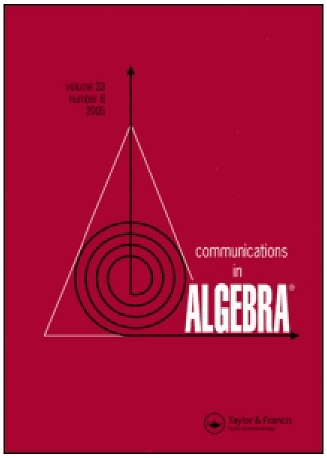


Communications in Algebra
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597239

A presentation for a crossed embedding of finite solvable groups

N.R. Racca [a]
[a] Departamento de Matemática, Universidade de Brasília, Brasília, D.F, Brazil

Online Publication Date: 01 January 1994
To cite this Article: Racca, N.R. (1994) 'A presentation for a crossed embedding of finite solvable groups', Communications in Algebra, 22:6, 1975 - 1998
To link to this article: DOI: 10.1080/00927879408824951
URL: http://dx.doi.org/10.1080/00927879408824951

COMMUNICATIONS IN ALGEBRA, 22(6), 1975-1998 (1994)

A Presentation for a Crossed Embedding of Finite Solvable Groups

by N.R. Rocco

Departamento de Matemática
Universidade de Brasília
70.910 - Brasília - D.F., Brazil

1. Introduction

We introduced in [11] a group construction as an operator in the class of groups, which involves intrinsically invariants of the argument such as the Non-abelian Tensor Square and the Schur Multiplier, among others. More specifically, given groups G and G^φ, isomorphic through an isomorphism $\varphi : G \to G^\varphi$, $g \mapsto g^\varphi$ for all g in G, then we defined the group

$$\mathcal{V}(G) \;=\; \langle\, G, G^\varphi \mid [g, h^\varphi]^k \;=\; [g^k, (h^k)^\varphi] \;=\; [g, h^\varphi]^{k^\varphi},\ \forall\, g,\ h,\ k\ \in G \rangle,$$

that is, $\mathcal{V}(G)$ is the quotient of the free product $G * G^\varphi$ by its normal subgroup generated by all the words $[g,h^\varphi]^k \cdot [g^k,(h^k)^\varphi]^{-1}$ and $[g,h^\varphi]^{k^\varphi} \cdot [g^k,(h^k)^\varphi]^{-1}$ for $g,\ h,\ k\ \in G$ (we use standard notation for commutators and conjugation in a group; see below).

In this paper we give a presentation for $\mathcal{V}(G)$ when G is a finite solvable group given by one of its AG-systems (see section 2 for details). This main result can be stated as

Theorem. Let G and G^φ be distinct isomorphic finite solvable groups given by AG-systems $\{a_1,...,a_n\}$ and $\{b_1,...,b_n\}$ respectively, where $\varphi : G \to G^\varphi$ is an isomorphism such that $a_i \mapsto b_i$, $1 \le i \le n$. Then the group

$$\begin{aligned}\delta(G) := \langle a_1,...,a_n, b_1,...,b_n \mid G-relations,\ G^\varphi-relations,\\ [a_i,b_j]^{a_k} = [a_i^{a_k}, b_j^{b_k}] = [a_i,b_j]^{b_k}, 1 \le i,j,k \le n\rangle\end{aligned}$$

is a presentation of $\mathcal{V}(G)$.

Such a presentation is obtained (Theorem 2.1) by mean of a convenient set of generators for the subgroup $[G, G^\varphi]$, so that the computation of those invariants of G mentioned above seems to be much easier to perform inside $\mathcal{V}(G)$ in this case, once $[G,G^\varphi]$ is isomorphic to the non-abelian tensor square $G \otimes G$ (cf. [11]). The relationship between $\mathcal{V}(G)$ and covering questions in groups is also explored in section 2, for arbitrary G. This section ends with an isomorphism (Theorem 2.11) between $\mathcal{V}(G)/\Delta(G)$ and a certain natural factor of a group introduced by Sidki ([12]), where $\Delta(G)$ is the (central) subgroup generated by all commutators $[g, g^\varphi]$, $g \in G$.

In Section 3 we study in some detail the subgroup $\Delta(G)$ as it plays an important role in the context. A section $\mu(G)/\Delta(G)$ is isomorphic to $\mathcal{H}_2(G)$ and thus, as an application, we use our approach to compute the Schur Multiplier of an arbitrary finite metacyclic group.

Section 4 is mainly concerned with some computational aspects of our results, including some tables for $\mathcal{V}(G)$ constructed with help of the GAP system [4]. A couple of open problems is left in Section 5.

Most of the work presented here was carried out during a visit of the author to the Lehrstuhl D für Mathematik, RWTH - Aachen, supported by a stipend from the German-Brazilian scientific agreement between GMD and CNPq. I want to express my gratitude to Professor J.Neubüser, and to all members of this Department for their help, the warm hospitality and for providing me with all necessary computing facilities. I am also especially grateful to Professor Larry C. Grove who was a visitor at this Department during the same occasion, and shared with me his experience and friendship.

Notation. Most of the notation utilized in these notes is standard. For elements x, y, z in a group G the conjugate of x by y is $x^y := y^{-1}xy$ and the commutator of x and y is $[x,y] := x^{-1}x^y$. Our commutators are left normed, $[x,y,z] := [[x,y],z]$, and the expression *commutator calculus* used in many places is mainly concerned with the use of the following identities (see e.g. [9]):

$$[x,y] = [x,y^{-1}]^{-y} = [x^{-1},y]^{-x};$$
$$[xy,z] = [x,z]^y[y,z] = [x,z][x,z,y][y,z];$$
$$[x,yz] = [x,z][x,y]^z = [x,z][x,y][x,y,z].$$

An expression of the type $1+x+\cdots+x^{n-1}$ for some natural number n is frequently denoted by $\Gamma(x^n)$ when it appears in the formal computation of a commutator $[x^n,y]$. A similar expression involving also some power of y is sometimes denoted by $W(x,y)$ in the same context.

2. The Presentation

We recall that a finite solvable group $G \neq \{1\}$ has a subnormal series $G = G_0 > G_1 > \ldots > G_n = \{1\}$ where $G_i \lhd G_{i-1}$ and G_{i-1}/G_i is cyclic of order r_i, $1 \leq i \leq n$. This means that $G_{i-1} = < a_i, ..., a_n >$ and $a_i^{r_i} \in G_i = < a_{i+1}, ..., a_n >$. The sequence $(a_1, ..., a_n)$ is called an *AG-system* of generators for G ([6]), with the following defining relations

$$a_i^{r_i} = w'_{ii}(a_{i+1}, ..., a_n), \quad 1 \leq i \leq n;$$
$$a_i^{a_j} = w'_{ij}(a_{j+1}, ..., a_n), \quad 1 \leq j < i \leq n.$$

which are called respectively *power-relations* and *conjugate-relations*. For our purposes we shall rewrite the power-conjugates relations by collecting the generators $a_i, 1 \leq i \leq n$, in decreasing order from left to right, so that for the given AG-system the relations are

$$G - relations: \begin{cases} a_i^{r_i} = w_{ii}(a_n, ..., a_{i+1}), & 1 \leq i \leq n; \\ a_i^{a_j} = w_{ij}(a_n, ..., a_{j+1}), & 1 \leq j < i \leq n. \end{cases}$$

We see that every element $g \in G$ has a normal expression

$$g = a_n^{\nu_n} \cdot a_{n-1}^{\nu_{n-1}} \cdots a_1^{\nu_1}, \ 0 \leq \nu_i < r_i, \ 1 \leq i \leq n,$$

and, by the conjugate-relations, the *special relations*

$$a_j^{a_i} = u_{ij}(a_n, ..., a_{j+1}) \cdot a_j, \ 1 \leq j < i \leq n \qquad (*)$$

follow.

Now let G and G^φ be distinct isomorphic finite solvable groups given by AG-systems $\{a_1, ..., a_n\}$ and $\{b_1, ..., b_n\}$ respectively, where $\varphi : G \to G^\varphi$ is an isomorphism such that $a_i \mapsto b_i, 1 \leq i \leq n$. The corresponding power-conjugates relations satisfied by these systems we call $G-$*relations* and $G^\varphi-$*relations*.

2.1. Theorem. *Let G and G^φ be as above and define the group*

$$\delta(G) := \langle a_1, ..., a_n, b_1, ..., b_n \mid G - relations, \ G^\varphi - relations, \\ [a_i, b_j]^{a_k} = [a_i^{a_k}, b_j^{b_k}] = [a_i, b_j]^{b_k}, 1 \leq i, j, k \leq n \rangle.$$

Then

(i) *the subgroup $[G, G^\varphi]$ of $\delta(G)$ is generated by the set*

$$T := \{[a_i, b_j] \mid 1 \leq i, j \leq n\};$$

(ii) $[g, h^\varphi]^f = [g^f, (h^f)^\varphi] = [g, h^\varphi]^{f^\varphi}, \ \ \forall \ f, g, h \in G.$

Proof. We proceed by induction on the polycyclic length n of the $AG-$system. For $n = 1$, $G = < a_1 >$ is a cyclic group of order r_1, and by definition of $\delta(G)$, with $i = j = k = 1$, we have $[a_1, b_1]^{a_1} = [a_1, b_1] = [a_1, b_1]^{b_1}$. These equalities imply that in this case $\delta(G)$ is a 2-generator nilpotent group of class at most 2, so that $[G, G^\varphi]$ is generated by $[a_1, b_1]$, which is central in $\delta(G)$. Therefore (i) and (ii) are proved for $n = 1$.

Suppose $n \geq 2$ and let N be the (normal) subgroup of G generated by $\{a_2, \ldots, a_n\}$. By induction we can assume that

- (i') The subgroup $H_1 := [N, N^\varphi]$ of $< N, N^\varphi >$ is generated by the set $X := \{ [a_i, b_j] \mid 2 \leq i, j \leq n \}$, and
- (ii') $[u, v^\varphi]^w = [u^w, (v^w)^\varphi] = [u, v^\varphi]^{w^\varphi}$, $\forall \ u, v, w \in N$.

Claim 1. *The subgroup H_1 is normal in $\delta(G)$.* In fact, we already know H_1 is normal in $\langle N, N^\varphi \rangle$. Now by (i') any commutator $[u, v^\varphi]$ in H_1 is a product of elements of $X \cup X^{-1}$, and from our relations a conjugate by a_1 (or b_1) of any such element is again in H_1, for $N \lhd G$. Thus a_1, and hence b_1, also normalizes H_1.

Part(i). To compute $[G, G^\varphi]$ we write a generic element of G in the form $g \cdot a_1^{\alpha_1}$, where $g \in N$ and $0 \leq \alpha < r_1$. Then by commutator calculus we have :

$$\begin{aligned}[g \cdot a_1^{\alpha_1}, h^\varphi \cdot b_1^{\beta_1}] &= [g, h^\varphi \cdot b_1^{\beta_1}]^{a_1^{\alpha_1}} \cdot [a_1^{\alpha_1}, h^\varphi \cdot b_1^{\beta_1}] \\ &= [g, b_1^{\beta_1}]^{a_1^{\alpha_1}} \cdot [g, h^\varphi]^{b_1^{\beta_1} \cdot a_1^{\alpha_1}} \cdot [a_1^{\alpha_1}, b_1^{\beta_1}] \cdot [a_1^{\alpha_1}, h^\varphi]^{b_1^{\beta_1}} \qquad (1)\end{aligned}$$

In expression (1) we separate three types of commutators :

Type 0. The commutator $[a_1^{\alpha_1}, b_1^{\beta_1}]$ is an element of the subgroup of $\delta(G)$ generated by $\{a_1, b_1\}$. By the defining relations this subgroup is nilpotent of class ≤ 2, so that $[a_1^{\alpha_1}, b_1^{\beta_1}] = [a_1, b_1]^{\alpha_1\beta_1}$.

Type 1. The commutator $[g, h^{\varphi}]$ is in H_1. Thus by the inductive assumptions and the normality of H_1 in $\delta(G)$, $[g, h^{\varphi}]$ is a product of elements of $X \cup X^{-1} \subseteq T \cup T^{-1}$.

Type 2. The last type of commutator to consider in (1), taking into account the symmetric behavior of our relations, is $[g, b_1^{\beta_1}]^{a_1^{\alpha_1}}$. Now $g \in N$ and thus $g = x \cdot a_2^{\alpha_2}$, where $x \in < a_3, \ldots, a_n >$. Hence

$$\begin{aligned} [g, b_1^{\beta_1}]^{a_1^{\alpha_1}} &= [x a_2^{\alpha_2}, b_1^{\beta_1}]^{a_1^{\alpha_1}} \\ &= [x a_2^{\alpha_2}, b_1]^{(1+b_1+\ldots+b_1^{\beta_1-1})a_1^{\alpha_1}} \quad \text{(by commutator calculus)} \\ &= [x a_2^{\alpha_2}, b_1]^{\Gamma(b_1^{\beta_1})a_1^{\alpha_1}} \\ &= [x, b_1]^{a_2^{\alpha_2}\Gamma(b_1^{\beta_1})a_1^{\alpha_1}} \cdot [a_2^{\alpha_2}, b_1]^{\Gamma(b_1^{\beta_1})a_1^{\alpha_1}} \\ &= [x, b_1]^{a_2^{\alpha_2}\Gamma(b_1^{\beta_1})a_1^{\alpha_1}} \cdot [a_2, b_1]^{(a_2^{\alpha_2-1}+\ldots+a_2+1)\Gamma(b_1^{\beta_1})a_1^{\alpha_1}} \\ &= [x, b_1]^{a_2^{\alpha_2}\Gamma(b_1^{\beta_1})a_1^{\alpha_1}} \cdot [a_2, b_1]^{\Gamma(a_2^{\alpha_2})\Gamma(b_1^{\beta_1})a_1^{\alpha_1}} \end{aligned} \tag{2}$$

A simple induction now shows that if $g = a_n^{\alpha_n} \cdots a_2^{\alpha_2}$ then

$$[g, b_1^{\beta_1}]^{a_1^{\alpha_1}} = [a_n, b_1]^{\Gamma(a_n^{\alpha_n})a_{n-1}^{\alpha_{n-1}}\cdots a_2^{\alpha_2}\Gamma(b_1^{\beta_1})a_1^{\alpha_1}} \cdots [a_2, b_1]^{\Gamma(a_2^{\alpha_2})\Gamma(b_1^{\beta_1})a_1^{\alpha_1}} \tag{3}$$

Claim 2. *For $2 \leq k \leq n$ and $0 \leq \alpha_k < r_k$, we have*

$$[a_k, b_1]^{a_k^{\alpha_k}} \equiv [a_k, b_1] \quad (\text{mod } H_1) \tag{4}$$

Consequently,

$$[a_k, b_1]^{\Gamma(a_k^{\alpha_k})} \equiv [a_k, b_1]^{\alpha_k} \quad (\text{mod } H_1) \tag{4'}$$

For,

$$\begin{aligned} [a_k, b_1]^{a_k} &= [a_k, b_1^{b_k}] \\ &= [a_k, w_{1k}(b_n, \ldots, b_2) \cdot b_1] \qquad \text{(by relations (*))} \\ &= [a_k, b_1] \cdot [a_k, w_{1k}(b_n, \ldots, b_2)]^{b_1} \\ &\equiv [a_k, b_1] \qquad \text{(by normality of } H_1\text{).} \end{aligned}$$

This proves (4), and keeping in mind that $[a_k, b_1]^{\Gamma(a_k^{\alpha_k})} = [a_k, b_1]^{a_k^{\alpha_k - 1}} \cdots [a_k, b_1]^{a_k} \cdot [a_k, b_1]$, we obtain (4').

By (4') we can rewrite (3) as

$$\begin{aligned} &[g, b_1^{\beta_1}]^{a_1^{\alpha_1}} \equiv \\ &[a_n, b_1]^{\alpha_n a_{n-1}^{\alpha_{n-1}}\cdots a_2^{\alpha_2} W(b_1^{\beta_1}, a_1^{\alpha_1})} \cdot [a_{n-1}, b_1]^{\alpha_{n-1} a_{n-2}^{\alpha_{n-2}}\cdots a_2^{\alpha_2} W(b_1^{\beta_1}, a_1^{\alpha_1})} \cdots \\ &\cdots [a_2, b_1]^{\alpha_2 W(b_1^{\beta_1}, a_1^{\alpha_1})} \quad (\text{mod } H_1) \end{aligned} \tag{5}$$

The arguments used so far also show that if $2 \leq k \leq n$ and if $y_k = a_n^{\gamma_n} \cdots a_k^{\gamma_k}$ is the normal expression of an element of G_{k-1} $(=< a_k, \cdots, a_n >)$, then

$$[y_k, b_1] \equiv [a_n, b_1]^{\gamma_n a_{n-1}^{\gamma_{n-1}} \cdots a_k^{\gamma_k}} \cdots [a_k, b_1]^{\gamma_1} \pmod{H_1} \tag{6}$$

For $2 \leq k \leq n$ let us set $H_k := \langle [a_k, b_1], \ldots, [a_2, b_1], X \rangle$, so that $H_1 \leq H_2 \leq \ldots \leq H_k$. Our goal is to show that $[g, b_1^{\beta_1}]^{a_1^{\alpha_1}} \in H_n$ and to this end we first need to control each factor $[a_k, b_1]^{\alpha_k a_{k-1}^{\alpha_{k-1}} \cdots a_2^{\alpha_2} W(b_1^{\beta_1}, a_1^{\alpha_1})}$ appearing in (5), for $2 \leq k \leq n$.

<u>Claim 3</u>. *For $1 \leq j < k \leq n$ each element of the form $[a_k, b_1]^{a_{k-1}^{\alpha_{k-1}} \cdots a_j^{\alpha_j}}$ can be reduced*, mod H_1, *to a product where each factor has the form $[a_i, b_1]^{\gamma_i a_{i-1}^{\alpha_{i-1}} \cdots a_{j+1}^{\alpha_{j+1}}}$ with $j+1 \leq i \leq n$.*

To see this we first collect $a_j^{\alpha_j}$ on the left using the normal form of the elements of G_j $(= \langle a_{j+1}, \ldots, a_n \rangle)$ and the fact that a_j normalizes G_j. Thus,

$$\begin{aligned} [a_k, b_1]^{a_{k-1}^{\alpha_{k-1}} \cdots a_j^{\alpha_j}} &= [a_k, b_1]^{a_j^{\alpha_j} a_n^{\alpha'_n} \cdots a_{j+1}^{\alpha'_{j+1}}} \\ &= ([a_k, b_1]^{a_j})^{a_j^{\alpha_j - 1} a^{\alpha'_n}{}_n \cdots a_{j+1}^{\alpha'_{j+1}}}. \end{aligned}$$

Now conjugation of $[a_k, b_1]$ by a_j gives

$$\begin{aligned} [a_k, b_1]^{a_j} &= [a_k^{a_j}, b_1^{b_j}] && \text{(by defining relations)} \\ &= [y_{j+1}, w_{j1}(b_n, \ldots, b_2) \cdot b_1] && \text{(by G-relations)} \\ &\equiv [y_{j+1}, b_1] \pmod{H_1} && (\text{since } j+1 \geq 2 \text{ and } H_1 \lhd \delta(G)), \end{aligned}$$

where $y_{j+1} \in G_j$. If $y_{j+1} = a_n^{\nu_n} \cdots a_{j+1}^{\nu_{j+1}}$ is the normal expression of y_{j+1} then by (5) we get

$$[a_k, b_1]^{a_j} \equiv [a_n, b_1]^{\nu_n \cdot a_{n-1}^{\nu_{n-1}} \cdots a_{j+1}^{\nu_{j+1}}} \cdots [a_{j+1}, b_1]^{\nu_{j+1}} \pmod{H_1}.$$

It should be observed that this last expression only involves "basic" commutators $[a_\ell, b_1]$ with $j+1 \leq \ell \leq n$ and the exponents conjugating such commutators only involve integer multiples of elements of G_j. Hence, by successive applications of the above procedure we will certainly remove the factor $a_j^{\alpha_j}$ from the conjugating exponent of $[a_k, b_1]$, that is, $[a_k, b_1]^{a_j^{\alpha_j}}$ is congruent, mod H_1, to a product where each factor has the desired form. Since $a_n^{\alpha'_n} \cdots a_{j+1}^{\alpha'_{j+1}} \in G_j$ we finally obtain the claimed form of $[a_k, b_1]^{a_{k-1}^{\alpha_{k-1}} \cdots a_j^{\alpha_j}}$ by mean of the normal expression of elements of G_j.

The reduction criterion provided by Claim 3 may be considered the crucial step for the proof of our theorem. In fact, upon successive applications of this criterion to the factors on the right side of (5) we can write each such factor as a product of elements, each of which belonging to a left coset of H_n determined by a representative of the form

$$[a_n, b_1]^{\gamma_n \cdot a_{n-1}^{\gamma_{n-1}} \cdots a_2^{\gamma_2} \cdot W(b_1^{\beta_1}, a_1^{\alpha_1})} \tag{7}$$

Our final step is then to show that these representatives are themselves elements of H_n. To this end we can now apply a reverse induction argument. In effect, by using the reduction provided by claim 3 and then (4) and (4') we see that $[a_n, b_1]^{a_{n-1}^{\gamma_{n-1}}}$ is an element of the subgroup of H_n generated by $X \cup \{[a_n, b_1]\}$. Thus we are done in case G has polycyclic length $n = 2$.

Suppose $n > 2$ and, by induction, that $[a_n, b_1]^{\gamma_n \cdot a_{n-1}^{\gamma_{n-1}} \cdots a_{n-k}^{\gamma_{n-k}}}$ is an element of the subgroup of H_n generated by $X \cup \{[a_n, b_1], \ldots, [a_{n-k+1}, b_1]\}$, with $1 \leq k \leq n-2$. But for $n - k + 1 \leq$

$\ell \leq n$ we can apply Claim 3 again to reduce, mod H_1, each conjugate $[a_\ell, b_1]^{a_{n-k-1}^{\gamma_{n-k-1}}}$ to a product where the factors have the form $[a_i, b_1]^{\lambda_i \cdot a_{i-1}^{\lambda_{i-1}} \cdots a_{n-k}^{\lambda_{n-k}}} \cdot [a_{n-k}^{\lambda_{n-k}}, b_1]$, for $n-k+1 \leq i \leq n$. Successive applications of this procedure to the left side factors $[a_i, b_1]^{\lambda_i \cdot a_{i-1}^{\lambda_{i-1}} \cdots a_{n-k}^{\lambda_{n-k}}}$ using arguments similar to those used to reach the representatives (7), and taking into account (6), show that each such factor is writable as a product each of its factors being in a left coset, determined by a representatives of the form $[a_n, b_1]^{\delta_n \cdot a_{n-1}^{\delta_{n-1}} \cdots a_{n-k}^{\delta_{n-k}}}$, of the subgroup of H_n generated by $X \cup \{[a_n, b_1], \ldots, [a_{n-k+1}, b_1]\}$. Hence by our present inductive assumption we conclude that for $n-k+1 \leq i \leq n$, $[a_i, b_1]^{\lambda_i \cdot a_{i-1}^{\lambda_{i-1}} \cdots a_{n-k}^{\lambda_{n-k}}}$ belongs to the above subgroup. On the other hand, by (4'), $[a_{n-k}^{\lambda_{n-k}}, b_1] \equiv [a_{n-k}, b_1]^{\lambda_{n-k}} \pmod{H_1}$. Consequently, $[a_n, b_1]^{\gamma_n \cdot a_{n-1}^{\gamma_{n-1}} \cdots a_{n-k-1}^{\gamma_{n-k-1}}}$ is an element of the subgroup $\langle\, X,\ [a_n, b_1], \ldots, [a_{n-k}, b_1]\,\rangle$, proving our present induction. Therefore part (i) is finally proved.

Part (ii) To prove this second part we first observe that we can use the result in part (i) to write $[g, h^\varphi] = \prod_{i,j} [a_i, b_j]^{\epsilon(i,j)}$, where $\epsilon(i,j) \in \{0, 1, -1\}$, $\forall\, i, j$. It then follows from our relations that

$$[g, h^\varphi]^{a_k} = [g, h^\varphi]^{b_k} \tag{8}$$

for all $g \in G$, $h^\varphi \in G^\varphi$, $1 \leq k \leq n$. Now by definition of $\delta(G)$ we have $[a_i, b_j]^{a_k} = [a_i^{a_k}, b_j^{b_k}]$ for all i, j, k. If a_ℓ (resp. b_ℓ) is any generator of G (resp. G^φ), then:

$$\begin{aligned}
[a_i a_\ell, b_j]^{a_k} &= ([a_i, b_j]^{a_\ell} \cdot [a_\ell, b_j])^{a_k} \\
&= [a_i, b_j]^{a_k a_\ell^{a_k}} \cdot [a_\ell^{a_k}, b_j^{b_k}] \\
&= [a_i^{a_k}, b_j^{b_k}]^{a_\ell^{a_k}} \cdot [a_\ell^{a_k}, b_\ell^{b_k}] \\
&= [a_i^{a_k} a_\ell^{a_k}, b_j^{b_k}] \\
&= [(a_i a_\ell)^{a_k}, b_j^{b_k}]
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
[a_i, b_j b_\ell]^{a_k} &= [a_i, b_j b_\ell]^{b_k} \qquad \text{(by (8))} \\
&= ([a_i, b_\ell] \cdot [a_i, b_j]^{b_\ell})^{b_k} \\
&= [a_i, b_\ell]^{b_k} \cdot [a_i, b_j]^{b_k b_\ell^{b_k}} \\
&= [a_i^{a_k}, b_j^{b_k}] \cdot [a_i^{a_k}, b_j^{b_k}]^{b_\ell^{b_k}} \\
&= [a_i^{a_k}, b_j^{b_k} b_\ell^{b_k}] \\
&= [a_i^{a_k}, (b_j b_\ell)^{b_k}].
\end{aligned} \tag{10}$$

Taking into account the normal form of the elements in G (resp. G^φ), identities (9) and (10) provide us with a recursive criterion to prove that

$$[g, h^\varphi]^{a_k} = [g^{a_k}, (h^\varphi)^{b_k}] \quad \forall\, g \in G,\ h^\varphi \in G^\varphi,\ 1 \leq k \leq n,$$

which in turn proves (ii). ∎

We recall that for arbitrary isomorphic groups G, G^φ, where $\varphi : G \to G^\varphi, g \mapsto g^\varphi, \forall g \in G$ is an isomorphism, a group $\mathcal{V}(G)$ has been defined as (see [11]):

$$\mathcal{V}(G) := \langle G, G^\varphi \mid [g_1, g_2^\varphi]^{g_3} = [g_1^{g_3}, (g_2^{g_3})^\varphi] = [g_1, g_2^\varphi]^{g_3^\varphi}, \quad \forall\, g_1, g_2, g_3 \in G \rangle$$

The given isomorphism φ extends uniquely to an automorphism (also denoted by φ) of $\mathcal{V}(G)$ such that $g \mapsto g^\varphi, g^\varphi \mapsto g$ and $[g_1, g_2^\varphi] \mapsto [g_2, g_1^\varphi]$.

2.2 Corollary. *Let G and G^φ be distinct isomorphic finite solvable groups given by AG-systems $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ respectively, as in Theorem 2.1. Then $\delta(G)$ is a presentation of $\mathcal{V}(G)$.*

Proof. Immediate by Theorem 2.1, since the set $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ actually generates the group $\mathcal{V}(G)$. ∎

For easy of reference we reproduce in the next lemma some of the relations satisfied by $\mathcal{V}(G)$, for a general group argument G (the reader is referred to [11], Lemma 2.1 for a proof).

2.3 Lemma. *$\mathcal{V}(G)$ satisfies the following relations:*

(i) $[g_1, g_2^\varphi]^{[g_3, g_4^\varphi]} = [g_1, g_2^\varphi]^{[g_3, g_4]}, \quad \forall\, g_1, g_2, g_3, g_4 \in G;$

(ii) $[g_1, g_2^\varphi, g_3] = [g_1, g_2, g_3^\varphi] = [g_1, g_2^\varphi, g_3^\varphi]$ *and*

$[g_1^\varphi, g_2, g_3] = [g_1^\varphi, g_2, g_3^\varphi] = [g_1^\varphi, g_2^\varphi, g_3], \quad \forall\, g_1, g_2, g_3 \in G;$

(iii) $[g, g^\varphi]$ *is central in* $\mathcal{V}(G), \quad \forall\, g \in G;$

(iv) $[g_1, g_2^\varphi][g_2, g_1^\varphi]$ *is central in* $\mathcal{V}(G), \quad \forall\, g_1, g_2 \in G;$

(v) $[g, g^\varphi] = 1, \quad \forall\, g \in G'.$

As we observed in the introduction, the subgroup $\Upsilon(G) := [G, G^\varphi]$ of $\mathcal{V}(G)$ is isomorphic to the non-abelian *tensor square* $G \otimes G$, such an isomorphism being defined by : $[h, g^\varphi] \mapsto h \otimes g, \ \forall\ g,\ h \in G$. (see [11] and [3] for references). Its subgroup $\Delta(G) := < [g, g^\varphi] \mid g \in G >$, which by Lemma 2.3 is central in $\mathcal{V}(G)$, is such that the quotient $\Upsilon(G)/\Delta(G)$ is isomorphic to the *exterior square* $G \wedge G$ (cf. Miller [7]; see also [2]).

Remark 1. It is appropriate to note that, modulo $\Delta(G)$, we have

$$\begin{aligned} 1 &\equiv [hg, (hg)^\varphi] \\ &\equiv [h, h^\varphi][g, g^\varphi][h, g^\varphi][g, h^\varphi] \qquad (\Delta(G) \text{ is central}) \\ &\equiv [h, g^\varphi][g, h^\varphi]. \end{aligned}$$

Therefore $[h, g^\varphi] \equiv [h^\varphi, g] \pmod{\Delta(G)}$ or, which is to say, *the extended automorphism φ of $\mathcal{V}(G)$ centralizes $\Upsilon(G)$ modulo $\Delta(G)$.*

Let $\Theta(G)$ denote the subgroup of $\mathcal{V}(G)$ generated by all the elements $g^{-1}g^\varphi, \ \forall\ g \in G$. (This subgroup is also usually written $[G, \varphi] :=< [g, \varphi] \mid g \in G >$, where $[g, \varphi]$

means $g^{-1}g^{\varphi}$). It follows from the relations $[h,g^{\varphi}]^k = [h,g^{\varphi}]^{k^{\varphi}}$ that $[h,g^{\varphi}]^{k^{-1}k^{\varphi}} = [h,g^{\varphi}]$, $\forall\, h,g,k \in G$. Therefore we have

Remark 2. $\Theta(G)$ *centralizes* $\Upsilon(G)$.

The role of $\Theta(G)$ is shown in the

2.4 Proposition. (i) $\Theta(G) \lhd \mathcal{V}(G)$;

(ii) $\mathcal{V}(G) = \Theta(G)\cdot G$, *a semidirect product;*

(ii)' *There is an epimorphism* $\rho : \mathcal{V}(G) \to G$, $g \mapsto g$, $g^{\varphi} \mapsto g$, $\forall\, g \in G$, *such that* $Ker(\rho) = \Theta(G)$.

Proof. (i) Denoting $g^{-1}g^{\varphi}$ by $[g,\varphi]$ then the identity $[hg,\varphi] = [h,\varphi]^g\cdot[g,\varphi]$ shows that $[h,\varphi]^g \in \Theta(G)$, $\forall\, h, g \in G$. Also

$$[h,\varphi]^{g^{\varphi}} = [h,\varphi]^{g(g^{-1}g^{\varphi})} = ([hg,\varphi]\cdot[\varphi,g])^{[g,\varphi]} \in \Theta(G),\ \forall h,g \in G.$$

These prove that $\Theta(G) \lhd \mathcal{V}(G)$.

(ii), (ii)' From $g^{\varphi} = g\cdot g^{-1}\cdot g^{\varphi} = g\cdot[g,\varphi]$ we obtain by part (i) that

$$< G, G^{\varphi} > = \mathcal{V}(G) = [G,\varphi]\cdot G \quad (= \Theta(G)\cdot G).$$

Now the map $g \mapsto g$, $g^{\varphi} \mapsto g$, $\forall g \in G$ extends naturally to an epimorphism $\rho : \mathcal{V}(G) \to G$ (since the defining relations of $\mathcal{V}(G)$ are the commutator relations on G), whose restriction to G is the identity map. As $\Theta(G) \leq Ker(\rho)$, $\Theta(G)\cap G = \{1\}$ and $\Theta(G) = Ker(\rho)$. ∎

The restriction of the epimorphism ρ to $\Upsilon(G)$ gives the *derived map*

$$\rho' : \Upsilon(G) \to G',\ [h,g^{\varphi}] \mapsto [h,g],\ \forall\, h,g \in G.$$

As a consequence of Theorem 2.1 and its Corollary 2.2 we then obtain the following well known result for finite solvable groups :

2.5 Proposition. *Let* G *be a finite solvable group given by an AG-system* $\{a_1,\ldots,a_n\}$. *Then the derived group* G' *is generated by the set*

$$T' = \{[a_i,a_j] \mid 1 \leq i < j \leq n\}.$$

Proof. We just apply ρ' to the set T of Theorem 2.1 (i) which, by Corollary 2.2, generates the subgroup $\Upsilon(G)$. ∎

Let us denote by $\mu(G)$ the kernel of ρ'. Hence $\Delta(G) \lhd \mu(G)$ and $\mu(G) = \Upsilon(G)\cap\Theta(G)$. The following relations will be useful in the study of $\mu(G)$.

2.6 Lemma. *For elements* $g,\ h,\ k \in G$ *the following identities hold*

(i) $[h,g^{\varphi}] = [\varphi,g,h]\cdot[h,g]$;

(ii) $[\varphi,g,h,k^{\varphi}] = 1$.

Proof. For (i) we have:

$$\begin{aligned}[h, g^{\varphi}] &= [h, g \cdot g^{-1} g^{\varphi}] \\ &= [h, g \cdot [g, \varphi]] \\ &= [h, [g, \varphi]][h, g]^{[g, \varphi]}.\end{aligned}$$

On conjugating the above identity by $[g, \varphi]^{-1}$, and using the fact that $\Theta(G)$ centralizes $\Upsilon(G)$, we get

$$\begin{aligned}[h, g^{\varphi}] &= [h, [g, \varphi]]^{[g, \varphi]^{-1}} \cdot [h, g] \\ &= [h, [\varphi, g]]^{-1} \cdot [h, g] \\ &= [\varphi, g, h] \cdot [h, g].\end{aligned}$$

As for (ii), use of (i) and Lemma 2.3 (ii) give

$$\begin{aligned}[\varphi, g, h, k^{\varphi}] &= [[h, g^{\varphi}] \cdot [g, h], k^{\varphi}] \\ &= [h, g^{\varphi}, k^{\varphi}]^{[g,h]} \cdot [g, h, k^{\varphi}] \\ &= [h, g, k^{\varphi}]^{[g,h]} \cdot [g, h, k^{\varphi}] = 1.\end{aligned}$$

∎

2.7 Proposition. (i) *$\mu(G)$ consists of all elements of $\Upsilon(G)$ of the form $[h_1, g_1^{\varphi}]^{\epsilon_1} \cdots [h_s, g_s^{\varphi}]^{\epsilon_s}$ such that $[h_1, g_1]^{\epsilon_1} \cdots [h_s, g_s]^{\epsilon_s} = 1$, where s is a natural number, $h_i, g_i \in G$, $\epsilon_i \in \{1, -1\}$, $1 \leq i \leq s$;*

(ii) *$\mu(G)$ is central in $\mathcal{V}(G)$.*

Proof. (i): Let $\gamma = [h_1, g_1^{\varphi}]^{\epsilon_1} \cdots [h_s, g_s^{\varphi}]^{\epsilon_s}$ be a generic element of $\Upsilon(G)$ with h_i, $g_i \in G$, $\epsilon_i \in \{1, -1\}$, $1 \leq i \leq s$. By Lemma 2.6,

$$\begin{aligned}\gamma &= ([\varphi, g_1, h_1] \cdot [h_1, g_1])^{\epsilon_1} \cdots ([\varphi, g_s, h_s] \cdot [h_s, g_s])^{\epsilon_s} \\ &= u \cdot [h_1, g_1]^{\epsilon_1} \cdots [h_s, g_s]^{\epsilon_s},\end{aligned}$$

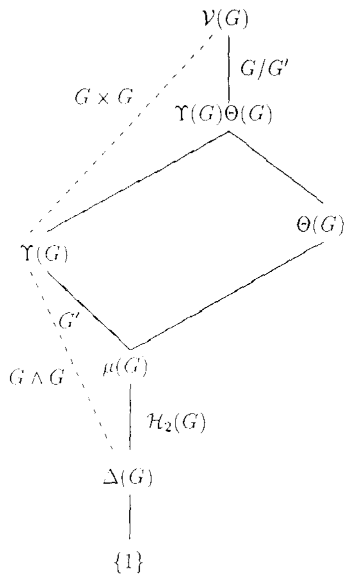
where $u \in \Theta(G) = Ker(\rho)$. Therefore $\gamma^{\rho'} = [h_1, g_1]^{\epsilon_1} \cdots [h_s, g_s]^{\epsilon_s}$, so that $\gamma \in \mu(G)$ if and only if $[h_1, g_1]^{\epsilon_1} \cdots [h_s, g_s]^{\epsilon_s} = 1$.

(ii) Let $\gamma = [h_1, g_1^{\varphi}]^{\epsilon_1} \cdots [h_s, g_s^{\varphi}]^{\epsilon_s} \in \mu(G)$ and $h \in G$. Commutator calculus yields :

$$\begin{aligned}[\gamma, h] &= [[h_1, g_1^{\varphi}]^{\epsilon_1} \cdots [h_s, g_s^{\varphi}]^{\epsilon_s}, h] \\ &= [[h_1, g_1^{\varphi}]^{\epsilon_1}, h]^{[h_2, g_2^{\varphi}]^{\epsilon_2} \cdots [h_s, g_s^{\varphi}]^{\epsilon_s}} \cdot [[h_2, g_2^{\varphi}]^{\epsilon_2} \cdots [h_s, g_s^{\varphi}]^{\epsilon_s}, h] \\ &= \ldots\ldots\ldots\ldots\ldots\ldots \\ &= [[h_1, g_1^{\varphi}]^{\epsilon_1}, h]^{[h_2, g_2^{\varphi}]^{\epsilon_2} \cdots [h_s, g_s^{\varphi}]^{\epsilon_s}} \cdots [[h_s, g_s^{\varphi}]^{\epsilon_s}, h] \\ &= [[h_1, g_1]^{\epsilon_1}, h^{\varphi}]^{[h_2, g_2]^{\epsilon_2} \cdots [h_s, g_s]^{\epsilon_s}} \cdots [[h_s, g_s]^{\epsilon_s}, h^{\varphi}] \qquad \text{(Lemma 2.3 (i),(ii))} \\ &= [[h_1, g_1]^{\epsilon_1} \cdots [h_s, g_s]^{\epsilon_s}, h^{\varphi}] \\ &= 1 \qquad \text{(by part (i))}.\end{aligned}$$

The above shows that G centralizes $\mu(G)$. Since by definition of $\mathcal{V}(G)$ the action of an element $h^{\varphi} \in G^{\varphi}$ on $\Upsilon(G)$ is the same as that of the corresponding $h \in G$, part (ii) is also proved. ∎

The diagram below summarizes some of those informations concerning the structure of $\mathcal{V}(G)$ we obtained in Propositions 2.4 - 2.8. The top section G/G' is not but the image of the composite map $\rho\pi$, where $\pi : G \to G/G'$ is the canonical epimorphism.



Remark 3. With the isomorphism between $\Upsilon(G)$ and the non-abelian tensor square $G \otimes G$, we observe that our subgroup $\mu(G)$ corresponds to the subgroup $J_2(G)$ of [3]. Part (ii) of Proposition 2.7 agrees with ([3], Proposition 2.5).

The following result is a consequence of our previous remarks and C. Miller's description of the second homology group ([7]) :

2.8 Proposition. *The section $\mu(G)/\Delta(G)$ of $\mathcal{V}(G)$ is isomorphic to the second homology group $\mathcal{H}_2(G)$.*

Remark 4. Some of the results concerning the subgroups $\Upsilon(G)$, $\Theta(G)$, and $\mu(G)$ are appropriate adaptations of similar results of [12], where S.Sidki studies the group

$$\chi(G) = \langle G, G^{\varphi} \mid [g, g^{\varphi}] = 1, \forall\, g \in G \rangle.$$

As we pointed out in [11], $\chi(G)$ contains a subgroup $R(G)$ such that the relations $[g_1, g_2^{\varphi}]^{g_3^{\varphi}} = [g_1^{g_3}, (g_2^{g_3})^{\varphi}] = [g_1, g_2^{\varphi}]^{g_3}$ hold in $\chi(G)/R(G)$ for all $g_1, g_2, g_3 \in G$. That subgroup is defined by $R(G) := [G, L(G), G^{\varphi}]$, where $L(G)$ is the subgroup of $\chi(G)$ generated by all $g^{-1}g^{\varphi}$, $g \in G$.

It results that on introducing in $\mathcal{V}(G)$ the relations $[g, g^{\varphi}] = 1$, $\forall\, g \in G$, we get an epimorphism $\zeta : \mathcal{V}(G)/\Delta(G) \to \chi(G)/R(G)$ such that $g\Delta(G) \mapsto gR(G)$, $g^{\varphi}\Delta(G) \mapsto g^{\varphi}R(G)$, $\forall\, g \in G$, $g^{\varphi} \in G^{\varphi}$. It is opportune to mention that for a finite group G, the order of $\chi(G)/R(G)$ is given by $|\chi(G)/R(G)| = |G|^2 \cdot |G'| \cdot |M(G)|$, where $M(G)$ denotes the Schur Multiplier of G ([10], Lemmas 2.2, 2.3 and [12], Lemma 4.1.11). Consequently, by a quick look over the diagram above we deduce, as a matter of order, that ζ is an isomorphism in case $|G|$ is finite.

On the other hand it is clear that $\mathcal{V}(G)/\Delta(G)$ is a homomorphic image of $\chi(G)$, where $g \mapsto g\Delta(G)$, $g^\varphi \mapsto g^\varphi\Delta(G)$, $\forall g \in G$ defines an epimorphism $\xi : \chi(G) \to \mathcal{V}(G)/\Delta(G)$. For the remainder of this section we evolve to show that ξ induces an isomorphism inverse of ζ, for any group G.

2.9 Lemma. *Let g, h, u, k be arbitrary elements of a group G. Then*

(i) $[h, g^\varphi, [g,h]] = 1$;

(ii) $[[\varphi, g, h], [g,h]] = 1$;

(iii) $[\varphi, g, h, u, k^\varphi] = [\varphi, g, h, u^\varphi, k^\varphi]$;

(iv) $[\varphi, g, h, u, k^\varphi] = 1$.

Proof. (i) follows from

$$\begin{aligned}[h, g^\varphi, [g,h]] &= [h, g, [g,h]^\varphi] && \text{(by Lemma 2.3 (ii))}\\ &= [[g,h]^{-1}, [g,h]^\varphi] \\ &= [g, h, [g,h]^\varphi]^{-1} && \text{(by Lemma 2.3 (iii))}\\ &= 1 && \text{(by Lemma 2.3 (v))};\end{aligned}$$

(ii) Since by Lemma 2.6 $[\varphi, g, h] = [h, g^\varphi][g,h]$, we have

$$\begin{aligned}1 &= [\varphi, g, h, [h, g^\varphi]\cdot[g,h]]\\ &= [\varphi, g, h, [g,h]]\cdot[\varphi, g, h, [h, g^\varphi]]^{[g,h]}.\end{aligned}$$

But $[\varphi, g, h] \in \Theta(G)$, which is centralized by $\Upsilon(G)$. Hence $[[\varphi, g, h], [g,h]] = 1$.

(iii) We refer to Lemmas 2.3 and 2.6 for the following identities :

$$\begin{aligned}[\varphi, g, h, u, k^\varphi] &= [[h, g^\varphi]\cdot[g,h], u, k^\varphi]\\ &= [[h, g^\varphi, u]^{[g,h]}\cdot[g,h,u], k^\varphi]\\ &= [[h, g^\varphi, u^\varphi]^{[g,h]}\cdot[g,h,u], k^\varphi]\\ &= [[h, g^\varphi, u^\varphi]^{[g,h]}, k^\varphi]^{[g,h,u]}\cdot[g,h,u,k^\varphi].\end{aligned}$$

Now $[h, g^\varphi, u^\varphi] \in \Upsilon(G) \lhd \mathcal{V}(G)$ and for $\gamma \in \Upsilon(G)$ it holds $\gamma^{[x,y]} = \gamma^{[x,y^\varphi]}$, $\forall\ x,\ y\ \in\ G$. Thus, by Lemma 2.3 (ii) again,

$$\begin{aligned}[\varphi, g, h, u, k^\varphi] &= [[h, g^\varphi, u^\varphi]^{[g,h]}, k^\varphi]^{[g,h,u^\varphi]}\cdot[g,h,u^\varphi,k^\varphi].\\ &= [[h, g^\varphi, u^\varphi]^{[g,h]}\cdot[g,h,u^\varphi], k^\varphi]\\ &= [[h, g^\varphi]\cdot[g,h], u^\varphi, k^\varphi].\end{aligned}$$

(iv) is a direct consequence of (iii) and Lemma 2.6 (ii). ∎

As a consequence we obtain the following interesting identity

2.10 Proposition. *Let $g,\ h,\ k,\ u_1,\ v_1,\ \ldots, u_n,\ v_n,\ n \geq 1$, be elements of a group G. Then*

$$[\varphi,\ g,\ h,\ u_1 v_1^\varphi \cdots u_n v_n^\varphi,\ k^\varphi] \ =\ 1.$$

Proof. For $n = 1$ this follows from

$$\begin{aligned}[\varphi, g, h, u_1 v_1^\varphi, k^\varphi] &= [[\varphi, g, h, v_1^\varphi]\cdot[\varphi, g, h, u_1]^{v_1^\varphi}, k^\varphi]\\ &= [[\varphi, g, h, u_1]^{v_1^\varphi}, k^\varphi] && \text{(by Lemma 2.6 (ii))}\\ &= [\varphi, g, h, u_1, (k^{v_1^{-1}})^\varphi]^{v_1^\varphi}\\ &= 1 && \text{(by Lemma 2.9 (iv)).}\end{aligned}$$

Suppose the assertion is true for some $n \geq 1$ and let $x := u_1 v_1^{\varphi} \cdots u_n v_n^{\varphi}$. Then

$$
\begin{aligned}
&[\varphi, g, h, x \cdot u_{n+1} v_{n+1}^{\varphi}, k^{\varphi}] = \\
&\quad [[\varphi, g, h, u_{n+1} v_{n+1}^{\varphi}] \cdot [\varphi, g, h, x]^{u_{n+1} v_{n+1}^{\varphi}}, k^{\varphi}] \\
&\quad = [[\varphi, g, h, u_{n+1} v_{n+1}^{\varphi}] \cdot [\varphi, g, h, x]^{u_{n+1} v_{n+1}^{\varphi}}, k^{\varphi}] \\
&\quad = [[\varphi, g, h, x]^{u_{n+1} v_{n+1}^{\varphi}}, k^{\varphi}] \qquad \text{(by commutator calculus and case } n = 1) \\
&\quad = [[\varphi, g, h, x]^{u_{n+1}}, (k^{v_{n+1}^{-1}})^{\varphi}]^{v_{n+1}^{\varphi}} \\
&\quad = [[\varphi, g, h, x] \cdot [\varphi, g, h, x, u_{n+1}], (k^{v_{n+1}^{-1}})^{\varphi}]^{v_{n+1}^{\varphi}} \\
&\quad = [\varphi, g, h, x, (k^{v_{n+1}^{-1}})^{\varphi}]^{[\varphi, g, h, x, u_{n+1}] v_{n+1}^{\varphi}} \cdot [\varphi, g, h, x, u_{n+1}, (k^{v_{n+1}^{-1}})^{\varphi}]^{v_{n+1}^{\varphi}} \\
&\quad = [\varphi, g, h, x, u_{n+1}^{\varphi}, k^{v_{n+1}^{-1}}]^{v_{n+1}^{\varphi}} \qquad \text{(this by case } n = 1 \text{ and Lemma 2.9 (iii))} \\
&\quad = 1 \qquad \text{(by Lemma 2.9 (iv))}.
\end{aligned}
$$

The proof is thus concluded. ∎

Finally we have

2.11 Theorem. *If G is any group, then*

(i) $[\Theta(G), G, G^{\varphi}] = 1$;

(ii) $\chi(G)/R(G) \cong \mathcal{V}(G)/\Delta(G)$.

Proof. (i): From $\Theta(G) = \langle [\varphi, g] \mid g \in G \rangle$ we see that $[\Theta(G), G]$ is the normal closure :

$$
\begin{aligned}
[\Theta(G), G] &= \langle [\varphi, g, h] \mid g, h \in g \rangle^{\mathcal{V}(G)} \\
&= \langle [\varphi, g, h], [\varphi, g, h, u] \mid g, h \in G, u \in < G, G^{\varphi} > \rangle.
\end{aligned}
$$

Hence

$$[\Theta(G), G, G^{\varphi}] = \langle [\varphi, g, h, k^{\varphi}], [\varphi, g, h, u, k^{\varphi}] \mid g, h, k \in G, u \in < G, G^{\varphi} > \rangle^{\mathcal{V}(G)}$$

and thus $[\Theta(G), G, G^{\varphi}] = 1$ by Lemmas 2.6 (ii) and 2.9 (iv).

(ii): As in the discussion preceding Lemma 2.9, let $\xi : \chi(G) \to \mathcal{V}(G)/\Delta(G)$ be the epimorphism given by $g \mapsto g\Delta(G)$, $g^{\varphi} \mapsto g^{\varphi}\Delta(G)$, $\forall g \in G$. By composing ξ and ζ it is then obvious that $Ker(\xi) \leq R(G)$ $(= [G, L(G), G^{\varphi}])$. On the other hand, ξ maps $R(G)$ to $[G, \Theta(G), G^{\varphi}]$ (mod $\Delta(G)$), which is trivial by part (i). Hence $ker(\xi) = R(G)$ and consequently ξ induces on $\chi(G)/R(G)$ an inverse of ζ. ∎

3. The Subgroup $\Delta(G)$

In this section we set some more results concerning the subgroup $\Delta(G)$ for an arbitrary group G. A convenient set of generators for it is found in Proposition 3.3. The following Lemma and its Corollary are easy consequences of Lemma 2.3 and commutator calculus.

3.1 Lemma. *Let G be any group and g, h generic elements of G. Then*

(i) $[g, h^{\varphi}][h, g^{\varphi}] = [gh, (gh)^{\varphi}] \cdot [h, h^{\varphi}]^{-1} \cdot [g, g^{\varphi}]^{-1}$ $(\in \Delta(G))$;

(ii) $[g, h^{\varphi}][h, g^{\varphi}] = [h, g^{\varphi}][g, h^{\varphi}]$;

(iii) *If* $h \in G'$ *then* $[g, h^{\varphi}][h, g^{\varphi}] = 1$;

(iv) *if* $g\,G' = h\,G'$ *then* $[g, g^{\varphi}] = [h, h^{\varphi}]$;

(v) *Denote by* $o'(x)$ *the order of a coset* xG', $x \in G$. *If* $o'(g)$ *or* $o'(h)$ *is finite, then* $[g, h^{\varphi}][h, g^{\varphi}]$ *has order dividing* $gcd(o'(g), o'(h))$ (by abuse of notation we set $gcd(n, \infty) := n$ for a natural number n);

(vi) *If* $o'(h)$ *is finite, then* $[h, h^{\varphi}]$ *has order dividing* $gcd(o'(h)^2, 2o'(h))$.

3.2 Corollary. *Let* G *be a finite group of odd order and* $g \in G$ *be an element with* $o'(g) = s$. *Then* $[g, g^{\varphi}]$ *has order dividing* s.

3.3 Proposition. *Let* $X = \{x_i\}_{i\in I}$ *be a set of generators of a group* G, *where we assume that* I *is a totally ordered set. Then* $\Delta(G)$ *is generated by the set*

$$\Delta := \{s_i := [x_i, x_i^{\varphi}]\,,\ t_{jk} := [x_j, x_k^{\varphi}][x_k, x_j^{\varphi}] \mid i,\ j,\ k \in I,\ j < k\}.$$

Proof. Let $g = h\cdot x_i^{\epsilon_i}$ and $h = w\cdot x_j^{\epsilon_j}$ be elements of G with $x_i\,,\ x_j \in X,\ \epsilon_i\,,\ \epsilon_j \in \{1, -1\}$ and w a word on $X \cup X^{-1}$. Computations like those performed in the proof of part (i) of Lemma 3.1 show that

$$\begin{aligned}[g, g^{\varphi}] &= [hx_i^{\epsilon_i}, (hx_i^{\epsilon_i})^{\varphi}] \\ &= [h, h^{\varphi}]\cdot[x_i, x_i^{\varphi}]^{\epsilon_i}\cdot([h, x_i^{\varphi}]\cdot[x_i, h^{\varphi}])^{\epsilon_i}\end{aligned}$$

and

$$\begin{aligned}[h, x_i^{\varphi}]\cdot[x_i, h^{\varphi}] &= [wy_j^{\epsilon_j}, x_i^{\varphi}]\cdot[x_i, w^{\varphi}(y_j^{\epsilon_j})^{\varphi}] \\ &= [w, x_i^{\varphi}]^{y_j^{\epsilon_j}}\cdot[y_j^{\epsilon_j}, x_i^{\varphi}]\cdot[x_i, (y_j^{\epsilon_j})^{\varphi}]\cdot[x_i, w^{\varphi}]^{(y_j^{\epsilon_j})^{\varphi}} \\ &= ([y_j, x_i^{\varphi}][x_i, y_j^{\varphi}])^{\epsilon_j}\cdot([w, x_i^{\varphi}][x_i, w^{\varphi}]).\end{aligned}$$

Using these identities we can easily complete the proof by induction on the length of g and h in the elements of $X \cup X^{-1}$, once $\Delta(G)$ is generated by all $[g, g^{\varphi}]$, $g \in G$. The choice $j < k$ in Δ is guaranteed by Lemma 3.1. ∎

3.4 An Application. We shall now use our approach to compute the Schur Multiplier of an arbitrary finite metacyclic group ([13] and [1]).

It is well known that such a group G has an AG-presentation

$$G = \langle\, a_1,\ a_2 \mid a_2^m = 1,\ a_1^s = a_2^t,\ a_2^{a_1} = a_2^r\,\rangle,$$

where m, s, t, r are integers such that $m,\ s > 0$, $r^s \equiv 1 \pmod m$ and m *divides* $t(r-1)$, so that G has order ms. Using the notation previously established, let $G^{\varphi} = \langle b_1, b_2 \rangle$ be the other copy of G, with the corresponding G^{φ}-relations. Hence $\Upsilon(G)$ is generated by $\{[a_1, b_1],\ [a_2, b_2],\ [a_1,\ b_2],\ [a_2, b_1]\}$, by Corollary 2.2, while $\Delta(G)$ is generated by $\{[a_1, b_1],\ [a_2, b_2],\ [a_1, b_2][a_2, b_1]\}$, by the last proposition. From this information we see that the factor group $\Upsilon(G)/\Delta(G)$ is cyclic, generated by the coset $[a_1, b_2]\,\Delta(G)$. The Schur Multiplier $M(G)$ is by Proposition 2.8 the quotient $\mu(G)/\Delta(G)$. Given that $\mu(G)$ is the kernel of the derived map ρ', we then get $\Upsilon(G)/\mu(G) \cong G'$. But from the presentation of

G it is readily seen that G' is generated by $[a_2, a_1] = a_2^{r-1}$, so that G' has order $\frac{m}{(m,r-1)}$, where $(m, r-1)$ denotes the g.c.d.$(m, r-1)$. On the other hand $G' = < [a_1, b_2]^{\rho'} >$, from which it follows that $\mu(G)/\Delta(G)$ is generated by the coset $[a_1, b_2]^{\frac{m}{(m,r-1)}}\Delta(G)$. So far we have got the information that $M(G)$ is cyclic, having order a divisor of the order of $[a_1, b_2]^{\frac{m}{(m,r-1)}}$ modulo $\Delta(G)$.

Now using G-relations and Lemma 2.3 we get by commutator calculus (note that $a_1^{\varphi} = b_1$ and $a_2^{\varphi} = b_2$) :

$$\begin{aligned}
[a_1, b_2]^{b_2} &= [a_1, b_2]^{a_2} \\
&= [a_1, b_2][a_1, b_2, a_2] \\
&= [a_1, b_2][a_1, a_2, b_2] \qquad \text{(by Lemma 2.3 (ii))} \\
&= [a_1, b_2][a_2^{m-(r-1)}, b_2] \qquad \text{(by G-relations)} \\
&= [a_1, b_2][a_2, b_2]^{m-(r-1)} \qquad \text{(by Lemma 2.3 (v))} \\
&\equiv [a_1, b_2] \qquad (mod\ \Delta(G))
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
[a_1, b_2]^{a_1} &= [a_1, b_2^{b_1}] \\
&= [a_1, b_2^r] \\
&\equiv [a_1, b_2]^r \qquad (mod\ \Delta(G)).
\end{aligned} \tag{13}$$

Since $a_1^s = a_2^t$ we have

$$[a_1^s, b_2] = [a_2^t, b_2] = [a_2, b_2]^t \equiv 1 \ \ (mod\ \Delta(G)).$$

On the other hand, by expression (13)

$$\begin{aligned}
[a_1^s, b_2] &= [a_1, b_2]^{1+a_1+\cdots+a_1^{s-1}} \\
&\equiv [a_1, b_2]^{1+r+\cdots+r^{s-1}} \qquad (mod\ \Delta(G)).
\end{aligned}$$

This shows that $[a_1, b_2]^{1+r+\cdots+r^{s-1}} \equiv 1 \ \ (mod\ \Delta(G))$.

Therefore $o([a_1, b_2]\Delta(G))$ divides $1 + r + \cdots + r^{s-1}$. (14)

Also, $[a_1, b_2^t] = [a_1, b_1^s] = [a_1, b_1]^s \equiv 1 \ \ (mod\ \ \Delta(G))$, while by (12),

$[a_1, b_2^t] = [a_1, b_2]^{1+b_2+\cdots+b_2^{t-1}} \equiv [a_1, b_2]^t \qquad (mod\ \Delta(G))$.

Thus $o([a_1, b_2]\Delta(G))$ divides t (15)

and, since $b_2^m = 1$, clearly we have by (12)

$$o([a_1, b_2]\Delta(G)) \text{ divides } m. \tag{16}$$

Consequently, $o([a_1, b_2]^{\frac{m}{(m,r-1)}}\Delta(G))$ ($= |M(G)|$) divides

$$k := \frac{(m, r-1)}{m} \cdot (m,\ t,\ 1 + r + \cdots + r^{s-1}).$$

To see that k is the precise order of $M(G)$ we can use the classical argument of constructing a *covering group* $\tilde{G}$ of G, in the sense that $\tilde{G}$ contains a subgroup Z such that $Z \leq \tilde{G}' \cap Z(\tilde{G})$ and $\tilde{G}/Z \cong G$. By a well known property of the Multiplier, such a subgroup Z is then a homomorphic image of $M(G)$.

But, doing computations similar to those performed above, it is not hard to check that, with the foregoing integers m, s, t, r and k, the group presented by

$$\tilde{G} = \langle a, b, c \mid a^m = 1, b^s = a^t, c^k = 1, a^b = a^r \cdot c, c^a = c, c^b = c \rangle,$$

satisfies the desired property, with $Z = \langle c \rangle$ of order k (see [5], page 301).

We conclude that $M(G)$ is in fact a *cyclic group of order* k, which agrees e.g. with [1], [13].

Remark 5. In order to analyse the subgroup $\Delta(G)$ more closely, let us digress for a moment. When G is a direct product, $G = N \times M$, the subgroup $\Upsilon(G)$ of $\mathcal{V}(G)$ is given by

$$\Upsilon(G) = \Upsilon(N) \times \Upsilon(M) \times [N, M^{\varphi}] \cdot [M, N^{\varphi}],$$

with $[N, M^{\varphi}]$ (resp. $[M, N^{\varphi}]$) being isomorphic to the usual tensor product $N \otimes_{\mathbb{Z}} M$ (resp. $M \otimes_{\mathbb{Z}} N$). We make evident here that the above decomposition of $\Upsilon(G)$ is found in ([11], Proposition 3.6 (iii)) where by a misprint it appeared with the missing factor $[N, M^{\varphi}] \cdot [M, N^{\varphi}]$ (see also [2], Remark 2). In this case

$$\Delta(G) = \Delta(N) \times \Delta(M) \times U$$

where U is the subgroup of $[N, M^{\varphi}][M, N^{\varphi}]$ generated by all $[x, y^{\varphi}][y, x^{\varphi}]$ with $x \in N$ and $y \in M$. By the isomorphism between $N \otimes_{\mathbb{Z}} M$ and $M \otimes_{\mathbb{Z}} N$ it results that U is isomorphic to $N \otimes_{\mathbb{Z}} M$. In fact, let V denote the subgroup of $(N \otimes_{\mathbb{Z}} M) \times (M \otimes_{\mathbb{Z}} N)$ generated by all $(x \otimes y)(y \otimes x)$ with $x \in N$, $y \in M$. It is clear that there is an epimorphism

$$\phi : N \otimes_{\mathbb{Z}} M \to V,\ x \otimes y \mapsto (x \otimes y)(y \otimes x), \forall x \in N,\ y \in M.$$

On the other side the isomorphism $f : M \otimes_{\mathbb{Z}} N \to N \otimes_{\mathbb{Z}} M$ given by $y \otimes x \mapsto x \otimes y$ yields the isomorphism :

$$(1 \times f) : (N \otimes_{\mathbb{Z}} M) \times (M \otimes_{\mathbb{Z}} N) \to (N \otimes_{\mathbb{Z}} M) \times (N \otimes_{\mathbb{Z}} M),$$
$$(x_1 \otimes y_1, y_2 \otimes x_2) \mapsto (x_1 \otimes y_1, x_2 \otimes y_2).$$

Restriction of $(1 \times f)$ to the *"diagonal"* composed with the projection on the first coordinate gives an inverse of ϕ. Since $U \cong V$ we get the assertion and thus :

$$\Delta(N \times M) \cong \Delta(N) \times \Delta(M) \times (N \otimes_{\mathbb{Z}} M).$$

Taking into account that for an abelian group A, $\Upsilon(A)$ is isomorphic to $A \otimes_{\mathbb{Z}} A$ (see [11], Remark 5), we can describe $\Delta(A)$ for all finitely generated abelian groups A, once $\Delta(A)$ corresponds to the *diagonal subgroup* D of $A \otimes_{\mathbb{Z}} A$, generated by all $a \otimes a$ with $a \in A$.

Let then G be a finitely generated group, $\pi : G \to G/G'$ the canonical epimorphism and denote by $\bar{G}$ this last factor group. Then π extends to an epimorphism $\bar{\pi} : \mathcal{V}(G) \to \mathcal{V}(\bar{G})$, such that $h \mapsto \bar{h}$, $h^\varphi \mapsto \bar{h}^\varphi$, $\forall h \in G$. By [11, Proposition 2.5], the kernel of $\bar{\pi}$ is given by :

$$Ker\bar{\pi} = < G', G'^\varphi > [G', G^\varphi][G, G'^\varphi]$$

In particular $Ker(\bar{\pi}) \cap \Upsilon(G) = [G', G^\varphi][G, G'^\varphi]$.

Denoting by π_o the restriction of $\bar{\pi}$ to $\Delta(G)$ then π_o is an epimorphism from $\Delta(G)$ to $\Delta(G/G')$, such that $[g, g^\varphi] \mapsto [\bar{g}, \bar{g}^\varphi]$, $\forall g \in G$.

Now suppose that $G/G' = B \times L$ where B denotes the torsion subgroup and L the free part of G/G'. Assume that $B = \prod_{i=1}^r < u_i >$, a direct product of the cyclic subgroups $< u_i > \cong C_{n_i}$ of order n_i, $1 \le i \le r$, and let $L = \prod_{j=1}^t < v_j >$, each $< v_j > \cong C_\infty$, $1 \le j \le t$. By the above observations we then have :

$$\Delta(G/G') \cong \Delta(B) \times \Delta(L) \times (B \otimes_{\mathbb{Z}} L);$$

$$\begin{aligned} \Delta(B) &= \prod_{i=1}^r < [u_i, u_i^\varphi] > \times \prod_{j<k} < [u_j, u_k^\varphi][u_k, u_j^\varphi] > \\ &\cong \prod_{i=1}^r C_{n_i} \times \coprod_{j<k} C_{(n_j, n_k)}; \end{aligned}$$

$$\begin{aligned} \Delta(L) &= \prod_{l=1}^t < [v_l, v_l^\varphi] \times \prod_{p<q} < [v_p, v_q^\varphi][v_q, v_p^\varphi] > \\ &\cong (C_\infty)^{\times \frac{t(t+1)}{2}} \qquad \text{(this is the free part of } \Delta(G/G')); \end{aligned}$$

$$\begin{aligned} B \otimes_{\mathbb{Z}} L &\cong \prod_{(j,k)=(1,1)}^{(r,t)} < [u_j, v_k^\varphi][v_k, u_j^\varphi] > \\ &\cong (\prod_{j=1}^r C_{n_j})^{\times t}. \end{aligned}$$

For each $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, t\}$ we choose a pre-image x_i and y_j in G of the corresponding u_i (resp. v_j) in G/G'. Thus $\bar{x}_i = u_i$, $\bar{y}_j = v_j$, $o'(x_i) = n_i$ and $o'(y_j) = \infty$, for $1 \le i \le r$ and $1 \le j \le t$. Set $X := \{x_i \mid 1 \le i \le r\}$ and $Y := \{y_j \mid 1 \le j \le t\}$. Since G is generated by $X \cup Y \cup G'$, by force of Proposition 3.3 and Lemmas 2.3 and 3.1, $\Delta(G)$ is generated by $\Delta := \Delta_X \cup \Delta_Y \cup \Delta_{XY}$, where

$$\begin{aligned} \Delta_X &:= \{[x_i, x_i^\varphi] \mid 1 \le i \le r\} \cup \{[x_j, x_k^\varphi][x_k, x_j^\varphi] \mid 1 \le j < k \le r\}, \\ \Delta_Y &:= \{[y_j, y_j^\varphi] \mid 1 \le j \le t\} \cup \{[y_j, y_k^\varphi][y_k, y_j^\varphi] \mid 1 \le j < k \le t\}, \\ \Delta_{XY} &:= \{[x_i, y_j^\varphi][y_j, x_i^\varphi] \mid 1 \le i \le r,\ 1 \le j \le t\}. \end{aligned}$$

Set $n_{jk} := (n_j, n_k)\ (= gcd(n_j, n_k))$. Lemma 3.1 gives again $([x_j, x_k^\varphi][x_k, x_j^\varphi])^{n_{jk}} = 1$ and $([x_i, y_j^\varphi][y_j, x_i^\varphi])^{n_i} = 1$, while $[x_i, x_i^\varphi]^{n_i} \in Ker(\pi_o)$, $\forall i = 1, \ldots, r$, $\forall j = 1, \ldots, t$.

3.5 Proposition. *With the above notation, $Ker(\pi_o)$ is generated by the set $\{[x_i, x_i^\varphi]^{n_i} \mid 1 \le i \le r\}$.*

Proof. Let N denote the subgroup $< [x_i, x_i^\varphi]^{n_i} \mid 1 \le i \le r >$. Since $N \le Ker(\pi_o)$ we have an epimorphism $\pi_o^* : \Delta(G)^* \to \Delta(G/G')$ given by $[h, h^\varphi]^* \mapsto [\bar{h}, \bar{h}^\varphi]$, $\forall h \in G$, where $\Delta(G)^* = \Delta(G)/N$ and $[h, h^\varphi]^* = [h, h^\varphi]N$.

On the other side,we can make use of the direct decomposition of $\Delta(G/G')$ to define an inverse $\psi : \Delta(G/G') \to \Delta(G)^*$. Obviously we set

$$\psi : [u_i, u_i^{\varphi}] \mapsto [x_i, x_i^{\varphi}]^*, \ 1 \le i \le r;$$

$$\psi : [u_j, u_k^{\varphi}][u_k, u_j^{\varphi}] \mapsto ([x_j, x_k^{\varphi}][x_k, x_j^{\varphi}])^*, \ 1 \le j < k \le r;$$

$$\psi : [v_j, v_j^{\varphi}] \mapsto [y_j, y_j^{\varphi}]^*, \ 1 \le j \le t;$$

$$\psi : [v_j, v_k^{\varphi}][v_k, v_j^{\varphi}] \mapsto ([y_j, y_k^{\varphi}][y_k, y_j^{\varphi}])^*, \ 1 \le j < k \le t;$$

$$\psi : [u_i, v_j^{\varphi}][v_j, u_i^{\varphi}] \mapsto ([x_i, y_j^{\varphi}][y_j, x_i^{\varphi}])^*, \ 1 \le i < k \le r, \ 1 \le j \le t.$$

That ψ is well defined on the generators of $\Delta(G/G')$ follows from Lemma 3.1 (iii),(iv). Thus we only need to check for orders. But our previous analysis yields that $[y_i, y_i^{\varphi}]$ has order n_i and $[x_i, x_i^{\varphi}]^{n_i} \in N$, while $([x_j, x_k^{\varphi}][x_k, x_j^{\varphi}])^{n_{jk}} = 1$, $([x_i, y_j^{\varphi}][y_j, x_i^{\varphi}])^{n_i} = 1$ and the rest correspond to the free generators of $\Delta(L)$. Therefore ψ defines a homomorphism. It is straightforward to check that $\pi^*\psi = I_{\Delta(G)^*}$, so that π^* is an isomorphism. ∎

Let $r_2(A)$ denote the *2-rank* of an abelian group A, that is, the cardinality of a maximal independent set of elements of 2-power order. In view of Corollary 3.2 we can resume the foregoing analysis as

3.6 Corollary. *Let G be a finitely generated group. Then $Ker(\pi_o)$ is an elementary abelian 2-group of rank at most $r_2(G/G')$. In particular, if G is a free group of rank n then $\Delta(G)$ is a free abelian group of rank $n(n+1)/2$.*

Remark 6. We observe that the results established above are also associated with the relationship between $\Delta(G)$ and the *Whitehead group* $\Gamma(G/G')$ ([15]). For an abelian group A, $\Gamma(A)$ is defined to be the (abelian) group generated by all symbols $\gamma(a)$, $a \in A$, subject to the relations

$$\gamma(a^{-1}) = \gamma(a), \ \forall a \in A;$$

$$\gamma(abc)\gamma(a)\gamma(b)\gamma(c) = \gamma(ab)\gamma(bc)\gamma(ca), \ \forall\, a, b, c \in A.$$

On setting $w(a,b) := \gamma(ab)\gamma(a)^{-1}\gamma(b)^{-1}$ then for all $a_1, \ldots, a_n$ in A we have (see [15]):

$$w(a_1, a_2) = w(a_2, a_1),$$

$$w(a_1, a_2a_3) = w(a_1, a_2) \cdot w(a_1, a_3),$$

$$w(a_1, a_1) = \gamma(a_1)^2,$$

$$\gamma(a_1 \cdots a_n) = \prod_{i=1}^{n} \gamma(a_i) \cdot \prod_{j<k} w(a_j, a_k),$$

and from these relations one gets that $\Gamma(C_n)$ is isomorphic to C_n or C_{2n} according to n odd or even, $\Gamma(C_\infty) \cong C_\infty$, and $\Gamma(A \times B) \cong \Gamma(A) \times \Gamma(B) \times A \otimes_{\mathbb{Z}} B$ (cf. [15]).

Making use of Lemma 3.1 we see that there is a well-defined epimorphism $\tau : \Gamma(G/G') \to \Delta(G)$ such that $\gamma(\bar{h}) \mapsto [h, h^{\varphi}]$, $\forall\, h \in G$ (consequently, $w(\bar{h}, \bar{g}) \mapsto [h, g^{\varphi}][g, h^{\varphi}]$) where $\bar{h}$ denotes h^{π} (see also [3]).

The composite map $\tau_o := \tau\pi_o$ thus gives an epimorphism $\tau_o : \Gamma(G/G') \to \Delta(G/G')$ and we can show by similar arguments that in the above situation, where G is a finitely generated group, $Ker(\tau_o)$ is an elementary abelian 2-group of rank precisely $r_2(G/G')$.

4. Some Computational Aspects

Our results provide a procedure to compute $\mathcal{V}(G)$, $G \otimes G$ and, in certain cases, $M(G)$ for finite solvable groups G given by an AG-system. The presentation of $\mathcal{V}(G)$ in Theorem 2.1 (Corollary 2.2) gives a small set of generators for $\Upsilon(G)$ and thus, since this subgroup is isomorphic to the non-abelian tensor square $G \otimes G$, we can for instance make use of a Reidemeister-Schreier process to write down a presentation for the last group or even for the exterior square $G \wedge G$, as we know a set of (central) generators for $\Delta(G)$ (Proposition 3.3).

The most confortable way to compute $\mathcal{V}(G)$ and its relevant subgroups in the present case yet, should be using an implementation of a *solvable quotient algorithm* (cf. Plesken [8]) to first compute an *AG-presentation* of $\mathcal{V}(G)$ and then make use for instance of the *AgGroup Functions* in the GAP system [4] to manipulate inside $\mathcal{V}(G)$. An implementation of such an algorithm following [8] has been carried out by A. Wegner [14] in Aachen. We acknowledge his efforts to send us his programm; however, we haven't been able to make use of it during the preparation of these notes. Nevertheless, making use of an implementation of the *nilpotent quotient algorithm* we followed the above suggestion using GAP to compute the tables at the end of this section for some finite non-abelian p-groups.

Computation of $\mathcal{V}(G)$ for finitely generated abelian groups G can be easely dealt with:

1. cyclic groups

a. Let $G = < a \mid a^n > (\cong C_n)$ be the cyclic group of order n. Then we have (refer to the picture on page 10 as well) $\Upsilon(G) = \Delta(G) = < [a, a^\varphi] >$. Now $[a, a^\varphi]$ being central in $\mathcal{V}(G)$, it satisfies $[a, a^\varphi]^n = 1$. To certify that $\mathcal{V}(G)$ is in fact a 2-generator nilpotent of class-2 group of order n^3 we can construct it as follows:

   starting with an abelian group V of type $C_n \times C_n$, say $V = < u, v \mid u^n = 1, v^n = 1, [u, v] = 1 >$, we extend V by an automorphism α of order n which maps $u \mapsto uv$, $v \mapsto v$. This extension, of order n^3, has the presentation

$$\mathcal{E}(n) = \langle u, v, \alpha \mid u^n = 1, \alpha^n = 1, v^n = 1, [u, \alpha] = v, [u, v] = [\alpha, v] = 1 \rangle.$$

   On mapping $a \mapsto u$, $a^\varphi \mapsto \alpha$ one sees that $\mathcal{E}(n)$ is a homomorphic image of $\mathcal{V}(C_n)$ and thus they are isomorphic, by comparing orders.

   In particular, for $n = 2$, $\mathcal{V}(C_2) \cong D_4$, the dihedral group of order 8 and for $n = 3$, $\mathcal{V}(C_3) \cong B(2,3)$, the 2-generator exponent-3 group of order 27.

b. Let $G = < a \mid \quad > (\cong C_\infty)$ be the infinite cyclic group. Then removing the orders of the elements, a similar argument shows that in this case $\mathcal{V}(C_\infty) \cong F_2(2)$, the 2-generator free nilpotent group of class 2.

2. direct products

Let $G = H \times K$ be a direct product of arbitrary groups H and K. In [11], Proposition 3.6, we prove:

(i) $\mathcal{V}(G) = \langle H, H^\varphi \rangle \cdot \langle K, K^\varphi \rangle \cdot [H, K^\varphi] \cdot [K, H^\varphi]$, (a direct product);

(ii) $\langle H, H^\varphi \rangle \cong \mathcal{V}(H)$; $\langle K, K^\varphi \rangle \cong \mathcal{V}(K)$;

(iii)
$$\begin{aligned}\Upsilon(G) &= \Upsilon(H) \cdot \Upsilon(K) \cdot [H, K^\varphi] \cdot [K, H^\varphi] \\ &\cong (H \otimes H) \times (K \otimes K) \times (H \otimes_{\mathbb{Z}} K) \times (K \otimes_{\mathbb{Z}} H).\end{aligned}$$

In particular, for abelian groups H and K we have also $\Upsilon(H) \cong (H \otimes_{\mathbb{Z}} H)$ and $\Upsilon(K) \cong (K \otimes_{\mathbb{Z}} K)$ (see also Remark 5).

It should be interesting to look more closely at the very particular case of $\mathcal{V}(C_2 \times C_2)$. For, let $C_2 \times C_2 = < a, b \mid a^2 = 1, b^2 = 1 [a,b] = 1 >$. Then by the above we have $\mathcal{V}(< a >) = < a, a^{\varphi} > \cong D_4 \cong < b, b^{\varphi} > = \mathcal{V}(< b >)$ and $< [a, b^{\varphi}] > \cong C_2 \cong < [b, a^{\varphi}] >$. Hence $\mathcal{V}(C_2 \times C_2) \cong (D_4 \times C_2)^{\times 2}$, of order 2^8. Here, the subgroup $< a, b^{\varphi} >$ is isomorphic to a *covering group* of $C_2 \times C_2$, namely D_4. In the last section we leave an open question concerning this point.

Next one may be faced for instance with the computation of $\mathcal{V}(G)$ for finite nilpotent groups G, which by item 2 above is then reduced to the case of finite p-groups, p: prime (see also section 3 of [11] for some concerning results). As observed in the second paragraph of the present section, our results give rise to performe computer assisted calculations with large groups in this case as well. To exemplify we inserted in Table 1 those informations obtained following this procedure, having for groups arguments G the non-abelian p-groups of order $\leq p^4 : p = 2, 3$. Each such group is given in the list below by a PAG-*system*, that is, an AG- system where r_i is a prime number, $1 \leq i \leq n$ (cf. section 2).

Non-abelian p-Groups of Order $\leq p^4$, $p = 2, 3$

(i) $D_4 = \langle a, b, c \mid a^2,\ b^2 = c,\ c^2,\ [b,a] = c,\ [c,a],\ [c,b] \rangle$;
(ii) $Q_8 = \langle a, b, c \mid a^2 = c,\ b^2 = c,\ c^2,\ [b,a] = c,\ [c,a],\ [c,b] \rangle$;
(iii) $H_1\ (= D_8) = \langle a, b, c, d \mid a^2,\ b^2 = cd,\ c^2 = d,\ d^2,\ [b,a] = c,\ [c,a] = d,\ [c,b],\ [d,a],\ [d,b],\ [d,c] \rangle$;
(iv) $H_2 = \langle a, b, c, d \mid a^2,\ b^2 = c, c^2 = d,\ d^2,\ [b,a] = c,\ [c,a] = d,\ [a,b],\ [d,a],\ [d,b],\ [d,c] \rangle$;
(v) $H_3\ (= Q_{16}) = \langle a, b, c, d \mid a^2 = c,\ b^2 = d,\ c^2 = d,\ d^2,\ [b,a] = c,\ [c,a],\ [c,b] = d,\ [d,a],\ [d,b],\ [d,c] \rangle$;
(vi) $H_4 = \langle a, b, c, d \mid a^2 = c,\ b^2,\ c^2 = d,\ d^2,\ [b,a] = d,\ [c,a],\ [c,b],\ [d,a],\ [d,b],\ [d,c] \rangle$;
(vii) $H_5\ \langle a, b, c, d \mid a^2 = c,\ b^2 = d,\ c^2,\ d^2,\ [b,a] = c,\ [c,a],\ [c,b],\ [d,a],\ [d,b],\ [d,c] \rangle$;
(viii) $H_6\ (= C_2 \times D_4) = \langle a, b, c, d \mid a^2 = d,\ b^2,\ c^2,\ d^2,\ [b,a] = d,\ [c,a],\ [c,b],\ [d,a],\ [d,b],\ [d,c] \rangle$;
(ix) $H_7 = \langle a, b, c, d \mid a^2 = d,\ b^2,\ c^2,\ d^2,\ [b,a],\ [c,a],\ [c,b] = d,\ [d,a],\ [d,b],\ [d,c] \rangle$;
(x) $H_8 = \langle a, b, c, d \mid a^2 = d,\ b^2,\ c^2,\ d^2,\ [b,a] = c,\ [c,a],\ [c,b],\ [d,a],\ [d,b],\ [d,c] \rangle$;
(xi) $H_9\ (= C_2 \times Q_8) = \langle a, b, c, d \mid a^2,\ b^2 = d,\ c^2 = d,\ d^2,\ [b,a],\ [c,a],\ [c,b] = d,\ [d,a],\ [d,b],\ [d,c] \rangle$;
(xii) $B(2,3) = \langle a, b, c \mid a^3,\ b^3,\ c^3,\ [b,a] = c,\ [c,a],\ [c,b] \rangle$;
(xiii) $K = \langle a, b, c \mid a^3,\ b^3 = c,\ c^3,\ [b,a] = c,\ [c,a],\ [c,b] \rangle$;
(xiv) $G_1 = \langle a, b, c, d \mid a^3,\ b^3 = c,\ c^3 = d,\ d^3,\ [b,a] = d,\ [c,a],\ [c,b],\ [d,a],\ [d,b],\ [d,c] \rangle$;
(xv) $G_2 = \langle a, b, c, d \mid a^3 = b,\ b^3,\ c^3 = d,\ d^3,\ [b,a],\ [c,a] = d,\ [c,b],\ [d,a],\ [d,b],\ [d,c] \rangle$;
(xvi) $G_3 = \langle a, b, c, d \mid a^3,\ b^3,\ c^3 = d,\ d^3,\ [b,a],\ [c,a] = b,\ [c,b],\ [d,a],\ [d,b],\ [d,c] \rangle$;
(xvii) $G_4 = \langle a, b, c, d \mid a^3,\ b^3,\ c^3 = d,\ d^3,\ [b,a] = d,\ [c,a],\ [c,b],\ [d,a],\ [d,b],\ [d,c] \rangle$;
(xviii) $G_5 = \langle a, b, c, d \mid a^3,\ b^3 = c,\ c^3,\ d^3,\ [b,a] = c,\ [c,a],\ [c,b],\ [d,a],\ [d,b],\ [d,c] \rangle$;
(xix) $G_6 = \langle a, b, c, d \mid a^3,\ b^3,\ c^3 = d,\ d^3,\ [b,a],\ [c,a] = b,\ [c,b] = d,\ [d,a],\ [d,b],\ [d,c] \rangle$;
(xx) $G_7 = \langle a, b, c, d \mid a^3 = d,\ b^3,\ c^3 = d,\ d^3,\ [b,a] = d,\ [c,a] = b,\ [c,b],\ [d,a],\ [d,b],\ [d,c] \rangle$;
(xxi) $G_8 = \langle a, b, c, d \mid a^3 = d,\ b^3,\ c^3 = d^2,\ d^3,\ [b,a] = d,\ [c,a] = b,\ [c,b],\ [d,a],\ [d,b],\ [d,c] \rangle$;
(xxii) $G_9 = \langle a, b, c, d \mid a^3,\ b^3,\ c^3,\ d^3,\ [b,a] = c,\ [c,a],\ [c,b],\ [d,a],\ [d,b],\ [d,c] \rangle$;
(xxiii) $G_{10} = \langle a, b, c, d \mid a^3,\ b^3,\ c^3 = d,\ d^3,\ [b,a] = d^2,\ [c,a] = b,\ [c,b],\ [d,a],\ [d,b],\ [d,c] \rangle$.

Reading the table. Each *entry* in Table 1 gives informations on $\mathcal{V}(G)$ corresponding to the group argument G numbered *according to the list* above. Since these groups are given by a *PAG*-system of length at most four, we have

generators for G: a subset of $\{a, b, c, d\}$;

generators for G^φ: a subset $\{x, y, z, w\}$;

conventions:

- $a^\varphi = x$, $b^\varphi = y$, $c^\varphi = z$ and $d^\varphi = w$, with the corresponding G^φ- relations;
- In order to save space we put $xa := [x, a]$, $xb := [x, b], \ldots, wd := [w, d]$; here the opposite order gives the inverse, $ax = [a, x]$, etc.
- In each column:
  1. "**nº**" gives the entry number according to the list above; for example entry (i) corresponds to the dihedral group D_4.
  2. "**Cl**" gives the nilpotency class of $\mathcal{V}(G)$; thus e.g., $\mathcal{V}(D_4)$ has nilpotency class 3, one more then D_4.
  3. "$|\mathcal{V}(G)|$" is just the order of $\mathcal{V}(G)$.
  4. "$\Upsilon(G)$.rel" gives the power- relations satisfied by those generators of $\Upsilon(G)$ extracted from the set T of Theorem 2.1 (ii), which afford a PAG-system for $\Upsilon(G)$; here a simple power, e.g. xa^2, means as usual that such power is the identity, while the absence of commutators means commutativity. All groups Υ found in Table 1 are abelian. Hence, reading for instance on entry (i) we get an exponent-2 power-commutator presentation for $\Upsilon(D_4)$:

$$\begin{aligned}\Upsilon(D_4) = \ &\langle\, xa, xb, ya, yb, cy \mid xa^2 = 1, [xb, xa] = 1, xb^2 = cy, [ya, xa] = 1,\\ &[ya, xb] = 1, ya^2 = cy, [yb, xa] = 1, [yb, xb] = 1, [yb, ya] = 1,\\ &yb^2 = 1, [cy, xa] = 1, [cy, xb] = 1, [cy, ya] = 1, [cy, yb] = 1, cy^2 = 1\,\rangle.\end{aligned}$$

  5. "$\Upsilon(G)$" displays just the isomorphism type of $\Upsilon(G)$, as they are in case abelian groups. Hence, from the first entry (i), $\Upsilon(D_4) \cong C_4 \times C_2^3$.
  6. "G.**action on**$\Upsilon(G)$" describes the action of each relevant generator of G on the generators of the (normal) subgroup $\Upsilon(G)$. The generators which are being acted here are those related in the above mentioned $\Upsilon(G)$.rel column. It should be noted that they are here ordered in a list, according to their appearance as the p-powered left hand side of the relations in column $\Upsilon(G)$.**rel**. Then their images under conjugation by a generator $g \in G$ is the corresponding list displayed following the symbol $\wedge g$. Thus for example, the actions of the relevant generators a and b of D_4 on the generators xa, xb, ya, yb, cy of $\Upsilon(D_4)$ read of entry (i) are:

$$\wedge a : \begin{cases} xa \longmapsto xa \\ xb \longmapsto xb \\ ya \longmapsto ya \\ yb \longmapsto yb \\ cy \longmapsto cy \end{cases} \qquad \wedge b : \begin{cases} xa \longmapsto xa \\ xb \longmapsto xb \cdot cy \\ ya \longmapsto ya \cdot cy \\ yb \longmapsto yb \\ cy \longmapsto cy \end{cases}$$

  It seems appropriate to mention that the knowledge of a presentation of $\Upsilon(G)$ and the compatible action of G on it suffice to construct $\mathcal{V}(G)$, once $\mathcal{V}(G) = \Upsilon(G) \cdot G \cdot G^\varphi$ and G^φ, due to our relations, acts on $\Upsilon(G)$ in the same way as G does.
  7. "$\Delta(G)$.**rel**" describes the power-relations for a PAG-system of $\Delta(G)$ where the generators are extracted from the set Δ of Proposition 3.3. These relations can be read of those in column $\Upsilon(G)$.**rel**. Reading on entry (i) we then see that $\Delta(D_4)$ is the elementary abelian 2-group of order 8, with generators $xa, yb, xb \cdot ya$.

8. The last column, "$M(G)$", displays the generators of $\mu(G)/\Delta(G)$, obtained as cosets of the kernel of the derived map ρ' (cf. Proposition 2.8). The orders of these generators can be read from their power-relations in column $\Upsilon(G)$.**rel**, modulo $\Delta(G)$. Therefore from entry (i) we get $M(D_4) \cong \langle \overline{cy} \mid \overline{cy}^2 = 1 \rangle$, the cyclic group of order 2, as it should be.

TABLE 1

nº	Cl	$\lvert\mathcal{V}(G)\rvert$	$\Upsilon(G)$.rel	$\Upsilon(G)$	G.action on $\Upsilon(G)$	$\Delta(G)$.rel	$M(G)$
(i)	3	2^{11}	xa^2, $xb^2 = cy$, $ya^2 = cy$, yb^2, cy^2	$C_4 \times C_2^3$	$\wedge a$: xa, xb, ya, yb, cy; $\wedge b$: xa, $xb \cdot cy$, $ya \cdot cy$, yb, cy	xa^2, yb^2, $(xb \cdot ya)^2$	$\overline{cy}$
(ii)	3	2^{12}	$xa^2 = cx$, $xb^2 = cx \cdot cy$, $ya^2 = cx \cdot cy$, $yb^2 = cy$, cx^2, cy^2	$C_4^2 \times C_2^2$	$\wedge a$: xa, $xb \cdot cx$, $ya \cdot cx$, yb, cx, cy; $\wedge b$: xa, $xb \cdot cy$, $ya \cdot cy$, yb, cx, cy	$xa^2 = cx$, $yb^2 = cy$, $(xb \cdot ya)^2$	$\overline{1}$
(iii)	4	2^{14}	xa^2, $xb^2 = cx$, $ya^2 = cx \cdot dx$, yb^2, $cx^2 = dx$, dx^2	$C_8 \times C_2^3$	$\wedge a$: xa, $xb \cdot cx \cdot dx$, $ya \cdot cx$, yb, $cx \cdot dx$, dx; $\wedge b$: xa, xb, ya, yb, cx, dx	xa^2, yb^2, $(xb \cdot ya)^2$	$\overline{dx}$
(iv)	4	2^{14}	$xa^2 = dy$, $xb^2 = cy$, $ya^2 = cy \cdot dy$, yb^2, $cy^2 = dy$, dy^2	$C_8 \times C_2^3$	$\wedge a$: xa, $xb \cdot dy$, $ya \cdot dy$, yb, cy, dy; $\wedge b$: xa, $xb \cdot cy \cdot dy$, $ya \cdot cy$, yb, $cy \cdot dy$, dy	$xa^2 = dy$, yb^2, dy^2 $(xb \cdot ya)^2$	$\overline{1}$
(v)	4	2^{14}	xa^2, $xb^2 = cy$, $ya^2 = cy \cdot dy$, $yb^2 = dy$, $cy^2 = dy$, dy^2	$C_8 \times C_2^3$	$\wedge a$: xa, xb, ya, yb, cy, dy; $\wedge b$: xa, $xb \cdot cy \cdot dy$, $ya \cdot cy$, yb, $cy \cdot dy$, dy	xa^2, $yb^2 = dy$, dy^2 $(xb \cdot ya)^2$	$\overline{1}$
(vi)	3	2^{14}	$xa^2 = cx \cdot dx$, xb^2, ya^2, yb^2, $cx^2 = dx$, dx^2	$C_8 \times C_2^3$	$\wedge a$: xa, $xb \cdot dx$, $ya \cdot dx$, yb, cx, dx; $\wedge b$: xa, xb, ya, yb, cx, dx	$xa^2 = cx \cdot dx$, $cx^2 = dx$, yb^2, dx^2 $(xb \cdot ya)^2$	$\overline{1}$
(vii)	3	2^{15}	$xa^2 = cx$, $xb^2 = cx \cdot cy$, $ya^2 = cx \cdot cy$, $yb^2 = dy$, cx^2, cy^2, dy^2	$C_4^3 \times C_2$	$\wedge a$: xa, $xb \cdot cx$, $ya \cdot cx$, yb, cx, cy, dy; $\wedge b$: xa, $xb \cdot cy$, $ya \cdot cy$, yb, cx, cy, dy	$xa^2 = cx$, $yb^2 = dy$, cx^2, dy^2 $(xb \cdot ya)^2$	$\overline{cx \cdot cy}$
(viii)	3	2^{18}	xa^2, $xb^2 = dy$, xc^2, yc^2, dy^2 $ya^2 = dy$, yb^2, za^2, zb^2, zc^2	$C_4 \times C_2^8$	$\wedge a$: trivial; $\wedge b$: xa, $xb \cdot dy$, xc, $ya \cdot dy$, yb, yc, za, zb, zc, dy; $\wedge c$: trivial	xa^2, $(xb \cdot ya)^2$, yb^2, $(xc \cdot za)^2$, zc^2, $(yc \cdot zb)^2$	$\overline{za}$, $\overline{zb}$, $\overline{dy}$

(continued)

TABLE 1 - Cont.

n°	Cl	$\lvert\mathcal{V}(G)\rvert$	$\Upsilon(G)$.rel	$\Upsilon(G)$	G.action on $\Upsilon(G)$	$\Delta(G)$.rel	$M(G)$
(ix)	2	2^{17}	$xa^2,\ xb^2,\ xc^2,$ $ya^2,\ yb^2,\ yc^2,$ $za^2,\ zb^2,\ zc^2$	C_2^9	$\wedge a$: *trivial*; $\wedge b$: *trivial*; $\wedge c$: *trivial*	$xa^2,$ $(xb\cdot ya)^2,$ $yb^2,$ $(xc\cdot za)^2,$ $zc^2,$ $(yc\cdot zb)^2$	$\overline{xb},$ $\overline{xc}$
(x)	3	2^{15}	$xa^2 = dx,$ $xb^2 = cy,$ $ya^2 = cy,\ yb^2,$ $yc^2, cx^2,$ $cy^2,\ dx^2$	$C_4^2 \times C_2^3$	$\wedge a : xa,\ xb\cdot cx,$ $ya\cdot cx,\ yb,$ $cx,\ cy,\ dx;$ $\wedge b : xa,\ xb\cdot cy,$ $ya\cdot cy,\ yb,$ $cx,\ cy,\ dx$	$xa^2 = dx,$ $yb^2,$ $(xb\cdot ya)^2,$ dx^2	$\overline{cx},$ $\overline{cy}$
(xi)	3	2^{19}	$xa^2,\ xb^2,$ $xc^2,\ ya^2,$ $yb^2 = dy,$ $yc^2 = dy\cdot dz,$ $za^2,\ dz^2,$ $zb^2 = dy\cdot dz,$ $zc^2 = dz,\ dy^2$	$C_4^2 \times C_2^7$	$\wedge a$: *trivial* $\wedge b : xa,\ xb,\ xc,$ $ya,\ yb,\ yc\cdot dy,$ $za,\ zb\cdot dy,\ zc,$ $dy,\ dz;$ $\wedge c : xa,\ xb,\ xc,$ $ya,\ yb,\ yc\cdot dz,$ $za,\ zb\cdot dz,\ zc,$ $dy,\ dz;$	$xa^2,$ $yb^2 = dy,$ $zc^2 = dz,$ $(xb\cdot ya)^2,$ $(xc\cdot za)^2,$ $(yc\cdot zb)^2,$ $dy^2,\ dz^2$	$\overline{xb},$ $\overline{xc}$
(xii)	3	3^{12}	$xa^3,\ xb^3,\ ya^3,$ $yb^3,\ cx^3,\ cy^3$	C_3^6	$\wedge a : xa,\ xb\cdot cx^2,$ $ya\cdot cx,\ yb,$ $cx,\ cy;$ $\wedge b : xa,\ xb\cdot cy^2,$ $ya\cdot cy,\ yb,$ $cx,\ cy$	$xa^3,$ $yb^3,$ $(xb\cdot ya)^3$	$\overline{cx},$ $\overline{cy},$
(xiii)	2	3^{10}	$xa^3,\ xb^3,$ $ya^3,\ yb^3$	C_3^4	$\wedge a$: *trivial* $\wedge b$: *trivial*	$xa^3,$ $yb^3,$ $(xb\cdot ya)^3$	$\overline{1}$
(xiv)	2	3^{13}	$xa^3,\ xb^3,\ ya^3,$ $yb^3 = cy,\ cy^3$	$C_3^3 \times C_9$	$\wedge a$: *trivial* $\wedge b$: *trivial*	$xa^3,$ $yb^3,$ $(xb\cdot ya)^3,$ cy^3	$\overline{1}$
(xv)	3	3^{14}	$xa^3 = az^2,$ $xb^3 = dx^2,$ $ya^3 = dx,\ yb^3,$ $dx^3,\ az^3$	$C_3^2 \times C_9^2$	$\wedge a : xa,\ xb\cdot dx^2,$ $ya\cdot dx,\ yb,$ $dx,\ az;$ $\wedge b : xa,\ xb,\ ya,$ $yb,\ dx,\ az$	$xa^3,$ $yb^3,$ $(xb\cdot ya)^3$	$\overline{dx}$
(xvi)	3	3^{15}	$xa^3,\ xc^3,\ za^3,$ $zc^3 = dz^2,\ bx^3,$ $bz^3,\ dz^3$	$C_3^5 \times C_9$	$\wedge a : xa,\ xc\cdot bx^2,$ $za\cdot bx,\ zc,$ $bx,\ bz,\ dz;$ $\wedge c : xa,\ xc\cdot bz^2,$ $za\cdot bz,\ zc,$ $bx,\ bz,\ dz;$	$xa^3,$ $zc^3,$ $(xc\cdot za)^3$	$\overline{bx}$ $\overline{bz}$
(xvii)	2	3^{17}	$xa^3,\ xb^3,\ xc^3,$ $ya^3,\ yb^3,\ yc^3,$ $za^3,\ zb^3,\ zc^3$	C_3^9	$\wedge a$: *trivial*; $\wedge b$: *trivial*; $\wedge c$: *trivial*	$xa^3,\ yb^3,$ $zc^3,$ $(xb\cdot ya)^3,$ $(xc\cdot za)^3,$ $(yc\cdot zb)^3$	$\overline{xc}$ $\overline{yc}$
(xviii)	3	3^{17}	$xa^3,\ xb^3,\ xc^3,$ $ya^3,\ yb^3,\ yc^3,$ $za^3,\ zb^3,\ zc^3$	C_3^9	$\wedge a$: *trivial* $\wedge b$: *trivial* $\wedge c$: *trivial*	$xa^3,\ yb^3,$ $zc^3,$ $(xb\cdot ya)^3,$ $(xc\cdot za)^3,$ $(yc\cdot zb)^3$	$\overline{xc}$ $\overline{yc}$

TABLE 1 - Cont.

nº	Cl	$\|\mathcal{V}(G)\|$	$\Upsilon(G)$.rel	$\Upsilon(G)$	G.action on $\Upsilon(G)$	$\Delta(G)$.rel	$M(G)$
(xix)	4	3^{14}	xa^3, xc^3, za^3, zc^3, bx^3, bz^3	C_3^6	$\wedge a : xa$, $xc \cdot bx^2$, $za \cdot bx$, zc, bx, bz; $\wedge c : xa$, $xc \cdot bz^2$, $za \cdot bz$, zc, bx, bz	xa^3, zc^3, $(xc \cdot za)^3$	$\overline{bz}$
(xx)	4	3^{14}	xa^3, xc^3, za^3, zc^3, bx^3, bz^3	C_3^6	$\wedge a : xa$, $xc \cdot bx^2$, $za \cdot bx$, zc, bx, bz; $\wedge c : xa$, $xc \cdot bz^2$, $za \cdot bz$, zc, bx, bz	xa^3, zc^3, $(xc \cdot za)^3$	$\overline{bz}$
(xxi)	4	3^{14}	xa^3, xc^3, za^3, zc^3, bx^3, bz^3	C_3^6	$\wedge a : xa$, $xc \cdot bx^2$, $za \cdot bx$, zc, bx, bz; $\wedge c : xa$, $xc \cdot bz^2$, $za \cdot bz$, zc, bx, bz	xa^3, zc^3, $(xc \cdot za)^3$	$\overline{bz}$
(xxii)	3	3^{19}	xa^3, xb^3, xd^3, ya^3, yb^3, yd^3, wa^3, wb^3, wd^3, cx^3, cy^3	$C^1 1_3$	$\wedge a : xa$, $xb \cdot cx^2$, xd, $ya \cdot cx$, yb, yd, wa, wb, wd, cx, cy; $\wedge b : xa$, $xb \cdot cy^2$, xd, $ya \cdot cy$, yb, yd, wa, wb, wd, cx, cy; $\wedge d$: *trivial*	xa^3, yb^3, wd^3, $(xb \cdot ya)^3$, $(xd \cdot wa)^3$, $(yd \cdot wb)^3$	$\overline{cx}$ $\overline{cy}$ $\overline{xd}$ $\overline{yd}$
(xxiii)	4	3^{15}	xa^3, $xc^3 = dz$, $za^3 = dz^2$, zc^3, bx^3, bz^3, dz^3	$C_3^5 \times C_9$	$\wedge a : xa$, $xc \cdot bx^2$, $za \cdot bx$, zc, bx, bz, dz; $\wedge c : xa$, $xc \cdot bz^2$, $za \cdot bz$, zc, bx, $bz \cdot dz$, dz;	xa^3, zc^3, $(xc \cdot za)^3$	$\overline{bz}$ $\overline{dz}$

5. Further Remarks and Open Problems

Remark 7. We can see by Table 1 above that e.g., $\Gamma(G/G')$ and $\Delta(G)$ are isomorphic for $G = Q_8$ and non-isomorphic for $G = Q_8 \times C_2$. Also, in [11] (Theorem 3.11) we found an upper bound for the order of $\mathcal{V}(G)$ when G is a finite p-group:

$$\textit{If } |G| = p^n \textit{ and } |G'| = p^m \textit{ then } |\mathcal{V}(G)| \textit{ divides } p^{n^2+2n-mn}.$$

This bound is attained for instance for $G = Q_8$.

Problem 1. To characterize those indecomposable finite 2-groups G for which the above bound is attained.

Remark 8. In section 4 we found a subgroup of $\Theta(C_2 \times C_2)$ which is a covering group of $C_2 \times C_2$, namely D_4. In general, for any abelian group G, $\Theta(G)/\Delta(G)$ is a covering group of G; this follows from our results in section 2. On the other hand, by ([2], Corollary 1), when G is perfect then $\Upsilon(G)$ is the (unique) covering group of G.

Problem 2. Given an arbitrary group G, is there a section of $\Upsilon(G) \cdot \Theta(G)$ containing a covering group of G ?

References

[1] R. Beyl, *The Schur Multiplicator of Metacyclic Groups*, Proc. Amer. Math. Soc. **40** (1973), 413-418.

[2] R. Brown, D.L. Johnson and E. F. Robertson, *Some Computations of Non-Abelian Tensor Products of Groups*, J. Algebra **111** (1987), 177-202.

[3] R. Brown and J.-L. Loday, *Van Kampen Theorems for Diagrams of Spaces*, Topology **26** (1987), 311-335.

[4] F. Celler, A. Niemeyer, M. Schönert et al., *"GAP - Users Manual"*, Lehrstuhl D für Mathematik - R.W.T.H. - Aachen, 1992.

[5] C.W. Curtis and I. Reiner, *"Methods of Representation Theory with applications to finite Groups and Orders"*, vol.1, Wiley, New York, 1981.

[6] H. Jürgensen, *Calculation with the Elements of a Finite Group Given by Generators and Defining Relations*. In *"Computational Problems in Abstract Algebra"* (J.Leech, ed.), (1970) 47-57. Pergamon Press, Oxford.

[7] C. Miller, *The Second Homology Group of a Group; Relations Among Commutators*, Proc. Amer. Math. Soc. **3** (1952), 588-595.

[8] W. Plesken, *Towards a Soluble Quotient Algorithm*, J. of Symbolic Comp. **4** (1987), 111-122.

[9] D. J. S. Robinson, *"A Course in the Theory of Groups"*, Springer, New York, 1982.

[10] N. R. Rocco, *On Weak Commutativity Between Finite p-Groups, p:odd*, J. Algebra **76** (1982), 471-488.

[11] __________, *On a Construction Related to the Non-abelian Tensor Square of a Group*, Bol. Soc. Bras. Mat. **22** (1991), 63-79.

[12] S. Sidki, *On Weak Permutability Between Groups*, J. Algebra **63** (1980), 186-225.

[13] J. W. Wamsley, *The Deficiency of Metacyclic Groups*, Proc. Amer. Math. Soc. **24** (1970), 724-726.

[14] A.Wegner, *Implementation eines Algorithmus zur Berechnung endlicher auflösbarer Faktorgruppen einer endlich präsentierten Gruppe im gruppentheoretischen Programmsystem GAP*, Diplomarbeit, R.W.T.H.-Aachen, 1989;

[15] J.H.C. Whitehead, *A Certain Exact Sequence*, Ann. of Math. **52** (1950), 51-110.

Received: February 1993