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Hypergraph matching complexes and Quillen complexes of symmetric groups

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Abstract

We investigate Quillen complexes of symmetric groups using hypergraph matching complexes and p-cycle complexes. We determine the homotopy type of the Quillen complex of S_{3p} at the prime p. We show that there is torsion in the homology of the Quillen complex of S_{13} at the prime 3, thereby providing the first example of a Quillen complex of a finite group whose homology is not free.

Keywords: Quillen complex; Symmetric group; Hypergraph matching complex

1. Introduction

Let G be a finite group and let p be a prime dividing $|G|$. The *Quillen complex* of G at p is the order complex of the poset $\mathscr{A}_p(G)$ of nontrivial elementary abelian p-subgroups of G. (Recall that a p-subgroup of a group G is simply a subgroup whose order is a power of p, and that such a subgroup is elementary abelian if it is isomorphic to a direct product of copies of the cyclic group $\mathbb{Z}_p$. Recall also that for any poset P, the order complex $\Delta(P)$ is the abstract simplicial complex whose k-dimensional faces are the chains $x_0 < \cdots < x_k$ from P.) Interest in Quillen complexes of finite groups was ignited by the paper of Quillen [Qu]. (The study of p-group complexes for a large class of not necessarily finite groups was initiated in the papers [Br1,Br2] of Brown.)

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This paper is about Quillen complexes of symmetric groups at odd primes. It is an outgrowth of the work of Ksontini in [Ks1], see also [Ks2]. Among the results in Ksontini's work is one which says that $\Delta(\mathscr{A}_p(S_n))$ is simply connected if and only if $3p+2 \leqslant n < p^2$ or $n \geqslant p^2+p$. In addition, in [Ks1] the fundamental group $\pi_1(\Delta(\mathscr{A}_p(S_n)))$ is determined in all cases except those where $p \geqslant 5$ and $n \in \{3p, 3p+1\}$ and that where $p=3$ and $n=10$. (The study of simple connectivity of Quillen complexes was initiated in the paper of Aschbacher [As].) One of the results in this paper is the determination of the homotopy type, and therefore the fundamental group, of $\Delta(\mathscr{A}_p(S_{3p}))$ when $p \geqslant 5$, see Theorem 3.1.

Important in Ksontini's work and the work herein is the poset $\mathscr{T}_p(n)$, which is the subposet of $\mathscr{A}_p(S_n)$ consisting of all nontrivial elementary abelian subgroups of S_n which are generated by p-cycles (a p-cycle is simply an element of order p whose support has size p), along with the poset $\mathscr{D}_p(n)$ of all partitions of $[n]$ into subsets of size 1 and p (ordered by refinement). Note that if $P \in \mathscr{T}_p(n)$ then the partition of $[n]$ into the orbits of P lies in $\mathscr{D}_p(n)$. (The use of these posets in the study of $\Delta(\mathscr{A}_p(S_n))$ when $p=2$, in which case $\mathscr{T}_p(n) \cong \mathscr{D}_p(n)$, was initiated in the paper [Bo] of Bouc. As we shall see, the posets $\mathscr{D}_p(n)$ are face posets of simplicial complexes, called hypergraph matching complexes. The study of such complexes (with p not necessarily prime) was initiated in the paper [BLVZ] of Björner, Lovász, Vrécica and Živaljević, and the most recent results are surveyed in the paper [Wa] of Wachs.)

The value of studying $\mathscr{T}_p(n)$ comes from the fact that when n is small with respect to p, the complexes $\Delta(\mathscr{A}_p(S_n))$ and $\Delta(\mathscr{T}_p(n))$ have similar topology. In [Ks1,Ks2], Quillen's fiber lemma (see [Qu, Proposition 1.6]) is used to obtain the following result.

Lemma 1.1 (Ksontini [Ks1, Corollary 3.3, Ks2, Corollary 3.2]). *If $n < p^2$ then $\Delta(\mathscr{A}_p(S_n))$ and $\Delta(\mathscr{T}_p(n))$ are homotopy equivalent.*

Close examination of $\mathscr{A}_p(S_n)$ when $p^2 \leqslant n < p^2+p$ combined with the fiber lemma gives the next result.

Lemma 1.2 (Ksontini [Ks1, Proposition 8.8]). *Assume $p^2 \leqslant n < p^2+p$ and set*

$$w := \frac{n!}{p^2(p-1)^2(p+1)(n-p^2)!}.$$

Then

$$\Delta(\mathscr{A}_p(S_n)) \simeq \Delta(\mathscr{T}_p(n)) \vee \bigvee_w S^1.$$

Here $\simeq$ indicates homotopy equivalence, $\vee$ means wedge and

$$\bigvee_a S^b$$

is a wedge of a spheres of dimension b.

We will provide an extension of Lemmas 1.1 and 1.2, showing that if $p>2$ and $p^2+p\leqslant n<2p^2$, then $\Delta(\mathscr{A}_p(S_n))$ is homotopy equivalent to a wedge of $\Delta(\mathscr{T}_p(n))$ and another (explicitly described) complex Γ (see Theorem 4.13). If $n<p^2+2p$ then Γ is (homotopy equivalent to) a wedge of spheres of dimension two (an explicit formula for the number of spheres is provided, see Corollary 4.14).

In order to use the results just mentioned to obtain information about $\Delta(\mathscr{A}_p(S_n))$, it is necessary to understand the topology of $\mathscr{T}_p(n)$, and this is where the smaller (when $p>3$) complex $\mathscr{D}_p(n)$ enters the picture. It was shown in [Ks1] that information on the topology of $\mathscr{D}_p(n)$ can be used to obtain information on that of $\mathscr{T}_p(n)$. Here, using a recent result of Björner et al. in [BWW], we will make precise the relationship between $\mathscr{D}_p(n)$ and $\mathscr{T}_p(n)$. In the language of [BWW], $\Delta(\mathscr{T}_p(n))$ is (homotopy equivalent to) an inflation of (a complex homotopy equivalent to) $\Delta(\mathscr{D}_p(n))$, and we will use this fact to show that $\Delta(\mathscr{T}_p(n))$ is homotopy equivalent to a complex built from the complexes $\Delta(\mathscr{D}_p(m))$, where $m\leqslant n$ and $m\equiv n \bmod p$, using the wedge, join and suspension operations (see Theorem 2.2). In particular, the homology of $\Delta(\mathscr{D}_p(n))$ embeds in that of $\Delta(\mathscr{T}_p(n))$. In the cases $n<2p^2$ mentioned above, it follows from the results described that the homology of $\Delta(\mathscr{T}_p(n))$ embeds in that of $\Delta(\mathscr{A}_p(S_n))$.

Although the homology of the complexes $\Delta(\mathscr{D}_p(n))$ seems by no means simple to determine, some results are available (see [Bo,Ks1,Wa]). In addition, if p and n are small enough, the integral homology of $\Delta(\mathscr{D}_p(n))$ can be computed using software developed by Dumas, Heckenbach, Saunders and Welker. (This software works with the GAP package, and can be used online at http://www-lmc.imag.fr/SimpHom/.) Dumas has used this software to examine the complex $\Delta(\mathscr{D}_3(13))$, and the computations show that $\widetilde{H}_2(\Delta(\mathscr{D}_3(13)))$ contains an element whose order is a power of two [Du]. It follows that $\Delta(\mathscr{A}_3(S_{13}))$ does not have torsion free homology and therefore does not have the homotopy type of a wedge of spheres (see Corollary 4.17). This torsion in the homology of a Quillen complex seems to be the first such example, and it provides a negative answer to a question attributed to Thévenaz in the paper [PW] of Pulkus and Welker. There are various results exhibiting classes of groups G and primes p such that the homology of $\Delta(\mathscr{A}_p(G))$ is free (see for example [Qu, Sections 3,10–12]; [D1,D2]), and the contrasting conclusion of Corollary 4.17 makes it (in the author's opinion) the most interesting result in this paper. Consequently, it seems desirable to have a proof which does not rely on computer calculations. Moreover, such a proof might indicate how one could obtain similar results on torsion in Quillen complexes of various symmetric groups, using Corollary 4.15.

2. Comparing $\mathscr{T}_p(n)$ and $\mathscr{D}_p(n)$

We begin by examining the relationship between $\Delta(\mathscr{T}_p(n))$ and $\Delta(\mathscr{D}_p(n))$. As noted in [Ks1], each of $\mathscr{T}_p(n)$ and $\mathscr{D}_p(n)$ is a simplicial poset. That is, there exist simplicial complexes $\mathscr{C}\mathscr{T}_p(n)$ and $\mathscr{C}\mathscr{D}_p(n)$ whose face posets are $\mathscr{T}_p(n)$ and $\mathscr{D}_p(n)$, respectively. Indeed, the k-simplices of $\mathscr{C}\mathscr{T}_p(n)$ are sets

$$\{\langle\sigma_0\rangle, \dots, \langle\sigma_k\rangle\},$$

where the $\langle\sigma_i\rangle$ are groups generated by p-cycles from S_n whose supports are pairwise disjoint, and the k-simplices of $\mathscr{C}\mathscr{D}_p(n)$ are sets

$$\{X_0, \dots, X_k\}$$

of pairwise disjoint subsets of $[n]$ all of which have size p. (The complexes $\mathscr{C}\mathscr{D}_p(n)$ are the hypergraph matching complexes mentioned in the introduction.)

Now $\Delta(\mathscr{T}_p(n))$ and $\Delta(\mathscr{D}_p(n))$ are homeomorphic to the barycentric subdivisions of $\mathscr{C}\mathscr{T}_p(n)$ and $\mathscr{C}\mathscr{D}_p(n)$, respectively (see [Bj, p. 1844]), so we have

$$\Delta(\mathscr{T}_p(n)) \simeq \mathscr{C}\mathscr{T}_p(n)$$

and

$$\Delta(\mathscr{D}_p(n)) \simeq \mathscr{C}\mathscr{D}_p(n).$$

The key concept from [BWW] for our purposes is that of an *inflation* of a simplicial complex. Let Δ be a simplicial complex on vertex set $X = \{x_1, \dots, x_t\}$ and let $\mathbf{m} = (m_1, \dots, m_t)$ be a t-tuple of positive integers. The $\mathbf{m}$-*inflation* of Δ is the simplicial complex $\Delta_{\mathbf{m}}$ whose vertex set is

$$X_{\mathbf{m}} := \{(x_i, j) : i \in [t], j \in [m_i]\}$$

with

$$\{(x_{i_1}, j_1), \dots, (x_{i_r}, j_r)\} \subseteq X_{\mathbf{m}}$$

a face of $\Delta_{\mathbf{m}}$ if and only if

- $x_{i_k} \neq x_{i_l}$ for $1 \leqslant k < l \leqslant r$, and
- $\{x_{i_1}, \dots, x_{i_r}\}$ is a face of Δ.

Before stating the key result from [BWW] on inflations, we must introduce some more (standard) notation. The reader may consult [Bj] for definitions of any terms not defined here. For a face F of a simplicial complex Δ, $\mathrm{lk}_\Delta F$ is the link of F in Δ. If Δ and Γ are complexes then $\Delta * \Gamma$ is the join of Δ and Γ and for any positive integer m, $\Sigma^m(\Delta)$ is the m-fold suspension of Δ. We write $\Sigma(\Delta)$ for $\Sigma^1(\Delta)$.

Theorem 2.1 (Björner et al. [BWW, Theorem 6.2]). *Let Δ be a simplicial complex on vertex set $\{x_1, \dots, x_t\}$ and let $\mathbf{m} = (m_1, \dots, m_t) \in \mathbb{N}^t$. For a face $F \in \Delta$, set*

$$\nu(F, \mathbf{m}) := \prod_{x_i \in F} (m_i - 1).$$

If Δ is connected then

$$\Delta_{\mathbf{m}} \simeq \bigvee_{F \in \Delta} \left(\bigvee_{\nu(F,\mathbf{m})} \Sigma^{|F|}(lk_{\Delta} F) \right).$$

If Δ has connected components $\Delta^{(1)}, \dots, \Delta^{(r)}$ then

$$\Delta_{\mathbf{m}} \simeq \biguplus_{i=1}^{r} \bigvee_{F \in \Delta^{(i)}} \left(\bigvee_{\nu(F,\mathbf{m})} \Sigma^{|F|}(lk_{\Delta^{(i)}} F) \right).$$

Now if $A \subseteq [n]$ with $|A| = p$ then there are exactly $(p-1)!$ p-cycles in S_n with support A. Since each group generated by a p-cycle contains $p-1$ p-cycles, there are exactly $(p-2)!$ cyclic subgroups of S_n with order p whose support is A. It follows that if we set

$$\mathbf{m} := ((p-2)!, (p-2)!, \dots, (p-2)!) \in \mathbb{N}^{\binom{n}{p}}$$

then

$$\mathscr{CT}_p(n) \cong \mathscr{CD}_p(n)_{\mathbf{m}},$$

and we can apply Theorem 2.1. For a prime p and a positive integer k, set

$$\psi(p,k) := ((p-2)! - 1)^{k+1}.$$

If $F \in \mathscr{CD}_p(n)$ has dimension k (size $k+1$) then

$$\mathrm{lk}_{\mathscr{CD}_p(n)} F \cong \mathscr{CD}_p(n - p(k+1)),$$

and, in the language of Theorem 2.1,

$$\nu(F, \mathbf{m}) = \psi(p,k).$$

For $-1 \leqslant k \leqslant \lfloor \frac{n}{p} \rfloor - 1$, the complex $\mathscr{CD}_p(n)$ contains

$$\phi(n,p,k) := \frac{1}{(k+1)!} \prod_{i=0}^{k} \binom{n-ip}{p}$$

k-simplices (the empty set is the unique (-1)-simplex).

Applying Theorem 2.1, we get the following result.

Theorem 2.2. *If $\mathscr{CD}_p(n)$ is connected then*

$$\mathscr{CT}_p(n) \simeq \bigvee_{k=-1}^{\lfloor \frac{n}{p} \rfloor - 1} \left(\bigvee_{\phi(n,p,k)\psi(p,k)} \Sigma^{k+1}(\mathscr{CD}_p(n - p(k+1))) \right).$$

Before continuing, we make the following remarks:

(1) As noted in [Ks1, Proposition 4.3], if $\mathscr{CD}_p(n)$ is not connected then $p \leqslant n \leqslant 2p$. It is easy to determine the homotopy types of both $\mathscr{CT}_p(n)$ and $\mathscr{CD}_p(n)$ in this case (see [Ks1, Examples 3.7, 4.7] or [Ks2, Example 3] for $\mathscr{CT}_p(n)$).

(2) When $k = -1$ we have $\phi(n,p,k) = \psi(p,k) = 1$, $S^k = \{\emptyset\}$ and $\mathscr{CD}_p(n-p(k+1)) = \mathscr{CD}_p(n)$. Therefore one of the factors in the wedge described in the theorem is a copy of $\mathscr{CD}_p(n)$. It follows that (as shown in [Ks1, Proposition 4.5]) there is some graded module M such that

$$\widetilde{H}_*(\mathscr{CT}_p(n)) = \widetilde{H}_*(\mathscr{CD}_p(n)) \oplus M.$$

Also, there is some group G such that

$$\pi_1(\mathscr{CT}_p(n)) \cong \pi_1(\mathscr{CD}_p(n)) * G.$$

Moreover, the factors in the wedge given in the theorem which arise from all $k > 1$ are all simply connected, and if $n > 3p$ then so are the factors which arise from $k \in \{0, 1\}$. (This follows from [Bj, 9.20] and simple observations about $\mathscr{CD}_p(m)$ when m is small.) Therefore, if $n > 3p$ then $\pi_1(\mathscr{CT}_p(n)) \cong \pi_1(\mathscr{CD}_p(n))$.

(3) If $p \in \{2, 3\}$ then for each p-set $X \subseteq [n]$ there is a unique group generated by a p-cycle with support X. Therefore $\mathscr{CD}_p(n)$ and $\mathscr{CT}_p(n)$ are isomorphic. This makes the formula of the theorem appear somewhat disturbing until one notices that in this case $\psi(p,k) = 0$ for all $k > -1$, so all the factors in the wedge associated with any $k > -1$ are just the point at which the wedge is formed.

A more precise statement about homology than the one made in the second remark above is given in the next corollary, which follows easily from Theorem 2.2 and basic facts about the homology of joins and suspensions. Corollary 6.3 of [BWW] gives the general version of this result. For an abelian group A and a nonnegative integer t, we write tA for the t-fold direct sum of A (so $tA = 0$ if $t = 0$).

Corollary 2.3. *If $\mathscr{CD}_p(n)$ is connected then for each $l \geqslant 0$ we have*

$$\widetilde{H}_l(\mathscr{CT}_p(n)) = \bigoplus_{k=-1}^{\lfloor \frac{n}{p} \rfloor - 1} \phi(n,p,k)\psi(p,k)\widetilde{H}_{l-k-1}(\mathscr{CD}_p(n-p(k+1))).$$

3. The case $n = 3p$

Using Theorem 2.2, we can determine the homotopy type of $\mathscr{CT}_p(3p)$ once we know the homotopy type of $\mathscr{CD}_p(n)$ for $n \in \{0, p, 2p, 3p\}$. It is not hard to determine the required information, all of which appears in [Ks1, Example 4.7]. We summarize the numerical and topological information which we need in the table below. We write $\widetilde{\chi}$ for the reduced Euler characteristic $\widetilde{\chi}(\mathscr{CD}_p(3p))$ and set

$$\tau(p) := \frac{1}{2}\binom{2p}{p}.$$

k	$\phi(3p,p,k)$	Homotopy type of $\mathscr{CD}_p(p(3-k-1))$	$\psi(p,k)$
-1	1	$\bigvee_{-\widetilde{\chi}} S^1$	1
0	$\binom{3p}{p}$	$\bigvee_{\tau(p)-1} S^0$	$(p-2)!-1$
1	$\frac{1}{2}\binom{3p}{p}\binom{2p}{p}$	point	$((p-2)!-1)^2$
2	$\frac{1}{6}\binom{3p}{p}\binom{2p}{p}$	S^{-1}	$((p-2)!-1)^3$

Note that

$$\widetilde{\chi}=\sum_{k=-1}^{2}(-1)^k\phi(3p,p,k).$$

Applying Theorem 2.2 and Lemma 1.1, along with the facts that

- $\Sigma^m(\bigvee_a S^b)\simeq\bigvee_a S^{b+m}$, and
- $\Sigma^m(\text{point})\simeq\text{point}$,

we get the following result.

Theorem 3.1. *Let $p>3$ be prime and let $n=3p$. Then both $\mathscr{CT}_p(n)$ and $\Delta(\mathscr{A}_p(S_n))$ have the homotopy type of a wedge of spheres. More precisely, if we set*

$$a:=\frac{(3p)!}{6}\left(\frac{(p-2)!-1}{p!}\right)^3$$

and

$$b:=\binom{3p}{p}\left(\left[\frac{1}{2}\binom{2p}{p}-1\right][(p-2)!-1]+\frac{1}{3}\binom{2p}{p}-1\right)+1$$

then

$$\Delta(\mathscr{A}_p(S_n))\simeq\mathscr{CT}_p(n)\simeq\left(\bigvee_a S^2\right)\vee\left(\bigvee_b S^1\right).$$

In particular, $\pi_1(\Delta(\mathscr{A}_p(S_n)))$ is free of rank b.

Again some remarks are in order.

(1) In [Ks1, Corollary 8.3] it is shown that if $n=dp$ with $d\geqslant 3$ then $\widetilde{H}_{d-1}(\mathscr{A}_p(S_n))$ is free of rank

$$\frac{(dp)!}{d!}\left(\frac{(p-2)!-1}{p!}\right)^d,$$

which agrees with the number of 2-spheres in our wedge when $d=3$.

(2) The homotopy types of $\Delta(\mathscr{A}_2(S_6))$ and $\Delta(\mathscr{A}_3(S_9))$ are given in [Ks1, Example 2.12 and Proposition 15.3], see also [Ks2, Example 2] for $\Delta(\mathscr{A}_2(S_6))$.
(3) In order to determine the homotopy type of $\Delta(\mathscr{A}_p(S_{3p+1}))$ for $p \geqslant 3$ we must "only" determine the homotopy type of $\mathscr{CD}_p(3p+1)$, as it is easy to find the homotopy type of $\mathscr{CD}_p(n)$ for $n \in \{1, p+1, 2p+1\}$. In fact, one can easily compute the number a such that

$$\mathscr{CT}_p(3p+1) \simeq \mathscr{CD}_p(3p+1) \vee \bigvee_a S^2.$$

However, the determination of the homotopy type of $\mathscr{CD}_p(3p+1)$ seems likely to be difficult. Indeed, one of the most striking results on matching complexes is that of Bouc, (see [Bo, Proposition 3]), which says that $\pi_1(\mathscr{CD}_2(7))$ is cyclic of order three. By Lemmas 1.1 and 1.2 (and Theorem 2.2), if the integral homology of $\mathscr{CD}_p(3p+1)$ has torsion for some $p \geqslant 3$ then so does that of $\Delta(\mathscr{A}_p(S_{3p+1}))$. However, computer calculations show that the integral homology of $\mathscr{CD}_3(10)$ is free.

4. The case $p^2 + p \leqslant n < 2p^2$

Now we examine the case where $p^2 + p \leqslant n < 2p^2$. Our first step is to determine the maximal elements of $\mathscr{A}_p(S_n)$. Let n be arbitrary and let $P \in \mathscr{A}_p(S_n)$ have orbits $\Omega_1, \dots, \Omega_r$. For each $i \in [r]$ the action of P on Ω_i determines a homomorphism $\omega_i : P \to S_{\Omega_i}$. Set

$$P^* := \prod_{i=1}^{r} \omega_i(P).$$

Now $P \leqslant P^*$ and $P^* \in \mathscr{A}_p(S_n)$, so if P is maximal then $P = P^*$. Each $\omega_i(P)$ is a transitive abelian subgroup of S_{Ω_i}. Therefore (see for example [DM, Theorem 4.2A(ii,v)] and recall that a transitive permutation group is called regular if each nonidentity element is a derangement),

- $\omega_i(P)$ acts regularly on Ω_i, and
- the centralizer of $\omega_i(P)$ in S_{Ω_i} is $\omega_i(P)$.

Now $C_{S_n}(P^*)$ must preserve (setwise) each nontrivial orbit of P^*. It follows that if $\mathscr{F}$ is the set of fixed points of P^* then

$$C_{S_n}(P^*) = P^* \times S_{\mathscr{F}}.$$

If $|\mathscr{F}| < p$ then P^* is a Sylow p-subgroup of $C_{S_n}(P^*)$ and therefore maximal in $\mathscr{A}_p(S_n)$, while if $|\mathscr{F}| \geqslant p$ then there is some nontrivial $Q \leqslant S_{\mathscr{F}}$ such that $P^* \times Q \in \mathscr{A}_p(S_n)$. We now have the following result.

Lemma 4.1. *Let $P \in \mathscr{A}_p(S_n)$. Then P is maximal in $\mathscr{A}_p(S_n)$ if and only if $P = P^*$ and P fixes at most $p-1$ points from $[n]$. In particular, if $p^2 + p \leqslant n < 2p^2$ then P is maximal in $\mathscr{A}_p(S_n)$ if and only if one of the following conditions holds.*

(1) *P is a maximal element of $\mathscr{T}_p(n)$, or*
(2) *$P = X \times Y$, where $|X| = p^2$ and X acts regularly on its support, while $Y \in \mathscr{T}_p(n)$ has rank $\lfloor \frac{n-p^2}{p} \rfloor$ and support disjoint from that of X.*

Now assume that $p^2 + p \leqslant n < 2p^2$. Set

$$\mathscr{X}_n := \{X \in \mathscr{A}_p(S_n) : |X| = p^2, X \text{ acts regularly on } \mathrm{supp}(X)\}.$$

For $P \in \mathscr{A}_p(S_n)$, set

$$I_P := \{Q \in \mathscr{A}_p(S_n) : Q \leqslant P\}.$$

For $X \in \mathscr{X}_n$, set

$$\mathscr{M}(X) := \{P \in \mathscr{A}_p(S_n) : X < P \text{ and } P \text{ is maximal in } \mathscr{A}_p(S_n)\}$$

and

$$\mathscr{I}(X) := \bigcup_{P \in \mathscr{M}(X)} I_P.$$

Note that if $X \in \mathscr{X}_n$ then no element of $\mathscr{T}_p(n)$ contains X. Therefore, $\mathscr{M}(X)$ consists of all maximal elements of $\mathscr{A}_p(S_n)$ which contain X and satisfy condition (2) of Lemma 4.1. Now, continuing to borrow the notation from [Ks1,Ks2], we define (for arbitrary n)

$$\mathscr{T}\mathscr{A}_p(n) := \bigcup_{P \in \mathscr{T}_p(n)} I_P.$$

Note that $\mathscr{T}\mathscr{A}_p(n)$ consists of those $Q \in \mathscr{A}_p(S_n)$ such that all orbits of Q on $[n]$ have size 1 or p. The next lemma follows immediately from the definitions and the fact that both $\mathscr{T}\mathscr{A}_p(n)$ and $\bigcup_{X \in \mathscr{X}_n} \mathscr{I}(X)$ are ideals in $\mathscr{A}_p(S_n)$.

Lemma 4.2. *If $p^2 + p \leqslant n < 2p^2$ then*

$$\mathscr{A}_p(S_n) = \mathscr{T}\mathscr{A}_p(n) \cup \bigcup_{X \in \mathscr{X}_n} \mathscr{I}(X),$$

so

$$\Delta(\mathscr{A}_p(S_n)) = \Delta(\mathscr{T}\mathscr{A}_p(n)) \cup \bigcup_{X \in \mathscr{X}_n} \Delta(\mathscr{I}(X)).$$

We want to use the decomposition given in Lemma 4.2 to determine the homotopy type of $\Delta(\mathscr{A}_p(S_n))$. The next lemma, which is a slight generalization of [Bj, Lemma 10.4(ii)], is the key one. A proof appears in [Sh]. Recall (see for example [Bj, p. 1846]) that for a nonnegative integer k, a complex Δ is called *k-connected* if $\pi_i(\Delta)$ is trivial for all $i \leqslant k$. Also, a complex Δ is (-1)-connected if Δ contains a nonempty face.

Lemma 4.3. *Let*

$$\Delta = \bigcup_{i=0}^{r} \Delta_i$$

be a simplicial complex with each Δ_i a subcomplex. Assume there is some $k \geqslant 0$ such that

- *Δ_0 is k-connected,*
- *$\dim(\Delta_i \cap \Delta_0) \leqslant k$ if $1 \leqslant i \leqslant r$, and*
- *$\Delta_i \cap \Delta_j \subseteq \Delta_0$ if $1 \leqslant i < j \leqslant r$.*

Then

$$\Delta \simeq \Delta_0 \vee \bigvee_{i=1}^{r} \Delta_i/(\Delta_i \cap \Delta_0).$$

If Δ_i is contractible for $1 \leqslant i \leqslant r$ then

$$\Delta \simeq \Delta_0 \vee \bigvee_{i=1}^{r} \Sigma(\Delta_i \cap \Delta_0).$$

Our immediate goal is to apply Lemma 4.3 with

$$\Delta_0 = \Delta(\mathscr{T}\mathscr{A}_p(n))$$

and

$$\{\Delta_i : 1 \leqslant i \leqslant r\} = \{\Delta(\mathscr{I}(X)) : X \in \mathscr{X}_n\}.$$

The following result is Lemma 3.2 of [Ks1] and Lemma 3.1 of [Ks2]. One can prove it using Quillen's fiber lemma after noting that $\mathscr{T}_p(n)$ consists of all those groups which can be obtained by intersecting the elements of some collection of maximal elements of $\mathscr{T}\mathscr{A}_p(n)$.

Lemma 4.4. *For all primes p and all n we have*

$$\Delta(\mathscr{T}\mathscr{A}_p(n)) \simeq \Delta(\mathscr{T}_p(n)).$$

For positive integers n, l define

$$\mu(n,l) := \left\lfloor \frac{n-l}{l+1} \right\rfloor.$$

Lemma 4.5. *For all positive integers n and all primes p, the complex* $\Delta(\mathcal{TA}_p(n))$ *is* $(\mu(n,p)-1)$*-connected.*

Proof. Since $\Delta(\mathcal{TA}_p(n)) \simeq \Delta(\mathcal{T}_p(n)) \simeq \mathcal{CT}_p(n)$, it suffices to show that $\mathcal{CT}_p(n)$ is $(\mu(n,p)-1)$-connected. Athanasiadis has shown (see [At]) that the $\mu(m,p)$-skeleton of $\mathcal{CD}_p(m)$ is vertex decomposable and therefore shellable for all positive integers m, p. It follows that $\mathcal{CD}_p(m)$ is $(\mu(m,p)-1)$-connected for all m, p. Note that since $p < p+1$, we have

$$\mu(m-p,p) \geqslant \mu(m,p) - 1$$

for all m. Also (see [Bj, 9.20]), if Γ is k-connected then $\Sigma^m(\Gamma)$ is $(k+m)$-connected. Certainly a wedge of k-connected spaces is k-connected. Given these facts, the lemma follows from Theorem 2.2. $\square$

Lemma 4.6. *Assume* $p^2 + p \leqslant n < 2p^2$. *Let* $X \in \mathcal{X}_n$. *Then*

$$\dim(\Delta(\mathcal{I}(X)) \cap \Delta(\mathcal{TA}_p(n))) = \left\lfloor \frac{n-p^2}{p} \right\rfloor.$$

Proof. Let $P = X \times Y$ be a maximal element of $\mathcal{I}(X)$, and let $Q \leqslant P$. Let $\pi : P \to X$ be the standard projection. Every proper subgroup of X has order p and therefore lies in $\mathcal{TA}_p(n)$. So $Q \in \mathcal{TA}_p(n)$ if and only if $\pi(Q) \neq X$. Since X has rank two and Y has rank $\lfloor \frac{n-p^2}{p} \rfloor$, we see that every maximal element of $\mathcal{I}(X) \cap \mathcal{TA}_p(n)$ has rank $1 + \lfloor \frac{n-p^2}{p} \rfloor$ and the lemma follows. $\square$

Lemma 4.7. *If* $p \geqslant 3$ *is prime and* $n < 2p^2$ *then*

$$\mu(n,p) - 1 \geqslant \left\lfloor \frac{n-p^2}{p} \right\rfloor.$$

Proof. When $p = 3$ the lemma can be proved by inspection. If $p \geqslant 5$ we have $\frac{n-p}{p+1} - 1 \geqslant \frac{n-p^2}{p}$ when $n < 2p^2$. $\square$

Lemma 4.8. *If* $p^2 + p \leqslant n < 2p^2$ *and* V, X *are distinct elements of* $\mathcal{X}_n$ *then*

$$\Delta(\mathcal{I}(V)) \cap \Delta(\mathcal{I}(X)) \subseteq \Delta(\mathcal{TA}_p(n)).$$

Proof. It suffices to show that if $P = V \times W$ and $R = X \times Y$ are maximal elements of $\mathcal{I}(V)$ and $\mathcal{I}(X)$, respectively, then $T := P \cap R \in \mathcal{TA}_p(n)$. Let π, ρ, τ be the elements of the partition lattice Π_n determined by the orbits of P, R, T, respectively (so each of these partitions lies in $\mathcal{D}_p(n)$). Let π_1, ρ_1 be the unique parts of size p^2

in π, ρ, respectively. Note that

$$\tau \leqslant_{\Pi_n} \pi \wedge \rho.$$

Since every part of τ has size $1, p$ or p^2, we see that if $T \notin \mathcal{TA}_p(n)$ then $\pi_1 = \rho_1$ is the unique orbit of T with size greater than p. If $\pi_1 = \rho_1$ then both V, X act regularly on π_1, which means that for distinct $i, j \in \pi_1$ there is a unique element of V which sends i to j, and the same holds for X. Since $V \neq X$ there exist some $i, j \in \pi_1$ such that no element of $V \cap X$ maps i to j. Since W, Y act trivially on π_1 there is no element of T which maps i to j, so π_1 is not an orbit of T and T has no orbit of size p^2. □

Lemma 4.9. *If $p^2 + p \leqslant n < 2p^2$ and $X \in \mathcal{X}_n$ then $\Delta(\mathcal{I}(X))$ is contractible.*

Proof. Let $\mathcal{N}$ be the nerve of the covering of $\Delta(\mathcal{I}(X))$ by the set

$$\{\Delta(I_P) : P \in \mathcal{M}(X)\}$$

(see [Bj, pp. 1849–1850] for a discussion of nerves). Note that for $P, Q \in \mathcal{A}_p(S_n)$ we have

$$\Delta(I_P) \cap \Delta(I_Q) = \begin{cases} \Delta(I_{P \cap Q}) & P \cap Q \neq 1, \\ \emptyset, & P \cap Q = 1. \end{cases}$$

For each $P \in \mathcal{A}_p(S_n)$, $\Delta(I_P)$ is a cone (with apex P), and therefore contractible. So, we have (see [Bj, Theorem 10.6])

$$\Delta(\mathcal{I}(X)) \simeq \mathcal{N}.$$

Since each $P \in \mathcal{M}(X)$ contains X, we see that $\mathcal{N}$ is a simplex and therefore contractible. □

Collecting our results from Lemmas 4.2 through 4.9, we have the following result.

Corollary 4.10. *If $p \geqslant 3$ and $p^2 + p \leqslant n < 2p^2$ then*

$$\Delta(\mathcal{A}_p(S_n)) \simeq \Delta(\mathcal{TA}_p(n)) \vee \bigvee_{X \in \mathcal{X}_n} \Sigma(\Delta(\mathcal{I}(X) \cap \mathcal{TA}_p(n))).$$

As noted above, we have

$$\Delta(\mathcal{TA}_p(n)) \simeq \mathcal{CT}_p(n).$$

Finally, we obtain further information about the remaining factors in the wedge of Corollary 4.10.

Lemma 4.11. *If $p^2 + p \leqslant n < 2p^2$ and $X \in \mathcal{X}_n$ then*

$$\Delta(\mathcal{I}(X) \cap \mathcal{TA}_p(n)) \simeq \mathcal{CT}_p(n - p^2) * \bigvee_p S^0.$$

Proof. We may assume that the unique nontrivial orbit of X is $[n]\setminus[n-p^2]$. For $P \in \mathscr{I}(X) \cap \mathscr{T}\mathscr{A}_p(n)$ there are homomorphisms $\pi_1 : P \to X$ and $\pi_2 : P \to S_{[n-p^2]}$ determined by the action of P. Let P_1, P_2 be the images of these homomorphisms, respectively. Note that P_1 is a proper subgroup of X and $P_2 \in \mathscr{T}\mathscr{A}_p(n-p^2)$. Define

$$P^+ = P_1 \times (P_2)^*.$$

Then $P \leqslant P^+$ and $P^+ \in \mathscr{I}(X) \cap \mathscr{T}\mathscr{A}_p(n)$. Also, if $Q \leqslant P$ then $Q^+ \leqslant P^+$. It follows (see [Qu, Section 1] or [Bj, Corollary 10.12]) that if we set

$$\mathscr{P}^+ := \{P^+ : P \in \mathscr{I}(X) \cap \mathscr{T}\mathscr{A}_p(n)\}$$

then

$$\Delta(\mathscr{I}(X) \cap \mathscr{T}\mathscr{A}_p(n)) \simeq \Delta(\mathscr{P}^+).$$

Let Γ be a discrete complex whose points are the nontrivial proper subgroups of X, and set

$$\Lambda = \Gamma * \mathscr{C}\mathscr{T}_p(n-p^2).$$

Let $\mathscr{Q}$ be the poset of nonempty faces of Λ, and let $f : \mathscr{Q} \to \mathscr{P}^+$ map each face of Λ to the group generated by its elements. Then f is an isomorphism. Since $\Delta(\mathscr{Q})$ is the barycentric subdivision of Λ, we have

$$\Delta(\mathscr{P}^+) \simeq \Lambda.$$

Finally, since X has rank two, there are $p+1$ nontrivial proper subgroups of X, so

$$\Gamma = \bigvee_p S^0. \qquad \square$$

Lemma 4.12. *If* $p^2 + p \leqslant n < 2p^2$ *then*

$$|\mathscr{X}_n| = \frac{n!}{(n-p^2)!p^3(p-1)(p^2-1)}.$$

Proof. An element of $\mathscr{X}_n$ is uniquely determined by first choosing a subset Ω of size p^2 from $[n]$ and then choosing a regular elementary abelian subgroup of S_Ω. A regular elementary abelian subgroup of S_Ω is determined by choosing a bijection $f : Z_p \times Z_p \to \Omega$. Since S_Ω acts transitively on the set of all such bijections, all regular elementary abelian subgroups of S_Ω are conjugate in S_Ω. Let X be one such subgroup. Since X is transitive and abelian, we have $C_{S_\Omega}(X) = X$ and $N_{S_\Omega}(X)$ is isomorphic to the holomorph of X, that is,

$$N_{S_\Omega}(X) \cong AGL_2(p)$$

(see for example [DM, Section 4.7]). Thus there are

$$\frac{(p^2)!}{p^2|GL_2(p)|} = \frac{(p^2)!}{p^3(p-1)(p^2-1)}$$

conjugates of X in S_Ω and the lemma follows. $\square$

We can now state our main result.

Theorem 4.13. *Let $p \geqslant 3$ be prime and assume $p^2 + p \leqslant n < 2p^2$. Set*

$$u := \frac{n!}{(n-p^2)!p^3(p-1)(p^2-1)}.$$

Then

$$\Delta(\mathscr{A}_p(S_n)) \simeq \mathscr{CT}_p(n) \vee \bigvee_u \left(\mathscr{CT}_p(n-p^2) * \bigvee_p S^1 \right).$$

Proof. This follows from Corollary 4.10 and Lemmas 4.11 and 4.12, along with the facts that

- $\Sigma(\Gamma) = \Gamma * S^0$ for any complex Γ,
- $S^0 * \bigvee_a S^0 \simeq \bigvee_a S^1$ for any number a, and
- the join operation is commutative and associative. $\square$

When $a := n - p^2$ is small enough so that we know the homotopy type of $\mathscr{CT}_p(a)$, we can make the formula in Theorem 4.13 more specific. For example, we have the following result.

Corollary 4.14. *Let $p \geqslant 3$ be prime and let $n = p^2 + a$ with $p \leqslant a < 2p$. Let u be as defined in Theorem* 4.13 *and set*

$$v = p\left[\binom{a}{p}(p-2)! - 1\right].$$

Then

$$\Delta(\mathscr{A}_p(S_n)) \simeq \mathscr{CT}_p(n) \vee \bigvee_{uv} S^2.$$

Proof. In this case $\mathscr{CT}_p(n-p^2)$ is a discrete set of size $\frac{v}{p} + 1$, that is,

$$\mathscr{CT}_p(n-p^2) \simeq \bigvee_{v/p} S^0.$$

The corollary now follows from the fact (see [BW, Lemma 2.5(ii)]) that for any a, b, c, d we have

$$\bigvee_a S^c * \bigvee_b S^d \simeq \bigvee_{ab} S^{c+d+1}. \qquad \square$$

The next corollary is more interesting.

Corollary 4.15. *Let $p \geqslant 3$ be prime and assume $p^2 + p \leqslant n < 2p^2$.*

(1) *If $\widetilde{H}_i(\mathscr{CD}_p(n))$ is not free then neither is $\widetilde{H}_i(\Delta(\mathscr{A}_p(S_n)))$.*
(2) *If $p \geqslant 5$ and there is some $m \leqslant n$ such that $m \equiv n \bmod p$ and $\widetilde{H}_i(\mathscr{CD}_p(m))$ is not free then neither is $\widetilde{H}_{i+\frac{n-m}{p}}(\Delta(\mathscr{A}_p(S_n)))$.*

Proof. By Theorem 4.13, $\widetilde{H}_*(\mathscr{CT}_p(n))$ is a direct summand in $\widetilde{H}_*(\mathscr{A}_p(S_n))$. The corollary now follows from Corollary 2.3. $\square$

According to computer calculations performed by Dumas [Du], we have the following result

Proposition 4.16. *$\widetilde{H}_2(\mathscr{CD}_3(13))$ contains an nonidentity element whose order is a power of two.*

If we accept this proposition then we have the following result.

Corollary 4.17. *$\widetilde{H}_2(\Delta(\mathscr{A}_3(S_{13})))$ contains an nonidentity element whose order is a power of two.*

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