an expression $\mathbf{yw}$ such that $\mathbf{yw} \equiv \mathbf{x}$, $\mathbf{y}$ is a basic invertator product consisting, if it is not $\mathbf{1}$, of basic invertators of depth 0, and $\mathbf{w}$ is a (possibly empty) working product whose factors are all in $\mathbf{J}_1$. Now $(\mathbf{yw})\rho = \mathbf{x}\rho = x$ and so $\mathbf{y}\rho = x$ modulo $\gamma_{k_1}(F)$. But then $\mathbf{y}\rho = 1$ modulo $\gamma_{k_1}(F)$ and by Lemma 2(i) $\mathbf{y}$ is a basic expression consisting, if it is not $\mathbf{1}$, of basic commutators of weight $< k_1$. Thus $\mathbf{y} = \mathbf{1}$ and so, provided $\mathbf{w}$ is not empty, $\mathbf{w}\rho = x$ as required. If $\mathbf{w}$ is empty then $x = 1$ and the result is trivial.

5. Free generators for $P_{K_r}(F)$

It has been assumed throughout that we are working with a fixed polyweight range Q which automatically defines a polyweight π: $\mathbf{A} \to Q$. Now define a sequence $K' = (k'_i)_{i=1}^{\infty}$ by $k'_i = k_{i+1}$ for all $i \geqslant 1$, and write $Q' = Q^{K'}$, denoting the least and greatest elements of Q' by $1'$ and ∞' respectively. Construct a new algebra of expressions $\mathbf{A}'$ upon an ordered set $\mathbf{G}'$ of generators chosen to be in one-to-one order-preserving correspondence with the subset $\mathbf{J}_1$ of $\mathbf{A}$: there is no reason, other than notational convenience, not to choose $\mathbf{G}' = \mathbf{J}_1$. Extend this correspondence to a homomorphism ρ': $\mathbf{A}' \to \mathbf{A}$ (see the remarks concerning this kind of construction in the introductory paragraph of [**2**] § 15 and the proof of [**2**] Lemma 15.1). Then ρ' is in fact a description of a subalgebra of $\mathbf{A}$, namely that generated by $\mathbf{J}_1$. Denote the identity of $\mathbf{A}'$ by $\mathbf{1}'$ and let $\pi' = \pi^{K'}$ be the polyweight defined on $\mathbf{A}'$ by Q' (the unique polyweight $\mathbf{A}' \to Q'$). For any function $\varphi \in Q$, define a function φ^*: $\omega \to \omega$ by

$$\varphi^*(j) = \varphi(j+1) \quad \text{for all } j \in \omega,$$

and write $\infty^* = \infty'$. This construction and the accompanying notation will remain fixed throughout this section.

LEMMA 9. (i) *For each $\varphi \in Q$ of depth $\geqslant 1$, in the sense of* [**2**] *Definition* 14.2(A), *$\varphi^* \in Q'$ and $d_{\varphi^*} = d_\varphi - 1$.*

(ii) *For any $\varphi, \psi \in Q$, both of depth $\geqslant 1$,*

$$\varphi \leqslant \psi \Rightarrow \varphi^* \leqslant \psi^*$$

and

$$(\varphi+\psi)^* = \varphi^* + \psi^*.$$

Proof. This follows immediately from the definition of φ^* and ψ^*, using [**2**] Definition 14.2.

LEMMA 10. *For any* $\mathbf{x}' \in \mathbf{A}'$, *$\pi(\mathbf{x}'\rho')$ is of depth $\geqslant 1$ and*

$$(\pi(\mathbf{x}'\rho'))^* = \pi'(\mathbf{x}').$$

Proof. This is by induction over the height of $\mathbf{x}'$. If $\mathbf{x}' = \mathbf{1}'$ then $\pi'(\mathbf{x}') = \pi'(\mathbf{1}') = \infty'$ by [**2**] Definition 2.3(i). Also

$$\pi(\mathbf{x}'\rho') = \pi(\mathbf{1}'\rho') = \pi(\mathbf{1}) \quad \text{since } \rho' \text{ is a homomorphism,}$$
$$= \infty \quad \text{by [2] Definition 2.3(i).}$$

Thus $\pi(\mathbf{x}'\rho')$ is of depth $\infty \geqslant 1$ and $(\pi(\mathbf{x}'\rho'))^* = \infty^* = \infty'$. If $\mathbf{x}' \in \mathbf{G}'$ then $\pi'(\mathbf{x}') = 1'$ by [**2**] Definition 2.3(i). Also $\mathbf{x}'\rho' \in \mathbf{J}_1$ by the definition of ρ', which means that $\mathbf{x}'\rho'$ is of 1-weight 1. Hence

$$\pi(\mathbf{x}'\rho')(1) = 1$$

and thus

$$\pi(\mathbf{x}'\rho')(j) = 0 \quad \text{for all } j > 1$$

by [**2**] Definition 14.2(A)(i). Then $\pi(\mathbf{x}'\rho')$ is of depth 1 and $(\pi(\mathbf{x}'\rho'))^* = 1'$ as required.

It may now be assumed inductively that $\mathrm{ht}(\mathbf{x}') > 0$ and the lemma is true for all expressions of smaller height than $\mathbf{x}'$.

Suppose that $\mathbf{x}' = (\mathbf{y}')^{-1}$, where $\mathbf{y}'$ is of smaller height than $\mathbf{x}'$. Then

$$\pi(\mathbf{x}'\rho') = \pi((\mathbf{y}'\rho')^{-1}) \quad \text{since } \rho' \text{ is a homomorphism,}$$
$$= \pi(\mathbf{y}'\rho') \quad \text{by [2] Definition 2.3(ii),}$$

and hence is of depth $\geqslant 1$ by the inductive assumption. Further

$$\begin{aligned}(\pi(\mathbf{x}'\rho'))^* &= (\pi(\mathbf{y}'\rho'))^* \\ &= \pi'(\mathbf{y}') \quad \text{by the inductive assumption,} \\ &= \pi'((\mathbf{y}')^{-1}) \quad \text{by [2] Definition 2.3(ii),} \\ &= \pi'(\mathbf{x}') \quad \text{as required.}\end{aligned}$$

Now suppose that $\mathbf{x}' = \mathbf{x}_1'\mathbf{x}_2'$, where $\mathbf{x}_1'$ and $\mathbf{x}_2'$ are both of smaller height than $\mathbf{x}'$. Then

$$\begin{aligned}\pi(\mathbf{x}'\rho') &= \pi(\mathbf{x}_1'\rho'.\mathbf{x}_2'\rho') \quad \text{since } \rho \text{ is a homomorphism,} \\ &= \min\{\pi(\mathbf{x}_1'\rho'), \pi(\mathbf{x}_2'\rho')\} \quad \text{by [2] Definition 2.3(ii).}\end{aligned}$$

But both $\pi(\mathbf{x}_1'\rho')$ and $\pi(\mathbf{x}_2'\rho')$ are of depth $\geqslant 1$ by the inductive assumption and so the same is true of $\pi(\mathbf{x}'\rho')$. Suppose further that $\pi(\mathbf{x}_1'\rho') \leqslant \pi(\mathbf{x}_2'\rho')$. Then $\pi(\mathbf{x}'\rho') = \pi(\mathbf{x}_1'\rho')$ so that

$$\begin{aligned}(\pi(\mathbf{x}'\rho'))^* &= (\pi(\mathbf{x}_1'\rho'))^* \\ &= \min\{(\pi(\mathbf{x}_1'\rho'))^*, (\pi(\mathbf{x}_2'\rho'))^*\}\end{aligned}$$

since $(\pi(\mathbf{x}_1'\rho') \leqslant (\pi(\mathbf{x}_2'\rho'))^*$ by Lemma 9(ii). Thus

$$\begin{aligned}(\pi(\mathbf{x}'\rho'))^* &= \min\{\pi'(\mathbf{x}_1'), \pi'(\mathbf{x}_2')\} \quad \text{by the inductive assumption,} \\ &= \pi'(\mathbf{x}_1'\mathbf{x}_2') \quad \text{by [2] Definition 2.3(iii),} \\ &= \pi'(\mathbf{x}'),\end{aligned}$$

as required. If $\pi(\mathbf{x}_1'\rho') \geqslant \pi(\mathbf{x}_2'\rho')$ the proof is similar.

Finally, suppose that $\mathbf{x}' = [\mathbf{x}_1', \mathbf{x}_2']$ where $\mathbf{x}_1'$ and $\mathbf{x}_2'$ are both of smaller height than $\mathbf{x}'$. Then

$$\begin{aligned}\pi(\mathbf{x}'\rho') &= \pi([\mathbf{x}_1'\rho', \mathbf{x}_2'\rho']) \quad \text{since } \rho' \text{ is a homomorphism} \\ &= \pi(\mathbf{x}_1'\rho') + \pi(\mathbf{x}_2'\rho') \quad \text{by [2] Definition 2.3(iv).}\end{aligned}$$

But both $\pi(\mathbf{x}_1'\rho')$ and $\pi(\mathbf{x}_2'\rho')$ are of depth $\geqslant 1$ by the inductive assumption and so the same is true of $\pi(\mathbf{x}'\rho')$. Then

$$\begin{aligned}(\pi(\mathbf{x}'\rho'))^* &= (\pi(\mathbf{x}_1'\rho'))^* + (\pi(\mathbf{x}_2'\rho'))^* \quad \text{by Lemma 9(ii),} \\ &= \pi'(\mathbf{x}_1') + \pi'(\mathbf{x}_2') \quad \text{by the inductive assumption,} \\ &= \pi'([\mathbf{x}_1', \mathbf{x}_2']) \quad \text{by [2] Definition 2.3(iv),} \\ &= \pi'(\mathbf{x}')\end{aligned}$$

as required. This completes the proof of the lemma.

COROLLARY 1. *If* $\mathbf{x}'$ *and* $\mathbf{y}'$ *are expressions in* $\mathbf{A}'$ *and* $\pi'(\mathbf{x}') < \pi'(\mathbf{y}')$ *then* $\pi(\mathbf{x}'\rho') < \pi(\mathbf{y}'\rho')$.

Proof. This is by contradiction: $\pi(\mathbf{x}'\rho') \geqslant \pi(\mathbf{y}'\rho')$ implies

$$(\pi(\mathbf{x}'\rho'))^* \geqslant (\pi(\mathbf{y}'\rho'))^*$$

by Lemma 9(ii), which in turn implies $\pi'(\mathbf{x}') \geqslant \pi'(\mathbf{y}')$ by this lemma.

COROLLARY 2. *If* $\mathbf{x}'$ *is an expression in* $\mathbf{A}'$ *then* $d(\mathbf{x}'\rho') = d(\mathbf{x}') + 1$.

Proof. This follows from Lemma 10, using Lemma 9(i).

Observe that, if $\mathbf{x}'$ is an invertator in $\mathbf{A}'$ then $\mathbf{x}'\rho'$ is an invertator in $\mathbf{A}$: this follows from the fact that ρ' is a homomorphism.

LEMMA 11. *If* $\mathbf{a}'$ *and* $\mathbf{b}'$ *are invertators in* $\mathbf{A}'$, $\alpha = \pm 1$, $\beta = \pm 1$, *and* $\mathbf{a}'^{\alpha} < \mathbf{b}'^{\beta}$ *then* $(\mathbf{a}'\rho')^{\alpha} < (\mathbf{b}'\rho')^{\beta}$.

Proof. If $\mathbf{a}'^{\alpha} < \mathbf{b}'^{\beta}$ then one of the parts of Definition 1 must obtain: these are checked one by one.

(i)(a) Suppose that $\pi'(\mathbf{a}') = \pi'(\mathbf{b}') = 1'$ so that $\mathbf{a}', \mathbf{b}' \in \mathbf{G}'$ and $\mathbf{a}' < \mathbf{b}'$ under the ordering assumed on $\mathbf{G}'$. Then $\mathbf{a}'\rho' < \mathbf{b}'\rho'$ by the definition of ρ' so that $(\mathbf{a}'\rho')^{\alpha} < (\mathbf{b}'\rho')^{\beta}$ by Definition 1(i)(a).

(i)(b) If $\pi'(\mathbf{a}') = \pi'(\mathbf{b}') = 1$ so that $\mathbf{a}', \mathbf{b}' \in \mathbf{G}'$, $\mathbf{a}' = \mathbf{b}'$, $\alpha = +1$, and $\beta = -1$ then $\mathbf{a}'\rho' = \mathbf{b}'\rho'$, and then $(\mathbf{a}'\rho')^{\alpha} < (\mathbf{b}'\rho')^{\beta}$ by Definition 1(iii)(d).

(ii) If $\pi'(\mathbf{a}') < \pi'(\mathbf{b}')$ then $\pi(\mathbf{a}'\rho') < \pi(\mathbf{b}'\rho')$ by Lemma 10, Corollary 1, and thus $(\mathbf{a}'\rho')^{\alpha} < (\mathbf{b}'\rho')^{\beta}$ by Definition 1(ii).

(iii) Suppose that $\pi'(\mathbf{a}') = \pi'(\mathbf{b}') > 1$. The proof proceeds by induction over $\pi'(\mathbf{a}')$, the inductive assumption being that the lemma is true for all pairs of invertators in $\mathbf{A}'$ of polyweight $< \pi'(\mathbf{a}')$. From this assumption it follows that $\mathrm{ld}(\mathbf{a}'\rho') = (\mathrm{ld}(\mathbf{a}'))\rho'$, $\mathrm{tr}(\mathbf{a}'\rho') = (\mathrm{tr}(\mathbf{a}'))\rho'$, $\mathrm{ld}(\mathbf{b}'\rho') = (\mathrm{ld}(\mathbf{b}'))\rho'$, and $\mathrm{tr}(\mathbf{b}'\rho') = (\mathrm{tr}(\mathbf{b}'))\rho'$. There are now four possibilities. If (a), $\mathrm{ld}(\mathbf{a}') < \mathrm{ld}(\mathbf{b}')$ then $\mathrm{ld}(\mathbf{a}'\rho') < \mathrm{ld}(\mathbf{b}'\rho')$ by the inductive assumption so that $(\mathbf{a}'\rho')^{\alpha} < (\mathbf{b}'\rho')^{\beta}$ by Definition 1(iii)(a). If (b), $\mathrm{ld}(\mathbf{a}') = \mathrm{ld}(\mathbf{b}')$ and $\mathrm{tr}(\mathbf{a}') < \mathrm{tr}(\mathbf{b}')$ or (c), $\mathrm{ld}(\mathbf{a}') = \mathrm{ld}(\mathbf{b}')$, $\mathrm{tr}(\mathbf{a}') = \mathrm{tr}(\mathbf{b}')$, and $\mathbf{a}_1'^{\alpha_1} > \mathbf{b}_1'^{\beta_1}$, where $\mathbf{a}' = [\mathbf{a}_1'^{\alpha_1}, \mathbf{a}_2'^{\alpha_2}]$ and $\mathbf{b}' = [\mathbf{b}_1'^{\beta_1}, \mathbf{b}_2'^{\beta_2}]$, the proof is similar. Finally, if (d), $\mathbf{a}' = \mathbf{b}'$, $\alpha = +1$, and $\beta = -1$ then $\mathbf{a}'\rho' = \mathbf{b}'\rho'$ and then $(\mathbf{a}'\rho')^{\alpha} < (\mathbf{b}'\rho')^{\beta}$ by Definition 1(iii)(d).

LEMMA 12. ρ' *maps the set of all* Q'*-basic invertators in* $\mathbf{A}'$ *one-to-one onto the set of all* Q*-basic invertators in* $\mathbf{A}$ *of depth* $\geqslant 1$.

Proof. As a consequence of Lemmas 10 and 11, ρ' maps the set of Q'-basic invertators in $\mathbf{A}'$ one-to-one into the set of invertators in $\mathbf{A}$ of depth $\geqslant 1$. It remains to show that if $\mathbf{x}'$ is a Q'-basic invertator in $\mathbf{A}'$ then $\mathbf{x}'\rho'$ is Q-basic and that if $\mathbf{x}$ is a Q-basic invertator of depth $\geqslant 1$ in $\mathbf{A}$ then there exists a Q'-basic invertator $\mathbf{x}'$ in $\mathbf{A}'$ such that $\mathbf{x} = \mathbf{x}'\rho'$.

Suppose then that $\mathbf{x}$ is a Q-basic invertator of depth $\geqslant 1$ in $\mathbf{A}$. If $\mathbf{x}$ is of depth 1 and 1-weight 1 then $\mathbf{x} \in \mathbf{J}_1$ and so, by the definition of ρ', there exists an $\mathbf{x}' \in \mathbf{G}'$ such that $\mathbf{x} = \mathbf{x}'\rho'$; but then $\mathbf{x}'$ is a Q'-basic invertator in $\mathbf{A}'$. There remain two possibilities: firstly, $\mathbf{x}$ is of depth $d > 1$ and d-weight 1 and, secondly, $\mathbf{x}$ is of depth $d \geqslant 1$ and d-weight > 1. The proof is by induction over $\pi(\mathbf{x})$. In the first case, by Lemma 2(ii), $\mathbf{x}$

is of the form

$$\mathbf{x} = [\mathbf{b}, n_1\mathbf{a}_1^{\alpha_1}, n_2\mathbf{a}_2^{\alpha_2}, \ldots, n_t\mathbf{a}_t^{\alpha_t}],$$

where $t \geqslant 1$, each $n_i \geqslant 1$, $\mathbf{b}$, $\mathbf{a}_1, \ldots,$ and $\mathbf{a}_t$ are basic invertators of depth $d-1$ but $[\mathbf{b}, \mathbf{a}_1^{\alpha_1}]$ is of depth d, $\mathbf{b} > \mathbf{a}_1 < \mathbf{a}_2 < \ldots < \mathbf{a}_t$ and each $\alpha_i = \pm 1$. But $d-1 \geqslant 1$ so, by the inductive assumption, there exist Q'-basic invertators $\mathbf{b}', \mathbf{a}_1', \mathbf{a}_2', \ldots, \mathbf{a}_t'$ in $\mathbf{A}'$ such that $\mathbf{b}'\rho' = \mathbf{b}$ and $\mathbf{a}_i'\rho' = \mathbf{a}_i$ $(i = 1, 2, \ldots, t)$. Thus

$$\mathbf{x} = [\mathbf{b}', n_1\mathbf{a}_1'^{\alpha_1}, n_2\mathbf{a}_2'^{\alpha_2}, \ldots, n_t\mathbf{a}_t'^{\alpha_t}]\rho',$$

$\mathbf{b}'$ and each $\mathbf{a}_i'$ is of depth $d-2$ but $[\mathbf{b}', \mathbf{a}_1'^{\alpha_1}]$ is of depth $d-1$ by Lemma 10, Corollary 2, and $\mathbf{b}' > \mathbf{a}_1' < \mathbf{a}_2' < \ldots < \mathbf{a}_t'$ by Lemma 11. Thus

$$[\mathbf{b}', n_1\mathbf{a}_1'^{\alpha_1}, n_2\mathbf{a}_2'^{\alpha_2}, \ldots, n_t\mathbf{a}_t'^{\alpha_t}]$$

is a Q'-basic invertator in $\mathbf{A}'$ as required. In the second case, by Lemma 2(iii), $\mathbf{x} = [\mathbf{b}, \mathbf{a}]$ where $\mathbf{a}$ and $\mathbf{b}$ are Q-basic invertators of depth d, $\mathbf{a} < \mathbf{b}$, and, if $\mathbf{b} = [\mathbf{b}_1, \mathbf{b}_2^{\beta_2}]$, then $\mathbf{b}_2 \leqslant \mathbf{a}$. But then, by the inductive hypothesis, there exist Q'-basic invertators $\mathbf{a}'$ and $\mathbf{b}'$ in $\mathbf{A}'$ such that $\mathbf{a}'\rho' = \mathbf{a}$ and $\mathbf{b}'\rho' = \mathbf{b}$. Now $\mathbf{a}' < \mathbf{b}'$ by Lemma 11 and if $\mathbf{b}' = [\mathbf{b}_1', \mathbf{b}_2'^{\beta_2}]$ then $\mathbf{b} = [\mathbf{b}_1'\rho, (\mathbf{b}_2'\rho')^{\beta_2}]$ so that $\mathbf{b}_2'\rho \leqslant \mathbf{a} = \mathbf{a}'\rho'$ and then $\mathbf{b}_2' \leqslant \mathbf{a}'$ by Lemma 11 again. Thus $[\mathbf{b}', \mathbf{a}']$ is a Q'-basic invertator in $\mathbf{A}'$ and $[\mathbf{b}', \mathbf{a}']\rho = \mathbf{x}$, as required.

Now suppose that $\mathbf{x}'$ is a Q'-basic invertator in $\mathbf{A}'$. The proof that $\mathbf{x}'\rho'$ is Q-basic proceeds by induction over $\pi'(\mathbf{x}')$. If $\mathbf{x}' \in \mathbf{G}'$ then $\mathbf{x}'\rho'$ is a Q-basic invertator in $\mathbf{A}$ by the definition of ρ'. Otherwise, by Definition 2(A), $\mathbf{x}' = [\mathbf{b}', \mathbf{a}'^{\alpha}]$ where both $\mathbf{a}'$ and $\mathbf{b}'$ are Q'-basic invertators and then $\mathbf{x}'\rho' = [\mathbf{b}'\rho', (\mathbf{a}'\rho')^{\alpha}]$ where $\mathbf{b}'\rho'$ and $\mathbf{a}'\rho'$ are Q-basic invertators by the inductive assumption. By Definition 2(A) again, $\mathbf{a}' < \mathbf{b}'$ and so $\mathbf{a}'\rho' < \mathbf{b}'\rho'$ by Lemma 11. If $\alpha = -1$ then $d(\mathbf{a}') < d(\mathbf{x}')$ and thus $d(\mathbf{a}'\rho') < d(\mathbf{x}'\rho')$ by Lemma 10, Corollary 2. If $\mathbf{b}'\rho' = [\mathbf{b}_1, \mathbf{b}_2^{\beta}]$ then, since $\mathbf{b}'$ is a Q-basic invertator, so are $\mathbf{b}_1$ and $\mathbf{b}_2$. By the first part of this proof then, there exist Q'-basic invertators $\mathbf{b}_1'$ and $\mathbf{b}_2'$ in $\mathbf{A}'$ such that $\mathbf{b}_1 = \mathbf{b}_1'$ and $\mathbf{b}_2 = \mathbf{b}_2'$. Then $\mathbf{b}'\rho = [\mathbf{b}_1', \mathbf{b}_2'^{\beta}]\rho'$ and therefore, since ρ' is one-to-one on invertators, $\mathbf{b}' = [\mathbf{b}_1', \mathbf{b}_2'^{\beta}]$. But then, by Definition 2(A) applied to $\mathbf{x}'$, $\mathbf{b}_2' \leqslant \mathbf{a}'$ and thus $\mathbf{b}_2 = \mathbf{b}_2'\rho' \leqslant \mathbf{a}'\rho'$ by Lemma 11. If $\mathbf{b}'\rho' = [\mathbf{b}_1, (\mathbf{a}'\rho')^{\beta}]$ and $d(\mathbf{a}'\rho') < d(\mathbf{b}_1'\rho')$ then, as above, there exists $\mathbf{b}_1' \in \mathbf{A}'$ such that $\mathbf{b}_1 = \mathbf{b}_1'\rho'$ and $\mathbf{b}' = [\mathbf{b}_1', \mathbf{a}'^{\beta}]$ and, by Lemma 10, Corollary 2, $d(\mathbf{a}') < d(\mathbf{b}_1')$ so that, applying Definition 2(A) to $\mathbf{x}'$ once more, $\beta = \alpha$. Definition 2(A)(ii) applied to $\mathbf{x}'\rho'$ now shows that this invertator is Q-basic.

This, with Lemma 10, gives

COROLLARY. *For each integer $d \geqslant 1$, ρ' maps the set $\mathbf{J}_{d-1}'$ of all Q'-basic invertators in $\mathbf{A}'$ of $(d-1)$-weight* 1 *one-to-one onto the set $\mathbf{J}_d$ of all Q-basic invertators in $\mathbf{A}$ of d-weight* 1.

THEOREM 2. *Let $\rho: \mathbf{A} \to F$ be a free description of the absolutely free group F. Then, for any non-negative integer d, ρ maps the set $\mathbf{J}_d$ of all basic invertators of d-weight* 1 *one-to-one onto a set $\mathbf{J}_d\rho$ of free generators for the subgroup $P_{K_d}(F)$ of F (see* [2] *Definition* 14.1).

Proof. This is by induction over d. If $d = 0$, $\mathbf{J}_d = \mathbf{J}_0 = \mathbf{G}$, the generating set of $\mathbf{A}$. Since ρ is a free description, this is mapped one-to-one onto a free generating set for $F = P_{K_0}(F)$ by ρ. Now suppose that $d \geqslant 1$. Then ρ' maps the generating set $\mathbf{G}'$ of $\mathbf{A}'$ one-to-one onto the set $\mathbf{J}_1$ which, by Lemma 8, is in turn mapped one-to-one onto a set of free generators for $\gamma_{k_1}(F)$. Hence $\rho'\rho: \mathbf{A}' \to \gamma_{k_1}(F)$ is a free description of $\gamma_{k_1}(F)$. By the inductive hypothesis then, $\rho'\rho$ maps $\mathbf{J}_{d-1}'$ one-to-one onto a set of free generators for $P_{K_{d-1}'}(\gamma_{k_1}(F)) = P_{K_d}(F)$. Thus, by Lemma 12, Corollary, ρ maps $\mathbf{J}_d = \mathbf{J}_{d-1}'\rho'$ one-to-one onto that set of free generators.

THEOREM 3. *Let G be a free polynilpotent group of type K_d $(d \geqslant 0)$ and let $\rho: \mathbf{A} \to G$ be a free description of G. Then, to each $x \in G$ there exists a basic invertator product $\mathbf{x}$ such that $x = \mathbf{x}\rho$ and which consists, if it is not* $\mathbf{1}$, *of basic invertators of depth $< d$. Furthermore, for any given $x \in G$, the expression $\mathbf{x}$ with these properties is unique.*

Proof. Let $x \in G$. Then there exists an expression $\mathbf{z} \in \mathbf{A}$ such that $\mathbf{z}\rho = x$. By Theorem 1, then, there exists an expression $\mathbf{xw} \in \mathbf{A}$ such that $\mathbf{z} \equiv \mathbf{xw}$ and hence $x = \mathbf{z}\rho = (\mathbf{xw})\rho$, $\mathbf{x}$ is a basic invertator product consisting, if it is not $\mathbf{1}$, of basic invertators of depth $< d$ and $\mathbf{w}$ is a (possibly empty) working product whose factors are all of d-weight 1, which means that they belong to $\mathbf{J}_d$. Thus, if $\mathbf{w}$ is not empty, $\mathbf{w}\rho \in P_{K_d}(F) = \{1\}$ and in any case $x = \mathbf{x}\rho$.

Now suppose that $\mathbf{x}$ and $\mathbf{y}$ are basic invertator products which consist, if they are not $\mathbf{1}$, of basic invertators of depth $< d$ and that $\mathbf{x}\rho = \mathbf{y}\rho$: it is shown that $\mathbf{x} = \mathbf{y}$ by induction over d. If $d = 0$ the result is trivial for then $\mathbf{x} = \mathbf{1} = \mathbf{y}$. Now suppose that $d \geqslant 1$. In $\mathbf{x}$ any factors of depth 0 precede those of depth $\geqslant 1$. Hence $\mathbf{x} = \mathbf{x}_1\mathbf{x}_2$ where either $\mathbf{x}_1$ or $\mathbf{x}_2$ may be $\mathbf{1}$ or empty but, if they are not, are basic invertator products consisting of basic invertators of depth 0 and depth $\geqslant 1$ respectively. For $i = 1, 2$, define $\mathbf{u}_i$ by $\mathbf{u}_i = \mathbf{x}_i$ if $\mathbf{x}_i$ is not empty and $\mathbf{u}_i = 1$ otherwise. Then $\mathbf{u}_1$ is a genuine basic invertator product consisting, if it is not $\mathbf{1}$, of basic invertators of depth 0 and is thus, by Lemma 2(i), a basic product consisting of basic commutators of weight $< k_1$, $\mathbf{u}_2$ is a basic invertator product consisting, if it is not $\mathbf{1}$, of basic invertators of depth $\geqslant 1$ and $< d$, $\mathbf{x} \equiv \mathbf{u}_1\mathbf{u}_2$ and $\mathbf{x}$ can be reconstituted from the expression $\mathbf{u}_1\mathbf{u}_2$ by deleting any spurious occurrences of $\mathbf{1}$ as a factor. In the same manner,

$y \equiv \mathbf{v}_1\mathbf{v}_2$, where $\mathbf{v}_1$ is a basic product consisting, if it is not **1**, of basic commutators of weight $< k_1$, $\mathbf{v}_2$ is a basic invertator product consisting, if it is not **1**, of basic invertators of depth $\geqslant 1$ and $< d$, and $\mathbf{y}$ can be reconstituted from $\mathbf{v}_1\mathbf{v}_2$ by deleting any spurious occurrence of **1** as a factor. It is now enough to show that $\mathbf{u}_1 = \mathbf{v}_1$ and $\mathbf{u}_2 = \mathbf{v}_2$.

Now $\mathbf{u}_1\rho = \mathbf{x}\rho$ modulo $\gamma_{k_1}(G)$ since the factors of $\mathbf{u}_2$ are all of depth $\geqslant 1$ and $\mathbf{x} \equiv \mathbf{u}_1\mathbf{u}_2$. Hence $\mathbf{u}_1\rho = \mathbf{v}_1\rho$ modulo $\gamma_{k_1}(G)$. But $G/\gamma_{k_1}(G)$ is free nilpotent of class $k_1 - 1$ by the assumption that G is free polynilpotent of type K_d and $d \geqslant 1$. Thus, by the Basis Theorem ([**2**] Theorem 9.1(D)(iii) with $\alpha = 0$, $\beta = \infty$, and $c = k_1 - 1$), $\mathbf{u}_1 = \mathbf{v}_1$. But then $\mathbf{u}_2\rho = \mathbf{v}_2\rho$. Since $\mathbf{u}_2$ is a basic invertator product consisting of invertators of depth $\geqslant 1$, it is either **1** or of the form

$$\mathbf{u}_2 = \mathbf{b}_1^{\beta_1}\mathbf{b}_2^{\beta_2}\ldots\mathbf{b}_r^{\beta_r},$$

where $r \geqslant 1$, the $\mathbf{b}_i$ are basic invertators of depth $\geqslant 1$, each β_i is a non-zero integer and $\mathbf{b}_1 < \mathbf{b}_2 < \ldots < \mathbf{b}_r$. Then, by Lemma 12, there exist Q'-basic commutators $\mathbf{b}_1', \mathbf{b}_2', \ldots, \mathbf{b}_r'$ in $\mathbf{A}'$ such that $\mathbf{b}_i'\rho' = \mathbf{b}_i$ $(i = 1, 2, \ldots, r)$. Write $\mathbf{u}_2' = \mathbf{b}_1'^{\beta_1}\mathbf{b}_2'^{\beta_2}\ldots\mathbf{b}_r'^{\beta_r}$ so that $\mathbf{u}_2 = \mathbf{u}_2'\rho'$. By Lemma 11, then,

$$\mathbf{b}_1' < \mathbf{b}_2' < \ldots < \mathbf{b}_r'$$

and thus $\mathbf{u}_2'$ is a Q'-basic invertator product in $\mathbf{A}'$ and, by Lemma 10, Corollary 2, each $\mathbf{b}_i'$ is of depth $< d - 1$. Similarly, there exists a Q'-invertator product $\mathbf{v}_2'$ in $\mathbf{A}'$ consisting, if it is not **1**, of Q'-basic invertators of depth $< d - 1$ and such that $\mathbf{v}_2 = \mathbf{v}_2'\rho'$. But then

$$\mathbf{v}_2'\rho'\rho = \mathbf{v}_2\rho = \mathbf{u}_2\rho = \mathbf{u}_2'\rho'\rho$$

and $\rho'\rho$ is a free description of $\gamma_{k_1}(F)$ which is a free polynilpotent group of type K_{d-1}'. Hence, by the inductive hypothesis, $\mathbf{u}_2' = \mathbf{v}_2'$ and thus $\mathbf{u}_2 = \mathbf{u}_2'\rho' = \mathbf{v}_2'\rho' = \mathbf{v}_2$.

Corollary 1. *With the hypotheses of the theorem, the image of the set of basic invertators of depth $< d$ under ρ, equipped with the order carried over from that set by ρ, forms a basis for G.*

Corollary 2. *In addition to the hypotheses of the theorem, let d' be an integer $(0 \leqslant d' \leqslant d)$. Then to each $x \in P_{K_{d'}}(G)$ there exists a basic invertator product* $\mathbf{x}$ *such that $x = \mathbf{x}\rho$ and which consists, if it is not* **1**, *of basic invertators of depth $\geqslant d'$ and $< d$. Furthermore, for any such x, the expression* $\mathbf{x}$ *with these properties is unique.*

Corollary 3. *With the hypotheses of Corollary 2, the image of the set of basic invertators of depth $\geqslant d'$ and $< d$ under ρ, equipped with the order carried over from that set by ρ, forms a basis for $P_{K_{d'}}(G)$.*

6. The collecting process as an algorithm

The proof of Theorem 1 is constructive. This theorem, together with the lemmas leading up to it, contains an algorithm which effectively calculates the expressions $\mathbf{y}$ and $\mathbf{w}$ whose existence it asserts; but this algorithm is a good deal simpler than would appear from a casual inspection of the proof of the theorem. The algorithm itself is therefore stated here as a recipe for the convenience of those readers who may wish to apply it. The usefulness of the algorithm is made clear by Theorems 2 and 3.

Assume that we are given an expression $\mathbf{x}$ and a non-negative integer d. The algorithm generates a sequence $\mathbf{y}_1\mathbf{w}_1, \mathbf{y}_2\mathbf{w}_2, \ldots$, of expressions; the $\mathbf{y}_i$ will turn out to be basic invertator products and the $\mathbf{w}_i$ to be working products; however, this will occur automatically and does not affect the description of the algorithm. It is assumed that redundant **1**'s will be deleted whenever they occur.

Initialization. Put $\mathbf{y}_1 = \mathbf{1}$. Let $\mathbf{w}$ be a product of generators and their inverses such that $\mathbf{w} \equiv \mathbf{x}$.

Recursive step. Suppose that $\mathbf{y}_n$ and $\mathbf{w}_n$ have been calculated $(n \geqslant 1)$. If $\mathbf{w}_n$ is empty or of depth d, terminate the process. Otherwise let $\mathbf{a}$ be the earliest invertator (under the order of Definition 1) which occurs as a factor of $\mathbf{w}_n$ and let $\varepsilon = \varepsilon_{\mathbf{a}}(\mathbf{w}_n)$. Set $\mathbf{y}_{n+1} = \mathbf{y}_n\mathbf{a}^{\varepsilon}$. Let $\mathbf{w}' = \mathbf{a}^{-\varepsilon}\mathbf{w}_n$ and suppose that the first (leftmost) occurrence of $\mathbf{a}$ as a factor of $\mathbf{w}'$ has exponent $-\alpha = \pm 1$. Then $\mathbf{w}'$ can be written in the form

$$\mathbf{w}' = \mathbf{x}_1\mathbf{a}^{-\alpha}\mathbf{b}_1^{\beta_1}\mathbf{b}_2^{\beta_2}\ldots\mathbf{b}_k^{\beta_k}\mathbf{a}^{\alpha}\mathbf{x}_3,$$

where $\mathbf{x}_1$ and $\mathbf{x}_3$ are (possibly empty) products, each $\mathbf{b}_i$ is a basic invertator, each $\beta_i = \pm 1$, if $\mathbf{a}$ occurs as a factor of $\mathbf{x}_1$ then it does so with exponent $-\alpha$ and none of the $\mathbf{b}_i$ is $\mathbf{a}$. Put

$$\mathbf{w}_{n+1} = \mathbf{x}_1\chi(\mathbf{b}_1^{\beta_1}, \mathbf{a}^{\alpha})\chi(\mathbf{b}_2^{\beta_2}, \mathbf{a}^{\alpha})\ldots\chi(\mathbf{b}_k^{\beta_k}, \mathbf{a}^{\alpha})\mathbf{x}_3.$$

Upon termination, $\mathbf{y}_n$ and $\mathbf{w}_n$ are the $\mathbf{y}$ and $\mathbf{w}$ of Theorem 1.

REFERENCES

1. K. W. Gruenberg, 'Residual properties of infinite soluble groups', *Proc. London Math. Soc.* (3) 7 (1957) 29–62.
2. M. A. Ward, 'Basic commutators', *Philos. Trans. Roy. Soc. London Ser. A* 264 (1969) 343–412.

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