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PRIME GRAPH COMPONENTS OF FINITE ALMOST SIMPLE GROUPS

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Abstract. In this paper we describe the almost simple groups G such that the prime graph $\Pi(G)$ is not connected. We construct the prime graph $\Pi(G)$ of a finite group G as follows: its vertices are the primes dividing the order of G and two vertices p, q are joined by an edge, and we write $p \sim q$, if there is an element in G of order pq.

Prime graph of almost simple groups

If G is a finite group, we define its prime graph, $\Gamma(G)$, as follows: its vertices are the primes dividing the order of G and two vertices p, q are joined by an edge, and we write $p \sim q$, if there is an element in G of order pq.
We denote the set of all the connected components of the graph $\Gamma(G)$ by $\{\Pi_i(G)$, for $i = 1, 2,..., t(G)\}$ and, if the order of G is even, we denote the component containing 2 by $\Pi_1(G) = \Pi_1$. We also denote by $\Pi(n)$ the set of all primes dividing n, if n is a natural number, and by $\Pi(G)$ the set of vertices of $\Gamma(G)$.
The concept of prime graph arose during the investigation of certain cohomological questions associated with integral representations of finite groups. It turned out that $\Gamma(G)$ is not connected if and only if the augmentation ideal of G is decomposable as a module. (see [**4**]). In addition, nonconnectedness of $\Gamma(G)$ has relations also with the existence of isolated subgroups of G. A proper subgroup H of G is *isolated* if $H \cap H^g = 1$ or H for every $g \in G$ and $C_G(h) \leq H$ for all $h \in H$. It was proved in [**10**] that G has a nilpotent isolated Hall π-subgroup whenever G is non-soluble and $\pi = \Pi_i(G)$, $i > 1$. We have in fact the following equivalences:

Theorem [**8**]
If G is a finite group, then the following are equivalent:
(1) the augmentation ideal of G decomposes as a module,
(2) the group G contains an isolated subgroup,
(3) the prime graph of G has more than one component.

It is therefore interesting to know when the prime graph of a group G is not connected, i. e. has more than one component. The first classification is a result of Gruenberg and Kegel.

Theorem A [**8**]
If G is a finite group whose prime graph has more than one component, then G has one of the following structures:
(a) Frobenius or 2-Frobenius;
(b) simple;
(c) an extension of a π_1*-group by a simple group;*
(d) simple by π_1*;*
(e) π_1 *by simple by* π_1*.*

The case of solvable groups has been completely determined by Gruenberg and Kegel:

Corollary [**8**]
If G is solvable with more than one prime graph component, then G is either Frobenius or 2-Frobenius and G has exactly two components, one of which consists of the primes dividing the lower Frobenius complement.

Also the case *(b)* of a simple group has been described by Williams in [**10**], by Kondratev in [**8**] and by Iiyori and Yamaki in [**7**]. A complete list of the simple groups with more than one component can also be found in [**9**].

In this paper we determine the case *(d)*. The case *(d)* is in fact the case of an almost simple group with more than one component. A group G is *almost simple* if there exists a finite simple non abelian group S such that $S \leq G \leq Aut(S)$. First we observe that if $\Gamma(G)$ is not connected then also $\Gamma(S)$ is not connected. Then,

using the Classification of Finite Simple Groups and the results of Williams and Kondratev, respectively in **[10]** and **[8]**, we consider the various cases.
The main results are then **Lemma 2** and **Theorem 3**, concerning sporadic and alternating groups, and **Theorem 5**, concerning finite simple groups of Lie type. In particular we observe that almost simple groups which are not simple have at most 4 components (see Table IV).

Remark 1
If $\Gamma(G)$ is not connected, and G has a non-nilpotent normal subgroup N, then $\Gamma(N)$ is not connected.

Proof.
We suppose that $\Gamma(N)$ is connected. As $\Gamma(G)$ is not connected, there must be p in $\Pi(G)$ such that $p \not\sim q$ for any q in $\Pi(N)$. Let P be a p-Sylow subgroup of G. If we consider $K = NP$, where P acts on N by conjugation, then K is a Frobenius group with kernel N, that must be nilpotent, against our hypothesis. $\square$

In order to describe almost simple groups with prime graph non-connected it is therefore enough to consider groups G such that $G \leq Aut(S)$ and S is a simple group with prime graph non-connected. A complete description of the simple groups with prime graph non-connected can be found in **[10]** and **[8]**. We suppose that $S < G$.
We use the Classification of Finite Simple Groups. Before beginning with a general study we want to treat a particular case.

Lemma 2
If $S = A_6$, there are four groups $G_1 = S_6, G_2, G_3, G_4 = Aut(A_6)$ such that $S < G_i \leq Aut(S)$. Then

$$\Gamma(G_1) = \overset{2}{\cdot}\!-\!\overset{3}{\cdot} \quad \overset{5}{\cdot}\ ; \qquad \Gamma(G_2) = \overset{2}{\cdot}\!-\!\overset{5}{\cdot} \quad \overset{3}{\cdot}\ ; \qquad \Gamma(G_3) = \overset{2}{\cdot} \quad \overset{3}{\cdot} \quad \overset{5}{\cdot}\ ;$$

$$\Gamma(G_4) = \overset{3}{\cdot}\!-\!\overset{2}{\cdot}\!-\!\overset{5}{\cdot}\ .$$

Proof. Since $Aut(S)/Inn(S)$ is isomorphic to the Klein group with 4 elements, there are three almost simple groups over S of order $2|S|$. Let G_1, G_2, G_3 be such groups. Then $G_1 \cong S_6$ and $2 \sim 3$ in $\Pi(S_6)$. Let θ be an outer automorphism of S_6 of order 2. Since $Aut(A_6) = Aut(S_6)$, we can consider G_2, the subgroup of $Aut(A_6)$ generated by $Inn(A_6)$ and θ. Then G_2 is a splitting extension of $Inn(A_6)$. Moreover θ centralizes an element of order 5 of A_6, and θ can not centralize any element of order 3 of A_6 because θ exchange the two conjugacy classes of elements of order 3 of A_6. Therefore $2 \sim 5$ in $\Pi(G_2)$.
Let G_3 be the other subgroup of $Aut(A_6)$ of order $2|A_6|$. Then G_3 is a non-split extension of A_6 and there is not any involution outside $Inn(A_6)$. Therefore its prime graph is the same as that of A_6. Let G_4 be $Aut(A_6)$, then $\Gamma(G_4)$ is connected. From these observations we can deduce the structure of the prime graph of the groups G_i, $i = 1, 2, 3, 4$. $\square$

Theorem 3
If $G = Aut(S)$ with S an alternating group or a sporadic group, $S \neq A_6$, then $\Gamma(G)$ is not connected if and only if G is one of the following groups and the connected

components are as follows:

$S = A_n,\ n = p, p+1, p$ prime	$\Pi_1(G) = \{2, 3, ..., q\}\ q < n-1$	$\Pi_2(G) = \{p\}$
$S = M_{12}$	$\Pi_1(G) = \{2, 3, 5\}$	$\Pi_2(G) = \{11\}$
$S = M_{22}$	$\Pi_1(G) = \{2, 3, 5, 7\}$	$\Pi_2(G) = \{11\}$
$S = J_3$	$\Pi_1(G) = \{2, 3, 5, 17\}$	$\Pi_2(G) = \{19\}$
$S = HS$	$\Pi_1(G) = \{2, 3, 5, 7\}$	$\Pi_2(G) = \{11\}$
$S = Sz$	$\Pi_1(G) = \{2, 3, 5, 7, 11\}$	$\Pi_2(G) = \{13\}$
$S = He$	$\Pi_1(G) = \{2, 3, 5, 7\}$	$\Pi_2(G) = \{17\}$
$S = O'N$	$\Pi_1(G) = \{2, 3, 5, 7, 11, 19\}$	$\Pi_2(G) = \{31\}$
$S = Fi_{22}$	$\Pi_1(G) = \{2, 3, 5, 7, 11\}$	$\Pi_2(G) = \{13\}$
$S = Fi'_{24}$	$\Pi_1(G) = \{2, 3, 5, 7, 11, 13, 17, 23\}$	$\Pi_2(G) = \{29\}$
$S = HN$	$\Pi_1(G) = \{2, 3, 5, 7, 11\}$	$\Pi_2(G) = \{19\}$

Proof. If $S = A_n$, the alternating group on n letters, $n \neq 6$, we know that $G = S_n$. From **[10]** we observe that $\Gamma(S)$ is not connected if and only if $n = p, (p+1), (p+2)$ for some prime p. If $n = p+2$ the element $(1, 2, ..., p)$ commutes with the element $(p+1, p+2)$ in S_n and then $p \sim 2$ in $\Pi(S_n)$. On the other hand if $n = p, p+1$ the centralizers of all the elements of order p are exactly the cyclic subgroups of order p that they generate.
If S is a sporadic group, then $|G/S| = 2$ and the result easily follows from the Atlas **[2]**. $\square$

We now suppose that $S = {}^dL_l(\bar{q})$ is a finite simple group of Lie type on the field with $\bar{q} = p^f$ elements. We recall that, in this case, the connected components $\Pi_i(S)$ for $i > 1$ of $\Gamma(S)$ are exactly sets of type $\Pi(|T|)$ for some maximal torus T such that T is isolated (see Lemma 5 of **[10]**).
We also observe that $\Pi_1(S)$ is obviously contained in $\Pi_1(G)$.
We use the notation and theorems of **[3]**. We consider $Inndiag(S)$: it is also a group of Lie type $\tilde{S}$, in which the maximal tori $\tilde{T}$ have order $|T|d$, if $T = \tilde{T} \cap S$ and $d = |Inndiag(S)/S|$. Then d divides $\bar{q}^2 - 1$ and, if $S \neq A_1(q), A_2(4)$, then $\Pi(\bar{q}^2 - 1) \subseteq \Pi_1(S)$.
As $\tilde{T}$ is abelian, it is therefore clear that if $t \in \Pi(|T|) = \Pi_i(S)$ for $i > 1$, we have $t \sim s$ for $s \in \Pi(\bar{q}^2 - 1)$.
If $S \neq A_1(q), A_2(4)$ and G contains an element of $Inndiag(S) \setminus S$, then by the above argument we can conclude that $\Gamma(G)$ is connected.

We can now consider G such that $Inndiag(S) \cap G = S$. Let α be in $G \setminus S$, then α does not belong to $Inndiag(S)$; we denote by Π_α the set of primes dividing $|C_S(\alpha)|$. We also recall from paragraph 9 of **[3]** that $p \in \Pi_\alpha$ for any $\alpha \in G \setminus S$ and therefore $\Pi_\alpha \subseteq \Pi_1(G)$ (if $S \neq A_1(q)$, $A_2(4)$). Therefore, if r is a prime dividing $|G/S|$, then r divides $|\alpha|$ for some $\alpha \in G \setminus S$ and we have $r \sim p$, and so $r \in \Pi_1(G)$.

Let $\Pi_i = \Pi(|T|)$ be a component of $\Gamma(S)$, for some isolated torus T. Then Π_i remains a component of $\Gamma(G)$ if and only if $(|T|, |C_S(\gamma)|) = 1$ for any $\gamma \in G \setminus S$. Then we only have to consider $\Pi(|T|)$ and check if $\Pi(|T|) \subseteq \Pi_1(G)$.
If $\alpha \in G \setminus S$ and $|\alpha| = r$ a prime, then, by theorem 9.1 of **[3]**, α is a field, a graph-field or a graph automorphism (in the sense of paragraph 7 of **[3]**); in

the same theorem $C_S(\alpha)$ is described for α a field or a graph-field automorphism. When α is a graph automorphism, $C_S(\alpha)$ is described in the paper **[1]**, if S is over a field of even characteristic, and in the paper **[5]**, if S is over a field of odd characteristic. If $(|T|, |C_S(\alpha)|) \neq 1$, then $r \sim t$ for some prime $t \in \Pi(|T|)$ and therefore $\Pi_i(S) = \Pi(|T|) \subseteq \Pi_1(G)$.
We now consider $\gamma \in G \setminus S$ and m a positive integer such that $\gamma^m = \alpha$ is an automorphism of order a prime r. If $(|T|, |C_S(\alpha)|) = 1$, then $C_S(\alpha) \cap T = 1$ and, as $C_S(\gamma) \leq C_S(\alpha)$, we also have $C_S(\gamma) \cap T = 1$. Thus γ does not centralize any element x in T. If for any $\alpha \in G \setminus S$ of order a prime r we have that $(|T|, |C_S(\alpha)|) = 1$, then, for any $\gamma \in G \setminus S$, we have $C_S(\gamma) \cap T = 1$. We conclude that, in this case, $\Pi_i(S) = \Pi(|T|)$ is a connected component $\Pi_j(G)$ for some $j > 1$.

We want to state now a number theoretical lemma:

Lemma 4
If $q = p^f$ and p and $l+1$ are prime numbers, then
i) $(q^m - 1, q^n - 1) = q^{(m,n)} - 1$;
ii) $((q^{l+1}-1)/(q-1), q-1) = (l+1, q-1)$ *and* $((q^{l+1}-1)/(q-1)(l+1, q-1), q-1) = 1$

Proof.
i) See Hilfsatz 2 a) of **[6]**.
ii) Let t be a prime dividing $q-1$, then

$$\frac{(q^{l+1}-1)}{(q-1)} = q^l + q^{l-1} + ... + q + 1 \equiv l + 1 \equiv 0 \quad (t) \quad \Longleftrightarrow \quad t = l + 1.$$

For the second statement we observe that, if $(l+1, q-1) = 1$, we can conclude by applying the first statement. Otherwise, as $l+1$ is a prime, $(l+1, q-1) = l+1$ and therefore if $q = 1 + (l+1)m$, for a positive integer m, we have

$$\frac{q^l + q^{l-1} + ... + q + 1}{l+1} = \frac{(1 + (l+1)m)^l + ... + (1 + (l+1)m) + 1}{l+1} =$$

$$\frac{(l+1) + (l+1)m(\sum_{j=1}^{l} \binom{j}{1}) + (l+1)^2 s}{(l+1)} = 1 + m(l+1)l/2 + (l+1)s \not\equiv 0 \quad (l+1),$$

where s is a positive integer. $\square$

Theorem 5
Let $S < G \leq Aut(S)$ with S a finite simple group of Lie type, then $\Gamma(G)$ is not connected if and only if G is one of the groups described in the Tables I, II, III, IV.

Proof.
The proof is made by a case by case analysis. For the connected components of $\Gamma(S)$, S a finite simple group of Lie type, we refer, without further reference, to **[10]** and **[8]**. From the remarks preceding Lemma 4, it is therefore enough to consider automorphisms α of S of order a prime r.

Type A_l

If $S = A_l(\bar{q})$ with $l > 1$, $S \neq A_2(2), A_2(4)$, then $\Gamma(S)$ is not connected if and only if
i) $l+1$ is a prime and in this case $\Pi_2(S) = \Pi(|T|) = \Pi((\bar{q}^l + \bar{q}^{l-1} + ... + 1)/d)$ where $d = (\bar{q} - 1, l + 1)$;
ii) l is an odd prime and $(\bar{q} - 1)|(l + 1)$ and in this case

$\Pi_2(S) = \Pi(|T|) = \Pi(\bar{q}^{l-1} + \bar{q}^{l-2} + ... + 1)$.
We study the different automorphisms of A_l.

i) If α is a field automorphism, then by theorem 9.1 of **[3]**, Π_α=$\Pi(|A_l(q)|)$, where $\bar{q} = q^r$.
If $r \neq l+1$, then r and $l+1$ are two distinct primes and therefore

$$\frac{(q^{l+1}-1)}{(q-1)(q-1,l+1)} \quad \text{divides} \quad \frac{(q^{r(l+1)}-1)}{(q^r-1)d} = \frac{(\bar{q}^l + \bar{q}^{l-1} + ... + \bar{q} + 1)}{d}.$$

This proves that $\Gamma(G)$ is connected.
If $r = l+1$, then $(|T|, |A_l(q)|) = 1$. In fact by lemma 4 i), we know that, for any $i \leq r$,

$$\left(\frac{(q^{rl} + q^{r(l-1)} + ... + 1)}{d} = \frac{(q^{(l+1)^2}-1)}{(q^{l+1}-1)d}, q^i - 1\right) = 1.$$

Since $(q^{(l+1)^2}-1)/(q^{l+1}-1)d = (\bar{q}^{l+1}-1)/(\bar{q}-1)(l+1,\bar{q}-1)$ by Lemma 4 ii), we have

$$\left(\frac{(\bar{q}^{l+1}-1)}{(\bar{q}-1)(l+1,\bar{q}-1)}, \bar{q}-1\right) = 1.$$

We can conlcude that $\Pi_2(S) = \Pi_2(G)$.

If α is a graph-field automorphism, then by theorem 9.1 of **[3]**, Π_α= $\Pi(|^2A_l(q)|)$, where $\bar{q} = q^r$ and $r = 2$. Then $(q^{l+1}+1)/(q+1)(q+1,l+1)$ divides both $|^2A_l(q)|$ and $|T| = (q^{2(l+1)}-1)/(q^2-1)(q^2-1,l+1)$ and so $\Gamma(G)$ is connected.

If α is a graph automorphism, then $r = 2$ and by theorems 19.9 of **[1]** and 4.27 of **[5]**, Π_α=$\Pi(|B_m(\bar{q})|) = \Pi(\bar{q}(\bar{q}^2-1)(\bar{q}^4-1)...(\bar{q}^{2m}-1))$ if $l+1 = 2m+1$.
We already know that $|T| = (\bar{q}^{l+1}-1)/(\bar{q}-1)(\bar{q}-1,l+1)$ is coprime with all the primes in Π_α because Π_α is contained in $\Pi_1(S)$. So in this case we have that $\Pi_2(S) = \Pi_2(G)$.

ii) The proof is similar to the one of *i)*. We obtain that $\Gamma(G)$ is connected, except in the cases in which α is a field automorphism and $r = l$, or α is a graph automorphism of order 2.

$S = A_2(2)$ admits only a graph automorphism α and Π_α=$\{2,3\}$ and then $\Pi_1(G) = \{2,3\}$, $\Pi_2(G) = \Pi(2^2+2+1) = \{7\}$.

$S = A_2(4)$: if α is a diagonal automorphism then $G \geq PGL(3,4)$ and in this case $\Gamma(G)$ is connected.
If α is a field automorphism, then $r = 2$ and Π_α=$\{2,3,7\}$ and so $\Pi_1(G) = \{2,3,7\}$, $\Pi_2(G) = \{5\}$.
If α is a graph-field or a graph automorphism, then $r = 2$ and Π_α=$\{2,3,5\}$ and so $\Pi_1(G) = \{2,3,5\}$, $\Pi_2(G) = \Pi((4^2+4+1)/3) = \{7\}$.

$S = A_1(\bar{q})$: if α is a diagonal automorphism of order 2, then $\bar{q}$ is odd. If $G = PGL(2,\bar{q})$ then $\Pi_1(G) = \Pi(G) \setminus \{p\}$, $\Pi_2(G) = \{p\}$, because 2 divides the order of every maximal torus T of G.
If α is a field automorphism, $\bar{q} = q^r$, Π_α=$\Pi(q(q^2-1))$. If $(q-1)/(2,q-1) \neq 1$ then

$$1 \neq \frac{q-1}{(2,q-1)} \quad \text{divides} \quad \frac{\bar{q}-1}{(2,\bar{q}-1)}.$$

If $(q-1)/(2,q-1)=1$, then $q=2$ or 3, and in this case $\Pi(|T_2|)=\Pi(\bar{q}-1)/(2,\bar{q}-1)=\Pi_2(G)$. Moreover if $r\neq 2$, then

$$1\neq \frac{q+1}{(2,q-1)} \quad \text{divides} \quad \frac{\bar{q}+1}{(2,\bar{q}-1)}.$$

Therefore if $q\neq 2,3$ and $r\neq 2$, $\Gamma(G)$ is connected, while if $r=2$ we have $\Pi_1(G)=\Pi(q(q^2-1))$ and $\Pi_2(G)=\Pi((q^2+1)/(2,q-1))$.
If α is a graph automorphism, then $r=2$ and Π_α={2}, so that $\Pi_i(G)=\Pi_i(S)$ for $i=1,2,3$.

Type B_l

If $S=B_l(\bar{q})$, then $\Gamma(S)$ is not connected if and only if
i) l is an odd prime and $\bar{q}=2,3$; in this case $\Pi_2(S)=\Pi(|T|)=\Pi(\bar{q}^{l-1}+\bar{q}^{l-2}...+1)$.
ii) $l=2^n$; in this case $\Pi_2(S)=\Pi(|T|)=\Pi((\bar{q}^l+1)/d)$ where $d=(\bar{q}-1,2)$;

i) In this case $Aut(S)=Inndiag(S)$ and so there is nothing else to prove.
ii) If α is a field automorphism, then Π_α=$\Pi(|B_l(q)|)$ and $\bar{q}=q^r$. If r is odd, $(q^l+1)/d$ divides both $|C_S(\alpha)|$ and $(\bar{q}^l+1)/d$ and then $\Gamma(G)$ is connected.
If $r=2$, $(\bar{q}^l+1)/d=(q^{2l}+1)/d$ is coprime with all the primes in Π_α, because Π_α is contained in $\Pi_1(S)$. So in this case we have that $\Pi_2(S)=\Pi_2(G)$.
If $l=2$ and $p=2$, then α can also be a graph automorphism of order 2. Then Π_α=$\Pi(|^2B_2(\bar{q})|)=\Pi(\bar{q}(\bar{q}^2-1)(\bar{q}^4+1))$ and then $\Gamma(G)$ is connected.
By proposition 19.5 of [**1**], we have thus described the centralizers of all $\alpha\in G\setminus S$.

Type D_l

If $S=D_l(\bar{q})$, then $\Gamma(S)$ is not connected if and only if
i) l is an odd prime and $\bar{q}=2,3,5$ and in this case
$\Pi_2(S)=\Pi(|T|)=\Pi((\bar{q}^l-1)/(4,\bar{q}^l-1))$;
ii) $l-1$ is an odd prime and $\bar{q}=2,3$ and in this case
$\Pi_2(S)=\Pi(|T|)=\Pi((\bar{q}^{l-1}-1)/(2,\bar{q}-1))$.

If $l\neq 4$, then the only automorphism α that we have to consider is a graph automorphism of order 2, then Π_α= $\Pi(|B_{l-1}(\bar{q})|)=\Pi(\bar{q}(\bar{q}^2-1)...(\bar{q}^{2(l-1)}-1))$. Therefore in case *i)* we have that $\Pi_2(S)=\Pi_2(G)$ and in case *ii)* $\Gamma(G)$ is connected.
If $l=4$, we have to consider also a graph automorphism of order 3. In this case, by Theorem 9.1(3) of [**3**], in $Aut(S)$ there are two conjugacy classes of subgroups of order 3 generated by a graph automorphism. We denote these two graph automorphisms by α and β. Then β is obtained from α by multiplying it with an element of order 3 of S, that is $\beta=g\alpha$, $g\in S$. Therefore, as $\Pi_\beta\subseteq$ Π_α=$\Pi(|G_2(\bar{q})|)=\Pi(\bar{q}(\bar{q}^6-1))$ and $\Pi_2(S)=\Pi(\bar{q}^3-1)$, we have that, in this case, $\Gamma(G)$ is connected.

Type E_6

If $S=E_6(\bar{q})$, then $\Gamma(S)$ is not connected and $\Pi_2(S)=\Pi(|T|)=\Pi((\bar{q}^6+\bar{q}^3+1)/d)$ where $d=(\bar{q}-1,3)$.

If α is a field automorphism, then $\bar{q}=q^r$ and Π_α=$\Pi(|E_6(q)|)$ ([**3**], Theorem 9.1). If $r\neq 3$, then

$$\frac{(q^6+q^3+1)}{d}=\frac{(q^9-1)}{d(q^3-1)} \quad \text{divides} \quad \frac{(q^{9r}-1)}{d(q^{3r}-1)}=\frac{(\bar{q}^9-1)}{d(\bar{q}^3-1)}=\frac{(\bar{q}^6+\bar{q}^3+1)}{d}.$$

Therefore $\Gamma(G)$ is connected.
If $r=3$, then Π_α=$\Pi(q(q^5-1)(q^8-1)(q^9-1)(q^{12}-1))$. As $(q^{18}+q^9+1)/d$ divides $(q^{27}-1)/(q^9-1)$, by Lemma 4 i) it is clear that $((q^{18}+q^9+1)/d,(q^5-1)(q^8-1))=1$. Moreover, if we apply Lemma 4 ii) to q^9, we have that $(q^{18}+q^9+1)/d$ is coprime with q^9-1. Finally $(q^{18}+q^9+1)/d=(\bar{q}^9-1)/(\bar{q}^3-1)d$, $\bar{q}^4-1=q^{12}-1$ and $((\bar{q}^9-1)/(\bar{q}^3-1)d,\bar{q}^4-1)=1$ again by Lemma 4 i). So in this case we have that $\Pi_2(S)=\Pi_2(G)$.

If α is a graph-field automorphism, then $r=2$, $\bar{q}=q^2$ and Π_α=$\Pi(|^2E_6(\bar{q})|)$ (**[3]**, Theorem 9.1). We observe that $|T|=(q^{12}+q^6+1)/d=(q^6+q^3+1)(q^6-q^3+1)/d$ and (q^6-q^3+1) divides $|^2E_6(q^2)|$. Then $\Gamma(G)$ is connected.

If α is a graph automorphism, by lemma 4.25 c) of **[5]** and 19.9 iii) of **[1]**, we have that $\Pi_\alpha\subseteq\Pi(\bar{q}(\bar{q}^8-1)(\bar{q}^{12}-1))$. Since $|T|$ is coprime with all the primes in Π_α, we have that $\Pi_2(S)=\Pi_2(G)$.

Type E_7

If $S=E_7(\bar{q})$, then $\Gamma(S)$ is not connected if and only if $\bar{q}=2,3$ and in this case S admits only a diagonal automorphism of order 2, and then there is nothing else to prove.

Type E_8

If $S=E_8(\bar{q})$, then $\Gamma(S)$ is not connected and
$\Pi_2(S)=\Pi(|T_0|)=\Pi(x(\bar{q})=\bar{q}^8-\bar{q}^4+1)$,
$\Pi_3(S)=\Pi(|T_1|)=\Pi(y_1(\bar{q})=\bar{q}^8+\bar{q}^7-\bar{q}^5-\bar{q}^4-\bar{q}^3+\bar{q}+1)$,
$\Pi_4(S)=\Pi(|T_2|)=\Pi(y_2(\bar{q})=\bar{q}^8-\bar{q}^7+\bar{q}^5-\bar{q}^4+\bar{q}^3-\bar{q}+1)$,
moreover, if $\bar{q}\equiv 0,1,4 \pmod{5}$,
$\Pi_5(S)=\Pi(|T_3|)=\Pi(z(\bar{q})=\bar{q}^8-\bar{q}^6+\bar{q}^4-\bar{q}^2+1)$
In this case α can only be a field automorphism (**[3]**, Theorem 9.1), $\bar{q}=q^r$ and

$$\Pi_\alpha=\Pi(|E_8(q)|)=\Pi(q(q^{14}-1)(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)).$$

We observe that

$$x(\bar{q})=\frac{(\bar{q}^{12}+1)}{(\bar{q}^4+1)};\quad y_1(\bar{q})=\frac{(\bar{q}^{10}-\bar{q}^5+1)}{(\bar{q}^2-\bar{q}+1)};\quad \text{and}$$

$$y_2(\bar{q})=\frac{(\bar{q}^{10}+\bar{q}^5+1)}{(\bar{q}^2+\bar{q}+1)};\quad z(\bar{q})=\frac{(\bar{q}^{10}+1)}{(\bar{q}^2+1)}.$$

If $r\neq 2,3,5$ we prove that $\Pi(G)$ is connected. In fact

$$r\neq 2,3\Longrightarrow\frac{(q^{12}+1)}{(q^4+1)}\ \text{divides}\ x(\bar{q});$$

$$r\neq 2,5\Longrightarrow\frac{(q^{10}+1)}{(q^2+1)}\ \text{divides}\ z(\bar{q});$$

$$r\neq 2,3\Longrightarrow\frac{(q^{15}+1)}{(q^5+1)}=q^{10}-q^5+1\ \text{divides}\ \bar{q}^{10}-\bar{q}^5+1.$$

Since $y_1(\bar{q}) = (\bar{q}^{10} - \bar{q}^5 + 1)/(\bar{q}^2 - \bar{q} + 1)$ we have to prove that $\bar{q}^2 - \bar{q} + 1 = q^{2r} - q^r + 1$ is coprime with $(q^{10} - q^5 + 1)/(q^2 - q + 1)$. In fact

$$r \neq 5 \Longrightarrow \left(\frac{(q^{3r}+1)}{(q^r+1)}, q^{15}+1 \right) = (q^{2r} - q^r + 1, q^3 + 1) = q^2 - q + 1 \Longrightarrow$$

$$\left(q^{2r} - q^r + 1, \frac{(q^{15}+1)}{q^2 - q + 1} \right) = (q^{2r} - q^r + 1, q + 1) = (3, q + 1).$$

Finally, since 3 does not divide $(q^{10} - q^5 + 1)/(q^2 - q + 1)$, we have proved the above statement and also that $(q^{10} - q^5 + 1)/(q^2 - q + 1)$ divides $y_1(\bar{q})$.
We can prove in a similar way that $(q^{10} + q^5 + 1)/(q^2 + q + 1)$ divides $y_2(\bar{q})$: in this case it is enough $r \neq 3, 5$. We can conclude that $\Gamma(G)$ is connected
We suppose now that $r = 2$. Then $y_2(\bar{q}) = (q^{20} + q^{10} + 1)/(q^4 + q^2 + 1)$ divides $q^{30} - 1$ and therefore $\Pi_4(S) \subseteq \Pi_1(G)$.
We want to prove that $x(\bar{q}) = (q^{24} + 1)/(q^8 + 1)$ is coprime with $|C_S(\alpha)|$. We observe that $(x(\bar{q}), q^{24} - 1) = 1$ and therefore

$$(x(\bar{q}), (q^{14} - 1)(q^{18} - 1)(q^{30} - 1)(q^{20} - 1)) = 1.$$

In a similar way we can prove that $z(\bar{q}) = (q^{20}+1)/(q^4+1)$ is coprime with $|C_S(\alpha)|$. Now we consider $y_1(\bar{q})$ which is a divisor of $(q^{30} + 1)/(q^{10} + 1)$. We observe that

$$(y_1(\bar{q}), (q^{14} - 1)(q^{18} - 1)(q^{30} - 1)(q^{20} - 1)) = 1.$$

Moreover $(y_1(\bar{q}), q^{24} - 1) = (y_1(\bar{q}), q^6 + 1)$ and therefore, since

$$y_1(\bar{q}) = (q^{30} + 1)/(q^{10} + 1)(q^4 - q^2 + 1) = (q^{30} + 1)/(q^6 + 1)s, \ s \text{ a positive integer,}$$

we have that $(y_1(\bar{q}), q^6 + 1) = 1$. We have thus proved that $y_1(\bar{q})$ is coprime with $|C_S(\alpha)|$. Therefore for $r = 2$, we have that $\Pi_2(G) = \Pi_2(S)$, $\Pi_3(G) = \Pi_3(S)$ and, if $\bar{q} \equiv 0, 1, 4 \quad (5)$, then $\Pi_4(G) = \Pi_5(S)$.

The proof for the cases $r = 3$ and 5 are similar to the previous one.

Type F_4

If $S = F_4(\bar{q})$, then $\Gamma(S)$ is not connected and
i) if $\bar{q}$ is odd, then $\Pi_2(S) = \Pi(|T|) = \Pi(\bar{q}^4 - \bar{q}^2 + 1)$
ii) if $\bar{q}$ is even, then $\Pi_2(S) = \Pi(|T|) = \Pi(\bar{q}^4 - \bar{q}^2 + 1))$ and $\Pi_3(S) = \Pi(|T_1|) = \Pi(\bar{q}^4 + 1)$.

i) α must be a field automorphism, $\bar{q} = q^r$ and Π_α=$\Pi(|F_4(q)|) = \Pi(q(q^8 - 1)(q^{12} - 1))$. We observe that $\bar{q}^4 - \bar{q}^2 + 1 = (q^{6r} + 1)/(q^{2r} + 1)$
If $r \neq 2, 3$, then $(q^6 + 1)/(q^2 + 1)$ divides $\bar{q}^4 - \bar{q}^2 + 1$. So in this case $\Gamma(G)$ is connected.
If $r = 2$ or 3, then $\Pi_\alpha \cap \Pi_2(S)$ is empty and therefore $\Pi_2(S) = \Pi_2(G)$.

ii) If α is a field automorphism, $\bar{q} = q^r$ and Π_α=$\Pi(|F_4(q)|) = \Pi(q(q^8 - 1)(q^{12} - 1))$ and if $r \neq 2$, then $q^4 + 1$ divides $q^{4r} + 1$ and so $\Pi_3(S) \subseteq \Pi_1(G)$.
For the component $\Pi_2(S)$, the proof is exactly the same as in part *i)*.
If $r = 2$, then $(\bar{q}^4 + 1) = q^8 + 1$ is coprime with all the primes in Π_α; so in this case $\Pi_3(S) = \Pi_3(G)$.

By proposition 19.5 of **[1]**, it is now enough to consider α a graph-field automorphism, $\bar{q} = q^2 = 2^m$ with m odd and $\Pi_\alpha = \Pi(|^2F_4(\bar{q})|) = \Pi(\bar{q}(\bar{q}^3+1)(\bar{q}^4-1)(\bar{q}^6+1))$. As $\bar{q}^4 - \bar{q}^2 + 1$ divides $\bar{q}^6 + 1$, we have that $\Pi_2(S) \subseteq \Pi_1(G)$, while $(\bar{q}^4 + 1) = q^8 + 1$ is coprime with all the primes in Π_α; so in this case $\Pi_3(S) = \Pi_2(G)$.

Type G_2

If $S = G_2(\bar{q})$, then $\Gamma(S)$ is not connected and
i) if $\bar{q} \equiv 1$ (3), then $\Pi_2(S) = \Pi(|T|) = \Pi(\bar{q}^2 - \bar{q} + 1)$;
ii) if $\bar{q} \equiv -1$ (3), then $\Pi_2(S) = \Pi(|T|) = \Pi(\bar{q}^2 + \bar{q} + 1)$;
iii) if $\bar{q} \equiv 0$ (3), then $\Pi_2(S) = \Pi(|T|) = \Pi(\bar{q}^2 - \bar{q} + 1)$ and $\Pi_3(S) = \Pi(|T|) = \Pi(\bar{q}^2 + \bar{q} + 1)$.

If α is a field automorphism, $\bar{q} = q^r$ and $\Pi_\alpha = \Pi(|G_2(q)|) = \Pi(q(q^6 - 1))$. If $r \neq 2, 3$ then $(q^3 + 1)/(q + 1)$ divides $(q^{3r} + 1)/(q^r + 1) = (\bar{q}^2 - \bar{q} + 1)$ and $(q^3 - 1)/(q - 1)$ divides $(q^{3r} - 1)/(q^r - 1) = (\bar{q}^2 + \bar{q} + 1)$ and so $\Gamma(G)$ is connected, in any of the three cases.
i) Let α be a field automorphism. If α has order $r = 2$, $\bar{q} = q^2$ and $\bar{q}^2 - \bar{q} + 1 = q^4 - q^2 + 1$ divides $q^6 + 1$ and is therefore coprime with $|G_2(q)|$. Similarly if α has order 3. So if $r = 2, 3$ we have that $\Pi_2(S) = \Pi_2(G)$.

ii) Let α be a field automorphism. If α has order $r = 2$, $\bar{q} = q^2$ and $\bar{q}^2 + \bar{q} + 1 = q^4 + q^2 + 1$. Since $q^2 + q + 1$ divides both $q^4 + q^2 + 1$ and $|G_2(q)|$, we have $\Pi_2(S) \subseteq \Pi_1(G)$. If α has order $r = 3$, we use the same argument of *i)* and conclude that $\Pi_2(S) = \Pi_2(G)$.

iii) Let α be a field automorphism. If α has order $r = 2$, as in case *i)*, we have that $\Pi_2(S) = \Pi_2(G)$, while, as in case *ii)*, $\Pi_3(S) \subseteq \Pi_1(G)$. If $r = 3$, we use the same argument of *i)* and conclude that $\Pi_2(S) = \Pi_2(G)$ and $\Pi_3(S) = \Pi_3(G)$.
If α is a graph-field automorphism of order $r = 2$, then $\bar{q} = q^2 = 3^n$, n an odd integer and $\Pi_\alpha = \Pi(|^2G_2(q^2)|) = \Pi(q(q^6 + 1)(q^2 - 1))$. As $\bar{q}^2 - \bar{q} + 1 = q^4 - q^2 + 1$ divides $q^6 + 1$, we have $\Pi_2(S) \subseteq \Pi_1(G)$, while $\Pi_3(S) = \Pi_2(G)$.
By lemma 4.22 of **[5]** and 19.2 of **[1]**, we have thus examined the centralizers of all automorphisms of S.

We now consider the twisted finite simple groups of Lie type. By the hypothesis that $G \cap Inndiag(S) = 1$, we obtain that, in this case, $G/S \cong <\gamma>$ and therefore we consider again an automorphism α of order a prime r. We suppose $S \neq {}^3D_4(\bar{q})$; if $r \neq 2$, α is a field automorphism, if $r = 2$ then α is a graph automorphism (in the sense of paragraph 7 of **[3]**). The same is true for $S = {}^3D_4(\bar{q})$, substituting the prime 3 to the prime 2.

Type ${}^2A_l$

If $S = {}^2A_l(\bar{q}^2)$ with $l > 1$, $S \neq {}^2A_3(2^2)$, ${}^2A_3(3^2)$, ${}^2A_5(2^2)$, then $\Gamma(S)$ is not connected if and only if
i) $l + 1$ is a prime and in this case
$\Pi_2(S) = \Pi(|T|) = \Pi(((-\bar{q})^l + (-\bar{q})^{l-1} + ... - \bar{q} + 1)/d)$ where $d = (\bar{q} + 1, l + 1)$;
ii) l is an odd prime and $(\bar{q} + 1)|(l + 1)$ and in this case
$\Pi_2(S) = \Pi(|T|) = \Pi((-\bar{q})^{l-1} + (-\bar{q})^{l-2} + ... - \bar{q} + 1)$.

i) If $r \neq 2$, then $\bar{q} = q^r$ and $\Pi_\alpha = \Pi(|^2A_l(q^2)|)$ (see Theorem 9.1 of **[3]**). If $r \neq l + 1$, then r and $l + 1$ are two distinct primes and therefore

$$\frac{(q^{l+1} + 1)}{(q + 1)(q + 1, l + 1)} \quad \text{divides} \quad \frac{(q^{r(l+1)} + 1)}{(q^r + 1)d} = \frac{(-\bar{q})^l + (-\bar{q})^{l-1} + ... - \bar{q} + 1)}{d}.$$

Therefore in this case $\Gamma(G)$ is connected.
If $r = l + 1$, the proof is similar to the one of A_l.

If $r = 2$, then by theorems 19.9 of **[1]** and 4.27 of **[5]** we have that Π_α=$\Pi(|B_m(\bar{q})|) = \Pi(\bar{q}(\bar{q}^2 - 1)(\bar{q}^4 - 1)...(\bar{q}^{2m} - 1))$,where $l + 1 = 2m + 1$.
We know that $|T| = (\bar{q}^{l+1} + 1)/(\bar{q} + 1)(\bar{q} + 1, l + 1)$ is coprime with all the primes in Π_α because Π_α is contained in $\Pi_1(S)$. So in this case $\Pi_2(S) = \Pi_2(G)$.

ii) The proof is similar to the one of *i)* and so $\Gamma(G)$ is connected, except when $r = l, 2$.

$S = {}^2A_3(2^2)$: it is enough to consider the automorphism of order 2 and so, as before, we have that $\Pi_2(S) = \Pi_2(G) = \{5\}$.
$S = {}^2A_3(3^2)$: it is enough to consider the automorphism of order 2 and so, as before, we have that $\Pi_2(S) = \Pi_2(G) = \{5\}$ and $\Pi_3(S) = \Pi_3(G) = \{7\}$.
$S = {}^2A_5(2^2)$: it is enough to consider the automorphism of order 2 and so, as before, we have that $\Pi_2(S) = \Pi_2(G) = \{7\}$ and $\Pi_3(S) = \Pi_3(G) = \{11\}$.

Type ${}^2B_2$

If $S = {}^2B_2(\bar{q}^2)$, then $\Gamma(S)$ is not connected and
$\Pi_2(S) = \Pi(|T|) = \Pi(\bar{q}^2 - 1)$,
$\Pi_3(S) = \Pi(|T_1|) = \Pi(\bar{q}^2 - \sqrt{2}\bar{q} + 1)$,
$\Pi_4(S) = \Pi(|T_2|) = \Pi(\bar{q}^2 + \sqrt{2}\bar{q} + 1)$.
We only have to consider the case in which r is an odd prime and $\bar{q}^2 = q^{2r} = 2^m$, m an odd integer. Then Π_α=$\Pi(|{}^2B_2(q^2)|) = \Pi(q(q^2 - 1)(q^4 - 1))$ (**[3]**, Theorem 9.1). As $q^2 - 1$ divides $q^{2r} - 1$, it is clear that $\Pi_2(S) \subseteq \Pi_1(G)$.
We observe that $q^4 + 1$ divides $q^{4r} + 1 = (\bar{q}^2 - \sqrt{2}\bar{q} + 1)(\bar{q}^2 + \sqrt{2}\bar{q} + 1)$ because r is odd.
It can be proved that, if $r \equiv 1, 7$ (8), $(q^2 - \sqrt{2}q + 1)$ divides $(\bar{q}^2 - \sqrt{2}\bar{q} + 1)$ and $(q^2 + \sqrt{2}q + 1)$ divides $(\bar{q}^2 + \sqrt{2}\bar{q} + 1)$;
or, if $r \equiv 3, 5$ (8), then $(q^2 - \sqrt{2}q + 1)$ divides $(\bar{q}^2 + \sqrt{2}\bar{q} + 1)$ and $(q^2 + \sqrt{2}q + 1)$ divides $(\bar{q}^2 - \sqrt{2}\bar{q} + 1)$.
So, in any case, we have that $\Gamma(G)$ is connected.

Type ${}^2D_l$

If $S = {}^2D_l(\bar{q}^2)$, then $\Gamma(S)$ is not connected if and only if
i) $l = 2^n$ and in this case $\Pi_2(S) = \Pi(|T|) = \Pi((\bar{q}^l + 1)/d)$ where $d = (\bar{q}^l + 1, 4)$;
ii) $\bar{q} = 2$ and $l = 2^n + 1$ and in this case $\Pi_2(S) = \Pi(|T|) = \Pi(2^{l-1} + 1)$;
iii) $\bar{q} = 3$ and
· $l = 2^n + 1$ and l is not a prime and in this case $\Pi_2(S) = \Pi(|T|) = \Pi((3^{l-1} + 1/2)$;
· $l \neq 2^n + 1$ and l is a prime and in this case $\Pi_2(S) = \Pi(|T_1|) = \Pi((3^l + 1/4)$;
· $l = 2^n + 1$ and l is a prime and in this case $\Pi_2(S) = \Pi(|T|) = \Pi((3^{l-1} + 1/2)$ and $\Pi_3(S) = \Pi(|T_1|) = \Pi((3^l + 1/4)$.

i) If $r \neq 2$, then $\bar{q} = q^r$, Π_α=$\Pi(|\,{}^2D_l(q^2)|)$ (**[3]**, Theorem 9.1) and $(q^l + 1)/d$ divides $(\bar{q}^l + 1)/d$; therefore $\Gamma(G)$ is connected.
If $r = 2$, then Π_α= $\Pi(|C_{l-1}(\bar{q})|) = \Pi(\bar{q}(\bar{q}^2 - 1)...(\bar{q}^{2(l-1)} - 1))$. Therefore, as Π_α is contained in $\Pi_1(G)$, we have that $\Pi_2(S) = \Pi_2(G)$.
ii) We only have to consider an automorphism of order $r = 2$.
Then Π_α= $\Pi(|B_{l-1}(\bar{q})|)$=$\Pi(\bar{q}(\bar{q}^2 - 1)...(\bar{q}^{2(l-1)} - 1))$ and, as $(2^{l-1} + 1)$ divides $|B_{l-1}(2)|$, we can conclude that $\Gamma(G)$ is connected.
iii) As in case *ii)*, we only have to consider the case $r = 2$.

Then $\Pi_\alpha = \Pi(|B_{l-1}(\bar{q})|) = \Pi(\bar{q}(\bar{q}^2-1)...(\bar{q}^{2(l-1)}-1))$ and $(3^{l-1}+1)$ divides $|B_{l-1}(3)|$, while $(3^l+1)/4$ is coprime with $|B_{l-1}(3)|$ by Lemma 4 i), when l is a prime. Therefore, when $l = 2^n + 1$, $\Pi(|T|) \subseteq \Pi_1(G)$ and, when l is a prime, $\Pi(|T_1|) = \Pi_2(G)$.

Type $^2E_6$

If $S = {}^2E_6(\bar{q}^2)$, then $\Gamma(S)$ is not connected and
i) if $\bar{q} = 2$ then $\Pi_2(S) = \Pi(|T|) = \Pi((2^6 - 2^3 + 1)/3) = \{19\}$,
$\Pi_3(S) = \Pi(|T_1|) = \{17\}$ and $\Pi_4(S) = \Pi(|T_2|) = \{13\}$.
ii) if $\bar{q} \neq 2$ then $\Pi_2(S) = \Pi(|T|) = \Pi((\bar{q}^6 - \bar{q}^3 + 1)/d)$ where $d = (\bar{q}+1, 3)$.

i) We only have to consider the case $r = 2$. Then, by 19.9 iii) of **[1]**, we have $\Pi_\alpha = \Pi(2(2^8-1)(2^{12}-1)) = \{2, 3, 5, 17, 13, 7\}$ and then $\Pi_3(S) \subseteq \Pi_1(G)$, $\Pi_4(S) \subseteq \Pi_1(G)$, while $\Pi_2(S) = \Pi_2(G)$.
ii) If $r \neq 2$, then $\bar{q} = q^r$ and $\Pi_\alpha = \Pi(|\,{}^2E_6(q^2)|)$ (see Theorem 9.1 of **[3]**). If $r \neq 3$, then $(q^6 - q^3 + 1)/d$ divides $(\bar{q}^6 - \bar{q}^3 + 1)/d$ and therefore $\Gamma(G)$ is connected.
If $r = 3$, then $\Pi_\alpha = \Pi(q(q^5+1)(q^8-1)(q^9+1)(q^{12}-1))$. It can be proved that $\Pi_\alpha \cap \Pi_2(S)$ is empty and therefore $\Pi_2(S) = \Pi_2(G)$.
If $r = 2$, then by lemma 4.25 c) of **[5]** and 19.9 iii) of **[1]**, we have $\Pi_\alpha \subseteq \Pi(\bar{q}(\bar{q}^8-1)(\bar{q}^{12}-1))$. As $(\bar{q}^6 - \bar{q}^3 + 1)/d = (\bar{q}^9+1)/(\bar{q}^3+1)d$ by Lemma 4 we can conclude that $|T|$ is coprime with all the primes in Π_α; so in this case we have that $\Pi_2(S) = \Pi_2(G)$.

Type $^2F_4$

If $S = {}^2F_4(2)'$, then $\Gamma(S)$ is not connected and $G = {}^2F_4(2)$ and $\Pi_2(G) = \{13\}$ (see **[2]**).
If $S = {}^2F_4(\bar{q}^2)$, then $\Gamma(S)$ is not connected and
$\Pi_2(S) = \Pi(|T_1|) = \Pi(\bar{q}^4 - \sqrt{2}\bar{q}^3 + \bar{q}^2 - \sqrt{2}\bar{q} + 1)$,
$\Pi_3(S) = \Pi(|T_2|) = \Pi(\bar{q}^4 + \sqrt{2}\bar{q}^3 + \bar{q}^2 + \sqrt{2}\bar{q} + 1)$.
We only have to consider the case in which r is an odd prime and $\bar{q}^2 = q^{2r} = 2^m$, m an odd integer. Then $\Pi_\alpha = \Pi(|{}^2F_4(q^2)|) = \Pi(q(q^8-1)(q^6+1)(q^{12}+1))$ (**[3]**, Theorem 9.1). We observe that

$$(\bar{q}^{12}+1)/(\bar{q}^4+1) = (\bar{q}^8-\bar{q}^4+1) = (\bar{q}^4-\sqrt{2}\bar{q}^3+\bar{q}^2-\sqrt{2}\bar{q}+1)(\bar{q}^4+\sqrt{2}\bar{q}^3+\bar{q}^2+\sqrt{2}\bar{q}+1).$$

If $r = 3$, $(\bar{q}^8 - \bar{q}^4 + 1) = q^{24} - q^{12} + 1 = (q^{36}+1)/(q^{12}+1)$ and it is therefore coprime with $(q^{36}-1)$. Moreover $(\bar{q}^8 - \bar{q}^4 + 1, q^8 - 1) = (\bar{q}^8 - \bar{q}^4 + 1, q^4 + 1)$ and (q^4+1) divides $(q^{12}+1)$; $(\bar{q}^8 - \bar{q}^4 + 1, q^{12}+1) = (3, q^{12}+1) = 1$. Therefore, in this case we have $\Pi_2(S) = \Pi_2(G)$ and $\Pi_3(S) = \Pi_3(G)$.
We can now suppose that $r \neq 3$. It can be proved that if $r \equiv 1, 7, 17, 23 \pmod{24}$, then

$$(q^4 - \sqrt{2}q^3 + q^2 - \sqrt{2}q + 1) \quad \text{divides} \quad (\bar{q}^4 - \sqrt{2}\bar{q}^3 + \bar{q}^2 - \sqrt{2}\bar{q} + 1) \quad \text{and}$$

$$(q^4 + \sqrt{2}q^3 + q^2 + \sqrt{2}q + 1) \quad \text{divides} \quad (\bar{q}^4 + \sqrt{2}\bar{q}^3 + \bar{q}^2 + \sqrt{2}\bar{q} + 1);$$

or, if $r \equiv 5, 11, 13, 19 \pmod{24}$, then

$$(q^4 - \sqrt{2}q^3 + q^2 - \sqrt{2}q + 1) \quad \text{divides} \quad (\bar{q}^4 + \sqrt{2}\bar{q}^3 + \bar{q}^2 + \sqrt{2}\bar{q} + 1) \quad \text{and}$$

$$(q^4 + \sqrt{2}q^3 + q^2 + \sqrt{2}q + 1) \quad \text{divides} \quad (\bar{q}^4 - \sqrt{2}\bar{q}^3 + \bar{q}^2 - \sqrt{2}\bar{q} + 1).$$

So, if $r \neq 3$, we have that $\Pi(G)$ is connected.

Type $^2G_2$

If $S = {}^2G_2(\bar{q}^2)$, then $\Gamma(S)$ is not connected and
$\Pi_2(S) = \Pi(|T_1|) = \Pi(\bar{q}^2 - \sqrt{3}\bar{q} + 1)$,
$\Pi_3(S) = \Pi(|T_2|) = \Pi(\bar{q}^2 + \sqrt{3}\bar{q} + 1)$.

We only have to consider the case in which r is an odd prime and $\bar{q}^2 = q^{2r} = 3^m$, m an odd integer. Then Π_α=$\Pi(|^2G_2(q^2)|) = \Pi(q(q^2-1)(q^6+1))$ ([**3**], Theorem 9.1). We observe that $(\bar{q}^2 - \sqrt{3}\bar{q}+1)(\bar{q}^2 + \sqrt{3}\bar{q}+1) = (\bar{q}^4 - \bar{q}^2 + 1) = (\bar{q}^6+1)/(\bar{q}^2+1)$. If $r = 3$, $(\bar{q}^4 - \bar{q}^2 + 1) = q^{12} - q^6 + 1 = (q^{18}+1)/(q^6+1)$ and it is therefore coprime with (q^2-1) and also with (q^6+1). Therefore, in this case we have $\Pi_2(S) = \Pi_2(G)$ and $\Pi_3(S) = \Pi_3(G)$.
If $r \neq 3$, the proof is similar to the one of $^2B_2$

Type $^3D_4$

If $S = {}^3D_4(\bar{q}^3)$, then $\Gamma(S)$ is not connected and $\Pi_2(S) = \Pi(\bar{q}^4 - \bar{q}^2 + 1)$.

If $r \neq 3$, then $\bar{q} = q^r$ and Π_α=$\Pi(|\ {}^3D_4(q)|) = \Pi(q(q^2-1)(q^8+q^4+1)$ ([**3**], Theorem 9.1). If $r \neq 2$, then $q^4 - q^2 + 1$ divides both $\bar{q}^4 - \bar{q}^2 + 1$ and $q^8 + q^4 + 1 = (q^4 - q^2 + 1)(q^4 + q^2 + 1)$, and then $\Gamma(G)$ is connected.
If $r = 2$, $\Pi_2(S) = \Pi(q^8 - q^4 + 1)$ and $(q^8 - q^4 + 1)$ is coprime with $(q^2-1)(q^8+q^4+1)$. So in this case we have that $\Pi_2(S) = \Pi_2(G)$.
If $r = 3$, then by Theorem 9.1(3) of [**3**], in $Aut(S)$ there are two conjugacy classes of subgroups generated by automorphisms of order 3. We denote these two automorphisms by α and β. Then β is obtained from α by multiplying it with an element of order 3 of S, that is $\beta = g\alpha$, $g \in S$. Therefore, as $\Pi_\beta \subseteq \Pi_\alpha$= $\Pi(|G_2(\bar{q})|) = \Pi(\bar{q}(\bar{q}^6 - 1))$ and $(\bar{q}^4 - \bar{q}^2 + 1)$ divides $\bar{q}^6 + 1$, we can conclude that $\Pi_2(S) = \Pi_2(G)$. □

We have thus examined all the almost simple groups.

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