On Cayley Isomorphisms of Finite Cayley Graphs – a Survey

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Abstract

The isomorphism problem for Cayley graphs have been extensively investigated over the past 30 years. Recently, substantial progress has been made on the study of this problem. The methods used in this area range from deep group theory, including the finite simple group classification, through to combinatorial techniques. Related to this problem, many long-standing open questions have been solved, and many new natural further research problems have arisen. The results and methods obtained in this area have also effectively been used to solve other problems regarding finite vertex-transitive graphs. This article is devoted to surveying results, open problems and methods in this area.

1 Introduction

The isomorphism problem for graphs, that is to decide whether or not two given graphs are isomorphic, is a fundamental problem in graph theory. Here we investigate this problem for Cayley graphs.

Let G be a finite group, and set $G^{\#} = G \setminus \{1\}$, the set of non-identity elements of G. For a subset S of $G^{\#}$, the associated *Cayley graph* $\mathrm{Cay}(G,S)$ is the directed graph Γ with vertex set G and with x adjacent to y if and only if $yx^{-1} \in S$. If $S = S^{-1} := \{s^{-1} \mid s \in S\}$, then the adjacency is symmetric and thus the Cayley graph $\mathrm{Cay}(G,S)$ may be viewed as an undirected graph.

The Cayley graph $\mathrm{Cay}(G,S)$ is determined completely by G and S. However, as for general graphs, it is a very difficult problem to decide whether or not any given two Cayley graphs or digraphs are isomorphic. If σ is an automorphism of G such that $S^{\sigma} = T$, then σ induces an isomorphism from $\mathrm{Cay}(G,S)$ to $\mathrm{Cay}(G,T)$. Such an isomorphism is called a *Cayley isomorphism*. However, it is of course possible for two Cayley graphs or digraphs $\mathrm{Cay}(G,S)$ and $\mathrm{Cay}(G,T)$ to be isomorphic but no Cayley isomorphisms map S to T. Here we investigate the conditions under which $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$ if and only if $S^\sigma = T$ for some $\sigma \in \mathrm{Aut}(G)$.

Definition 1.1 A Cayley graph $\mathrm{Cay}(G,S)$ is called a *CI-graph* of G if, for any Cayley graph $\mathrm{Cay}(G,T)$, whenever $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$ we have $S^\sigma = T$ for some $\sigma \in \mathrm{Aut}(G)$. (CI stands for *Cayley Isomorphism* or *Cayley Invariant.*)

One long-standing open question about Cayley graphs is the following question.

Question 1.2 Which Cayley graphs for a group G are CI-graphs?

One of the principal motivations of investigating CI-graphs is to determine isomorphism classes of Cayley graphs. By the definition, if $\mathrm{Cay}(G,S)$ is a CI-graph, then to decide whether or not $\mathrm{Cay}(G,S)$ is isomorphic to $\mathrm{Cay}(G,T)$, we only need to decide whether or not S is conjugate under $\mathrm{Aut}(G)$ to T, and usually, the latter is much easier.

Question 1.2 has received considerable attention, and has been investigated under various conditions in the literature. Interest in this question stems from a conjecture of Ádám (1967) that all circulant graphs were CI-graphs of the corresponding cyclic groups. This conjecture was disproved by Elspas and Turner (1970), and since then, a lot of work has been devoted to seeking CI-graphs. Recently, substantial progress in this area has been made: many important open problems have been solved, many new significant and deep results have been obtained, and many related new problems have naturally arisen. In this paper, we shall survey various aspects of this area by discussing the main results, methods and open problems.

The next definition collects some of the basic terms used in this topic.

Definition 1.3 Let m be a positive integer, and let G be a finite group.

(1) G is said to have the *m-DCI property* if all Cayley digraphs of G of valency m are CI-graphs; G is said to have the *m-CI property* if all undirected Cayley graphs of G of valency m are CI-graphs.

(2) G is called an *m-DCI-group* if all Cayley digraphs of G of valency at most m are CI-graphs; G is called an *m-CI-group* if all undirected Cayley graphs of G of valency at most m are CI-graphs. In particular, G is called a *DCI-group* or a *CI-group* if G is a $|G|$-DCI-group or a $|D|$-CI-group, respectively.

(3) G is called a *connected m-CI-group* if all undirected connected Cayley graphs of G of valency at most m are CI-graphs; G is called a *connected m-DCI-group* if all connected Cayley digraphs of G of valency at most m are CI-graphs.

By definition, the *valency* of a Cayley graph $\mathrm{Cay}(G,S)$ is equal to the cardinality $|S|$ of the connection subset S. Note that here 'valency' for digraphs means 'out-valency'. It easily follows from the definition that $\mathrm{Cay}(G,S)$ is connected if and only if $\langle S\rangle = G$, and that if $\mathrm{Cay}(G,S)$ is disconnected then all of its connected components are isomorphic.

Notation: Here a graph may be directed or undirected. For a positive integer n, C_n denotes the (directed or undirected) *cycle* of length n, and K_n the *complete graph* on n vertices. For a graph $\Gamma = (V,E)$, its *complement* $\overline{\Gamma} = (V,\overline{E})$ is the graph with vertex set V such that $(a,b) \in \overline{E}$ if and only if $(a,b) \notin E$. The *direct product* $\Gamma_1 \times \Gamma_2$ of two graphs $\Gamma_1 = (V_1,E_1)$ and $\Gamma_2 = (V_2,E_2)$ is the graph with vertex set $V_1 \times V_2$ such that $((a_1,a_2),(b_1,b_2))$ is an edge if and only if either $(a_1,b_1) \in E_1$ and $a_2 = b_2$, or $(a_2,b_2) \in E_2$ and $a_1 = b_1$. The *wreath product* $\Gamma_1[\Gamma_2]$ of a graph $\Gamma_2 = (V_2,E_2)$ by a graph $\Gamma_1 = (V_1,E_1)$ is the graph with vertex set $V_1 \times V_2$ such that $((a_1,a_2),(b_1,b_2))$ is an edge if and only if either $(a_1,b_1) \in E_1$ or $a_1 = b_1$ and $(a_2,b_2) \in E_2$. For a graph $\Gamma = (V,E)$, and a vertex $x \in V$, the set of vertices to which x is joined in Γ is denoted by $\Gamma(x) = \{y \mid (x,y) \in E\}$, that is the neighborhood of x. For a positive integer n and a graph Γ, denote by $n\Gamma$ a graph consisting of n copies of Γ.

For a subgroup H of a group G, $\mathbf{N}_G(H)$ and $\mathbf{C}_G(H)$ will denote the normalizer and the centralizer of H in G, respectively. A group is said to be *homocyclic* if it is a direct product of cyclic groups of the same order. Let π be a set of primes, and π' the complement of π in the set of all primes. A group G is called a *π-group* if all prime divisors of $|G|$ lie in π. If $\pi = \{p\}$ then p' denotes π'. For groups G, H, we use $G \rtimes H$ to denote a semi-direct product of G by H.

2 Some basic properties and examples

In this section, we shall discuss some basic properties about CI-graphs and some constructions of Cayley graphs that are not CI-graphs, which will demonstrate some of basic methods used in the area.

First we notice a simple but useful property: For a finite group G and a subset $S \subset G^{\#}$, if $\mathrm{Cay}(G,S)$ is disconnected then

$$\mathrm{Cay}(G,S) = \frac{|G|}{|\langle S\rangle|}\mathrm{Cay}(\langle S\rangle, S)$$

so that for any $T \subset G^{\#}$,

$$\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T) \text{ if and only if } \mathrm{Cay}(\langle S\rangle, S) \cong \mathrm{Cay}(\langle T\rangle, T).$$

Let H, L be two subgroups of G, and let $S \subset H^{\#}, T \subset L^{\#}$ be such that $\langle S\rangle = H$, $\langle T\rangle = L$ and $\mathrm{Cay}(H,S) \cong \mathrm{Cay}(L,T)$. Then $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$. If $\mathrm{Cay}(G,S)$ is a CI-graph, then $S^\sigma = T$ for some $\sigma \in \mathrm{Aut}(G)$. Thus $H^\sigma = \langle S\rangle^\sigma = \langle S^\sigma\rangle = \langle T\rangle = L$, that is, H is conjugate under $\mathrm{Aut}(G)$ to L, and in particular, $H \cong L$. Therefore, we have

Proposition 2.1 *Let G be a finite group, and let $\mathrm{Cay}(G,S)$ be a CI-graph of G.*

(1) *For each $T \subset G^{\#}$, if $\mathrm{Cay}(\langle S\rangle, S) \cong \mathrm{Cay}(\langle T\rangle, T)$, then $\langle S\rangle$ is conjugate under $\mathrm{Aut}(G)$ to $\langle T\rangle$, in particular, $\langle S\rangle \cong \langle T\rangle$.*

(2) *All subgroups of G isomorphic to $\langle S\rangle$ are conjugate under $\mathrm{Aut}(G)$.*

Thus, for example, if G has subgroups $\mathbb{Z}_2^2$ and $\mathbb{Z}_4$ then neither $\mathrm{Cay}(G,\{a,a^{-1}\})$ nor $\mathrm{Cay}(G,\{a,a^{-1},a^2\})$ is a CI-graph, where a is an element of G of order 4. An immediate consequence is that if a group G has the m-CI property for $m = 2$ or 3 then a Sylow 2-subgroup of G is elementary abelian, cyclic or generalized (see [49]). This property is used in [49, 63] to determine the simple groups with the m-CI property for $m = 2$ or 3. The following simple property provides a method to construct Cayley graphs which are not CI-graphs by using the wreath product.

Lemma 2.2 (see [57]) *Let $G = \langle a, H\rangle$ be an abelian group where $H < G$, and let $R = a^{n_1}H \cup \ldots \cup a^{n_k}H$, where $\langle R\rangle = G$ and the n_i are distinct positive integers such that $a^{n_i}H \neq H$. Set $\Sigma := \mathrm{Cay}(G/H, R/H)$. Then $\mathrm{Cay}(G,R) = \Sigma[\overline{K}_m]$ where $m = |H|$. Further, if $S = R \cup R_0$ for some $R_0 \subseteq H^{\#}$, then $\mathrm{Cay}(G,S) = \Sigma[\Sigma_0]$ where $\Sigma_0 = \mathrm{Cay}(H, R_0)$.*

Assume that G has two subgroups $H = \langle a\rangle \cong \mathbb{Z}_{p^2}$ and $K = \langle b\rangle \times \langle c\rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime. Let $S_1 = a\langle a^p\rangle$ and $T_1 = b\langle c\rangle$; and let $S_2 = a\langle a^p\rangle \cup a^{-1}\langle a^p\rangle$ and $T_2 = b\langle c\rangle \cup b^{-1}\langle c\rangle$. Then by Lemma 2.2, $\mathrm{Cay}(H,S_i) \cong C_p[\overline{K_p}] \cong \mathrm{Cay}(K,T_i)$ where $i = 1,2$. As $\langle S_i\rangle \ncong \langle T_i\rangle$, by Proposition 2.1, $\mathrm{Cay}(G,S_i)$ is not a CI-graph. Now a straightforward analysis leads to the following result.

Proposition 2.3 *Let G be a finite group, and let p be a prime divisor of $|G|$. Assume that G has the p-DCI property or the $2p$-CI property. Then a Sylow p-subgroup of G is of exponent p, cyclic or generalized quaternion.*

For a cyclic group $G \cong \mathbb{Z}_{p^r}$ with $p \geq 3$ and $r \geq 2$, let $S = a\langle a^p \rangle \cup \{a^p\}$ and $T = a\langle a^p \rangle \cup \{a^{2p}\}$. Then by Lemma 2.2, $\mathrm{Cay}(G,S) \cong C_p[C_p] \cong \mathrm{Cay}(G,T)$. If $\alpha \in \mathrm{Aut}(G)$ is such that $S^\alpha = T$ then $a^\alpha = a^{ip+1}$ for some i. Thus $(a^p)^\alpha = (a^\alpha)^p = (a^{ip+1})^p = a^{ip^2+p} = a^p \notin T$, which is a contradiction. Thus $\mathrm{Cay}(G,S)$ is not a CI-graph. Using similar constructions, we have the following conclusion, see for example [2] or [67].

Proposition 2.4 *Let p be a prime, and let $G = \mathbb{Z}_{p^r}$.*

(1) *If $p = 2$ and $r \geq 3$ then G is not a connected 3-DCI-group; if $p > 2$ and $r \geq 2$ then G is not a connected $(p+1)$-DCI-group.*

(2) *If $p = 2$ and $r \geq 4$ then G is not a connected 6-CI-group; if $p = 3$ and $r \geq 3$ then G is not a connected 8-CI-group; if $p > 3$ and $r \geq 2$ then G is not a connected $2(p+1)$-CI-group.*

All of the constructions of non-CI-graphs $\mathrm{Cay}(G,S)$ given above satisfies that S contains all non-identity elements of a subgroup or a coset of a subgroup of G. In particular, S is not minimal generating subset of $\langle S \rangle$. A Cayley graph $\mathrm{Cay}(G,S)$ is said to be *minimal* if it is connected but $\mathrm{Cay}(G, S \setminus \{s\})$ is disconnected for any $s \in S$, in other words, S is a minimal generating subset of G. It seems unlikely for minimal Cayley graphs not to be CI-graphs. For example, Ming Yao Xu (1993) conjectured that minimal Cayley graphs are all CI-graphs (refer to [97, 98]). The first example of minimal Cayley graphs which are not CI-graphs are constructed in [51]. The following is an extension of the example which is given in [68].

Proposition 2.5 *Let $G = \langle a \rangle \times \langle x \rangle \times \langle e \rangle$ where $o(a) > 1$ is odd, $o(x) = 2^r \geq 4$ and $o(e) = 2$, and let $S = \{x, xe, ax^2\}$ and $T = \{x, xe, ax^2e\}$. Then $\mathrm{Cay}(G,S)$ is a minimal Cayley graph of G, $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$, and $S^\sigma \neq T$ for any $\sigma \in \mathrm{Aut}(G)$. In particular, $\mathrm{Cay}(G,S)$ is not a CI-graph.*

It is believed that such examples are very rare, and the following problem is formulated in [51].

Problem 2.6 Characterize minimal Cayley graphs which are not CI-graphs.

There have been some partial results about the problem. Huang and Meng [41] proved that all minimal Cayley graphs of cyclic groups are CI-graphs. The author [51, Theorem 3.1(ii)] proved that all minimal Cayley graphs of abelian groups of odd order are CI-graphs. Then this result was extended by Feng and Xu [24] to prove that all minimum Cayley graphs of abelian groups are CI-graphs, and by Feng and Gao [26] to prove that all minimal Cayley graphs of abelian groups with cyclic Sylow 2-subgroups are CI-graphs. Very recently, a complete classification of finite abelian groups for which all minimal Cayley digraphs are CI-graphs is obtained in [68], that is, it is proved that all minimal Cayley digraphs of an abelian group G are CI-graphs if and only if either G is a 2-group, or the Sylow 2-subgroup cannot be written as $H \times \mathbb{Z}_2$ where $\exp(H) \geq 4$.

We conclude this section with discussing the isomorphism problem for Cayley graphs coming from different groups. By the discussions before Proposition 2.1, we know that even for non-isomorphic groups we can get isomorphic Cayley graphs. The trivial examples are complete graphs, k-partite graphs and their complements, which come from non-isomorphic groups of the same order. A less trivial example is as follows: let $G = \langle a, b\rangle \cong \mathrm{D}_{2n}$ and $H = \langle c\rangle \cong \mathbb{Z}_{2n}$, and let $S = \{a, b\}$ and $T = \{c, c^{-1}\}$. Then $G \ncong H$ but $\mathrm{Cay}(G, S) \cong C_{2n} \cong \mathrm{Cay}(H, T)$. A characterization result for the prime-square case was obtained by Joseph [43]. J. Morris [74] extended Joseph's result to circulant graphs of odd prime-power as follows.

Theorem 2.7 *Let p be an odd prime, and let $n = n_1 + n_2 + \ldots + n_k$. Then there exists a graph Γ which is a Cayley graph of $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{p^{n_1}} \times \ldots \times \mathbb{Z}_{p^{n_k}}$ if and only if $\Gamma = \Gamma_1[\Gamma_2[\ldots[\Gamma_k]\ldots]]$, where Γ_i is a circulant of order p^{n_i}.*

On the other hand, J. Dixon [14] enumerated the Cayley graphs isomorphic to small hypercubes.

3 A criterion for CI-graphs

The most useful tool for the study of isomorphisms of Cayley graphs is the criterion given in Theorem 3.3, which was obtained by Babai [5] and also by Alspach and Parsons [4]. It transforms the graph theoretic problem of deciding whether a Cayley graph Γ of a group G is a CI-graph to the group theoretic problem of deciding whether all regular representations of G in $\mathrm{Aut}\,\Gamma$ are conjugate. Some group theoretic results can therefore be applied to study CI-graphs. Actually, almost all proofs of important and deep results obtained so far in this area have involved using this criterion. We shall discuss the criterion in detail in Theorem 3.3.

Let G be a finite group, and let $\Gamma = \mathrm{Cay}(G, S)$. Let $\mathrm{Aut}\,\Gamma$ be the full automorphism group of Γ. By the definition, the group G acting by right multiplication (that is, $g : x \to xg$) is a subgroup of $\mathrm{Aut}\,\Gamma$ and acts regularly on $V\Gamma = G$. We shall often denote this regular subgroup by $\hat{G}$, but sometimes still denote it by G for convenience. Sabidussi [84] obtained a criterion for a graph to be a Cayley graph, also see [9, Lemma 16.3].

Proposition 3.1 *A graph Γ is a Cayley graph of a group G if and only if $\mathrm{Aut}\,\Gamma$ contains a subgroup which is isomorphic to G and regular on $V\Gamma$.*

We note that $G \leq \mathrm{Aut}\,\Gamma \leq \mathrm{Sym}(G)$, and that any $\sigma \in \mathbf{N}_{\mathrm{Sym}(G)}(G)$ induces an automorphism of G. The normalizer of G in $\mathrm{Aut}\,\Gamma$ is often used to characterize Γ because of the following simple property, see [29, Lemma 2.1].

Proposition 3.2 *Let $\Gamma = \mathrm{Cay}(G, S)$ be a Cayley graph, and let $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$. Then $\mathbf{N}_{\mathrm{Aut}\,\Gamma}(G) = G \rtimes \mathrm{Aut}(G, S)$.*

The next is the criterion of Babai for a Cayley graph to be a CI-graph we have mentioned. Since it is a fundamental tool in the study of isomorphisms of finite Cayley graphs, here we present a proof which is essentially the same as Babai's proof in [5].

Theorem 3.3 *Let Γ be a Cayley graph of a finite group G. Then Γ is a CI-graph of G if and only if all regular subgroups of $\mathrm{Aut}\,\Gamma$ isomorphic to G are conjugate.*

Proof. Assume that Γ is a CI-graph of G. We still use G to denote the right regular representation of G, and let $\tilde{G}$ be an arbitrary subgroup of $\mathrm{Aut}\,\Gamma$ which is isomorphic to G and acts regularly on $V\Gamma$. Let $\mathrm{Sym}(V\Gamma)$ be the symmetric group on $V\Gamma$. Since all regular representations of G are permutationally isomorphic (refer to [15]), there exists an isomorphism $\varphi :\ G \to \tilde{G}$ and a bijection $\rho \in \mathrm{Sym}(V\Gamma)$ such that, for all $g \in G$ and $x \in V\Gamma$, $(x^g)^\rho = (x^\rho)^{g^\varphi}$. Thus $g\rho = \rho g^\varphi$, and so $\rho^{-1} g \rho = g^\varphi$, that is, $G^\rho = \rho^{-1} G \rho = G^\varphi = \tilde{G}$. Let $\Sigma = \Gamma^{\rho^{-1}}$, that is, Σ is the graph with vertex set $V\Gamma$ such that x is adjacent to y in Σ if and only if x^ρ is adjacent to y^ρ in Γ (noting that ρ is a permutation on $V\Gamma$). Then $\rho^{-1}(\mathrm{Aut}\,\Sigma)\rho = \mathrm{Aut}\,\Gamma \geq G$, and so by Proposition 3.1, Σ is also a Cayley graph of G. Since

Γ is a CI-graph, there exists $\alpha \in \mathrm{Aut}(G)$ which induces an isomorphism from Γ to Σ. We still denote this graph isomorphism by α. Then $\Gamma^{\alpha} = \Sigma = \Gamma^{\rho^{-1}}$, and so $\Gamma^{\alpha\rho} = \Gamma$, that is $\beta := \alpha\rho$ is an automorphism of Γ. Now $G^{\beta} = G^{\alpha^{-1}\beta} = G^{\rho} = \tilde{G}$, namely, $\tilde{G}$ and G are conjugate. Since $\tilde{G}$ is arbitrary, all regular subgroups of $\mathrm{Aut}\,\Gamma$ isomorphic to G are conjugate.

Conversely, suppose that all regular subgroups of $\mathrm{Aut}\,\Gamma$ isomorphic to G are conjugate. Let Σ be a Cayley graph of G which is isomorphic to Γ, and let φ be an isomorphism from Σ to Γ. Then $\varphi \in \mathrm{Sym}(V\Gamma)$, and since $G \leq \mathrm{Aut}\,\Sigma$, we have $G^{\varphi} \leq \mathrm{Aut}\,\Gamma$. By the assumption, there exists $\beta \in \mathrm{Aut}\,\Gamma$ such that $G^{\varphi} = G^{\beta}$. Now $\varphi\beta^{-1} \in \mathrm{Sym}(V\Gamma)$ and $G^{\varphi\beta^{-1}} = G$, that is, $\gamma := \varphi\beta^{-1}$ normalizes G. Thus γ induces an automorphism of G. Also, since φ is an isomorphism from Σ to Γ and $\beta^{-1} \in \mathrm{Aut}\,\Gamma$, γ is also an isomorphism from Σ to Γ. Hence Γ is a CI-graph of G. $\square$

In the rest of this section, we give some examples to demonstrate the applications of the criterion. By the definition, both the complete graph $\mathrm{Cay}(G, G^{\#})$ and its complement $\mathrm{Cay}(G, \emptyset)$ are CI-graphs of G. We shall sometimes call them *trivial CI-graphs*. A digraph Γ is called a *Directed Graphical Regular Representation* (*DGRR*) of the group G if Γ is a Cayley graph of G and $\mathrm{Aut}\,\Gamma = \hat{G}$. Hence by Theorem 3.3, if Γ is a DGRR of G, then Γ is a CI-graph of G. The problem of classifying finite groups which have DGRR was solved, see [6, Theorem 2.1], and it was proved that if G does not have a DGRR then G is $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$, Q_8 or $\mathbb{Z}_3^2$. It is easy to see that all minimal Cayley graphs of the groups $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$, Q_8 and $\mathbb{Z}_3^2$ are CI-graphs. Thus we have the following result, which is observed in [67].

Proposition 3.4 *Any finite groups of order greater than 2 have non-trivial CI-graphs.*

Next we employ the Sylow's Theorem to study CI-graphs of prime-power order. Let G be a p-group where p is a prime. Suppose that Γ is a connected Cayley graph of G of valency less that p. Let $A = \mathrm{Aut}\,\Gamma$. Then $A = \hat{G}A_1$ such that $p \nmid |A_1|$. Thus $\hat{G}$ is a Sylow p-subgroup of A. By Sylow's Theorem, all regular subgroups of A are conjugate, and so by Theorem 3.3, Γ is a CI-graph. This simple property was first observed by Babai in [5, Theorem 3.1], and is slightly extended for undirected graphs in [54, Theorem 3.1] as follows.

Proposition 3.5 *Let p be a prime and let G be a p-group. Then G is a connected $(p-1)$-DCI- and $(2p-2)$-CI-group. In particular, the cyclic group $\mathbb{Z}_p$ is a DCI-group.*

We remark that the family of groups $\mathbb{Z}_p$ with p prime is the first known infinite family of CI-groups, which was obtained by several people using the spectrum of graphs, see [16, 20, 91]. We end this section with discussing an extreme case.

Corollary 3.6 *Let G be a finite group, and let Γ be a Cayley graph of G. If $\hat{G}$ is normal in* $\mathrm{Aut}\,\Gamma$*, then Γ is a CI-graph if and only if $\hat{G}$ is the unique regular subgroup of* $\mathrm{Aut}\,\Gamma$ *which is isomorphic to G.*

A Cayley graph $\mathrm{Cay}(G,S)$ is called a *normal Cayley graph* if $\hat{G}$ is normal in $\mathrm{Aut}\,\Gamma$ (refer to [98]). There indeed exist normal Cayley graphs which are not CI-graphs. Let $G = \langle a \rangle \cong \mathbb{Z}_{2^r}$ where $r \geq 3$. Set $S = \{a, a^{2^{r-1}+1}, a^2\}$, and let $\Gamma = \mathrm{Cay}(G,S)$. It is easily proved that $\mathrm{Aut}\,\Gamma = \hat{G} \rtimes \langle \sigma \rangle \cong \mathbb{Z}_{2^r} \rtimes \mathbb{Z}_2$, where $\hat{G} = \langle \hat{a} \rangle$ and $\hat{a}^\sigma = \hat{a}^{2^{r-1}+1}$, so that $\hat{G}$ is normal in $\mathrm{Aut}\,\Gamma$. Since $(\hat{a}\sigma)^2 = \hat{a}^{2^{r-1}+2}$, it follows that $\langle \hat{a}\sigma \rangle$ is a regular subgroup of $\mathrm{Aut}\,\Gamma$. Therefore, by Corollary 3.6, Γ is not a CI-graph. Actually, letting $T = \{a, a^{2^{r-1}+1}, a^{2^{r-1}+2}\}$, it is easily shown that $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$ but $S^\sigma \neq T$ for any $\sigma \in \mathrm{Aut}(G)$. However, normal Cayley graphs which are not CI-graphs seem to be very rare, so we would like to formulate a problem as follows.

Problem 3.7 Characterize normal Cayley graphs which are not CI-graphs.

4 Isomorphisms of circulants

A Cayley graph or digraph of a cyclic group is also called a *circulant.* One of the most remarkable achievements regarding the isomorphism problem for Cayley graphs is the complete classification of cyclic CI-groups, which was completed recently by M. Muzychuk (1995,1997), that is,

Theorem 4.1 (1) *A cyclic group of order n is a DCI-group if and only if $n = k, 2k$ or $4k$ where k is odd square-free.*

(2) *A cyclic group of order n is a CI-group if and only if either $n \in \{8, 9, 18\}$ or $n = k, 2k$ or $4k$ where k is odd square-free.*

The work of seeking such a classification had been lasted for 30 years. The necessary condition of the theorem is not hard to show. By Proposition 2.4, if G is a cyclic CI-group then a Sylow p-subgroup of G is $\mathbb{Z}_p$, $\mathbb{Z}_4$, $\mathbb{Z}_8$ or $\mathbb{Z}_9$. A similar construction shows that if $r > 1$, then $\mathbb{Z}_{8r}$ is not a 6-CI-group and $\mathbb{Z}_{9r}$ is not a 8-CI-group. The order of a cyclic CI-group G is therefore $8, 9, 18, k, 2k$ or $4k$, where k is odd and square-free.

However, verifying whether cyclic groups of such orders are CI-groups was difficult. Many special cases were settled separately by many people (where p and q are distinct primes): $n = p$, Djokovic (1970), Elspas and Turner (1970) and Turner (1967); $n \leq 37$ and $n \neq 16, 24, 25, 27, 36$ – McKay (unpublished); $n = 2p$ – Babai (1977); $(n, \varphi(n)) = 1$ – Pálfy (1979), $n = pq$ – Alspach and Parsons (1979), and Godsil (1983); $n = 4p$, $p > 2$ – Godsil (1983). The proof of Theorem 4.1 given by Muzychuk consists of his two papers, namely [75] and [76]. The first one is for the case n is odd, and the second one is for the case n is even. The proof is highly technical, and the main argument in the proof is based on the technique of Schur rings.

On a different direction, the circulants of small valency have been investigated by many people. Toida (1979) proved that cyclic groups are 3-CI-groups. Boesch and Tindell (1984) conjectured that cyclic groups are 4-CI-groups. Sun (1988) first proved this conjecture. Other proofs can be found in [13] and [25]. Cyclic groups are also 5-CI-groups, see [88] or [46]. For the directed case, Sun (1984) proved that cyclic groups are 2-DCI-groups. However, by Proposition 2.4, $\mathbb{Z}_8$ is not a 3-DCI-group and $\mathbb{Z}_{16}$ is not a 6-CI-group. A natural question is, for a positive integer m, which cyclic groups are m-DCI-groups, and which cyclic groups are m-CI-groups? From Proposition 2.4 it follows that if G is a cyclic m-DCI-group, $p \,\Big|\, |G|$ and $p < m$, then a Sylow p-subgroup of G is either $\mathbb{Z}_p$ or $\mathbb{Z}_4$; if G is a cyclic m-CI-group, $p \,\Big|\, |G|$ and $2p < m$, then a Sylow p-subgroup of G is $\mathbb{Z}_4$, $\mathbb{Z}_8$, $\mathbb{Z}_9$ or $\mathbb{Z}_p$. We have shown that $\mathbb{Z}_{8k}$ is not a 6-CI-group and $\mathbb{Z}_{9k}$ is not a 8-CI-group. Thus we can obtain the candidates for cyclic m-DCI-groups and cyclic m-CI-groups. Here we have used the k-(D)CI property for all values of k with $1 \leq k \leq m$ to deduce the conclusion, the author [50] proved that, for directed graphs, the conclusion can be deduced only from the m-DCI property:

Theorem 4.2 *Let G be a cyclic group, and let G_p be a Sylow p-group of G. Suppose that G has the m-DCI property. If $|G|$ is not a prime-square and $p + 1 \leq m \leq (|G| - 1)/2$, then $G_p \cong \mathbb{Z}_p$ or $\mathbb{Z}_4$.*

We remark that Theorem 1.2 of [50] was unfortunately mis-stated and the statement of Theorem 4.2 is a correct version of Theorem 1.2 of [50].

Theorem 4.2 is only regarding the directed graph case. We guess a similar conclusion should hold for the undirected graph case, see [50, Problem 1.4]. For the prime-square order case, a complete characterization is obtained in [50] and [34]: for $1 \leq m \leq \frac{p^2-1}{2}$, it is proved in [50, Theorem 1.1] that $\mathbb{Z}_{p^2}$ has the m-DCI property if and only if $m < p$ or $m \equiv 0$ or -1 $(\bmod p)$; while it is proved in [34] that $\mathbb{Z}_{p^2}$ has the m-CI property if and only if one of the following holds: $p = 2$ or 3, m is odd, $[\frac{m}{p}]$ is odd, $m \leq p-1$, or $m = kp$ or $kp + (p-1)$ for some even positive integer k.

By Theorem 4.2, for a positive integer n which is not a prime-square, if $\mathbb{Z}_n$ has the m-DCI property, then n can be written as $n = n_1 n_2$ such that $(n_1, n_2) = 1$, n_1 divides $4k$ where k is odd and square-free, and every prime divisor of n_2 is greater than m. If $n_1 = 1$ then every prime divisor of n is greater than m, and it follows from Theorem 6.1 that $\mathbb{Z}_n$ is an m-DCI-group. On the other hand, if $n_2 = 1$ then n divides $4k$ where k is odd and square-free, and consequently $\mathbb{Z}_n$ is a DCI-group by Theorem 4.1. In the general case, it is still open whether $\mathbb{Z}_n$ is an m-DCI-group. We are inclined to think that this is true and we therefore propose the following conjecture.

Conjecture 4.3 *Let $n = n_1 n_2$ be an integer such that $(n_1, n_2) = 1$ and n_1 divides $4k$ where k is odd and square-free.*

(1) *If every prime divisor of n_2 is greater than m, then $\mathbb{Z}_n$ is an m-DCI-group.*

(2) *If every prime divisor of n_2 is greater than $2m$, then $\mathbb{Z}_n$ is an m-CI-group.*

If this conjecture is true, it is a generalization of Muzychuk's theorem, that is Theorem 4.1.

We conclude this section with discussing a conjecture made by Toida (1977) regarding a special class of circulant graphs. Let $G = \mathbb{Z}_n$ be a cyclic group of order n, and let $\mathbb{Z}_n^*$ be the set of elements of G of order n. Toida [90] posed the following conjecture:

Conjecture 4.4 *If $S \subseteq \mathbb{Z}_n^*$ then $\mathrm{Cay}(G, S)$ is a CI-graph.*

This conjecture was proved for the case where n is a prime-power by different people: Klin and Pöschel [44, 45], Gol'lfand, Najmark and Pöschel [30]. Very recently, the conjecture was proved independently in [77] and [61].

5 Finite CI-groups and DCI-groups

A DCI-group is also a CI-group, but the converse is not true, for example, the Frobenius group of order $3p$ with p odd prime is a CI-group but not a DCI-group, refer to Theorem 5.6 and Theorem 5.8. A DCI-group has the m-DCI property and is an m-DCI-group for all $m \leq |G|$; while a CI-group has the m-CI property and is an m-CI-group for all $m \leq |G|$.

In 1977, Babai [5] initiated the study of general CI-groups. Then Babai and Frankl (1978,1979) proved the following remarkable result.

Theorem 5.1 *Let G be a finite CI-group.*

(1) *If G is of odd order, then a Sylow 3-subgroup is $\mathbb{Z}_3$, $\mathbb{Z}_9$ or $\mathbb{Z}_{27}$, and Sylow p-subgroups for $p \neq 3$ are elementary abelian. Moreover, either G is abelian, or G has an abelian normal subgroup of index 3.*

(2) *If G is insoluble, then $G = U \times V$ where $(|U|, |V|) = 1$, U is a direct product of elementary abelian Sylow groups, and $V \cong \mathrm{PSL}(2,5), \mathrm{SL}(2,5), \mathrm{PSL}(2,13)$ or $\mathrm{SL}(2,13)$.*

A crucial property for the proof of Theorem 5.1 is that the property of being CI-groups is inherited by subgroups and quotient groups to characteristic subgroups, that is,

Lemma 5.2 ([7, Lemmas 3.2 and 3.5]) *Let G be a CI-group. Then every subgroup of G is a CI-group, and if N is a characteristic subgroup of G then G/N is a CI-group.*

Proof. Let H be a subgroup of G, and let $S, T \subseteq H^{\#}$ satisfying $\mathrm{Cay}(H,S) \cong \mathrm{Cay}(H,T)$ so that $\mathrm{Cay}(G,S) \cong \mathrm{Cay}(G,T)$. Noting that either $\langle S \rangle = H$ or $\langle H^{\#} \setminus S \rangle = H$, and that $\mathrm{Cay}(H,S)$ is a CI-graph if and only if $\mathrm{Cay}(H, H^{\#} \setminus S)$ is a CI-graph, we may assume that $\langle S \rangle = \langle T \rangle = H$. Since G is a CI-group, there is $\sigma \in \mathrm{Aut}(G)$ such that $S^\sigma = T$. Now $H^\sigma = \langle S^\sigma \rangle = \langle T \rangle = H$, so σ induces an automorphism of H. Thus H is a CI-group.

Let N be a characteristic subgroup of G. Let S_1, T_1 be subsets of G/N which do not contain the identity of G/N such that $\Sigma := \mathrm{Cay}(G/N, S_1) \cong \Gamma := \mathrm{Cay}(G/N, T_1)$. Let S, T be the full preimages of S_1, T_1 under $G \to G/N$, respectively. Then $\mathrm{Cay}(G,S) = \Sigma[\overline{K}_{|N|}] \cong \Gamma[\overline{K}_{|N|}] = \mathrm{Cay}(G,T)$. Since G is a CI-group, there exists $\sigma \in \mathrm{Aut}(G)$ such that $S^\sigma = T$. As N is characteristic in G, σ fixes N, so σ induces an automorphism $\overline{\sigma}$ of G/N sending S_1 to T_1. Thus G/N is a CI-group. $\square$

Due to Theorem 5.1, one of the main problems regarding CI-groups has been to determine which groups listed in Theorem 5.1 are really CI-groups. In particular, Babai and Frankl [7] asked whether elementary abelian groups are all CI-groups:

Question 5.3 For a prime p and a positive integer d, is the elementary abelian group $\mathbb{Z}_p^d$ a CI-group ?

Recently, some ingenious graph constructions have been found which led to much progress on this topic. L. Nowitz (1992) constructed a 32-valent Cayley graph of $\mathbb{Z}_2^6$ that is not a CI-graph, which provides a negative answer to the Babai and Frankl's question.

Theorem 5.4 (Nowitz) *The elementary abelian group $\mathbb{Z}_2^6$ is not a CI-group.*

It is proved in [78] that there are two non-conjugate regular subgroups of $\mathrm{Aut}\,\Gamma$ which are isomorphic to $\mathbb{Z}_2^6$. Thus by Theorem 3.3, Γ is not a CI-graph. The proof in [78] is very technical and complicated. A simpler proof was given by Alspach [2]. With the assistance of computer, it was found in [11] that the Nowitz graph has the full automorphism group of order $2^{13}3$. $\square$

The status for Question 5.3 is that $\mathbb{Z}_p^d$ for $d \leq 4$ and $\mathbb{Z}_2^5$ have been proved to be CI-groups: Godsil [28] for $d = 2$; Dobson [17] for $d = 3$; Conder and Li [11] for $p = 2$ and $d = 4, 5$; Hirasaka and Muzychuk [37] for $p > 2$ and $d = 4$. The question for odd primes p is still open.

A natural question related to Theorem 5.1 is whether A_5, $\mathrm{SL}(2,5)$, $\mathrm{PSL}(2,13)$ and $\mathrm{SL}(2,13)$ are CI-groups, see for example [94]. It is easy to exclude $\mathrm{PSL}(2,13)$ and so $\mathrm{SL}(2,13)$ from information given in the Atlas [12], see for example [63]. The author [48] constructed a 29-valent graph which is a Cayley graph but not a CI-graph of A_5. Thus we have

Theorem 5.5 *The simple group A_5 is not a CI-group. Moreover, all finite CI-groups are soluble.*

Here we briefly describe the construction of the 29-valent graph. Let $X = \mathrm{PSL}(2,29)$. By the Atlas [12], it is easy to show that X has a subgroup $H \cong \mathbb{Z}_{29} \rtimes \mathbb{Z}_7$ and an involution g such that $\langle H, g\rangle = X$ and $H \cap H^g \cong \mathbb{Z}_7$. Let Γ be a graph with vertex set $V\Gamma = \{xH \mid x \in X\}$ and xH, yH adjacent if and only if $yx^{-1} \in HgH$. Then Γ is a connected X-arc transitive graph of order 60 and valency 29. It was proved in [58] that $\mathrm{Aut}\,\Gamma \cong \mathbb{Z}_2 \times \mathrm{PSL}(2,29)$.

By the Atlas [12], X has subgroups P, Q isomorphic to A_5 but not conjugate in X. Thus P and Q are not conjugate in $\mathrm{Aut}\,\Gamma$. Now $(|H|, |\mathrm{A}_5|) = 1$ and $|H||\mathrm{A}_5| = |X|$. It follows that $H \cap P = H \cap Q = 1$ and hence that $X = PH = QH$. Consequently, both P and Q are regular on $V\Gamma$. Thus, by Lemma 3.1, we may identify $V\Gamma$ with P in such a way that $\Gamma = \mathrm{Cay}(P, S)$ for some $S \subset P$. By Theorem 3.3, Γ is not a CI-graph of P.

Very recently, with the assistance of computer, Conder and the author [11] found a 5-valent Cayley graph of A_5 which is not a CI-graph. $\square$

Praeger and the author gave a description of arbitrary finite CI-groups as a consequence of a series of papers [63, 64, 65], which is dependent on the classification of finite simple groups, refer to Section 7. Very recently, Pálfy and the author [62] obtained a more precise description for such groups, using Lemma 5.2 and induction on the group order. Next we briefly describe the description. In our list of candidates for CI-groups, most members contain a direct factor of groups defined as follows. Let M be an abelian group for which all Sylow subgroups are elementary abelian, and let $n \in \{2, 3, 4, 8, 9\}$ such that $(|M|, n) = 1$. Let

$$E(M, n) = M \rtimes \langle z \rangle$$

such that $o(z) = n$, and if $o(z)$ is even then z inverts all elements of M, that is, $x^z = x^{-1}$ for all $x \in M$; while if $o(z) = 3$ or 9 then $x^z = x^l$ for all $x \in M$, where l is an integer satisfying $l^3 \equiv 1 \pmod{\exp(M)}$ and $(l(l-1), \exp(M)) = 1$. Let $\mathcal{CI}$ denote the class of the following finite groups G:

(1) $G = U \times V$ with $(|U|, |V|) = 1$, where all Sylow subgroups of G are elementary abelian, or isomorphic to $\mathbb{Z}_4$ or Q_8; moreover, U is abelian, and $V = 1, \mathrm{Q}_8, \mathrm{A}_4, \mathrm{Q}_8 \times E(M, 3)$, $E(M, n)$ with $n \in \{2, 3, 4\}$, or $E(M, n) \times E(M', 3)$ where $n \in \{2, 4\}$ and $|M|, |M'|, 6$ are coprime pairwise.

(2) G is one of the groups: $\mathbb{Z}_8$, $\mathbb{Z}_9$, $\mathbb{Z}_{18}$, $\mathbb{Z}_9 \rtimes \mathbb{Z}_2\,(= \mathrm{D}_{18})$, $\mathbb{Z}_9 \rtimes \mathbb{Z}_4$ with center of order 2, $\mathbb{Z}_2^2 \rtimes \mathbb{Z}_9$ with center of order 3, $E(M, 8)$, $\mathbb{Z}_2^k \times E(M, 9)$ with $k \geq 1$.

Then we have a description for finite CI-groups.

Theorem 5.6 *All finite CI-groups are in $\mathcal{CI}$.*

Due to Theorem 5.6, the problem of classifying CI-groups becomes the problem of determining which members of $\mathcal{CI}$ are CI-groups. We further note that if $G \in \mathcal{CI}$ then

$\mathrm{Cay}(G,S)$ is a CI-graph of G if and only if $\mathrm{Cay}(\langle S\rangle,S)$ is a CI-graph of $\langle S\rangle$. Unfortunately, even with this knowledge, it is still very difficult to obtain a complete classification of finite CI-groups. Actually, so far only very 'few' families of groups are proved to be CI-groups, which are listed in the next theorem.

Theorem 5.7 *The following groups are all known CI-groups: where p is a prime,*

(i) $\mathbb{Z}_n$, *where either $n \in \{8,9,18\}$, or $n \,\big|\, 4k$ and k is odd square-free;*

(ii) $\mathbb{Z}_p^2$, $\mathbb{Z}_p^3$, $\mathbb{Z}_p^4$;

(iii) $\mathbb{Z}_2^5$, Q_8, A_4, $\mathbb{Z}_2^2\times\mathbb{Z}_3$, D_{18}, $\mathbb{Z}_9\rtimes\mathbb{Z}_4$, $\mathbb{Z}_2^2\rtimes\mathbb{Z}_9$;

(iv) D_{2p}, F_{3p} *(a Frobenius group of order $3p$);*

(v) $\mathbb{Z}_p\rtimes\mathbb{Z}_4$ *(with center of order 2)*, $\mathbb{Z}_p\rtimes\mathbb{Z}_8$ *(with center of order 4).*

The studies on cyclic CI-groups is described in Section 4. C. Godsil [28] proved $\mathbb{Z}_p^2$ to be CI-groups. It is announced in [95] that $\mathbb{Z}_p^3$ are CI-groups, but the first proof was published by E. Dobson [17]. Dobson's proof is very technical, and a much simpler proof was given by B. Alspach and L. Nowitz, see [3]. Recently, Hirasaka and Muzychuk [37] proved that $\mathbb{Z}_p^4$ for $p>2$ is a CI-group; while with the assistance of a computer, Conder and the author [11] proved that both $\mathbb{Z}_2^4$ and $\mathbb{Z}_2^5$ are CI-groups. Thus $\mathbb{Z}_p^k$ for $k\le 4$ are CI-groups. It is easily checked that Q_8 is a CI-group, see [95]; it is proved that A_4 is a CI-group in [35]. The other groups in part (iii) are verified to be CI-groups in [11] with the assistance of computer. L. Babai [5] proved that D_{2p} is a CI-group. Very recently, Pálfy and the author [62] proved that the Frobenius group F_{3p} of order $3p$, the group $\mathbb{Z}_p\rtimes\mathbb{Z}_4$ with center of order 2 and the group $\mathbb{Z}_p\rtimes\mathbb{Z}_8$ with center of order 4 are all CI-groups.

Using Theorem 5.6, we may obtain a description of finite DCI-groups, which was first obtained by Praeger, Xu and the author in [66].

Theorem 5.8 *If G is a DCI-group, then all Sylow subgroups of G are elementary abelian or isomorphic to $\mathbb{Z}_4$ or Q_8. Moreover, $G=U\times V$, where $(|U|,|V|)=1$, U is abelian, and $V=1,\mathrm{Q}_8$, A_4, $H_2(M,2)$, or $H_2(M,4)$.*

The candidates for CI-groups and DCI-groups are so restricted that it would be possible to obtain a complete classification of finite (D)CI-groups.

Problem 5.9 Give a complete classification of finite CI-groups and finite DCI-groups.

For this problem, determining whether the following members $G \in \mathcal{CI}$ are CI-groups would be crucial: $G = \mathbb{Z}_p^5$ with odd p, $G = \mathbb{Z}_p^2 \times \mathbb{Z}_q$ where p, q are different primes, and the groups G of which all Sylow subgroups are cyclic.

6 Isomorphisms of connected Cayley graphs

It is easily shown that a disconnected Cayley graph is a disjoint union of isomorphic connected smaller Cayley graphs, that is, the Cayley graph $\mathrm{Cay}(G, S)$ is a union of $|G|/|\langle S\rangle|$ copies of $\mathrm{Cay}(\langle S\rangle, S)$. Thus, to determine isomorphic classes of Cayley graphs, we only need to consider the connected graph case. On the other hand, to determine whether a member of $\mathcal{CI}$ to be a CI-group, we also only need to consider connected graphs. With respect to these, some recent work has focused on isomorphisms of connected Cayley graphs.

The Sylow's Theorem is used in Proposition 3.5 to study CI-graphs of prime-power order. Such a study can be extended to the more general case. For a finite group F, by Wielandt [92], all nilpotent Hall π-subgroups (if exist) of F are conjugate; by Gross [33], all Hall π-subgroups (if exists) of F of odd order are conjugate. Let G be a finite group which is of odd order or nilpotent, and let Γ be a Cayley graph of G. Let $A = \mathrm{Aut}\,\Gamma$ and A_1 the stabilizer of 1 in A. Suppose that $(|G|, |A_1|) = 1$. Then G is a Hall π-subgroup of A, where π is the set of prime divisors of $|G|$. Since G is of odd order or nilpotent, all Hall π-subgroups of A are conjugate. By Theorem 3.3, Γ is a CI-graph of G. Thus we have

Theorem 6.1 ([51, Theorem 4.1]) *Let G be a finite group, and let Γ be a Cayley graph of G. Let $A = \mathrm{Aut}\,\Gamma$ and A_1 the stabilizer of 1 in A. Assume that G is of odd order or nilpotent. If $(|G|, |A_1|) = 1$ then Γ is a CI-graph of G.*

For the case where G is cyclic, the result was obtained independently by J. Morris [74]. We do not know whether, for arbitrary groups G, $(|G|, |A_1|) = 1$ implies that Γ is a CI-graph. A solution to this question would be interest for understanding better CI-graphs. An immediate consequence of Theorem 6.1 is the following result, see [51].

Theorem 6.2 *Suppose that G is a finite group which is of odd order or nilpotent. If p is the smallest prime divisor of $|G|$, then G is a connected m-(D)CI-group for any $m \leq p - 1$.*

However, this result cannot be extended to the case where $m = p$. An infinite family of groups G which are not connected p-DCI-groups, where p is the smallest prime divisor of $|G|$, was constructed in [55]. Here we briefly describe the construction.

Let p be a prime, and let $d = d_1 d_2$ be such that $d_1 \geq \max\{3, p\}$ and $d_2 \geq 2$. By [42, p. 508], $p^d - 1$ has a primitive prime q_1, and $p^{d_1} - 1$ has a primitive prime divisor q_2. Now $q_1 \geq d + 1 > p$ and $q_2 \geq d_1 + 1 > p$. Let $l = q_1 q_2$, and let $G = \mathbb{Z}_p^d \rtimes \mathbb{Z}_l \leq \mathrm{AGL}(1, p^d)$. Then p is the smallest prime divisor of $|G|$, and $\mathbb{Z}_p^d$ is a minimal normal subgroup of G. Further, $\varphi(l) = (q_1 - 1)(q_2 - 1) \geq d d_1 \geq 3d > 2d + 3$ (where $\varphi(l)$ is the number of positive integers less than l and coprime to l). Thus there exists $k \in \{2, 3, \ldots, l-2\} \setminus \{p^i, -p^i \pmod p) \mid 0 \leq i \leq d - 1\}$ such that k is coprime to l. Let a and z be elements of G of order p and l, respectively. Let

$$S_i = \{z^i, a^{-1} z^i a, \ldots, a^{-p+1} z^i a^{p-1}\},$$

and let $\Gamma_i = \mathrm{Cay}(G, S_i)$. It was proved in [55] that $\Gamma_1 \cong \Gamma_i$ and $\mathrm{Aut}(G) = \mathrm{A\Gamma L}(1, p^d)$. Suppose that Γ_1 is a CI-graph. Then there exists $\alpha \in \mathrm{Aut}(G)$ such that $S_1^\alpha = S_k$. Since $\mathrm{Aut}(G, S_1)$ is transitive on S_1, we may assume that $z^\alpha = z^k$. Thus $\alpha \in \mathbf{N}_{\mathrm{Aut}(G)}(\langle z \rangle) = \Gamma\mathrm{L}(1, p^d)$, and it follows that α is a Frobenius automorphism of G, that is, $z^\alpha = z^{p^j}$ for some $j \in \{0, 1, \ldots, d-1\}$. Therefore, $k = p^j \pmod l$, which contradicts the choice of k, and so Γ_1 is not a CI-graph.

Let $S = S_1 \cup S_1^{-1}$ and $T = S_k \cup S_k^{-1}$. Then $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$. Suppose that $\mathrm{Cay}(G, S)$ is a CI-graph. Then there exists $\alpha \in \mathrm{Aut}(G)$ such that $S^\alpha = T$. Since $\mathrm{Aut}(G, S_1)$ is transitive on S_1, we may assume that $z^\alpha = z^k$ or z^{-k}. Thus $\alpha \in \mathbf{N}_{\mathrm{Aut}(G)}(\langle z \rangle) = \Gamma\mathrm{L}(1, p^d)$, and it follows that α is a Frobenius automorphism of G, namely, $z^\alpha = z^{p^j}$ for some $j \in \{0, 1, \ldots, d-1\}$. Therefore, $k = p^j$ or $-p^j \pmod l$, which also contradicts the choice of k, and so Γ is not a CI-graph. Thus we have

Theorem 6.3 ([55]) *For any prime p, there exist infinitely many groups G such that p is the smallest prime divisor of $|G|$ and G is neither a connected p-DCI-group nor a connected $2p$-CI-group.*

In the case where $p = 2$, the graphs constructed above are minimum Cayley graphs of G since G is not cyclic and the graphs are of valency 2. Regarding Theorem 6.3, a natural question arises (see [55]).

Problem 6.4 (1) Characterize finite groups G which are not connected p-DCI-groups, where p is the smallest prime divisor of $|G|$.

(2) Characterize finite groups G which are not connected m-CI-groups, where $p \leq m \leq 2p$ and p is the smallest prime divisor of $|G|$. In particular, are all finite groups connected 3-CI-groups?

In [56], it was proved that the simple groups $\mathrm{PSL}(2,q)$ are connected 2-DCI-groups, and conjectured that finite simple groups are all connected 2-DCI-groups. The conjecture has been proved by Hirasaka and Muzychuk [38]:

Theorem 6.5 *Finite simple groups are connected* 2*-DCI-groups.*

It was shown in [48] that all symmetric cubic Cayley graphs of nonabelian simple groups are CI-graphs.

Let G be an abelian group, and let $\Gamma = \mathrm{Cay}(G,S)$ be connected and $A = \mathrm{Aut}\,\Gamma$. It is proved in [46] that either A_1 is faithful on S, or S contains a coset of some nontrivial subgroup of G. This result is applied in [54] to prove the following result, which is an extension of Theorem 6.1 for abelian groups.

Theorem 6.6 *Let G be an abelian group and let $\Gamma = \mathrm{Cay}(G,S)$. Let $A = \mathrm{Aut}\,\Gamma$. Suppose that G is abelian and $(|G|,|A_1|) = q$ where q is a prime. Then either Γ is a CI-graph of G, or S contains a coset of some nontrivial subgroup of G.*

However, the situation for abelian groups is different: we will see that abelian groups G are all connected p-DCI-groups where p is the smallest prime divisor of $|G|$. Let p be a prime, and let $G = \langle a \rangle \cong \mathbb{Z}_{p^r k}$, where $r \geq 2$ and k is even and coprime to p. Set $S = a\langle a^{p^{r-1}k}\rangle \cup \{a^{p^{r-1}k}, a^2\}$. Let $\Gamma = \mathrm{Cay}(G,S)$ and $A = \mathrm{Aut}\,\Gamma$. Then it is easily checked that $|A_1| = p$, so $(|G|,|A_1|) = p$. Let $T = a\langle a^{p^{r-1}k}\rangle \cup \{a^{p^{r-1}k}, a^{2+2^{r-1}k}\}$. Then a straightforward calculation shows that $\mathrm{Cay}(G,T) \cong \Gamma$. If there exists $\alpha \in \mathrm{Aut}(G)$ such that $S^\alpha = T$, then $a^\alpha = a$ or $a^{1+2^{r-1}k}$. Thus $(a^2)^\alpha = a^2$ or $(a^2)^\alpha = (a^{1+2^{r-1}k})^2 = a^2$, so $(a^2)^\alpha \notin T$, which is a contradiction. Hence Γ is not a CI-graph of G. Therefore, for any prime p, there exists a cyclic group G and a Cayley graph Γ of G such that $(|G|,|A_1|) = p$ and Γ is not a CI-graph of G. However we do not know any other examples. On the other hand, in all known examples, if G is abelian and S contains no cosets of a subgroup of G, then $\mathrm{Cay}(G,S)$ is a CI-graph of G. For these reasons, we would like to pose the following problem.

Problem 6.7 (1) Let G be an abelian group, and let $\Gamma = \mathrm{Cay}(G, S)$ and $A = \mathrm{Aut}\,\Gamma$. Determine Cayley graphs Γ such that $(|G|, |A_1|)$ is a prime and Γ is not a CI-graph.

(2) Does there exist an abelian group G and a Cayley graph $\mathrm{Cay}(G, S)$ such that S contains no cosets of a subgroup of G and $\mathrm{Cay}(G, S)$ is not a CI-graph of G?

The following theorem extends Theorem 6.2 to the case $m = p$ for abelian groups, and gives a complete classification of abelian connected $(p+1)$-DCI-groups, which was obtained by Meng and Xu in [99] for the case where $p = 2$, and by the author in [52] for the other case.

Theorem 6.8 *Let G be an abelian group and let p be the least prime divisor of $|G|$. Let G_p be a Sylow p-subgroup of G. Then G is a connected p-DCI-group. Further, G is a connected $(p+1)$-DCI-group if and only if one of the following holds:*

(i) *G is of rank at least 3;*

(ii) *G is of rank at most 2, and either G_p is homocyclic of rank 2, or $G_p \cong \mathbb{Z}_p$ or $\mathbb{Z}_4$.*

For the undirected case, it is proved in [13] that all abelian groups are connected 4-CI-groups, and abelian connected 5-CI-groups are completely classified in [72].

7 Finite m-DCI-groups and m-CI-groups

In 1979, Toida [90] proved that cyclic groups are 3-CI-groups. Then L. Sun proved that cyclic groups are 2-DCI-groups and 4-CI-groups in [87] and [88], respectively. In a series of papers [21, 22, 24], Fang and Xu completely classified abelian m-DCI-groups for $m \leq 3$. For $m \leq 5$, the abelian m-CI-groups are completely classified in [23, 25]. The methods for these results are mainly combinatorial. This work led Xu (1988) to propose the problem of studying m-(D)CI-groups. By employing the finite simple group classification and some other deep group theory results, the status of investigating m-(D)CI-groups has been significantly changed.

By definition, an m-DCI-group is also an m-CI-group; an m-DCI-group is a k-DCI-group, and an m-CI-group is a k-CI-group for all $k \leq m$. It easily follows from the definition that if G is a 1-DCI- or 2-CI-group then all cyclic subgroups of the same order are conjugate under $\mathrm{Aut}(G)$. The classes of p-groups (p a prime) with the latter property were investigated by quite a few people (see Higman (1963), Shult (1969), Gross (1976) and Wilkens (1996)).

Theorem 7.1 (see [93]) *Let p be a prime and let G be a p-group all subgroups of G order p are conjugate under* $\mathrm{Aut}(G)$. *Then either G is homocyclic, or $p = 2$ and for some $k \geq 1$,*

(i) $\mathbf{Z}(G) = G' = \Phi(G) = \mathbb{Z}_2^k$, *and* $\mathbf{Z}(G) \setminus \{1\}$ *consists of all the involutions of G;*

(ii) G/G' *is of order* 2^k *or* 2^{2k}.

Dependent on the classification of 2-transitive affine permutation groups, a description for general 1-DCI-groups was obtained by Zhang in [100]. A key property in Zhang's work is that if G is a 1-DCI-group and N is a characteristic subgroup of G, then N and G/N are also 1-DCI-groups. The following improved form was obtained in [64, Corollary 1.3].

Theorem 7.2 *Let G be a 1-DCI-group. Then all Sylow subgroups of G are 1-DCI-groups.*

(1) *If G is of odd order, then $G = U \times V$, where $(|U|, |V|) = 1$, U is abelian, and $V = 1$ or $M \rtimes \mathbb{Z}_n$ with $(|M|, n) = 1$ and M is abelian.*

(2) *If G is even order, then $G = U \times V$, where $(|U|, |V|) = 1$, U is a 1-DCI-group of odd order, and V is a Sylow 2-subgroup of G, or $V = \mathrm{PSL}(2, q)$ or $\mathrm{SL}(2, q)$ with $q \in \{5, 7, 8, 9\}$ or $\mathrm{PSL}(3, 4)$, or $V = (L \times M) \rtimes ((H \rtimes K) \times \mathbb{Z}_n)$ where 6, $|L|$, $|M|$ and n are pairwise coprime, and*

   (i) *$L \times M$ is abelian, $\langle M, H\rangle = M \times H$ and $\langle L, \mathbb{Z}_n\rangle = L \times \mathbb{Z}_n$,*

   (ii) *$L \rtimes H \rtimes K = \mathbb{Z}_{5^u}^2 \rtimes \mathbb{Z}_{3^t} \rtimes \mathbb{Z}_{2^r}$, $\mathbb{Z}_{3^u}^4 \rtimes \mathbb{Z}_{5^t} \rtimes \mathbb{Z}_{2^r}$, or $\mathbb{Z}_q^2 \rtimes \mathrm{Q}_8 \rtimes \mathbb{Z}_{3^s}$ where $q \neq 1$, each prime divisor of q lies in $\{3, 5, 7, 11, 23\}$, and if $3 \,\big|\, q$ then $M = 1$ and $s = 0$.*

We note that an m-DCI-group has the k-DCI property for all $k \leq m$. Applying the k-DCI property for $2 \leq k \leq m$ to the groups listed in Theorem 7.2, an explicit list of groups which contain finite m-DCI-groups for $m \geq 2$ is produced in [66]. Let M be an abelian group all Sylow subgroups of which are homocyclic, and let m be the exponent of M. We define certain nonabelian extensions of such homocyclic groups M. Let r, s be non-negative integers such that $r + s \geq 1$, and suppose that there exists an integer l such that $1 < l < m$ and l has *order e modulo m* (that is, e is the least positive integer such that $l^e \equiv 1 \pmod{m}$ and we write $o(l \bmod m) = e$) and in addition r, s, e are as one of the lines in the following table.

r, s	e
$r = 0,\ s \geq 1$	3
$r \geq 1,\ s = 0$	2 or 4
$r \geq 1,\ s \geq 1$	6

Definition 7.3 *Let*

$$H_e(M, 2^r 3^s) = M \rtimes \langle z \rangle = M \rtimes \mathbb{Z}_{2^r 3^s}, \quad \textit{where } x^z = x^l \textit{ for all } x \in M.$$

Let $m \geq 2$ be an integer, and let $\mathcal{DCI}(m)$ denote the class of finite groups $G = U \times V$ which satisfies the following conditions:

(i) a Sylow p-subgroup G_p of G is homocyclic or Q_8, and further, if $p = m$ then G_p is elementary abelian, cyclic or Q_8; if $p < m$ then G_p is elementary abelian, $\mathbb{Z}_4$ or Q_8;

(ii) $(|U|, |V|) = 1$, U is nilpotent, and V is one of the following: A_4, $\mathrm{Q}_8 \rtimes \mathbb{Z}_3$, $\mathbb{Z}_3^2 \rtimes \mathrm{Q}_8$, A_5 or $H_e(M, 2^r)$ with $r \geq 1$.

Then we have an explicit list for the candidates of m-DCI-groups for $m \geq 2$.

Theorem 7.4 (Li, Praeger and Xu [66]) *Let G be an m-DCI-group for some $m \geq 2$. Then G is a member of $\mathcal{DCI}(m)$.*

The problem of determining finite m-DCI-groups is therefore reduced to the problem of determining which members G of $\mathcal{DCI}(m)$ are m-DCI-groups. By definition, an m-DCI-group is a connected m-DCI-group. A further reduction for the study of m-DCI-groups is given in the next theorem.

Theorem 7.5 (Li [60]) *Let $m \geq 2$ be an integer, and let G be a finite group which is a member of $\mathcal{DCI}(m)$. Then G is an m-DCI-group if and only if all subgroups of G are connected m-DCI-groups.*

This result was used in [60] to prove that a finite group G is a 2-DCI-group if and only if G is a member of $\mathcal{DCI}(2)$. For special classes of groups, there are some further results: by Fang and Xu [21, 22, 24], an abelian group G is an m-DCI-group for $m \leq 3$ if and only if $G \in \mathcal{DCI}(m)$; by Qu and Yu [82], a dihedral group G is an m-DCI-group for $m \leq 3$ if and only if $G \in \mathcal{DCI}(m)$; by Ma [70], a di-cyclic group G is an m-DCI-group for $m \leq 3$ if and only if $G \in \mathcal{DCI}(m)$. Combining Theorem 7.5 and Theorem 6.8, an immediate consequence is that if G is an abelian member of $\mathcal{DCI}(m)$ then G is a $(p+1)$-DCI-group where p is the smallest prime divisor of $|G|$. However, it is still a very difficult problem to obtain a complete classification of general m-DCI-groups for $m \geq 3$.

For undirected graphs, dependent on the classification of affine primitive permutation groups of rank 2 and 3, a description of finite 2-CI-groups is given in [64, 65].

Theorem 7.6 *Suppose that G is a finite 2-CI-group. Then a Sylow p-subgroup G_p of G is either homocyclic, or $p = 2$ and G_2 is elementary abelian, cyclic or generalized quaternion.*

(1) *If $|G|$ is odd, then $G = U \times V$, where $(|U|, |V|) = 1$, U is abelian and $V = M \rtimes \mathbb{Z}_n$ with $(|M|, n) = 1$ and M is abelian.*

(2) *If $|G|$ is even and G is not nilpotent, then $G = U \times V$, where $(|U|, |V|) = 1$, $|U|$ is odd, and either $V = \mathrm{A}_5$, $\mathrm{PSL}(2, 8)$, $\mathrm{SL}(2, 5)$, $\mathrm{SL}(2, 7)$ or $\mathrm{SL}(2, 9)$, or $V = (L \times M) \rtimes ((H \rtimes K) \times \mathbb{Z}_n)$ where $|L|, |H|, |K|, |M|$ and n are pairwise coprime, and*

 (i) *$L \times M$ is abelian, $\langle M, H\rangle = M \times H$ and $\langle L, \mathbb{Z}_n\rangle = L \times \mathbb{Z}_n$,*

 (ii) *$H \rtimes K$ is indecomposable, neither H nor K centralizes a Sylow subgroup of L, and $(L, H \rtimes K)$ satisfies one of the lines of the following table.*

L	$H \rtimes K$	*conditions*
$\mathbb{Z}_2^3$	$\mathbb{Z}_{7^t} \rtimes \mathbb{Z}_{3^s}$	$t, s \geq 1$
$\mathbb{Z}_{5^{u_1}7^{u_2}11^{u_3}}^2$	$\mathbb{Z}_{3^t} \rtimes \mathbb{Z}_{2^r}$	$r, t \geq 1,\ u_1 + u_2 + u_3 \geq 1$
$\mathbb{Z}_{3^{u_1}}^4 \times \mathbb{Z}_{19^{u_2}}^2$	$\mathbb{Z}_{5^t} \rtimes \mathbb{Z}_{2^r}$	$r, t \geq 1,\ u_1 + u_2 \geq 1$
$\mathbb{Z}_{7^{u_1}11^{u_2}19^{u_3}}^2$	$\mathbb{Z}_{3^{t_1}5^{t_2}} \rtimes \mathbb{Z}_{2^r}$	$r, t_1, t_2 \geq 1,\ u_1 + u_2 \geq 1, u_3 \geq 1,$ $[\mathbb{Z}_{19^{u_3}}, \mathbb{Z}_{3^{t_1}}] = 1,\ [\mathbb{Z}_{7^{u_1}11^{u_2}}^2,\ \mathbb{Z}_{5^{t_2}}] = 1$
$\mathbb{Z}_{3^{u_1}}^6 \times \mathbb{Z}_{19^{u_2}}^2$	$\mathbb{Z}_{7^{t_1}5^{t_2}} \rtimes \mathbb{Z}_{2^r}$	$r, t_1 \geq 1,\ u_1 + u_2 \geq 1,\ t_2 = 0 \Leftrightarrow u_2 = 0$ $[\mathbb{Z}_{3^{u_1}}^6, \mathbb{Z}_{5^{t_2}}] = 1,\ [\mathbb{Z}_{19^{u_2}}^2, \mathbb{Z}_{7^{t_1}}] = 1$
$\mathbb{Z}_{5^{u_1}11^{u_2}}^2$	$\mathbb{Z}_{3^{t_1}7^{t_2}} \rtimes \mathbb{Z}_{2^r}$	$u_1, t_1, t_2, r \geq 1,\ u_2 \geq 0,\ [\mathbb{Z}_{11^{u_2}}^2, \mathbb{Z}_{7^{t_2}}] = 1$
$\mathbb{Z}_q^2$	$\mathrm{Q}_8 \rtimes \mathbb{Z}_{3^s}$	*$q \neq 1$, each prime divisor of q lies in $\{3, 5, 7, 11, 23\}$,* *and if 3 divides q then $M = 1$, $s = 0$ and $n = 1$*

Applying the k-CI property for $3 \leq k \leq m$ to the groups listed in Theorem 7.6, an explicit list of groups which contains all m-CI-groups for $m \geq 4$ was produced in [65].

Theorem 7.7 *Let G be an m-CI-group for some $m \geq 4$ and G_p a Sylow p-subgroup of G for some prime p. Then $G = U \times V$, where $(|U|, |V|) = 1$, U is nilpotent, and either $V = 1$ or V is one of the following:*

 (i) *$\mathbb{Z}_3^2 \rtimes \mathrm{Q}_8$, A_5, $\mathrm{SL}(2, 5)$;*

 (ii) *$\mathbb{Z}_2^2 \rtimes \mathbb{Z}_{3^s}$, $\mathrm{Q}_8 \rtimes \mathbb{Z}_{3^s}$ for some $s \geq 1$;*

 (iii) *$H_e(M, 2^r3^s)$, where $e \in \{2, 3, 4, 6\}$, $r, s \geq 0$ and $r + s \geq 1$.*

Moreover, if $p > [\frac{m}{2}]$ then G_p is homocyclic; if $p = [\frac{m}{2}]$ then G_p is elementary abelian, cyclic or Q_8; if $p < [\frac{m}{2}]$ then either G_p is elementary abelian, or $G_p \cong \mathbb{Z}_4, \mathbb{Z}_8, \mathbb{Z}_9$ or Q_8.

By Proposition 2.4, if $m \geq 6$ then $r = 1, 2$ or 3, and if $m \geq 8$ then $s = 1$ or 2. Further, with the assistance of computer, Conder and the author [11] proved that neither $\mathrm{Q}_8 \rtimes \mathbb{Z}_3$ nor $\mathrm{SL}(2,5)$ is a 7-CI-group. Therefore, in the case where $m \geq 8$, G is soluble and we may have a finer description for m-CI-groups which is similar to that for CI-groups (see Theorem 5.6). However, as for m-DCI-groups, even with this result, it is still a very hard problem to obtain a complete classification of m-CI-groups for $m \geq 4$. Let $\mathcal{CI}(m)$ denote the class of groups which satisfy Theorem 7.7. Then $\mathbb{Z}_2^6$ is a member of $\mathcal{CI}(m)$, but by [78], $\mathbb{Z}_2^6$ is not a 31-CI-group. From [23, 25], we known that for $m \leq 5$, an abelian group G is an m-CI-group if and only if $G \in \mathcal{CI}(m)$. We are particularly interested in the case of small values of m, and we think solving the following problem must be interest for understanding better m-(D)CI-groups.

Problem 7.8 (1) Completely determine m-DCI-groups for certain small values of $m \geq 3$. In particular, give a complete classification of 3-DCI-groups.

(2) Completely determine m-CI-groups for certain small values of $m \geq 4$. In particular, give a complete classification of 4-CI-groups.

(3) For given a prime p and an integer $d \geq 4$, find the largest value of m such that $\mathbb{Z}_p^d$ is an m-(D)CI-group.

8 Finite groups with the m-DCI or m-CI property

Finite m-DCI-groups have the i-(D)CI property, and m-CI-groups have the i-CI property for all $i \leq m$. They have very restricted structure shown as in Theorems 7.4 and 7.7. Naturally we would ask: if we only impose a single m-DCI or m-CI property on a group G, what can we say about the structure of G? In [67], Praeger, Xu and the author initiated the study of finite groups with the m-(D)CI property, and proposed

Problem 8.1 Characterize finite groups with the m-DCI property or the m-CI property.

By the definition, finite groups with the 1-DCI property are exactly the 1-DCI-groups, which were characterized in Theorem 7.2. Finite groups G with the 1-CI property are exactly the finite groups G in which all involutions are conjugate under $\mathrm{Aut}(G)$. Now let G be a finite group with the 2-CI property. From Proposition 2.1, it follows that a Sylow 2-subgroup of G is cyclic, generalized quaternion or elementary abelian. By [83, 10.2.2], a finite group with a cyclic or generalized quaternion Sylow 2-subgroup is not simple. Thus a Sylow 2-subgroup of a nonabelian simple 2-CI-group G must be elementary abelian. Therefore, by [89, p. 582], $G = \mathrm{J}_1$, $\mathrm{Ree}(3^{2n+1})$ (for some $n \geq 1$), $\mathrm{PSL}(2, 2^n)$ (for some $n \geq 2$) or $\mathrm{PSL}(2, q)$ with $q \equiv \pm 3 \pmod 8$. It was easily proved in [63] that, for any $z \in G$, $|\{\{z^i, z^{-i}\} \mid (o(z), i) = 1\}|$ divides $|\mathbf{N}_{\mathrm{Aut}(G)}(\langle z \rangle)/\mathbf{C}_{\mathrm{Aut}(G)}(\langle z \rangle)|$. Applying this divisibility condition to these simple groups G, calculations show that G must be A_5 or $\mathrm{PSL}(2, 8)$. Further, we have

Theorem 8.2 ([63, 11]) *Suppose that G is a nonabelian simple group. Then*

(i) *G has the 2-CI property if and only if $G = \mathrm{A}_5$ or $\mathrm{PSL}(2, 8)$;*

(ii) *G has the m-CI property for $m = 3, 4$ if and only if $G = \mathrm{A}_5$;*

(iii) *G does not have the 5-CI property.*

Several infinite families of finite groups are constructed in [48, 49] which show the m-(D)CI property does not necessarily imply the k-(D)CI property for $k < m$. Such groups involve Frobenius groups $F(M, n) = M \rtimes \langle z \rangle$, where

(i) M is an abelian group of odd order and all Sylow subgroups of M are homocyclic;

(ii) $\langle z \rangle \cong \mathbb{Z}_n$ where $n \geq 2$, and $(|M|, n) = 1$;

(iii) there exists an integer l such that for any $x \in M \setminus \{1\}$, $x^z = x^l$ and n is the least positive integer satisfying $l^n \equiv 1 \pmod{o(x)}$.

The following result is obtained in [48, 49].

Theorem 8.3 (1) *Let $G = F(M, q)$ and $m = q - 1$ where q is a prime. If $q \geq 3$ then G has the m-DCI property but does not have the k-DCI property for any $k < m$.*

(2) *Let $G = F(M, q)$ and $m = q - 1$, where $q \geq 5$ is a prime. Then G has the m-CI property but does not have the nontrivial k-CI property for any $k < m$.*

(3) *Let H be a group of odd order such that H has the 2-CI property and 3 divides $|H|$, and let $G = H \times \mathrm{A}_4$. Then G has the 3-CI property but does not have the 2-CI property.*

However, it is proved in [59] that the 3-DCI property implies the 1-DCI or the 2-DCI property. In the cases $m = 2$ and 4, we have a complete characterization.

Theorem 8.4 ([47, 59]) *For $m = 2$ or 4, a finite group G has the m-DCI property but does not have the k-DCI property for all $k < m$ if and only if $G = F(M, m+1)$ for some abelian group M.*

For $m \geq 5$, we formulate a problem as follows.

Problem 8.5 Let $m \geq 5$ be an integer.

(1) Characterize finite groups which have the m-DCI property but not the k-DCI property for any k with $5 \leq k < m$.

(2) Characterize finite groups which have the m-CI property but not the k-CI property for any k with $2 \leq k < m$.

This problem seems to be very difficult in general. It would be reasonable firstly to consider the case where G is of prime-power order or nilpotent. We conjecture that nilpotent groups which have the m-DCI property are homocyclic (refer to Theorem 8.8). Peter Neumann (1996, private communication) posed a conjecture that if G is a group of odd order and has the m-DCI property then G is a semidirect product of a nilpotent group by a cyclic group. For general groups, we only have the following theorems which describe the structure of Sylow subgroups of groups with the m-DCI property and the m-CI property for 'large' values of m.

Theorem 8.6 ([67]) *Let G be a finite group with the m-DCI property, let p be an odd prime dividing $|G|$ such that $p \leq m \leq 2p-2$, and let G_p be a Sylow p-subgroup of G.*

(1) If $m = p$, then either G_p is of exponent p or G_p is cyclic.

(2) If $p+1 \leq m \leq 2p-2$, then G_p is of exponent p.

Further, if $p \leq m \leq 2p-2$ and G_p is noncyclic, then all elements of G of order p are conjugate under $\mathrm{Aut}(G)$.

Theorem 8.7 ([67]) *Let G be a finite group with the $2m$-CI property, let p be an odd prime dividing $|G|$ such that $p \leq m \leq \frac{3p-1}{2}$, and let G_p be a Sylow p-subgroup of G.*

(1) If $m = p$ then either G_p is of exponent p or G_p is cyclic.

(2) If $p = 3$ and $m = 4$ then G_3 is of exponent 3 or $G_3 \cong \mathbb{Z}_9$;

if $p = 3$ and $m = 5$ then G_3 is of exponent at most 9;

if $p \geq 5$ and $p + 1 \leq m < \frac{3p-1}{2}$ then G_p is of exponent p.

For abelian groups with the m-DCI property, we have a further characterization.

Theorem 8.8 ([57]) *If an abelian group G has the m-DCI property, then all Sylow subgroups of G are homocyclic.*

By [67, Theorem 1.6], for abelian groups G and for $1 \leq m \leq 4$, the m-DCI property implies the k-DCI property for all $k < m$. On the other hand, by [57], $\mathbb{Z}_{25}$ has the 9-DCI property but does not have the k-DCI property for $k = 6, 7$ or 8. It is asked in [57] whether, for abelian groups and $5 \leq m \leq 8$, the m-DCI property imply the k-DCI property for all $k < m$? It is easy to see that an abelian group with all Sylow subgroups homocyclic has the 1-DCI property, and therefore, by Theorem 8.8, for abelian groups the m-DCI property implies the 1-DCI property for any positive integer m. It seems that for abelian groups, with a few exceptions, the m-DCI property implies the k-DCI property for all $k < m$. The following problem is naturally proposed in [57].

Problem 8.9 Classify the finite abelian groups which have the m-DCI property but do not have the k-DCI property for some $k < m$.

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