Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol. 22, No. 1, 63-79

On a Construction Related to the Non-abelian Tensor Square of a Group

N. R. Rocco

Abstract. Let G and G^{φ} be isomorphic groups. We introduce and study a quotient $\mathcal{V}(G)$ of the free product $G * G^{\varphi}$ which is a group extention of the non-abelian tensor square $G \otimes G$. This seems to bring computational advantages to calculate this last group. Looking over $\mathcal{V}$ as an operator in the class of groups we prove that it preserves properties of the argument G such as finiteness, set of prime divisors, nilpotency and solvability. For a finite p-group G we find a good polynomial bound for the order of $\mathcal{V}(G)$.

1. Introduction

The non-abelian tensor product $G \otimes H$ of the groups G and H, as introduced by R. Brown and J.-L. Loday [2], generalises the usual tensor product $\frac{G}{G'} \otimes_{\mathbb{Z}} \frac{H}{H'}$ of the abelianized groups, on the assumption that each of G and H acts on the other.

Specifically, given groups G, H each of which acts on the other (on the right)

$$G \times H \to G, (g,h) \mapsto g^h; H \times G \to H, (h,g) \mapsto h^g$$

in such a way that for all $g, g_1 \in G$ and $h, h_1 \in H$,

$$(1) \qquad g^{h^{g_1}} = g^{g_1^{-1} h g_1} \quad \text{and} \quad h^{g^{h_1}} = h^{h_1^{-1} g h_1}$$

where G and H acts on itself by conjugation, then the *non-abelian tensor product* $G \otimes H$ is defined to be the group generated by all symbols $g \otimes h, g \in G, h \in H$, subject to the relations

$$(2) \qquad gg_1 \otimes h = (g^{g_1} \otimes h^{g_1})(g_1 \otimes h)$$

$$(3) \qquad g \otimes hh_1 = (g \otimes h_1)(g^{h_1} \otimes h^{h_1})$$

Received 15 February 1991.

for all $g, g_1 \in G, h, h_1 \in H$, where the action of G on itself is the conjugation $g^{g_1} = g_1^{-1} g g_1$, and similarly for H.

In particular, as the conjugation action of a group G on itself satisfies (1), the *tensor square* $G \otimes G$ of a group G may always be defined. This tensor square is the focus of attention of [1] and [3], and constructions related to the general non-abelian tensor product are focused in [4].

The purpose of this article is to study a group which is also related to the above construction, defined as follows:

Let G and G^φ be isomorphic groups through $\varphi, g \mapsto g^\varphi, \forall g \in G$. We define the group

$$\mathcal{V}(G) := \langle G, G^\varphi \mid \ [g_1, g_2^\varphi]^{g_3} = [g_1^{g_3}, (g_2^{g_3})^\varphi] = [g_1, g_2^\varphi]^{g_3^\varphi}, \quad \forall g_1, g_2, g_3 \in G \rangle$$

(here we keep in mind that for elements h, k of any group, $h^k = k^{-1} h k$ and $[h, k] = h^{-1} h^k$).

Our motivation to introduce $\mathcal{V}(G)$ is that its *subgroup* $[G, G^\varphi]$ *is actually isomorphic to the non-abelian tensor square* $G \otimes G$ (Proposition 2.6).

Another construction related to $\mathcal{V}(G)$ is the one introduced by S. Sidki [10],

$$\chi(G) = \langle G, G^\varphi \mid \ [g, g^\varphi] = 1, \quad \text{for all} \quad g \in G \rangle,$$

which has, among other attributes, the property of being a finite group when G is finite. Considering the subgroup $\Delta(G)$ of $\mathcal{V}(G)$, generated by all $[g, g^\varphi], g \in G$, we obtain $\Delta(G) \leq \mathcal{V}(G)' \cap Z(\mathcal{V}(G))$. *The finiteness of* $\mathcal{V}(G)$ *then follows from the fact that* $\frac{\mathcal{V}(G)}{\Delta(G)}$ *is isomorphic to a certain natural factor of* $\chi(G)$ (Proposition 2.4).

By using techniques similar to those used in [5] and [9] we describe the lower central series and the derived series of $\mathcal{V}(G)$ in terms of the corresponding series of G. Our main results are the following:

Theorem A. *Let G be a nilpotent group of class c (resp. a solvable group of derived length ℓ). Then $\mathcal{V}(G)$ is a nilpotent group of class at most $c+1$ (resp. a solvable group of derived length at most $\ell+1$).*

Theorem B. *Let G be a finite p-group of order p^n with G' of order p^m. Then $\mathcal{V}(G)$ is a p-group of order dividing $p^{n^2+2n-mn}$.*

In particular we obtain bounds for $G \otimes G$ similar to those of Jones [6] for the Schur Multiplier:

$$p^{d^2} \leq |G \otimes G| \leq p^{n(n-m)}$$

where $d = d(G)$ denotes the minimal number of generators of G.

2. Basic Results

In this section we derive some properties of the group $\mathcal{V}(G)$ and identify $G \otimes G$ as a subgroup of it. We use some standard commutator identities without reference (see, for instance, D. Robinson [8]):

For elements x, y, z in a group G, the conjugate of x by y is $x^y := y^{-1}xy$; the commutator of x and y is $[x, y] := x^{-1}x^y$ and our commutators are left normed, $[x, y, z] = [[x, y], z]$. The following identities hold:

$$\begin{aligned}
&[x, y] = [x, y^{-1}]^{-y} = [x^{-1}, y]^{-x};\\
&[xy, z] = [x, z]^y[y, z] = [x, z][x, z, y][y, z];\\
&[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z];\\
&[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1.
\end{aligned}$$

We simplify the definition of $\mathcal{V}(G)$ as

$$\mathcal{V}(G) = \langle G, G^{\varphi} \mid \ [g, h^{\varphi}]^{k^{\epsilon}} = [g^k, (h^k)^{\varphi}], \quad \text{for all} \quad g, h, k \in G, \epsilon \in \{1, \varphi\}\rangle,$$

where $\varphi: G \to G^{\varphi}, g \mapsto g^{\varphi}$ is a group isomorphism.

2.1 Lemma. *The following relations hold in $\mathcal{V}(G)$:*

(i) $[g_1, g_2^{\varphi}]^{[g_3, g_4^{\varphi}]} = [g_1, g_2^{\varphi}]^{[g_3, g_4]}, \quad \forall g_1, g_2, g_3, g_4 \in G;$

(ii) $[g_1, g_2^{\varphi}, g_3] = [g_1, g_2, g_3^{\varphi}] = [g_1, g_2^{\varphi}, g_3^{\varphi}]$ *and*
$[g_1^{\varphi}, g_2, g_3] = [g_1^{\varphi}, g_2, g_3^{\varphi}] = [g_1^{\varphi}, g_2^{\varphi}, g_3], \quad \forall g_1, g_2, g_3 \in G;$

(iii) $[g, g^{\varphi}]$ *is central in* $\mathcal{V}(G), \quad \forall g \in G;$

(iv) $[g_1, g_2^{\varphi}][g_2, g_1^{\varphi}]$ *is central in* $\mathcal{V}(G), \quad \forall g_1, g_2 \in G;$

(v) $[g, g^{\varphi}] = 1, \quad \forall g \in G'$.

Proof. (i) The defining relations of $\mathcal{V}(G)$ yield:

$$\begin{aligned}
[g_1, g_2^{\varphi}]^{[g_3, g_4^{\varphi}]} &= [g_1, g_2^{\varphi}]^{g_3^{-1} g_4^{-\varphi} g_3 g_4^{\varphi}} \\
&= [g_1^{g_3^{-1}}, (g_2^{g_3^{-1}})^{\varphi}]^{g_4^{-\varphi} g_3 g_4^{\varphi}} \\
&= \ldots\ldots\ldots\ldots\ldots\ldots \\
&= [g_1^{g_3^{-1} g_4^{-1} g_3 g_4}, (g^{g_3^{-1} g_4^{-1} g_3 g_4})^{\varphi}] \\
&= [g, g_2^{\varphi}]^{[g_3, g_4]};
\end{aligned}$$

(ii) From $[x, y] = x^{-1} x^{y}$ and commutator calculus we get

$$\begin{aligned}
[g_1, g_2, g_3^{\varphi}] &= [g_1^{-1} g_1^{g_2}, g_3^{\varphi}] \\
&= [g_1^{-1}, g_3^{\varphi}]^{g_1^{g_2}} \cdot [g_1^{g_2}, g_3^{\varphi}] \\
&= [g_1^{-1}, g_3^{\varphi}]^{g_2^{-1} g_1 g_2} [g_1, (g_1^{g_2^{-1}})^{\varphi}]^{g_2} \\
&\qquad \text{(by defining relations of} \mathcal{V}(G)) \\
&= [g_1, g_3^{\varphi}]^{-g_1^{-1} g_2^{-1} g_1 g_2} [g_1, (g_2 g_3 g_2^{-1})^{\varphi}]^{g_2} \\
&= [g_1, g_3^{\varphi}]^{-[g_1, g_2]} \cdot [g_1, (g_2^{-1})^{\varphi}]^{g_2} [g_1, (g_2 g_3)^{\varphi}] \\
&= [g_1, g_3^{\varphi}]^{-[g_1, g_2]} [g_1, g_2^{\varphi}]^{-1} [g_1, g_3^{\varphi}] [g_1, g_2^{\varphi}]^{g_3} \\
&= [g_1, g_3^{\varphi}]^{-[g_1, g_2^{\varphi}]} [g_1, g_2^{\varphi}]^{-1} [g_1, g_3^{\varphi}] [g_1, g_2^{\varphi}]^{g_3} \quad \text{(by (i))} \\
&= [g_1, g_2^{\varphi}]^{-1} [g_1, g_3^{\varphi}]^{-1} [g_1, g_3^{\varphi}] [g_1, g_2^{\varphi}]^{g_3} \\
&= [g_1, g_2^{\varphi}]^{-1} [g_1, g_2^{\varphi}]^{g_3} \\
&= [g_1, g_2^{\varphi}, g_3];
\end{aligned}$$

Now we observe that

$$\begin{aligned}
[g_1, g_2^{\varphi}, g_3^{\varphi}] &= [g_1 g_2^{\varphi}]^{-1} [g_1, g_2^{\varphi}]^{g_3^{\varphi}} \\
&= [g_1, g_2^{\varphi}]^{-1} [g_1, g_2^{\varphi}]^{g_3} \quad \text{(by defining relations)} \\
&= [g_1, g_2^{\varphi}, g_3]
\end{aligned}$$

The last two relations in (ii) follow by a symmetric argument.

(iii) It follows from (ii) that for all $g, h \in G$,

$$[g, g^{\varphi}, h] = [g, g, h^{\varphi}] = 1;$$

But

$$\begin{aligned}
[g, g^{\varphi}, h^{\varphi}] &= [g, g^{\varphi}]^{-1} \cdot [g, g^{\varphi}]^{h^{\varphi}} \\
&= [g, g^{\varphi}]^{-1} [g, g^{\varphi}]^{h} \\
&= [g, g^{\varphi}, h],
\end{aligned}$$

so that (iii) is proved:

(iv) For $g_1, g_2 \in G$ we get

$$\begin{aligned}[][g_1g_2, (g_1g_2)^{\varphi}] &= [g_1, (g_1g_2)^{\varphi}]^{g_2}[g_2, (g_1g_2)^{\varphi}] \\ &= [g_1, g_2^{\varphi}]^{g_2}[g_1, g_1^{\varphi}]^{g_2^{\varphi}g_2}[g_2, g_2^{\varphi}][g_2, g_1^{\varphi}]^{g_2^{\varphi}} \\ &= [g_1, g_2^{\varphi}]^{g_2}[g_1, g_1^{\varphi}][g_2, g_2^{\varphi}][g_2, g_1^{\varphi}]^{g_2^{\varphi}} \quad \text{(by (iii))}\end{aligned}$$

Therefore, again by (iii), we can write

$$[g_1g_2, (g_1g_2)^{\varphi}][g_1, g_1^{\varphi}]^{-1}[g_2, g_2^{\varphi}]^{-1} = [g_1, g_2^{\varphi}]^{g_2}[g_2, g_1^{\varphi}]^{g_2^{\varphi}}$$

As the first member is central in $\mathcal{V}(G)$, on conjugating by $g_2^{-\varphi}$ and using the definition of $\mathcal{V}(G)$ we obtain

$$[g_1, g_2^{\varphi}][g_2, g_1^{\varphi}] = [g_1g_2, (g_1g_2)^{\varphi}][g_1, g_1^{\varphi}]^{-1}[g_2, g_2^{\varphi}]^{-1},$$

which belongs to the center of $\mathcal{V}(G)$;

As for (v), we first observe that when $g \in G'$ is a simple commutator, say $g = [x, y]$, then by (i) and (ii),

$$\begin{aligned}[][[x, y], [x, y]^{\varphi}] &= [x, y, (x^{-1}x^{y})^{\varphi}] \\ &= [x, y^{\varphi}, x^{-1}x^{y}] \\ &= [x, y^{\varphi}]^{-1}[x, y^{\varphi}]^{[x, y^{\varphi}]} \\ &= [x, y^{\varphi}]^{-1}[x, y^{\varphi}] = 1.\end{aligned}$$

Now for a general element $g \in G'$, say $g = [x_1, y_1]\dots[x_r, y_r]$, we use (i), (ii) and make induction on $r \geq 1$ to get

$$\begin{aligned}[][g, g^{\varphi}] &= [[x_1, y_1]\dots[x_r, y_r], [x_1, y_1]^{\varphi}\dots[x_r, y_r^{\varphi}]] \\ &= \dots\dots\dots\dots\dots\dots \\ &= [[x_1, y_1^{\varphi}]\dots[x_r, y_r^{\varphi}], [x_1, y_1^{\varphi}]\dots[x_r, y_r^{\varphi}]] = 1,\end{aligned}$$

proving (v). □

2.2 Lemma. *Let a, b, x be elements in G such that $[x, a] = 1 = [x, b]$. Then*

$$[a, b, x^{\varphi}] = 1 = [[a, b]^{\varphi}, x].$$

Proof. By Lemma 2.1 (ii) we obtain

$$\begin{aligned}[a,b,x^{\varphi}) &= [a,b^{\varphi},x] \\ &= [a,b^{\varphi}]^{-1}\cdot[a,b^{\varphi}]^{x} \\ &= [a,b^{\varphi}]^{-1}[a^{x},(b^{x})^{\varphi}] \\ &= [a,b^{\varphi}]^{-1}[a,b^{\varphi}]=1.\end{aligned}$$

The other identity follows by the symmetry in part (ii) of Lemma 2.1. □

2.3 Lemma. *Let x, y be elements of G such that $[x,y]=1$. Then*

(i) $[x^n, y^{\varphi}] = [x,y^{\varphi}]^n = [x,(y^{\varphi})^n]$, *for all $n \in \mathbb{Z}$;*

(ii) *If x and y are torsion elements of orders $o(x)$ and $o(y)$, then $o([x,y^{\varphi}])$ divides the g.c.d.$(o(x),o(y))$.*

Proof. (i) is proved by induction for $n \geq 0$, while

$$[x,y^{\varphi}]^{-1} = [x^{-1},y^{\varphi}]^{x} = [x^{-1},(y^{x})^{\varphi}] = [x^{-1},y^{\varphi}];$$

(ii) is a consequence of (i). □

Remark 1. By the symmetry between the defining relations of $\mathcal{V}(G)$ we note that the isomorphism φ extends uniquely to an automorphism Ψ of $\mathcal{V}(G)$ sending $g \mapsto g^{\varphi}, g^{\varphi} \mapsto g$ and $[g_1, g_2^{\varphi}] \mapsto [g_2, g_1^{\varphi}]^{-1}$, for all $g, g_1, g_2 \in G$.

Remark 2. For a finite group G, we can get the finiteness of $\mathcal{V}(G)$ making use of the finiteness of the following group $\chi(G)$ (cf. S. Sidki [10]):

For the given isomorphic pair G, G^{φ}, consider the group

$$\chi(G) := \langle G, G^{\varphi} \mid \ [g,g^{\varphi}]=1, \quad \forall g \in G\rangle.$$

Then we quote the following results [10] on $\chi(G)$ (see also [5,9]): "Let G be a finite π-group (π a set of primes), finite nilpotent or solvable of finite degree. Then $\chi(G)$ is also a finite π-group, finite nilpotent or solvable of finite degree". It should be noted that $\chi(G)$ has a subgroup $R(G)$ such that the relations $[g_1, g_2^{\varphi}]^{g_3^{\varphi}} = [g_1^{g_3}, (g_2^{g_3})^{\varphi}]$ hold in $\dfrac{\chi(G)}{R(G)}$ for all $g_1, g_2, g_3 \in G$ ([10], Lemma 4.11 (iii)). Here $R(G) = [G, L(G), G^{\varphi}]$, where $L(G)$ is given by $L(G) = [G,\varphi] := \langle g^{-1}g^{\varphi}, \ \forall g \in G\rangle$.

Returning to our group $\mathcal{V}(G)$ we note that on introducing the relations $[g, g^\varphi] = 1$ for all $g \in G$ it renders an epimorphism $\rho: \mathcal{V}(G) \to \dfrac{\chi(G)}{R(G)}$ defined by $g \mapsto gR(G), g^\varphi \mapsto g^\varphi R(G)$, $\forall g \in G$, $\forall g^\varphi \in G^\varphi$, whose Kernel $\Delta(G)$ is contained in $Z(\mathcal{V}(G)) \cap \mathcal{V}(G)'$, by Lemma 2.1 (iii). This implies that $\Delta(G)$ is a homomorphic image of the Schur Multiplier of $\dfrac{\chi(G)}{R(G)}$ which, together with the above quoted results, gives

2.4 Proposition. *Let G be a finite π-group (π a set of primes), finite nilpotent or solvable of finite degree. Then $\mathcal{V}(G)$ is also a finite π-group, finite nilpotent or solvable of finite degree.*

Let N be a normal subgroup of G. We set $\bar{G}$ for the quotient group $\dfrac{G}{N}$ and note that the canonical epimorphism $\pi = G \to \bar{G}$ gives raise to an epimorphism $\tilde{\pi}: \mathcal{V}(G) \to \mathcal{V}(\bar{G})$ such that $g \mapsto \bar{g}, g^\varphi \mapsto \overline{g^\varphi}$, where $\overline{G^\varphi} = \dfrac{G^\varphi}{N^\varphi}$ is identified with $\bar{G}^\varphi$.

2.5 Proposition. *With the above notation we have*

(i) $[N, G^\varphi] \trianglelefteq \mathcal{V}(G), [G, N^\varphi] \trianglelefteq \mathcal{V}(G)$;

(ii) $\operatorname{Ker} \tilde{\pi} = < N, N^\varphi > [N, G^\varphi] \cdot [G, N^\varphi]$.

Proof. (i) For elements $x \in N$ and $g, h \in G$, it follows that

$$\begin{aligned}[x, g^\varphi]^h &= [x, g^\varphi][x, g^\varphi, h] \\ &= [x, g^\varphi][x, g, h^\varphi] \quad \text{(by Lemma 2.1)}.\end{aligned}$$

This implies that G normalizes $[N, G^\varphi]$, and similarly G^φ normalizes $[N, G^\varphi]$, from what we get $[N, G^\varphi] \trianglelefteq \mathcal{V}(G)$. An analogous argument shows that $[G, N^\varphi] \trianglelefteq \mathcal{V}(G)$.

To prove (ii) we set $M = < N, N^\varphi > \cdot [N, G^\varphi] \cdot [G, N^\varphi]$, so that $M \leq \operatorname{Ker} \tilde{\pi}$. Furthermore M is a normal subgroup of $\mathcal{V}(G)$; thus we can define a function $\theta: \bar{G} \cup \bar{G}^\varphi \to \dfrac{\mathcal{V}(G)}{M}$ by setting $(\bar{g})\theta = Mg$ and $(\bar{g}^\varphi)\theta = Mg^\varphi$, which is well defined since $N, N^\varphi \subseteq M$. The restrictions of θ to $\bar{G}$ and $\bar{G}^\varphi$ are both homomorphisms, so that there is a unique homorphism θ^* which extends θ to the free product $\bar{G} * \bar{G}^\varphi$. We see that the relations

$$[\bar{g}_1\bar{g}_2, \bar{g}_3^\varphi] = [\overline{(g_1^{g_2})}, \overline{(g_3^{g_2})}^\varphi][\bar{g}_2, \bar{g}_3^\varphi]$$

and

$$[\bar{g}_1, (\bar{g}_2\bar{g}_3)^{\varphi}] = [\bar{g}_1, \bar{g}_3^{\varphi}][\overline{(g_1^{g_3})}, \overline{(g_2^{g_3})}^{\varphi}]$$

are preserved by θ^*. Consequently, θ induces a homomorphism $\tilde{\theta}\colon \mathcal{V}(\bar{G}) \to \dfrac{\mathcal{V}(G)}{M}$. Since $M \leq \mathrm{Ker}(\tilde{\pi})$ this yields an epimorphism $\bar{\pi}\colon \dfrac{\mathcal{V}(G)}{M} \to \mathcal{V}(\bar{G})$

$$\begin{tikzcd}
\mathrm{Ker}(\tilde{\pi}) \arrow[ddrr] & & & & \dfrac{\mathcal{V}(G)}{M} \arrow[dd, dashed, "\bar{\pi}"] \\
& \text{©} & \mathcal{V}(G) \arrow[urr, "q"] \arrow[drr, "\tilde{\pi}"'] & \text{©} & \\
M \arrow[uu] \arrow[urr] & & & & \mathcal{V}(\bar{G})
\end{tikzcd}$$

such that $(Mg)\bar{\pi} = \bar{g}$ and $(Mg^{\varphi})\bar{\pi} = \bar{g}^{\varphi}$. By composition of $\tilde{\theta}$ and $\bar{\pi}$ we get $(\bar{g})\tilde{\theta}\bar{\pi} = \bar{g}$ and $(\bar{g}^{\varphi})\tilde{\theta}\bar{\pi} = \bar{g}^{\varphi}$, $\forall g \in G$. Thus $\tilde{\theta}\bar{\pi} = 1_{\mathcal{V}(\bar{G})}$, and this in turn shows that $\tilde{\theta}$ is an isomorphism. □

Now we want to consider the subgroup

$$\Upsilon(G) = [G, G^{\varphi}]$$

which is normal in $\mathcal{V}(G)$.

By the early definition of the non-abelian tensor square $G \otimes G$ we see that the map $\tau\colon G \otimes G \to \Upsilon(G)$ defined on the generators by $(g_1 \otimes g_2)^{\tau} = [g_1, g_2^{\varphi}]$ extends to an epimorphism from $G \otimes G$ to $\Upsilon(G)$. In fact we have

2.6 Proposition. *τ is an isomorphism.*

Proof. Firstly we look at the free product $G * G^{\varphi}$. Its subgroup $[G, G^{\varphi}]$ is free, freely generated by the commutators $[g_1, g_2^{\varphi}]$ where $1 \neq g_1 \in G, 1 \neq g_2^{\varphi} \in G^{\varphi}$. (See for instance [7], chap. 4). As a normal subgroup of $G * G^{\varphi}$, $[G, G^{\varphi}]$ admits the actions of G and G^{φ} by conjugation and the following identities hold

$$(I)\begin{cases} [g_1, g_2^{\varphi}]^{g} = [g_1 g, g_2^{\varphi}][g, g_2^{\varphi}]^{-1} \\ [g_1, g_2^{\varphi}]^{g^{\varphi}} = [g_1, g^{\varphi}]^{-1} \cdot [g_1, (g_2 g)^{\varphi}], \end{cases}$$

for all $g, g_1, g_2 \in G$.

Now the map $\mu: [G, G^{\varphi}] \to G \otimes G$ defined on the free generator $[g_1, g_2^{\varphi}]$ by $[g_1, g_2^{\varphi}]^{\mu} = g_1 \otimes g_2$ extends to an epimorphism from the (free) group $[G, G^{\varphi}]$ ($\trianglelefteq G * G^{\varphi}$) onto $G \otimes G$. Consequently, the introduction in $G * G^{\varphi}$ of the defining relations of $\mathcal{V}(G)$ takes us to describe $\Upsilon(G)$ as the quotient of $[G, G^{\varphi}]$ (still a subgroup of $G * G^{\varphi}$) by the relations

$$(II)\begin{cases} [g_1 g_2, g_3^{\varphi}] = [g_1^{g_2}, (g_3^{g_2})^{\varphi}][g_2, g_3^{\varphi}] \\ [g_1, (g_2 g_3)^{\varphi}] = [g_1, g_3^{\varphi}] \cdot [g_1^{g_3}, (g_2^{g_3})^{\varphi}], \end{cases}$$

for all $g_1, g_2, g_3 \in G$. But relations (II) are mapped by μ in the defining relations of $G \otimes G$, from what we get that μ induces an epimorphism from $\Upsilon(G)$ onto $G \otimes G$. We have $\mu\tau = 1_{\Upsilon(G)}$ and $\tau\mu = 1_{G\otimes G}$, thus proving our assertion. □

Remark 3. An argument similar to that used in Proposition 2.5 (ii) may be used to show if N is a normal subgroup of G and $\tilde{\pi}: \mathcal{V}(G) \to \mathcal{V}\left(\frac{G}{N}\right)$ is the epimorphism induced by the projection $\pi: G \to \frac{G}{N}$, then $\mathrm{Ker}(\tilde{\pi}) \cap \Upsilon(G) = [N, G^{\varphi}] \cdot [G, N^{\varphi}]$.

We close this section by proving

2.7 Proposition. *Let*

$$G = G_1 \trianglerighteq G_2 (= G') \trianglerighteq \cdots \trianglerighteq G_j \trianglerighteq \cdots,$$

$$1 = \xi_0(G) \trianglelefteq \xi_1(G) (= Z(G)) \trianglelefteq \cdots \trianglelefteq \xi_j(G) \trianglelefteq \cdots,$$

and

$$G = \gamma_1(G) \trianglerighteq \gamma_2(G) \trianglerighteq \cdots \trianglerighteq \gamma_j(G) \trianglerighteq \cdots$$

be respectively the derived series, the upper central series and the lower central series of G. Then

(i) $[\xi_j(G), G_{j+1}^{\varphi}] = 1$, *for all* $j \geq 0$;

(ii) $[\xi_{j+1}(G), \gamma_j(G^{\varphi})] \cdot [\gamma_j(G), \xi_{j+1}(G^{\varphi})]$ *is central in* $\Upsilon(G)$ *for all* $j \geq 1$;

(iii) $[\xi_j(G), \gamma_j(G^{\varphi})]$ *is central in* $\mathcal{V}(G)$, *for all* $j \geq 1$.

Proof. (i) is trivial for $j = 0$ while the general case follows directly from Lemma 2.2, since $G_j \leq \gamma_j(G)$ and $[\xi_j(G), \gamma_j(G)] = 1$ for all $j \geq 1$.

(ii) for $j \geq 1, z \in \xi_{j+1}(G), g \in \gamma_j(G)$ and $g_1, g_2 \in G$ we have

$$\begin{aligned} [[z, g^\varphi], [g_1, g_2^\varphi]] &= [z, g^\varphi]^{-1}[z, g^\varphi]^{[g_1, g_2^\varphi]} \\ &= [z, g^\varphi]^{-1}[z, g^\varphi]^{[g_1, g_2]} \quad \text{(Lemma 2.1 (i))} \\ &= [z, g^\varphi, [g_1, g_2]] \\ &= [z, g, [g_1, g_2]^\varphi] \quad \text{(Lemma 2.1 (ii))} \\ &= 1 \quad \text{(by Lemma 2.2, since} [\xi_{j+1}(G), \gamma_j(G)] \leq \xi_1(G)). \end{aligned}$$

This implies that $\Upsilon(G)$ centralizes $[\xi_{j+1}(G), \gamma_j(G^\varphi)]$ and by symmetry $\Upsilon(G)$ also centralizes $[\gamma_j(G), \xi_{j+1}(G^\varphi)]$.

(iii) This part follows directly from Lemma 2.1 (ii) since $[\xi_j(G), \gamma_j(G)] = 1$, for all $j \geq 1$. □

3. The Main Results

The description of $\mathcal{V}(G)$ as the product $\mathcal{V}(G) = \Upsilon(G) \cdot G \cdot G^\varphi$, which comes from the fact that $\Upsilon(G) \trianglelefteq \mathcal{V}(G)$, gives an elegant description for the lower central series and the derived series of $\mathcal{V}(G)$.

3.1 Theorem. *For $i \geq 2$ the i-th term of the lower central series of $\mathcal{V}(G)$ is given by*

$$\gamma_i(\mathcal{V}(G)) = \gamma_i(G)\gamma_i(G^\varphi)[\gamma_{i-1}(G), G^\varphi][G, \gamma_{i-1}(G^\varphi)]$$

Proof. For $i = 2, \gamma_2(\mathcal{V}(G)) = [\mathcal{V}(G), \mathcal{V}(G)] = [\Upsilon(G) \cdot G \cdot G^\varphi, \Upsilon(G) \cdot G \cdot G^\varphi]$. By using the defining relations of $\mathcal{V}(G)$ together with Lemma 2.1 and Proposition 2.5 (i) we get

$$[\Upsilon(G) \cdot G \cdot G^\varphi, \Upsilon(G) \cdot G \cdot G^\varphi] \leq \Upsilon(G) \cdot \gamma_2(G) \cdot \gamma_2(G^\varphi).$$

This shows that $\gamma_2(\mathcal{V}(G)) = \gamma_2(G)\gamma_2(G^\varphi) \cdot \Upsilon(G)$. Suppose, by induction on $i \geq 2$, that

$$\gamma_i(\mathcal{V}(G)) \leq \gamma_i(G)\gamma_i(G^\varphi)[\gamma_{i-1}(G), G^\varphi] \cdot [G, \gamma_{i-1}(G^\varphi)].$$

Then by Proposition 2.5 (i),

$$[\gamma_i(\mathcal{V}(G)), G] \leq \gamma_{i+1}(G) \cdot [\gamma_i(G^\varphi), G] \cdot [\gamma_{i-1}(G), G^\varphi, G] \cdot [G, \gamma_{i-1}(G^\varphi), G],$$

and once more invoking Lemma 2.1 (i) we obtain

$$[\gamma_{i-1}(G), G^{\varphi}, G] = [\gamma_i(G), G^{\varphi}] = [G, \gamma_{i-1}(G^{\varphi}), G].$$

Therefore $[\gamma_i(\mathcal{V}(G)), G] \leq \gamma_{i+1}(G) \cdot [\gamma_i(G), G^{\varphi}] \cdot [G, \gamma_i(G^{\varphi})]$. By symmetry it follows that

$$[\gamma_i(\mathcal{V}(G)), G^{\varphi}] \leq \gamma_{i+1}(G^{\varphi})[\gamma_i(G), G^{\varphi}][G, \gamma_i(G^{\varphi})],$$

and these last two inclusions show that

$$\gamma_{i+1}(\mathcal{V}(G)) \leq \gamma_{i+1}(G) \cdot \gamma_{i+1}(G^{\varphi})[\gamma_i(G), G^{\varphi}][G, \gamma_i(G^{\varphi})],$$

so that our theorem is proved by induction. □

3.2 Corollary. *Let G be a nilpotent group of class c. Then $\mathcal{V}(G)$ is a nilpotent group of class at most $c+1$.*

The next theorem is proved using, step by step, similar arguments as in the proof of Theorem 3.2. We will omit its proof.

3.3 Theorem. *For $i \geq 2$ the i-th term of the derived series of $\mathcal{V}(G)$ is given by*

$$\mathcal{V}(G)_i = G_i G_i^{\varphi}[G_{i-1}, G_{i-1}^{\varphi}],$$

where G_i, denotes the i-th term of the derived series of G.

3.4 Corollary. *Let G be a solvable group of derived length ℓ. Then $\mathcal{V}(G)$ is solvable of derived length at most $\ell + 1$.*

3.5 Proposition. *Let $G = N \cdot H$ be a semidirect product of its subgroups $N \trianglelefteq G$ and $H \leq G$. Then*

(i) $\mathcal{V}(G) = \langle N, N^{\varphi}\rangle [N, H^{\varphi}][H, N^{\varphi}] \cdot \langle H, H^{\varphi}\rangle$;

(ii) $\langle H, H^{\varphi}\rangle \cong \mathcal{V}(H)$.

Proof. (i), (ii). It follows easily from Proposition 2.5 that $[N, H^{\varphi}]$ and $[H, N^{\varphi}]$ are both normal subgroups of $\mathcal{V}(G)$; also, $\langle N, N^{\varphi}\rangle[N, H^{\varphi}][H, N^{\varphi}]$ is actually the Kernel of $\tilde{\pi}\colon \mathcal{V}(G) \to \mathcal{V}\left(\frac{G}{N}\right)$ $(\cong \mathcal{V}(H))$. On writing $\mathcal{V}(G) = \mathcal{V}(NH) = [NH, N^{\varphi}H^{\varphi}] \cdot NH \cdot N^{\varphi}H^{\varphi}$ we see that

$$[NH, N^{\varphi}H^{\varphi}] \leq [N, N^{\varphi}][N, H^{\varphi}][H, N^{\varphi}]$$

and thus $\mathcal{V}(G)$ has the desired expression. As for (ii), $\langle H, H^{\varphi}\rangle^{\tilde{\pi}} = \mathcal{V}\left(\frac{G}{N}\right)$ ($\cong$ $\mathcal{V}(H)$), while on the other hand $\mathcal{V}(H)$ is mapped onto $\langle H, H^{\varphi}\rangle$. Therefore $\mathrm{Ker}(\tilde{\pi}) \cap \langle H, H^{\varphi}\rangle = \{1\}$ and $< H, H^{\varphi} > \cong \mathcal{V}(H)$. □

3.6 Proposition. *Let $G = N \times H$ be the direct product of its normal subgroups N and H. Then*

(i) $\mathcal{V}(G) = \langle N, N^{\varphi}\rangle \cdot [N, H^{\varphi}] \cdot [H, N^{\varphi}] \cdot < H, H^{\varphi} >$

(ii) $\langle N, N^{\varphi}\rangle \cong \mathcal{V}(N)$; $\quad \langle H, H^{\varphi}\rangle \cong \mathcal{V}(H)$

(iii) $\Upsilon(G) = \Upsilon(N) \times \Upsilon(H)$.

Proof. Parts (i) and (ii) follows from double application of Proposition 3.5. As for (iii), we get from Proposition 2.7 (i) that the four subgroups $[N, H^{\varphi}], [N, N^{\varphi}]$, $[H, N^{\varphi}]$ and $[H, H^{\varphi}]$ are mutually centralized in $\Upsilon(G)$, since $[N, H] = 1$. Also, normality of $[N, H^{\varphi}]$ and $[H, N^{\varphi}]$ in $\mathcal{V}(G)$ give

$$\Upsilon(G) = [N, N^{\varphi}] \cdot [N, H^{\varphi}][H, N^{\varphi}][H, H^{\varphi}].$$

Lastly we observe that part (ii) implies $[N, N^{\varphi}] \cong \Upsilon(N)$ and $[H, H^{\varphi}] \cong \Upsilon(H)$. □

Remark 4. The result in Part (iii) is Proposition 11 of [1].

In fact, by arguments similar to those used in Proposition 2.6 we can prove that when H and K are groups which act trivially on each other (but by conjugation on themselves) then the subgroup $[H, K^{\varphi}]$ of $\mathcal{V}(H \times K)$ is isomorphic to $H \otimes K$ which in turn is the usual tensor product $H \otimes_{\mathbb{Z}} K$ (this follows from Lemma 2.1; see also Remark 2 of [1]).

Remark 5. In case of abelian groups A and B we have therefore the known decomposition of the ordinary tensor product: $(A \times B) \otimes_{\mathbb{Z}} (A \times B) \cong \Upsilon(A \times B) \cong (A \otimes_{\mathbb{Z}} A) \times (A \otimes_{\mathbb{Z}} B) \times (B \otimes_{\mathbb{Z}} A) \times (B \otimes_{\mathbb{Z}} B)$.

3.7 Corollary. *Let $G = P_1 \times \cdots \times P_n$ be a finite nilpotent group where $\{P_1, \cdots, P_n\}$ is the set of distinct Sylow p-subgroups of G. Then,*

(i) $\mathcal{V}(G) \cong \mathcal{V}(P_1) \times \cdots \times \mathcal{V}(P_n)$

(ii) $\Upsilon(G) \cong \Upsilon(P_1) \times \ldots \times \Upsilon(P_n)$

Proof. For any prime p dividing $|G|$, let P be a Sylow p-subgroup of G and N be a normal p-complement in G. We have by Lemma 2.3 (ii) that $[N, P^{\varphi}] = [P, N^{\varphi}] = 1$.

The previous proposition then yields $\mathcal{V}(G) \cong \mathcal{V}(N) \times \mathcal{V}(P)$ and $\Upsilon(G) \cong \Upsilon(N) \times \Upsilon(P)$. Parts (i), (ii) now follow by induction on $n \geq 2$. □

From now on we restrict ourselves to the case of a finite p-group G.

3.8 Lemma. *Let G be a finite p-group and $c \in Z(G) \cap G'$ be an element of order p. If $\phi(G)$ denotes the Frattini subgroup of G, then*

$$|\mathcal{V}(G)| \quad \textit{divides} \quad p^2 \left|\frac{G}{\phi(G)}\right| \left|\mathcal{V}\left(\frac{G}{< c >}\right)\right|$$

Proof. By Proposition 2.7 (i) we get $[c, g^{\varphi}] = 1$ for all $g \in G'$. On the other hand, if $x \in G$ then, by Lemma 2.3 (i), $[c, (x^p)^{\varphi}] = [c, x^{\varphi}]^p = [c^p, x^{\varphi}] = 1$, so that $[c, g^{\varphi}] = 1$ for all $g \in G^p := \langle x^p | x \in G \rangle$. It follows that $[c, \phi(G)^{\varphi}] = 1$ since $\phi(G) = G'G^p$. If we set $\lambda: G \to [c, G^{\varphi}], g \mapsto [c, g^{\varphi}]$ then λ is an epimorphism, as $[c, G^{\varphi}]$ is central in $\mathcal{V}(G)$. Also, $\phi(G) \leq \mathrm{Ker}(\lambda)$. Let $\pi: G \to \dfrac{G}{< c >}$ be the canonical projection and $\tilde{\pi}$ its induced in $\mathcal{V}(G)$, whose kernel is $\mathrm{Ker}(\tilde{\pi}) =< c >< c^{\varphi} > [c, G^{\varphi}][G, c^{\varphi}]$. Let $\bar{a}$ be a generator of $\dfrac{G}{\phi(G)}$. If c is a simple commutator, say $c = [x, y]$, then we get

$$\begin{aligned}
[a, c^{\varphi}] &= [a, [x, y]^{\varphi}] \\
&= [[x, y]^{\varphi}, a]^{-1} \\
&= [x, y^{\varphi}, a]^{-1} \quad \text{(by Lemma 2.1 (ii))} \\
&= [a, [x, y^{\varphi}]].
\end{aligned}$$

In general, if c is a product of commutators, say $c = [x_1, y_1][x_2, y_2] \ldots [x_r, y_r]$, then by induction we get $[a, c^{\varphi}] = [a, [x_1, y_1^{\varphi}] \ldots [x_r, y_r^{\varphi}]]$.

Analogously, $[c, a^{\varphi}] = [[x_1, y_1^{\varphi}] \ldots [x_r, y_r^{\varphi}], a]$. Since $c \in Z(G) \cap G'$, it follows from the above identities that, in $[c, G^{\varphi}][G, c^{\varphi}]$, the elements of the form $[c, a^{\varphi}][a, c^{\varphi}]$ are all trivial.

On the other hand if $\dfrac{G}{\phi(G)}$ is generated by $\{\bar{a}_1, \ldots, \bar{a}_d\}$, we have

$$[c, (a_1^{i_1} \ldots a_d^{i_d})^{\varphi}] = [c, a_1^{\varphi}]^{i_1} \ldots [c, a_d^{\varphi}]^{i_d}$$

and

$$[(a_1^{j_1}\dots a_d^{j_d}), c^\varphi] = [a_1, c^\varphi]^{j_1}\dots[a_d, c^\varphi]^{j_d},$$

by Lemma 2.3 (i). This means that $[c, G^\varphi][G, c^\varphi]$ is generated by the $2d$ elements

$$[a_1, c^\varphi], \dots, [a_d, c^\varphi], [c, a_1^\varphi], \dots, [c, a_d^\varphi].$$

But since $[a_i, c^\varphi][c, a_i^\varphi] = 1$ for $i = 1, \dots, d$, it results that $[c, G^\varphi][G, c^\varphi] = [c, G^\varphi]$, which is generated by $[c, a_1^\varphi], \dots, [c, a_d^\varphi]$. This together with the fact that λ is an epimorphism gives

$$|\mathrm{Ker}(\tilde{\pi})| \le p^2 \cdot |[c, G^\varphi]| \le p^2 \cdot \left|\frac{G}{\phi(G)}\right|.$$

Therefore $|\mathcal{V}(G)|$ divides $p^2\left|\dfrac{G}{\phi(G)}\right| \cdot \left|\mathcal{V}\left(\dfrac{G}{< c >}\right)\right|$. □

3.9 Proposition. *Let G be a finite p-group of class 2. Then $|\Upsilon(G)|$ divides*

$$\left|G' \otimes_{\mathbb{Z}} \frac{G}{G'}\right| \cdot \left|\Upsilon\left(\frac{G}{G'}\right)\right|.$$

Proof. Let $\tilde{\pi}: \mathcal{V}(G) \to \mathcal{V}\left(\dfrac{G}{G'}\right)$ be the epimorphism induced by the canonical map $\pi: G \to \dfrac{G}{G'}$. By Remark 3 we have $\mathrm{Ker}(\tilde{\pi}) \cap \Upsilon(G) = [G', G^\varphi][G, (G')^\varphi]$, while $\Upsilon(G)^{\tilde{\pi}} = \Upsilon\left(\dfrac{G}{G'}\right)$. Thus it remains to evaluate $|\mathrm{Ker}(\tilde{\pi}) \cap \Upsilon(G)|$. Since $G' \le Z(G)$, Proposition 2.7 (i) gives $[G', G'^\varphi] = 1$. Hence, for $c \in G'$ and $g = dh \in G$, where $h \in G', [c, (dh)^\varphi] = [c, h^\varphi][c, d^\varphi]^{h^\varphi} = [c, d^\varphi]$.

As $[G', G^\varphi]$ is central in $\mathcal{V}(G)$ (Proposition 2.7 (iii)), this implies that $[G', G^\varphi]$ is a homomorphic image of $G' \otimes_{\mathbb{Z}} \dfrac{G}{G'}$ through the map $c \otimes \bar{d} \mapsto [c, d^\varphi]$, where $c \in G'$ e $\bar{d} = d^\pi$.

Therefore $|[G', G^\varphi]|$ divides $\left|G' \otimes_{\mathbb{Z}} \dfrac{G}{G'}\right|$. Suppose $G' = \langle c_1, \dots, c_m\rangle$ and $\dfrac{G}{G'} = \langle \bar{d}_1, \dots, \bar{d}_n\rangle$. Then $[G', G^\varphi]$ is generated by the set $\{[c_i, d_j^\varphi], 1 \le i \le m, 1 \le j \le n\}$ and similarly $[G, (G')^\varphi]$ is generated by $\{[d_j, c_i^\varphi], 1 \le j \le n, 1 \le i \le m\}$. But each c_i is a product of commutators so that we get, as in the proof of Lemma 3.8, $[c_i, d_j^\varphi][d_j, c_i^\varphi] = 1$, for all pairs (i, j). This in turn gives $[G', G^\varphi][G, (G')^\varphi] = [G', G^\varphi]$, and consequently $|\mathrm{Ker}(\tilde{\pi}) \cap \Upsilon(G)|$ divides $\left|G' \otimes \dfrac{G}{G'}\right|$. □

3.10 Corollary. *Let G be a p-group of class ≤ 2 with $|G| = p^n$ and $|G'| = p^m$. Then $|\Upsilon(G)|$ divides $p^{n(n-m)}$.*

Proof. We observe that $\left|G' \otimes_{\mathbb{Z}} \frac{G}{G'}\right|$ divides $p^{m(n-m)}$ and

$$\left|\Upsilon\left(\frac{G}{G'}\right)\right| = \left|\frac{G}{G'} \otimes_{\mathbb{Z}} \frac{G}{G'}\right|$$

divides $p^{(n-m)^2}$. □

3.11 Theorem. *Let G be a finite p-group with $|G| = p^n$ and $|G'| = p^m$. Then $|\mathcal{V}(G)|$ divides $p^{n^2+2n-mn}$.*

Proof. Since $|\mathcal{V}(G)| = |\Upsilon(G)| \cdot |G|^2$, all we need is to evaluate $|\Upsilon(G)|$. If G has nilpotence class ≤ 2 then we are done with Corollary 3.8.

Suppose G has class at least 3 and let $c \in \gamma_3(G) \cap Z(G)$ be an element of order p. We argument by induction on $|G|$. Since

$$\left|\frac{G}{< c >}\right| = p^{n-1} \quad \text{and} \quad \left|\left(\frac{G}{< c >}\right)'\right| = p^{m-1},$$

our hypothesis give that $\left|\Upsilon\left(\frac{G}{< c >}\right)\right|$ divides $p^{(n-1)(n-m)}$.

On the other hand $\left|\frac{G}{\phi(G)}\right|$ divides $\left|\frac{G}{G'}\right| = p^{n-m}$, so that by Lemma 3.8 we finally obtain $|\Upsilon(G)|$ divides $p^{n(n-m)}$. □

3.12 Corollary. *Let $|G| = p^n, |G'| = p^m$ and $d = d(G)$ be the minimal number of generators of G. Then*

$$p^{d^2} \leq |G \otimes G| \leq p^{n(n-m)}$$

Proof. We observe that on making $N = \phi(G)$ and

$$\tilde{\pi}\colon \mathcal{V}(G) \to \mathcal{V}\left(\frac{G}{\phi(G)}\right)$$

in Proposition 2.5, then by Remark 3 it results that

$$\operatorname{Ker}(\tilde{\pi}) \cap \Upsilon(G) = [\phi(G), G^{\varphi}][G, \phi(G)^{\varphi}],$$

so that the restriction of $\tilde{\pi}$ to $\Upsilon(G)$ renders

$$|\Upsilon(G)| \geq \left|\Upsilon\left(\frac{G}{\phi(G)}\right)\right| = \left|\left[\frac{G}{\phi(G)}, \left(\frac{G}{\phi(G)}\right)^{\varphi}\right]\right|.$$

But $\dfrac{G}{\phi(G)}$ is elementary abelian of order p^d and (as observed in Remarks 4 and 5.)

$$\left[\frac{G}{\phi(G)}, \left(\frac{G}{\phi(G)}\right)^{\varphi}\right]$$

is precisely the usual tensor product

$$\frac{G}{\phi(G)} \otimes_{\mathbb{Z}} \frac{G}{\phi(G)},$$

of order p^{d^2}.

On the other hand the last theorem gives the upper bound

$$|\Upsilon(G)| = \frac{|\mathcal{V}(G)|}{|G|^2} \leq p^{n^2-mn}.$$

Our proof is now finished by the isomorphism of Proposition 2.6. □

Acknowledgments

We are grateful to professor S. Sidki for his encouragement and helpful comments, and to professor J. Neubüser who kindly provided us with the SOGOS system, use of which was made in testing some of these ideas with finite p-groups.

References

[1] R. Brown, D.L. Johnson and E. F. Robertson, *Some Computations of Non-Abelian Tensor Products of Groups*, J. Algebra **111** (1987), 177-202.

[2] R. Brown and J.-L. Loday, *Van Kampen Theorems for Diagrams of Spaces*, Topology **26** (1987), 311-335.

[3] N.D. Gilbert, *The non-abelian tensor square of a free product of groups*, Arch. Math. **48** (1987), 369-375.

[4] N.D. Gilbert and P.J. Higgins, *The Non-Abelian Tensor Product of Groups and Related Constructions*, Glasgow Math. J. **31** (1989), 17-29.

[5] N. Gupta, N. Rocco and S. Sidki, *Diagonal Embeddings of Nilpotent Groups*, III. J. Math. **30** (1986), 274-283.

[6] M.R. Jones, *Some inequalities for the multiplicator of a finite group II*, Proc. Amer. Math. Soc. **45** (1974), 167-172.

[7] W. Magnus, Karras and Solitar, *"Combinatorial Group Theory"*, Dover, New York, 1966.

[8] D.J.S. Robinson, *"A Course in the Theory of Groups"*, Springer, New York, 1982.

[9] N.R. Rocco, *On Weak Commutativity between Finite p-Group, p:odd*, J. Algebra **76** (1982), 471-488.

[10] S. Sidki, *On Weak Permutability between Groups*, J. Algebra **63** (1980), 186-225.

N.R. Rocco
Departamento de Matemática
Universidade de Brasília
70.910 - Brasília - D.F., Brazil