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Topology of subgroup lattices of symmetric and alternating groups

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Abstract

We determine the homotopy type of the order complex of the subgroup lattice of the symmetric group S_n when n is a prime or a power of two. (The prime case has been treated previously in unpublished work of G. Ivanyos.) We do the same for alternating groups of prime degree. In addition, we show that, for any $n > 1$, the homology of the order complex of the subgroup lattice of S_n has rank at least $n!/2$ in dimension $n - 3$.

1. Introduction and statement of results

For a finite group G, let $\Delta(G)$ be the order complex of the poset obtained by removing 1 and G from the subgroup lattice of G, so the k-simplices in $\Delta(G)$ are the chains

$$H_0 < \cdots < H_k$$

of nontrivial proper subgroups of G. Kratzer and Thévenaz [KrTh2, Corollary 4.10] show that if G is solvable with chief series of length n then $\Delta(G)$ has the homotopy type of a wedge of spheres of dimension $n - 2$ and give a formula for the number of spheres in this wedge.

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However, not much is known about the topology of $\Delta(G)$ when G is not solvable. It is natural from the group-theoretic point of view to begin attacking this problem by considering $\Delta(G)$ when G is nonabelian simple. Standard theorems from topological combinatorics (in particular the homotopy complementation formula of Björner and Walker, see Lemma 2.3) also indicate that one should look first at simple groups, and, more generally, characteristically simple groups and their automorphism groups.

In this paper we obtain some partial results on the homotopy type of $\Delta(A_n)$ and $\Delta(S_n)$. The problem of determining the homotopy types of these complexes for all n seems quite difficult. In fact, the reduced Euler characteristic $\tilde{\chi}(\Delta(S_n))$, which is equal to the Möbius number $\mu(1, S_n)$ by a theorem of Hall (see [Ha, (2.21)] or [St, Propositions 3.8.5,3.8.6]), is known only when n is small or n is prime, twice a prime or a power of two (see [Sh1]). By [KrTh1, Thèorém 3.1], both $\tilde{\chi}(\Delta(S_n))$ and $\tilde{\chi}(\Delta(A_n))$ are divisible by $\frac{n!}{2}$ for all $n \geqslant 5$. In [Sh1] it is shown that $\tilde{\chi}(\Delta(S_n)) = (-1)^{n-1}\frac{|\mathrm{Aut}(S_n)|}{2}$ for infinitely many n but there exist some $n > 1$ for which this equality does not hold (the smallest is $n = 14$, and the smallest n for which $\tilde{\chi}(\Delta(S_n))$ is unknown is $n = 18$). Our first main result sheds some light on this phenomenon.

Theorem 1.1. *For each $n > 1$ there exist complexes Γ_n and Λ_n such that $\Delta(S_n)$ is homotopy equivalent to the wedge of Γ_n and Λ_n, and Γ_n is homotopy equivalent to a wedge of $\frac{n!}{2}$ spheres of dimension $n - 3$.*

The next corollary is immediate.

Corollary 1.2. *For every $n > 1$, the complex $\Delta(S_n)$ is not acyclic. In particular,*

$$\dim(\tilde{H}_{n-3}(\Delta(S_n))) \geqslant \frac{n!}{2}$$

for all $n > 1$.

It should be noted that although Theorem 1.1 is not stated explicitly in the unpublished work of Ivanyos [Iv], it follows easily from results therein. Our proof of Theorem 1.1 appears in Section 3, where a more precise version of the theorem is obtained. This more precise version, along with the results in [Sh1], leads to the next conjecture. Recall that a subgroup $G \leqslant S_n$ is called *primitive* if G is transitive and there is no partition π of $[n]$ into subsets $\pi_1, \ldots, \pi_r$ of equal size k with $1 < k < n$, such that G permutes the π_j. We will call an element of order two from S_n which is not contained in A_n an *odd involution*. Also, from now on,

$$\bigvee_a S^d$$

will denote a wedge of a spheres of dimension d. More generally,

$$\bigvee_{i \in I} \Delta_i$$

will denote the wedge of the spaces Δ_i ($i \in I$), $\Delta \vee \Gamma$ will denote the wedge of two spaces Δ, Γ and $\Delta \simeq \Gamma$ will be written to indicate that Δ and Γ are homotopy equivalent.

Conjecture 1.3. *If no primitive proper subgroup of S_n contains an odd involution then the complex Λ_n from Theorem* 1.1 *is contractible, so*

$$\Delta(S_n) \simeq \bigvee_{n!/2} S^{n-3}.$$

It is natural to ask which $n \in \mathbb{N}$ satisfy the condition on primitive subgroups from Conjecture 1.3. Define Ω to be the set of all $n \in \mathbb{N}$ such that some primitive proper subgroup of S_n contains an odd involution. For $x \in \mathbb{N}$, let $e(x)$ be the number of $n \leqslant x$ such that S_n contains a primitive subgroup other than A_n or S_n. Using the Aschbacher–O'Nan-Scott Theorem (see for example [DiMo, Theorem 4.1A]), Cameron et al. show in [CNT] (see also [DiMo, Theorem 4.8A]) that

$$e(x) \sim \frac{2x}{\log x}.$$

It follows that

$$\lim_{n \to \infty} \frac{|\Omega \cap [n]|}{n} = 0,$$

so if Conjecture 1.3 is correct, then the "typical" $\Delta(S_n)$ has the homotopy type of a wedge of $\frac{n!}{2}$ spheres of dimension $n-3$.

As mentioned above, it was shown in [Sh1] that when n is prime or n is a power of 2 then $\tilde{\chi}(\Delta(S_n)) = (-1)^{n-1}\frac{n!}{2}$. Examination of the complex Λ_n in these cases shows gives the following two results. The first of these results appears in [Iv], and our proof follows the same lines as than in [Iv], although the proofs to the key steps differ. The proof of the second result involves transferring results about Euler characteristic from [Sh1] to results on homotopy type. This procedure requires some care.

Theorem 1.4 (Ivanyos, 1995). *If p is prime then*

$$\Delta(S_p) \simeq \bigvee_{p!/2} S^{p-3}.$$

Theorem 1.5. *If $n = 2^a$ for some $a \in \mathbb{N}$ then*

$$\Delta(S_n) \simeq \bigvee_{n!/2} S^{n-3}.$$

The homotopy complementation formula cannot be used in any obvious way when examining the subgroup lattice of the alternating group A_n. However, using various standard techniques from topological combinatorics, we will also prove the following theorem.

Theorem 1.6. *Let $p \geqslant 5$ be a prime such that every transitive proper subgroup of A_p is solvable. Then*

$$\Delta(A_p) \simeq \left(\bigvee_{\frac{p-2}{2}(p-1)!} S^{p-4} \right) \vee \left(\bigvee_{(p-1)!} S^1 \right).$$

Note that the condition on transitive proper subgroups given in Theorem 1.6 holds for most primes p. Indeed, it was already known by Burnside (see [DiMo, Theorem 3.5B]) that every nonsolvable transitive subgroup of S_p is 2-transitive and almost simple. All 2-transitive almost simple groups are known (a list appears in [Ca], see also [DiMo, Section 7.7]), and upon examining this list one can see that if there is a nonsolvable transitive proper subgroup of A_p then either $p \in \{11, 23\}$ or $p = \frac{q^d - 1}{q-1}$, where q is a prime power and d is a prime.

In a forthcoming paper, we will use the theory of lexicographic shellability, due to A. Björner and M. Wachs, to show among other things that the homology group $\widetilde{H}_{n-4}(\Delta(A_n))$ is nontrivial for all $n > 2$.

The remainder of this paper is organized as follows. In Section 2 we record some basic results about the topology of order complexes. In Section 3 we prove Theorems 1.1, 1.4–1.6.

2. Basic results in poset topology

In this section we collect some fundamental results on topology of order complexes which we will use throughout the paper. The reader is referred to [Bj3], where many of the results listed below appear, for definitions of terms not defined here. We begin with some notation. For a finite lattice L, the unique minimum and maximum elements of L will be denoted by $\widehat{0}$ and $\widehat{1}$, respectively, unless otherwise indicated. Also, $\widehat{L}$ will denote the poset $L \setminus \{\widehat{0}, \widehat{1}\}$. The *atoms* and *coatoms* of L are the minimal and maximal elements of $\widehat{L}$, respectively. For any poset P, any $Q \subseteq P$ and any $x \in P$, we define

$$Q_{\leqslant x} := \{y \in Q : y \leqslant x\}$$

and

$$Q_{\geqslant x} := \{y \in Q : y \geqslant x\}.$$

The *order complex* ΔP of a poset P is the abstract simplicial complex whose k-dimensional simplices are the chains

$$x_0 < \cdots < x_k$$

from P. For $x, y \in P$ we define

$$[x, y] := \{z \in P : x \leqslant z \leqslant y\}$$

and

$$(x, y) := \{z \in P : x < z < y\},$$

and define $(x, y]$ and $[x, y)$ similarly.

Note that if $f : P \to Q$ is an order preserving map of posets then f determines a simplicial map (also called f) from ΔP to ΔQ. The next result, due to Quillen (see [Qu, Proposition 1.6]), is perhaps the most important one on topology of order complexes.

Lemma 2.1 (Fiber lemma). *Let $f : P \to Q$ be an order preserving map of posets. If either $\Delta f^{-1}(Q_{\leqslant q})$ is contractible for each $q \in Q$ or $\Delta f^{-1}(Q_{\geqslant q})$ is contractible for each $q \in Q$ then the simplicial map f is a homotopy equivalence, so*

$$\Delta P \simeq \Delta Q.$$

Note that each abstract simplicial complex Δ has a geometric realization, and all geometric realizations of Δ are homeomorphic. We make no distinction between an abstract simplicial complex and its geometric realization. One condition which implies that ΔP is contractible is that P has a unique minimum element or a unique maximum element, as if x is an element of either type then one can define a series of elementary collapses which reduce ΔP to the point x by pairing each nonempty chain C from $P \setminus \{x\}$ with $C \cup \{x\}$ (see also [Qu, Section (1.5)]).

For a lattice L, L^* will denote the subposet containing all elements of L which can be obtained by taking the meet of a set of coatoms of L. Note that $\widehat{1} \in L^*$, since it is the meet of the empty set of coatoms. The following result is well known and is implicit in [Bj2, Theorem 2.1], which is a homotopy version of Rota's cross-cut theorem (see [Ro]).

Lemma 2.2. *Let L be a finite lattice and let P be any subposet of L which contains $L^* \cup \{\widehat{0}\}$*

(1) *If $\widehat{0} \in L^*$ then*

$$\Delta \widehat{P} \simeq \Delta \widehat{L^*}.$$

(2) *If $\widehat{0} \notin L^*$ then $\Delta \widehat{P}$ is contractible.*

Proof. We prove Lemma 2.2 using the fiber lemma. Set

$$M := \begin{cases} \widehat{L^*}, & \widehat{0} \in L^*, \\ L^* \setminus \{\widehat{1}\}, & \widehat{0} \notin L^*. \end{cases}$$

Let $i : M \to \widehat{P}$ be the identity embedding. For each $x \in \widehat{P}$, let x^* be the meet of all coatoms of L which lie above x. Then $i^{-1}(\widehat{P}_{\geqslant x}) = M_{\geqslant x^*}$ is contractible. The first claim of the lemma follows immediately. If $\widehat{0} \notin L^*$ then M contains a unique minimum element, namely the meet of all coatoms of L, and the second claim follows. $\square$

If L is a finite lattice with meet and join operations $\wedge$ and $\vee$, respectively, and $x \in L$ then $a \in L$ is called a *complement* to x if $x \wedge a = \widehat{0}$ and $x \vee a = \widehat{1}$. The set of complements to x will be denoted by $x^{\perp}$. We will use the following result of Björner and Walker (see [BjWal, Theorem 4.2]). Here $\Sigma(\Delta)$ denotes the suspension of a complex Δ and $\Delta * \Gamma$ denotes the join of complexes Δ and Γ.

Lemma 2.3 (Homotopy complementation formula). *Let L be a finite lattice and let $x \in L$. If $x^{\perp}$ is an antichain in L then*

$$\Delta\widehat{L} \simeq \bigvee_{a \in x^{\perp}} \Sigma(\Delta(\widehat{0}, a) * \Delta(a, \widehat{1})).$$

In particular, if $x^{\perp} = \emptyset$ then $\Delta\widehat{L}$ is contractible.

Finally, we will need the following gluing lemma, which is a slight generalization of [Bj3, Lemma 10.4(ii)]. Recall that a complex Δ is k-connected if for every $l \leqslant k$, every continuous function from the l-sphere S^l to Δ can be extended to a continuous function from the $(l+1)$-ball B^{l+1} to Δ. (Equivalently, Δ is k-connected if $\pi_i(\Delta) = 0$ for all $i \leqslant k$.) In particular, a wedge of k-dimensional spheres is $(k-1)$-connected. If Γ is a subcomplex of the complex Δ, the quotient CW-complex obtained from Δ by identifying all points in Γ is denoted by Δ/Γ. (See, for example, [Hat, p. 8] for basic information on quotient complexes.)

Lemma 2.4. *Let Δ be a complex with subcomplexes $\Delta_0, \Delta_1, \dots, \Delta_r$ such that*

$$\Delta = \bigcup_{i=0}^{r} \Delta_i.$$

Assume that there is some $k \geqslant 0$ such that

- *Δ_0 is k-connected,*
- *$\dim(\Delta_i) \leqslant k$ for $1 \leqslant i \leqslant r$, and*
- *$\Delta_i \cap \Delta_j \subseteq \Delta_0$ for $1 \leqslant i < j \leqslant r$.*

Then

$$\Delta \simeq \Delta_0 \vee \bigvee_{i=1}^{r} \Delta_i/(\Delta_i \cap \Delta_0).$$

If Δ_i is contractible for each $i \in [r]$ then

$$\Delta \simeq \Delta_0 \vee \bigvee_{i=1}^{r} \Sigma(\Delta_i \cap \Delta_0).$$

Proof. Set

$$\Delta^+ = \bigcup_{i=1}^{r} \Delta_i$$

and

$$\Gamma = \bigcup_{i=1}^{r} (\Delta_i \cap \Delta_0),$$

so $\Gamma = \Delta^+ \cap \Delta_0$. Let $\iota : \Gamma \to \Delta_0$ be the identity embedding and let $c : \Gamma \to \Delta_0$ be any constant map. Since Δ_0 is k-connected and $\dim(\Gamma) \leqslant k$, we see that ι and c are homotopic (see [Hat, Lemma 4.6]). It follows (see [Hat, Proposition 0.18]) that

$$\Delta_0 \bigsqcup_{\iota} \Delta^+ \simeq \Delta_0 \bigsqcup_{c} \Delta^+.$$

(For spaces A, B, C with $A \subseteq B$ and a function $f : A \to C$, $B \bigsqcup_f C$ denotes the space obtained from the disjoint union of B and C by identifying a with $f(a)$ for each $a \in A$.) Now

$$\Delta_0 \bigsqcup_{\iota} \Delta^+ = \Delta,$$

and since $\Delta_i \cap \Delta_j \subseteq \Delta_0$ for $1 \leqslant i < j \leqslant r$, we have

$$\Delta_0 \bigsqcup_{c} \Delta^+ \simeq \Delta_0 \vee \bigvee_{i=1}^{r} \Delta_i / (\Delta_i \cap \Delta_0).$$

This gives the first claim. The second claim follows from the well-known fact (a proof appears in [Sh2, Lemma 2.5]) that if Λ is a contractible complex and Θ is a subcomplex of Λ then

$$\Lambda/\Theta \simeq \Sigma(\Theta). \qquad \square$$

3. Proofs of Theorems 1.1, 1.4, 1.5 and 1.6

In this section we prove Theorems 1.1, 1.4, 1.5 and 1.6. A key object in the proof of Theorem 1.1 and that of Theorem 1.6 is the partition lattice Π_n, which we examine below.

3.1. The partition lattice. The partition lattice Π_n is the lattice of all partitions π of the set $[n]$ into subsets, which are called the *parts* of π. The order on Π_n is the refinement order, so $\pi \leqslant \sigma$ if each part of π is a subset of some part of σ. Thus the minimum element $\hat{0}$ of Π_n is the partition into n singletons and the maximum element is the partition with one part $[n]$. If π has parts $\pi_1, \ldots, \pi_k$ we write $\pi = [\pi_1|\ldots|\pi_k]$. Note that in this case the interval $[\pi, \hat{1}]$ in Π_n is isomorphic to Π_k.

A partition π is called an *equipartition* if each part of π has the same size. An equipartition π is called *nontrivial* if $\pi \in \widehat{\Pi_n}$. Thus a transitive subgroup $H \leqslant S_n$ is primitive if and only if H stabilizes no nontrivial equipartition of $[n]$.

The partition lattice is ranked, with rank function r given by

$$r([\pi_1|\ldots|\pi_k]) = n - k.$$

For $X \subseteq [n-2]$ set

$$\Pi_n^X := \{\pi \in \Pi_n : r(\pi) \in X\}$$

and let μ_X be the Möbius function on $\Pi_n^X \cup \{\widehat{0}, \widehat{1}\}$ (see [St, Section 3.7] for the definition of the Möbius function). The next result follows easily from theorems of A. Björner on shellability and, with the possible exception of the final claim, is well known.

Proposition 3.1. *For any $X \subseteq [n-2]$, the order complex $\Delta(\Pi_n^X)$ has the homotopy type of a wedge of $|\mu_X(\widehat{0}, \widehat{1})|$ spheres of dimension $|X| - 1$. In particular,*

$$\Delta(\Pi_n^{[n-2]}) \simeq \bigvee_{(n-1)!} S^{n-3}$$

and

$$\Delta(\Pi_n^{[n-2]\setminus\{1\}}) \simeq \bigvee_{\frac{n-2}{2}(n-1)!} S^{n-4}.$$

Proof. As noted in [Bj1, Example 2.9], the poset Π_n is shellable. It follows that each $\Pi_n^X \cup \{\widehat{0}, \widehat{1}\}$ is shellable ([Bj1, Theorem 4.1]). Thus if $\widetilde{\chi}(X)$ is the reduced Euler characteristic of $\Delta(\Pi_n^X)$ then (see for example [Bj1, Appendix]) we have

$$\Delta(\Pi_n^X) \simeq \bigvee_{|\widetilde{\chi}(X)|} S^{|X|-1}.$$

The basic result of Hall mentioned in the introduction (see [Ha, (2.21)] or [St, Propositions 3.8.5,3.8.6]) gives

$$\widetilde{\chi}(X) = \mu_X(\widehat{0}, \widehat{1}),$$

and it remains to examine the cases $X = [n-2]$ and $X = [n-2]\setminus\{1\}$. It is well known (see [St, p. 128] or [Bj1, Example 2.9]) that $\mu_{[n-2]}(\widehat{0}, \widehat{1}) = (-1)^{n-1}(n-1)!$.

Now

$$\begin{aligned}\mu_{[n-2]\setminus\{1\}}(\widehat{0},\widehat{1}) &= -\sum_{r(\pi)>1}\mu_{[n-2]}(\pi,\widehat{1})\\ &= -\sum_{r(\pi)>0}\mu_{[n-2]}(\pi,\widehat{1})+\sum_{r(\pi)=1}\mu_{[n-2]}(\pi,\widehat{1})\\ &=(-1)^{n-1}(n-1)!+\binom{n}{2}(-1)^{n-2}(n-2)!\\ &=(-1)^{n}\frac{n-2}{2}(n-1)!.\end{aligned}$$

Indeed, the first equality follows from the definition of the Möbius function and the third equality follows from the definition and the fact that Π_n contains $\binom{n}{2}$ elements of rank one, with $[\pi,\widehat{1}]\cong\Pi_{n-1}$ whenever $r(\pi)=1$. $\square$

From now on, S_X will denote the group of all permutations of a set X (so $S_{[n]}=S_n$). For a partition $\pi=[\pi_1|\dots|\pi_k]\in\Pi_n$, S_π will denote the group

$$S_{\pi_1}\times\cdots\times S_{\pi_k}\leqslant S_n.$$

Also, for any $H\leqslant S_n$, we define $\mathbf{orb}(H)\in\Pi_n$ to be the partition whose parts are the orbits of H on $[n]$. We record some simple but important facts about the function **orb**, which will be used without reference, in the following lemma.

Lemma 3.2. *Let* $n\in\mathbb{N}$.

(1) *The map* **orb** *form the lattice of subgroups of* S_n *to* Π_n *is order preserving.*
(2) *For any* $\pi\in\Pi_n$, *we have* $\mathbf{orb}(S_\pi)=\pi$.
(3) *For* $H\leqslant S_n$ *and* $\pi\in\Pi_n$, *we have* $\mathbf{orb}(H)\leqslant\pi$ *if and only if* $H\leqslant S_\pi$.

3.2. The proof of Theorem 1.1. Here we prove a more precise version Theorem 1.1 and explain Conjecture 1.3. Let $\mathscr{L}(S_n)$ be the lattice of subgroups of S_n. The minimum and maximum elements of $\mathscr{L}(S_n)$ are 1 and S_n, respectively. Let

$$\mathscr{I}:=\{t\in S_n:\ |t|=2\}$$

be the set of involutions in S_n and let

$$\mathscr{I}_0:=\mathscr{I}\setminus(\mathscr{I}\cap A_n)$$

be the set of odd involutions in S_n. It follows immediately from the fact that $[S_n:\ A_n]=2$ that in the lattice $\mathscr{L}(S_n)$ we have

$$A_n^{\perp}=\{\langle t\rangle:\ t\in\mathscr{I}_0\}.$$

Since $\langle t\rangle$ is an atom in $\mathscr{L}(S_n)$ for each $t\in\mathscr{I}_0$, the homotopy complementation formula (Lemma 2.3) gives the following result.

Lemma 3.3. *For any $n>2$ we have*

$$\Delta(S_n) \simeq \bigvee_{t \in \mathscr{I}_0} \Sigma(\Delta(\langle t \rangle, S_n)).$$

We will determine the homotopy type of $\Delta(\langle t \rangle, S_n)$ when t is a transposition, and Theorem 1.1 will follow. Conjecture 1.3 reflects a conjecture on the homotopy type of $\Delta(\langle t \rangle, S_n)$ when $t \in \mathscr{I}_0$ is not a transposition. For $t \in \mathscr{I}_0$, set

$$\mathbf{NT}^t := \{H \in (\langle t \rangle, S_n) : H \text{ is not transitive}\}$$

and

$$\mathbf{NP}^t := \{H \in (\langle t \rangle, S_n) : H \text{ is not primitive}\}.$$

Note that both $\mathbf{NT}^t$ and $\mathbf{NP}^t$ are ideals in the poset $(\langle t \rangle, S_n)$ and that $\mathbf{NT}^t \subseteq \mathbf{NP}^t$. Our first step is to determine the homotopy type of $\Delta\mathbf{NT}^t$ for arbitrary $t \in \mathscr{I}_0$.

Lemma 3.4. *Assume $n>2$.*

(1) *If $t \in S_n$ is a transposition then*

$$\Delta\mathbf{NT}^t \simeq \bigvee_{(n-2)!} S^{n-4}.$$

(2) *If $t \in S_n$ is an odd involution which is not a transposition then $\Delta\mathbf{NT}^t$ is contractible.*

Proof. Consider the restriction $\mathbf{orb}_t$ of the orbit map $\mathbf{orb}$ to $\mathbf{NT}^t$. For any $H \in \mathbf{NT}^t$, we know that $\mathbf{orb}(\langle t \rangle)$ refines $\mathbf{orb}(H)$. Conversely, if $\mathbf{orb}(\langle t \rangle)$ strictly refines $\pi \in \widehat{\Pi_n}$ then $S_\pi \in (\langle t \rangle, S_n)$ and

$$\mathbf{orb}_t^{-1}((\Pi_n)_{\leqslant \pi}) = (\langle t \rangle, S_\pi].$$

If t is a transposition then the only subgroup of S_n with the same orbits as $\langle t \rangle$ is $\langle t \rangle$. It follows that

$$\mathbf{orb}_t^{-1}((\Pi_n)_{\leqslant \mathbf{orb}(\langle t \rangle)}) = \begin{cases} \emptyset & t \text{ a transposition,} \\ (\langle t \rangle, S_{\mathbf{orb}(t)}] & \text{otherwise.} \end{cases}$$

Therefore,

$$\mathrm{Image}(\mathbf{orb}_t) = \begin{cases} (\mathbf{orb}(t), \hat{1}) & t \text{ a transposition,} \\ [\mathbf{orb}(t), \hat{1}) & \text{otherwise.} \end{cases}$$

In addition, the preimage of each $\pi \in \mathrm{Image}(\mathbf{orb}_t)$ has a unique maximum element S_π. The Quillen fiber lemma (Lemma 2.1) gives

$$\Delta\mathbf{NT}^t \simeq \Delta\mathrm{Image}(\mathbf{orb}_t),$$

and the contractibility of $\Delta\mathbf{NT}^t$ when t is not a transposition follows from the presence of the unique minimum element $\mathbf{orb}(t)$ in Image$(\mathbf{orb}_t)$. Our claim when t is a transposition follows from Lemma 3.1, after noting that $[\mathbf{orb}(t), \widehat{1}] \cong \Pi_{n-1}$. $\square$

Since S_n contains $\binom{n}{2}$ transpositions and

$$\Sigma\left(\bigvee_a S^d\right) \simeq \bigvee_a S^{d+1}$$

for all a, d, we see that Theorem 1.1 will follow from Lemmas 3.3 and 3.4 once we show that if $t \in S_n$ is a transposition then

$$\Delta(\langle t \rangle, S_n) \simeq \Delta\mathbf{NT}^t.$$

A classical theorem of Jordan (see [DiMo, Theorem 3.3A]) says that if $G \leqslant S_n$ is primitive and contains a transposition then $G = S_n$. Therefore, we have

$$(\langle t \rangle, S_n) = \mathbf{NP}^t,$$

and the next lemma completes the proof of Theorem 1.1.

Lemma 3.5. *Let $t \in S_n$ be a transposition with $n > 2$. Then*

$$\Delta\mathbf{NP}^t \simeq \Delta\mathbf{NT}^t.$$

Before proving Lemma 3.5, we note that [Iv] also gives the homotopy type of $\Delta(\langle t \rangle, S_n)$ when t is a transposition.

Proof. We use the Quillen fiber lemma (Lemma 2.1) to show that the identity embedding of $\mathbf{NT}^t$ in $\mathbf{NP}^t$ determines a homotopy equivalence of order complexes. Fix a transposition t. We must show that for $H \in \mathbf{NP}^t$ the complex $\Delta\mathbf{NT}^t_{\leqslant H}$ is contractible. If $H \in \mathbf{NT}^t$ then $\mathbf{NT}^t_{\leqslant H}$ has a unique maximum element H and the desired conclusion follows.

So, assume that $H \notin \mathbf{NT}^t$, so H is transitive but stabilizes some nontrivial equipartition $\pi = [\pi_1|\cdots|\pi_k] \in \Pi_n$. We may assume that $t = (12)$, and since $t \in H$ we may assume that

$$\{1, 2\} \subseteq \pi_1.$$

Set

$$\mathbf{L} := \mathbf{NT}^t_{\leqslant H} \cup \{\langle t \rangle, H\}.$$

Then $\mathbf{L}$ is a lattice with minimum and maximum elements $\langle t \rangle, H$, respectively. Note that for $A, B \in \mathbf{L}$ we have $A \vee B = H$ if and only if $\langle A, B \rangle$ is transitive. Let X be the subgroup of H generated by all H-conjugates of t. We will show that there are no complements to X in $\mathbf{L}$, so Lemma 3.5 follows from the homotopy complementation formula (Lemma 2.3).

First note that if $(ab)\in H$ is a transposition then $\{a,b\}\subseteq\pi_i$ for some $i\in[r]$. Therefore, for all $x\in X$ and all $i\in[r]$ we have

$$\pi_i x = \pi_i.$$

Now assume $A\in\mathbf{L}$ with $A\cap X = \langle t\rangle$. Then A contains no H-conjugate of t other than t. In particular, $\{t\}$ is a conjugacy class of A, so $A\leqslant C_H(t)$. But then $\{1,2\}$ is an orbit of A. Since A permutes the π_i, we see that

$$\pi_1 a = \pi_1$$

for all $a\in A$. But now we have

$$\pi_1 g = \pi_1$$

for all $g\in\langle A,X\rangle$. Therefore $\langle A,X\rangle$ is not transitive and A is not a complement to X in $\mathbf{L}$. $\square$

Conjecture 1.3 is equivalent to the next conjecture.

Conjecture 3.6. *It $t\in S_n$ is an odd involution which is not a transposition then $\Delta\mathbf{NP}^t$ (the order complex of the poset of subgroups properly containing $\langle t\rangle$ which are not primitive) is contractible.*

Note that one cannot hope to prove Conjecture 3.6 for all $t\in\mathscr{I}_0$ using the same approach that was used to prove Lemma 3.5. For example, say p is an odd prime and let $n=2p$. Let $H\leqslant S_n$ be dihedral of order $2p$, acting regularly. Let $t\in H$ be any involution. Then $t\in\mathscr{I}_0$ (as t is the product of p transpositions) and $\mathbf{NT}^t_{\leqslant H}=\emptyset$, so $\Delta\mathbf{NT}^t_{\leqslant H}$ is not contractible. Also, if $n\equiv 3 \bmod 4$, $H\leqslant S_n$ is a dihedral group of order $2n$ acting naturally and $t\in H$ is an involution then we have $t\in\mathscr{I}_0$. If in addition n is squarefree but not prime then $H\in\mathbf{NP}^t$ and $\Delta\mathbf{NT}^t_{\leqslant H}$ is not contractible. It would be interesting to know (for arbitrary n) if there exist $t\in\mathscr{I}_0$ with more than one fixed point and $H\in\mathbf{NP}^t$ such that $\Delta\mathbf{NT}^t_{\leqslant H}$ is not contractible.

3.3. Proof of Theorem 1.4. Here we will see that Theorem 1.4 follows quite easily from the results in Section 3.2 and basic facts about transitive groups of prime degree. Let p be any prime. By Lemmas 3.3–3.5, Theorem 1.4 will follow if we show that if $t\in\mathscr{I}_0$ is not a transposition then $\Delta(\langle t\rangle, S_p)$ is contractible. As shown in [Sh1, Corollary 3.2] and [Sh1, proof of Theorem 3.3], if $t\in\mathscr{I}_0$ then

$$(t,S_p)=\mathbf{NT}^t$$

unless $p\equiv 3 \bmod 4$ and t has exactly one fixed point. In this exceptional case, there exist some Sylow p-subgroups $P\leqslant S_p$ such that $t\in N_{S_p}(P)$, and every transitive subgroup of S_p which contains t is contained in exactly one such $N_{S_p}(P)$. So, let

$$\mathscr{N} := \{N_{S_p}(P):\ P\in Syl_p(S_p), t\in N_{S_p}(P)\}.$$

Then

$$\Delta(\langle t\rangle, S_p) = \Delta\mathbf{NT}^t \cup \bigcup_{N\in\mathcal{N}} \Delta(\langle t\rangle, N].$$

By observation, we have

$$\Delta(S_3) \simeq \bigvee_3 S^0,$$

so we may assume that $p>3$. By Lemma 3.4, $\Delta\mathbf{NT}^t$ is contractible, and since each interval $(\langle t\rangle, N]$ contains a maximum element, each $\Delta(\langle t\rangle, N]$ is contractible. Lemma 10.4(ii) of [Bj3] (or our Lemma 2.4) gives

$$\Delta(\langle t\rangle, S_p) \simeq \bigvee_{N\in\mathcal{N}} \Sigma(\Delta(\mathbf{NT}^t \cap (\langle t\rangle, N])).$$

Now each poset $\mathbf{NT}^t \cap (\langle t\rangle, N]$ contains a unique maximum element, namely, the unique cyclic subgroup of order $p-1$ in N which contains t. Therefore, each $\Delta(\mathbf{NT}^t \cap (\langle t\rangle, N])$ is contractible and our proof is complete.

3.4. The proof of Theorem 1.5. Note first that $\Delta(S_2)$ contains only the empty face and is, by definition, a sphere of dimension -1. Either direct inspection or the theorem of Kratzer and Thévenaz ([KrTh2, Corollary 4.10]) mentioned in the introduction can be used to prove that $\Delta(S_4)$ has the homotopy type of a wedge of 12 spheres of dimension 1. Thus when proving Theorem 1.5, we may assume that $a>2$, that is, $n\geqslant 8$. The theorem will follow from Lemmas 3.3, 3.4(1) and 3.5 once we prove the following result.

Lemma 3.7. *Let $t\in S_{2^a}$ be an odd involution which is not a transposition. Then $\Delta(\langle t\rangle, S_{2^a})$ is contractible.*

Lemma 3.7 follows from Lemma 3.4(2) and the next two lemmas.

Lemma 3.8. *If $t\in S_{2^a}$ is an odd involution which is not a transposition then $\Delta\mathbf{NT}^t \simeq \Delta\mathbf{NP}^t$.*

Proof. We cannot show that the identity embedding of $\mathbf{NT}^t$ in $\mathbf{NP}^t$ induces a homotopy equivalence of order complexes, so we introduce some additional posets as follows. Recall that for a finite lattice L, we defined L^* to be the lattice consisting of all elements of L which can be obtained by taking the meet of some coatoms of L. Set

$$\mathbf{L}_t := \mathbf{NP}^t \cup \{t, S_n\}$$

(so $\widehat{\mathbf{L}_t} = \mathbf{NP}^t$) and set

$$\mathbf{M}_t := \mathbf{L}_t^* \cup \mathbf{NT}^t.$$

Thus $\widehat{\mathbf{M}_t}$ consists of those elements of $(\langle t\rangle, S_n)$ which are either intransitive or the intersection of some collection of maximal transitive but imprimitive subgroups (or both). Certainly $\mathbf{L}_t^* \subseteq \mathbf{M}_t$. It is straightforward to show that $\langle t \rangle \in \mathbf{L}_t^*$ (and unnecessary to do so for our purposes, since if $\langle t \rangle \notin \mathbf{L}_t^*$ then $\Delta\mathbf{NP}^t$ is contractible by Lemma 2.2(2)). So, by Lemma 2.2(1), we have

$$\Delta\mathbf{NP}^t \simeq \Delta\widehat{\mathbf{L}_t^*} \simeq \Delta\widehat{\mathbf{M}}_t.$$

We complete the proof by using the fiber lemma to show that the identity embedding of $\mathbf{NT}^t$ into $\widehat{\mathbf{M}}_t$ induces a homotopy equivalence of order complexes. To do this, it suffices to show that if $H \in \widehat{\mathbf{M}}_t \backslash \mathbf{NT}^t$ and $\mathbf{P} = (\langle t \rangle, H) \cap \mathbf{NT}^t$ then $\Delta\mathbf{P}$ is contractible. Let $\Pi(H) \leqslant \Pi_n$ be the image of the restriction $\mathbf{orb}_H$ of the orbit map $\mathbf{orb}$ to $\mathbf{P}$. For each $\pi \in \Pi(H)$, the poset $\mathbf{orb}_H^{-1}(\Pi(H)_{\leqslant \pi})$ has a unique maximum element, namely, $H \cap S_\pi$. By the fiber lemma (Lemma 2.1), we have

$$\Delta\mathbf{P} \simeq \Delta\Pi(H).$$

If we show that $\mathbf{orb}(\langle t \rangle) \in \Pi(H)$ then we are done, as $\mathbf{orb}(\langle t \rangle)$ will be the unique minimum element of $\Pi(H)$. Thus, it suffices to show that there is some $K \in (t, H)$ such that $\mathbf{orb}(K) = \mathbf{orb}(t)$.

This is shown in the proof of [Sh1, Lemma 6.2], but we resketch the proof here. By the definition of $\mathbf{M}_t$, there exist nontrivial equipartitions $\Psi_1, \ldots, \Psi_k$ of $[2^a]$ such that

$$H = \bigcap_{i=1}^{k} \mathrm{Stab}(\Psi_i).$$

(Here $\mathrm{Stab}(\Psi_i)$ is the stabilizer of Ψ_i in S_{2^a}, that is, the group of permutations which permute the parts of Ψ_i). Let P be a Sylow 2-subgroup of H which contains t. Since H is a transitive group of degree 2^a, the group P is also transitive. Let z be an element of order two in the center of P. Since P is transitive, z is fixed-point free. In particular, $z \in A_{2^a}$ so $z \neq t$. Let $\Psi_z = \mathbf{orb}(\langle z \rangle)$. Then Ψ_z is a P-invariant equipartition of $[2^a]$. Consider the group $Q = \langle t, z \rangle$. This group is abelian of order four and it follows that every orbit of Q has size two or four (Q has no fixed points since $z \in Q$). Since t is odd, there exist oddly many parts of size two in $\mathbf{orb}(Q)$ which are also parts of $\mathbf{orb}(\langle t \rangle)$. Now consider any Ψ_i, $i \in [k]$. Since the transitive group P stabilizes both Ψ_i and Ψ_z, there exists some $c \in \mathbb{N}$ such that if X is a part of Ψ_i and Y is a part of Ψ_z then $|X \cap Y| \in \{0, c\}$ (this is [Sh1, Lemma 5.1], which is easy to prove). Evidently $c \in \{1, 2\}$, and we claim that $c = 2$. Assume for contradiction that $c = 1$. Let Γ be the set of all parts of size two from $\mathbf{orb}(Q)$ which are also parts of $\mathbf{orb}(\langle t \rangle)$. As noted above, $|\Gamma|$ is odd. Now let X be any part of Ψ_i which intersects some element Y of Γ nontrivially. Since Y is a part of Ψ_z, we have $|X \cap Y| = 1$, so $|X^t \cap Y| = 1$. As Ψ_i is $\langle t \rangle$-invariant, we know $X \neq X^t$, so X contains no fixed point of t. Now for each part W of size four from $\mathbf{orb}(Q)$, we have $|W \cap X| = |W \cap X^t| = 2$. Any such part of $\mathbf{orb}(Q)$ is the union of two parts of size two from $\mathbf{orb}(\langle t \rangle)$. Since Ψ_i is a nontrivial equipartition of $[2^a]$, we know that $|X|$ is even. It follows that X intersects evenly many elements of Γ nontrivially. Since X is arbitrary, $|\Gamma|$ is even, which gives the desired contradiction. So, $c = 2$, which means that every part of $\mathbf{orb}(\langle z \rangle)$ is contained in some part of Ψ_i. Let $\{j, k\}$ be any element of Γ and let u be the transposition (j, k). Since Ψ_i was arbitrary in the argument above, u stabilizes each Ψ_i. Let $K = \langle t, u \rangle$. Since t is not a transposition, we have $u \neq t$ so $\langle t \rangle < K \leqslant H$. Since $\{j, k\}$ is a part of $\mathbf{orb}(\langle t \rangle)$ we have $\mathbf{orb}(K) = \mathbf{orb}(\langle t \rangle)$ as desired. $\square$

Lemma 3.9. *Let $t \in S_{2^a}$ be an odd involution. Then the identity embedding of $\mathbf{NP}^t$ into (t, S_n) induces a homotopy equivalence of order complexes.*

Proof. As noted in the proof of [Sh1, Lemma 6.10], if H is a primitive proper subgroup of S_{2^a} which contains an odd involution then $p = 2^a - 1$ is a prime and $H \cong \mathrm{PGL}_2(p)$ is embedded in S_{2^a} by its action on the $p+1$ points from its natural projective space. Thus our lemma will follow from the fiber lemma once we show that if $t \in \mathrm{PGL}_2(p) \backslash \mathrm{PSL}_2(p)$ is an involution then $\Delta(t, \mathrm{PGL}_2(p))$ is contractible.

Let t be such an involution and let g be a preimage of t in $\mathrm{GL}_2(q)$. Then g^2 is a scalar matrix while g is not, and since $t \notin \mathrm{PSL}_2(p)$ we see that g is conjugate to a diagonal matrix with eigenvalues $\pm\lambda$ for some $\lambda \in \mathbb{F}_p$. It follows that t fixes exactly two points in the projective space on which $\mathrm{PGL}_2(p)$ acts naturally.

Let $\mathbf{L}$ be the lattice of t-invariant subgroups of $\mathrm{PSL}_2(p)$. The map $K \mapsto K \cap \mathrm{PSL}_2(p)$ determines an isomorphism between $(\langle t \rangle, \mathrm{PGL}_2(p))$ and $\widehat{\mathbf{L}}$ (its inverse maps M to $\langle t \rangle M$). We will show that $\Delta\mathbf{L}$ is contractible. We need the following well-known facts about subgroups of $\mathrm{PSL}_2(p)$. The original reference for these facts is [Di]. See also [Do] or [Sh3] for these facts, and note that since $p = 2^a - 1$ with $a > 2$ we have $p \equiv 7 \bmod 8$.

(1) If K is a proper subgroup of $\mathrm{PSL}_2(p)$ then one of the following conditions holds.
 (a) K is cyclic of order dividing $\frac{p+1}{2}$ or $\frac{p-1}{2}$.
 (b) K is dihedral of order dividing $p+1$ or $p-1$.
 (c) K has a nontrivial normal p-subgroup and $|K|$ divides $\frac{p(p-1)}{2}$.
 (d) K is isomorphic to one of A_4, S_4 or A_5.

(2) If K is a maximal subgroup of $\mathrm{PSL}_2(p)$ then one of the following conditions holds.
 (a) K is dihedral of order $p+1 = 2^a$.
 (b) K is dihedral of order $p-1$. In this case, K is the stabilizer of a set of two points from the natural projective space.
 (c) K is a Borel subgroup, that is, the normalizer of a Sylow p-subgroup of $\mathrm{PSL}_2(p)$. In this case, $|K| = \frac{p(p-1)}{2}$ and K is the stabilizer of a point in the natural projective space.
 (d) $K \cong S_4$.
 (e) $K \cong A_5$. (Such K occur if and only if $a \equiv 1 \bmod 4$.)

(3) If $x \in \mathrm{PSL}_2(p)$ has order greater than two then the centralizer of x is cyclic of order p, $\frac{p+1}{2}$ or $\frac{p-1}{2}$ and the normalizer of $\langle x \rangle$ is either a Borel subgroup or is dihedral of order $p+1$ or $p-1$. If x has order two then the centralizer of x is dihedral of order $p+1$.

(4) The centralizer $C(t)$ of t in $\mathrm{PSL}_2(p)$ is dihedral of order $p-1$. It is the stabilizer of the set of the two points which are fixed by t.

Let $K \in \mathbf{L}$. If K is not isomorphic to one of A_4, S_4 or A_5 then K contains a unique maximal characteristic cyclic subgroup X, which must also lie in $\mathbf{L}$. Then the normalizer of X, which is maximal in $\mathrm{PSL}_2(p)$ and contains K, also lies in $\mathbf{L}$. If $K \in \mathbf{L}$ is isomorphic to A_4 then the normalizer of K, which is isomorphic to S_4, also lies in $\mathbf{L}$. However, the group S_4 has no outer automorphism, so t must induce an inner automorphism on any $X \cong S_4$ which lies in $\mathbf{L}$. However, the centralizer of any element of order at most two in S_4 contains a subgroup of order four, while the centralizer $C(t)$ of t in $\mathrm{PSL}_2(p)$ has order $p-1 = 2(2^{a-1}-1)$ which is not divisible by four. Thus $\mathbf{L}$ contains no group isomorphic to A_4 or S_4. Similarly, if some $K \in \mathbf{L}$ is isomorphic to A_5 then t acts on K as conjugation by some element of order at most two in S_5. Any such element either centralizes a subgroup of order four from A_5 or is a transposition. Thus if $\mathbf{L}$ contains a subgroup $X \cong A_5$ then $\langle t \rangle X \cong S_5$. However, $\mathrm{PGL}_2(p)$ contains no subgroup isomorphic to S_5. (This is well known to finite group theorists. One way to see it is to note that $\mathrm{PSL}_2(p)$ is isomorphic to the commutator subgroup $\Omega_3(p)$ of $\mathrm{SO}_3(p)$ (see for example [As, p. 253]). The action of $\mathrm{SO}_3(p)$ on $\Omega_3(p)$ by conjugation embeds $\mathrm{SO}_3(p)$ in $\mathrm{Aut}(\mathrm{PSL}_2(p)) \cong \mathrm{PGL}_2(p)$. Since $|\mathrm{SO}_3(p)| = |\mathrm{PGL}_2(p)|$, we have $\mathrm{PGL}_2(p) \cong \mathrm{SO}_3(p) \leqslant \mathrm{GL}_3(p)$. The irreducible (complex) character degrees of S_5 are $1,1,4,4,5,5,6$, so S_5 has no faithful complex representation of degree three. It follows that S_5 has no faithful representation of degree three in any characteristic $p > 5$). Therefore, $\mathbf{L}$ contains no subgroup isomorphic to A_5.

We now see that every coatom of $\mathbf{L}$ is a Borel subgroup or dihedral of order $p+1$ or $p-1$. A Borel subgroup B is t-invariant if and only if t fixes the point fixed by B. Thus $\mathbf{L}$ contains exactly two Borel subgroups, and the intersection of these two groups is the cyclic subgroup of $C(t)$ with order $\frac{p-1}{2}$. If K is a coatom of $\mathbf{L}$ which is neither a Borel subgroup nor $C(t)$ and M is any other coatom of $\mathbf{L}$ then the structure of centralizers of elements in $\mathrm{PSL}_2(p)$ described above forces $|K \cap M| \in \{1,2,4\}$.

We now see that if $K \in \widehat{\mathbf{L}^*}$, one of the following holds.

(1) K is cyclic of order $\frac{p-1}{2}$ and $K \leqslant C(t)$.
(2) K is a Borel subgroup.
(3) K is dihedral of order $p+1$ or $p-1$.
(4) $|K| \in \{2,4\}$.

In all cases just listed but the first, K contains an odd number of elements of order two. The automorphism t must fix one of these elements. Thus $K \cap C(t) \neq 1$. The group mentioned in the first case is contained in $C(t)$. Therefore, for each $K \in \widehat{\mathbf{L}^*}$, we have $K \cap C(t) \neq 1$. Thus $C(t)$ has no complement in $\mathbf{L}^*$ and $\Delta\mathbf{L}^*$ is contractible by the homotopy complementation formula (Lemma 2.3). Now $\Delta\mathbf{L}$ is contractible by Lemma 2.2. □

3.5. Proof of Theorem 1.6

The homotopy complementation formula is not available when we examine $\Delta(A_p)$. However, after careful examination of the subgroup lattice of A_p, Lemma 2.4 can be applied in order to prove Theorem 1.6. We call a prime $p>2$ "good" if every transitive proper subgroup of A_p is solvable. Set

$$\mathscr{N} := \{N_{A_p}(P) : P \in Syl_p(A_p)\}.$$

If p is good, the maximal transitive proper subgroups of A_p are the elements of $\mathscr{M}$. Each group $N \in \mathscr{M}$ is the split extension of a cyclic group P of order p (generated by a p-cycle) by a cyclic group C of order $\frac{p-1}{2}$ (generated by the square of a $(p-1)$-cycle). We record the following well-known facts.

(A) There are p conjugates of C in N and every intransitive subgroup of N is contained in one of these conjugates.
(B) Any two distinct conjugates of C in N intersect trivially.
(C) If M, N are distinct elements of $\mathscr{M}$ then $M \cap N$ is intransitive.
(D) $|\mathscr{M}| = (p-2)!$.

Let p be a good prime. In [Sh2] it is shown that

$$\Delta(A_5) \simeq \bigvee_{60} S^1.$$

The group $\mathrm{PSL}_2(7)$ contains a maximal subgroup isomorphic to S_4. This subgroup has index seven, and the action on its cosets gives an embedding of the nonsolvable group $\mathrm{PSL}_2(7)$ into A_7 as a transitive subgroup, so seven is not good. So, we may assume that $p \geqslant 11$. For $N \in \mathscr{M}$, set

$$\mathbf{L}_{\leqslant N} := \{H : 1 < H \leqslant N\}$$

and let $\mathbf{IN}_p$ be the poset of nontrivial intransitive subgroups of A_p. Then

$$\Delta(A_p) = \Delta\mathbf{IN}_p \cup \bigcup_{N \in \mathscr{M}} \Delta\mathbf{L}_{\leqslant N}.$$

By facts (A) and (B) above, for each $N \in \mathscr{M}$ the poset $\mathbf{L}_{\leqslant N} \cap \mathbf{IN}_p$ is the union of p disjoint components, each of which is isomorphic to the poset obtained by removing the minimum element from the lattice of divisors of $\frac{p-1}{2}$. The next three facts follow immediately.

(E) If $\frac{p-1}{2}$ has r prime divisors, counting multiplicities, then $\Delta(\mathbf{L}_{\leqslant N} \cap \mathbf{IN}_p)$ is pure of dimension $r-1$. Therefore,

$$\dim\left(\bigcup_{N \in \mathscr{M}} \Delta(\mathbf{L}_{\leqslant N} \cap \mathbf{IN}_p)\right) = r - 1.$$

(F) Since $p \geqslant 11$ and $r \leqslant \log_2(p-1) - 1$, we have $r < p - 6$.
(G) We have

$$\Delta(\mathbf{L}_{\leqslant N} \cap \mathbf{IN}_p) \simeq \bigvee_{p-1} S^0.$$

For any $n \in \mathbb{N}$ and any $\pi \in \Pi_n$, we have

$$\mathbf{orb}(S_\pi \cap A_n) = \begin{cases} \pi & r(\pi) \neq 1, \\ \widehat{0} & r(\pi) = 1. \end{cases}$$

Thus the image of $\mathbf{IN}_p$ under the orbit map **orb** is $\Pi_p^{[p-2]\setminus\{1\}}$. The fiber lemma (Lemma 2.1) and Proposition 3.1 give the following results.

(H) We have

$$\Delta\mathbf{IN}_p \simeq \bigvee_{\frac{p-2}{2}(p-1)!} S^{p-4}.$$

(I) In particular, $\Delta\mathbf{IN}_p$ is $(p-5)$-connected.

Since each $\mathbf{L}_{\leqslant N}$ contains the maximum element N, each $\Delta\mathbf{L}_{\leqslant N}$ is contractible. Using this fact along with facts (E),(F),(I) and Lemma 2.4, we get

$$\Delta(A_p) \simeq \Delta\mathbf{IN}_p \vee \bigvee_{N \in \mathcal{M}} \Sigma(\Delta(\mathbf{L}_{\leqslant N} \cap \mathbf{IN}_p)).$$

Theorem 1.6 now follows from facts (D),(G) and (H) since, as mentioned earlier, for any $a, d \geqslant 0$ we have

$$\Sigma\left(\bigvee_a S^d\right) \simeq \bigvee_a S^{d+1}.$$

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