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A CERTAIN EXACT SEQUENCE

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CONTENTS

Introduction

In this paper we give detailed proofs of the theorems announced in[1] [1]. These theorems concern, first of all, an exact sequence, $\Sigma(K)$, where K is a (connected)

[1] Numbers in square brackets refer to the list of references at the end of the paper. The two papers in [2] will be referred to as CH I and CH II.

complex.[2] The sequence $\Sigma(K)$ is the same as Σ in §1 below if

$$C_{n+1} = \pi_{n+1}(K^{n+1}, K^n), \qquad A_n = \pi_n(K^n) \qquad (n \geqq 2),$$

β, j are the (homotopy) boundary and injection operators and $C_2 = jA_2$, $C_n = A_n = 0$ if $n < 2$. It is shown to be a homotopy invariant and *a fortiori* a topological invariant of K.

Various realizability theorems are proved, which show that, if $\pi_1(K) = 1$ and dim $K \leqq 4$, the part of $\Sigma(K)$ which we call $\Sigma_4(K)$ is an algebraic equivalent of the homotopy type of K. Thus $\Sigma_4(K)$ may be used to replace the more complicated cohomology ring, $R(K)$, which was defined in [3]. Moreover $\Sigma_4(K)$, besides being simpler, is in some other ways better than $R(K)$. For $\Sigma_4(K)$, unlike $R(K)$, is defined for infinite complexes. Besides this, $\Sigma_4(K)$ includes $\pi_3(K)$ as a component part and therefore yields more information than $R(K)$ concerning homotopy classes of maps $K \to K'$. Thus the theory of $\Sigma_4(K)$, unlike that of $R(K)$, includes the homotopy classification of maps $S^3 \to S^2$, where S^n is an n-sphere.[3] But $\Sigma_4(K)$ does not include $\pi_4(K)$ and is incapable of distinguishing between the two classes of maps $S^4 \to S^3$. A step towards including $\pi_4(K)$ in a purely algebraic system would be to calculate $\pi_4(S_1^2 \cup S_2^2)$, where $S_1^2 \cap S_2^2$ is a single point.

On replacing $H_4(K)$ by $H_4(K)/\mathfrak{j}\pi_4(K)$ we obtain a very simple algebraic expression for the 4-type, as defined in CH *I*, of a simply connected complex.[4]

Another set of theorems concerns a certain group, $\Gamma(A)$, which is constructed from a given Abelian group A. We prove that $\Gamma(\Pi_2) \approx \Gamma_3$, where Π_2, Γ_3 are taken from $\Sigma(K)$. When K is a finite, simply connected complex we use $\Gamma(\Pi_2)$ to express the secondary modular boundary homomorphism,

$$\mathfrak{b}(m): H_4(m) \to \Gamma_3/m\Gamma_3$$

in terms of the Pontrjagin square map

$$\mathfrak{p}: H^2(K, A) \to H^4(K, \Gamma(A)),$$

which is defined in Chapter IV. Our expression for $\mathfrak{b}(m)$ shows that, if K is given in a suitable form (e.g. as a simplicial complex, which is known to be simply connected) $\Sigma_4(K)$ can be calculated constructively.

In Chapter V we show how the domain of definition of $\Sigma(K)$ can be extended from the category[5] of CW-complexes to the category of all arcwise connected

[2] All our complexes will be CW-complexes, as defined in CH I. Here we define $\Sigma(K)$ for complexes which need not be simply connected.

[3] Consider also the group of homotopy classes of maps, $\phi: K \to K$, where $K = S^2 \cup S^3$ and $S^2 \cap S^3$ is a single point, such that $\phi \mid S^n$ is of degree $+1$ over S^n for each $n = 2, 3$. All such maps induce the identical automorphism of $R(K)$. But the induced automorphism of $\pi_3(K)$ varies with the Hopf invariant of $\phi \mid S^3$ in S^2.

[4] An algebraic expression for the 3-type of a complex, which is not simply connected, is given in [9].

[5] The term category will mean the same as in [10]. We follow Eilenberg and MacLane in recognizing categories in which the objects are "all" groups etc. They indicate various means by which this can be justified.

spaces. The method used is to realize the singular complex of a space, X, by a CW-complex, $K(X)$, which is seen to be of the same homotopy type as X in case X is itself a CW-complex. We then define $\Sigma(X)$ as $\Sigma\{K(X)\}$, if X is any arcwise connected space.

In presenting these theorems we have, as far as possible, separated the purely algebraic part of the theory from the geometrical applications. The result is that Chapters I and II are purely algebraic. The geometrical applications are given in Chapters III, IV and V. In the latter we refer to certain "topological" and "homotopy" categories, which we define as follows. The *topological category* of all (topological) spaces will mean the one in which the objects are all spaces and the mappings are all maps of one space into another. The *homotopy category* of all spaces will mean the one in which the mappings are all homotopy classes of maps of one space into another. Similarly we define the topological and homotopy categories of all (geometrical) complexes of any specified kind. Here a complex means a pair (X, K), where X is the space which is covered by a complex K. A map of (X, K) into a complex (Y, L) means a triple (ϕ, K, L), where ϕ maps X into Y, and a homotopy class of maps, $(X, K) \to (Y, L)$, has a similar meaning. We shall denote (X, K) by the single letter K and $\phi: K \to L$ will stand for (ϕ, K, L).

We shall introduce a number of standard operators, β, j, k, l etc., which we shall denote by the same letters, with or without subscripts, in whatever system they occur. With the exception of the deformation operators, in §3 for example, a subscript attached to an operator, as in $\beta_n: C_n \to A_{n-1}$, will always agree with the one attached to the group which is being operated on. All our groups, except groups of operators, will be additive and we shall denote zero homomorphisms, $C \to 0$, and identical automorphisms $C \to C$, by 0 and 1.

Chapter I. The Sequence $\Sigma(C, A)$

1. Definition of $\Sigma(C, A)$

Let (C, A) denote a sequence of arbitrary Abelian[6] groups, C_n, A_n, together with a sequence of homomorphisms,

$$\cdots \xrightarrow{j} C_{n+1} \xrightarrow{\beta} A_n \xrightarrow{j} C_n \xrightarrow{\beta} A_{n-1} \xrightarrow{j} \cdots$$

such that $j_n A_n = \beta_n^{-1}(0)$. In general $\beta_n C_n \neq j_{n-1}^{-1}(0)$. We assume that C_n, A_n are defined for every $n = 0, \pm 1, \pm 2, \cdots$. Let

$$d_{n+1} = j_n \beta_{n+1}: C_{n+1} \to C_n .$$

Then $d_n d_{n+1} = 0$, since $\beta_n j_n = 0$. Let $Z_n = d_n^{-1}(0)$. Then $jA_n \subset Z_n$, since $d_n j_n = j_{n-1}\beta_n j_n = 0$, and $dC_{n+1} \subset jA_n$. Let

$$\Gamma_n = j_n^{-1}(0), \qquad \Pi_n = A_n/\beta C_{n+1}, \qquad H_n = Z_n/dC_{n+1}$$

[6] It is just as easy to define Σ and to prove Theorem 1 if A_n, C_n are non-Abelian, provided βC_{n+1}, dC_{n+1} are invariant sub-groups of A_n, Z_n.

and let

(1.1) $$i_n:\Gamma_n \to A_n\,, \qquad k_n:A_n \to \Pi_n\,, \qquad l_n:Z_n \to H_n$$

be the identical map of Γ_n and the natural homomorphisms of A_n, Z_n. Notice that the sequence

(1.2) $$\Gamma_n \xrightarrow{i} A_n \xrightarrow{j} C_n \xrightarrow{\beta} A_{n-1}$$

is (internally)[7] exact. Notice also that $j_n k_n^{-1}(0) = j_n\beta_{n+1}C_{n+1} = l_n^{-1}(0)$. Therefore a homomorphism, $\mathfrak{j}_n:\Pi_n \to H_n$, is defined by $\mathfrak{j}_n k_n = l_n j_n$.

Let $z \in Z_{n+1}$. Then $j_n\beta_{n+1}z = 0$, whence $\beta_{n+1}z \in \Gamma_n$. Since

$$\beta_{n+1}d_{n+2} = \beta_{n+1}j_{n+1}\beta_{n+2} = 0$$

it follows that $\beta_{n+1} \mid Z_{n+1}$ induces a homomorphism $\mathfrak{b}_{n+1}:H_{n+1} \to \Gamma_n$. Therefore a sequence of homomorphisms,

Σ: $$\cdots \xrightarrow{\mathfrak{j}} H_{n+1} \xrightarrow{\mathfrak{b}} \Gamma_n \xrightarrow{\mathfrak{i}} \Pi_n \xrightarrow{\mathfrak{j}} H_n \xrightarrow{\mathfrak{b}} \Gamma_{n-1} \xrightarrow{\mathfrak{i}} \cdots$$

is defined by

(1.3) $$\mathfrak{b}lz = \beta z, \qquad \mathfrak{i} = ki, \qquad \mathfrak{j}k = lj.$$

We describe $\mathfrak{b}$ as the *secondary boundary operator*. We shall sometimes write $\Sigma = \Sigma(C, A)$.

THEOREM 1. *The sequence Σ is exact.*

It follows from (1.3) and the exactness of (1.2) that

$$\mathfrak{i}\mathfrak{b}lz = \mathfrak{i}\beta z = k\beta z = 0 \qquad (z \in Z_{n+1})$$

$$\mathfrak{j}\mathfrak{i} = \mathfrak{j}ki = lji = 0$$

$$\mathfrak{b}\mathfrak{j}k = \mathfrak{b}lj = \beta j = 0.$$

Therefore $\mathfrak{i}\mathfrak{b} = 0$, $\mathfrak{j}\mathfrak{i} = 0$, $\mathfrak{b}\mathfrak{j} = 0$.

Let $\mathfrak{i}\gamma = k\gamma = 0$, where $\gamma \in \Gamma_n$. Then $\gamma = \beta c$, for some $c \in C_{n+1}$, since $k_n^{-1}(0) = \beta C_{n+1}$. Moreover $dc = j\beta c = j\gamma = 0$. Therefore $c \in Z_{n+1}$, $lc \in H_{n+1}$ and we have

$$\gamma = \beta c = \mathfrak{b}lc \in \mathfrak{b}H_{n+1}.$$

Therefore $\mathfrak{i}_n^{-1}(0) = \mathfrak{b}H_{n+1}$.

Let $\mathfrak{j}ka = lja = 0$, where $a \in A_n$. Then $ja = dc = j\beta c$, for some $c \in C_{n+1}$, since $l_n^{-1}(0) = dC_{n+1}$. Therefore $a = \gamma + \beta c$, where $\gamma \in \Gamma_n$, and

$$ka = k\gamma + k\beta c = \mathfrak{i}\gamma \in \mathfrak{i}\Gamma_n.$$

Therefore $\mathfrak{j}_n^{-1}(0) = \mathfrak{i}\Gamma_n$.

Let $\mathfrak{b}lz = \beta z = 0$, where $z \in Z_n$. Then $z = ja$, for some $a \in A_n$, since $\beta_n^{-1}(0) = jA_n$. Therefore

$$lz = lja = \mathfrak{j}ka \in \mathfrak{j}\Pi_n.$$

Therefore $\mathfrak{b}_n^{-1}(0) = \mathfrak{j}\Pi_n$ and the theorem is proved.

[7] I.e. the last homomorphism need not be onto nor the first an isomorphism into.

2. The secondary modular boundary operator

Let $m > 0$ and assume that, for some particular value of n,

$$(2.1)\qquad \begin{cases} \text{(a)}\quad mz = 0 \quad \textit{implies} \quad z = 0, \quad \text{where} \quad z \in Z_{n-1} \\ \text{(b)}\quad jA_{n-1} = Z_{n-1}. \end{cases}$$

Let

$$H_n(m) = \{d_n^{-1}(mZ_{n-1})/dC_{n+1}\}_m ,$$

where $G_m = G/mG$ if G is any additive Abelian group. We proceed to define what we call the secondary *modular* boundary homomorphism

$$(2.2)\qquad \mathfrak{b}_n(m){:}H_n(m) \rightarrow \Gamma_{n-1,m} = \Gamma_{n-1}/m\Gamma_{n-1} .$$

Let $c_* \in H_n(m)$ and let $c \in C_n$ be any representative of c_*. Then $dc = mz = mja$, for some $a \in A_{n-1}$, by (2.1b). Therefore $j(\beta c - ma) = 0$, whence $\beta c - ma \in \Gamma_{n-1}$. If $a' \in A_{n-1}$ is any other element such that $mja' = dc = mja$ it follows from (2.1a) that $j(a' - a) = 0$. Therefore $a' = \gamma + a$, where $\gamma \in \Gamma_{n-1}$, and $\beta c - ma' = (\beta c - ma) - m\gamma$. Therefore the coset, $(\beta c - ma)_m \in \Gamma_{n-1,m}$, which contains $\beta c - ma$, is uniquely determined by c, where $a \in A_{n-1}$ is an arbitrary element such that $mja = dc$.

If $c' \in C_n$ is any other representative of c_*, then $c' = c + mc_1 + dc_2$, where $c_1 \in C_n$, $c_2 \in c_{n+1}$, and

$$dc' = dc + dmc_1 = jm(a + \beta c_1).$$

Moreover $\beta c' - m(a + \beta c_1) = \beta c - ma$. Therefore a single-valued map (2.2), which is obviously a homomorphism, is defined by

$$\mathfrak{b}_n(m)c_* = (\beta c - ma)_m ,$$

where $c \in C_n$ is any representative of c_* and $jma = dc$.

As an alternative to (2.1a), let $j{:}A_{n-1} \rightarrow C_{n-1}$ have a right inverse,[8] $u{:}Z_{n-1} \rightarrow A_{n-1}$. Then $j(\beta - ud) = 0$, whence $(\beta - ud)C_n \subset \Gamma_{n-1}$. We define $\mathfrak{b}(m){:}H_n(m) \rightarrow \Gamma_{n-1,m}$ by $\mathfrak{b}(m)c_* = \{(\beta - ud)c\}_m$, where $c \in d^{-1}(mZ_{n-1})$. The homomorphism u is determined by j, modulo an arbitrary homomorphism $\theta{:}Z_{n-1} \rightarrow \Gamma_{n-1}$. If $dc = mz$ we have $\{\beta - (u + \theta)\, d\}\, c = (\beta - ud)c - m\theta z$. Therefore u and $u + \theta$ determine the same homomorphism $\mathfrak{b}(m)$, which is therefore determined uniquely by the system (C, A). If (2.1a) is also satisfied this definition of $\mathfrak{b}(m)$ is equivalent to the previous one.

3. Induced homomorphisms of Σ

Let Σ' be a sequence of the same sort as Σ. By a *homomorphism* (*isomorphism*),

$$F = (\mathfrak{h}, \mathfrak{g}, \mathfrak{f}){:}\Sigma \rightarrow \Sigma'$$

[8] This is always the case if C_{n-1}, and hence jA_{n-1} (See p. 50 of [18]) is free Abelian.

we mean a family of homomorphisms (isomorphisms)[9]

$$\mathfrak{h}_{n+1}:H_{n+1} \to H'_{n+1}, \qquad \mathfrak{g}_n:\Gamma_n \to \Gamma'_n, \qquad \mathfrak{f}_n:\Pi_n \to \Pi'_n,$$

such that

$$\mathfrak{b}\mathfrak{h} = \mathfrak{g}\mathfrak{b}, \qquad \mathfrak{i}\mathfrak{g} = \mathfrak{f}\mathfrak{i}, \qquad \mathfrak{j}\mathfrak{f} = \mathfrak{h}\mathfrak{j}, \tag{3.1}$$

where $\mathfrak{b}:H'_{n+1} \to \Gamma'_n$ etc. are the homomorphisms in Σ'.

Let (C', A') be a system of the same sort as (C, A). Then a homomorphism (isomorphism)

$$(h, f):(C, A) \to (C', A') \tag{3.2}$$

will mean a family of homomorphisms (isomorphisms),

$$h_{n+1}:C_{n+1} \to C'_{n+1}, \qquad f_n:A_n \to A'_n$$

such that

$$\beta h = f\beta, \qquad jf = hj. \tag{3.3}$$

Notice that $dh = j\beta h = jf\beta = hj\beta = hd$ in consequence of (3.3). Also $f\Gamma_n \subset \Gamma'_n$ since $jfi = hji = 0$, where $i:\Gamma_n \to A_n$ is the identical map.

Let (3.2) be a given homomorphism and let $\Sigma' = \Sigma(C', A')$. Then $kf\beta = k\beta h = 0$, $lhd = ldh = 0$. Therefore h, f induce homomorphisms

$$\mathfrak{h}:H_{n+1} \to H'_{n+1}, \qquad \mathfrak{g}:\Gamma_n \to \Gamma'_n, \qquad \mathfrak{f}:\Pi_n \to \Pi'_n,$$

according to the rules

$$\mathfrak{h}lz = lhz, \qquad i\mathfrak{g} = fi, \qquad \mathfrak{f}k = kf \qquad (z \in Z_{n+1}). \tag{3.4}$$

It follows from (1.3), (3.3) and (3.4) that

$$\begin{aligned} \mathfrak{b}\mathfrak{h}lz &= \mathfrak{b}lhz = \beta hz = f\beta z \\ &= \mathfrak{g}\beta z = \mathfrak{g}\mathfrak{b}lz, \\ \mathfrak{i}\mathfrak{g} &= ki\mathfrak{g} = kfi = \mathfrak{f}ki = \mathfrak{f}\mathfrak{i} \\ \mathfrak{j}\mathfrak{f}k &= \mathfrak{j}kf = ljf = lhj = \mathfrak{h}lj \\ &= \mathfrak{h}\mathfrak{j}k. \end{aligned}$$

Therefore $(\mathfrak{h}, \mathfrak{g}, \mathfrak{f}):\Sigma \to \Sigma'$ is a homomorphism. We call it the homomorphism *induced* by (h, f).

By a *deformation operator*, $\xi:C \to C'$, we mean a family of arbitrary homomorphisms, $\xi_{n+1}:C_n \to C'_{n+1}$. We describe two homomorphisms,

$$(h, f), \qquad (h^*, f^*):(C, A) \to (C', A'),$$

as *homotopic*, and write $(h, f) \simeq (h^*, f^*)$, if, and only if, there is a deformation operator, $\xi:C \to C'$, such that

[9] An isomorphism, without qualification, will always mean an isomorphism onto.

$$(3.5)\qquad \begin{cases} h_n^* - h_n = d_{n+1}\xi_{n+1} + \xi_n d_n \\ f_n^* - f_n = \beta_{n+1}\xi_{n+1}j_n . \end{cases}$$

Since $ld = 0$, $ji = 0$, $k\beta = 0$ it follows from (3.5) that $lh^*z = lhz$, $f^*i = fi$, $kf^* = kf$. Therefore (h, f) and (h^*, f^*) induce the same homomorphism $\Sigma(C, A) \to \Sigma(C', A')$.

Let (C, A) and (C', A') both satisfy (2.1). Let $c \in d_n^{-1}(mZ_{n-1})$ and let $dc = mz$. Let $Z'_{n-1} \subset C'_{n-1}$ and $H'_n(m)$ be defined in the same way as Z_{n-1} and $H_n(m)$. Then $hZ_{n-1} \subset Z'_{n-1}$, since $dh = hd$, and $dhc = hdc = mhz$. Therefore h_n induces a homomorphism

$$\mathfrak{h}(m): H_n(m) \to H'_n(m),$$

according to the rule $\mathfrak{h}(m)c_* = (hc)_*$, where c_* and $(hc)_*$ are the elements of $H_n(m)$ and $H'_n(m)$, which correspond to c and hc. Also $\mathfrak{g}$ induces a homomorphism

$$\mathfrak{g}(m): \Gamma_{n-1,m} \to \Gamma'_{n-1,m}$$

such that $\mathfrak{g}(m)\gamma_m = (\mathfrak{g}\gamma)_m$, where γ_m, $(\mathfrak{g}\gamma)_m$ are the cosets which contain γ, $\mathfrak{g}\gamma$. By the definition of $\mathfrak{b}(m)$ we have $\mathfrak{b}(m)c_* = (\beta c - ma)_m$ where $a \in A_{n-1}$ is such that $dc = mja$. Then $dhc = hdc = mhja = mjfa$ and

$$\begin{aligned} \mathfrak{b}(m)(hc)_* &= (\beta hc - mfa)_m \\ &= \{f(\beta c - ma)\}_m \\ &= \{\mathfrak{g}(\beta c - ma)\}_m \\ &= \mathfrak{g}(m)\mathfrak{b}(m)c_* . \end{aligned}$$

Therefore $\mathfrak{b}(m)$ is natural in the sense that

$$(3.6)\qquad \mathfrak{b}(m)\mathfrak{h}(m) = \mathfrak{g}(m)\mathfrak{b}(m).$$

Obviously (h, f) and (h^*, f^*) induce the same homomorphisms $\mathfrak{h}(m)$, $\mathfrak{g}(m)$ if $(h, f) \simeq (h^*, f^*)$.

Let $j: A_{n-1} \to C_{n-1}$ and $j: A'_{n-1} \to C'_{n-1}$ have right inverses, u and u', let $jA_{n-1} = Z_{n-1}$ and let $\mathfrak{b}(m)$ be defined by the second method in §2. Since $j(fu - u'h) = hju - h = 0$ it follows that $fu - u'h = \phi: Z_{n-1} \to \Gamma'_{n-1}$. Let $dc = mz$, where $c \in C_n$. Then $f(\beta - ud)c = (\beta - u'd)hc - m\phi z$, whence $\mathfrak{g}(m)\mathfrak{b}(m) = \mathfrak{b}(m)\mathfrak{h}(m)$.

4. Combinatorial realizability

By a *composite chain system* we shall mean a system (C, A), of the kind introduced in §1, such that

(a) $C_r = A_r = 0$ *if* $r < 2$

(b) *each* C_n *is a free Abelian group.*

Let (C, A) be a composite chain system. Then $\Sigma(C, A)$ terminates with $H_2 \to 0$, followed by a series of homomorphisms, $0 \to 0$, which we discard. Let

$$\Sigma' \colon \cdots \xrightarrow{\mathfrak{j}} H'_{n+1} \xrightarrow{\mathfrak{b}} \Gamma'_n \xrightarrow{\mathfrak{i}} \Pi'_n \xrightarrow{\mathfrak{j}} \cdots \xrightarrow{\mathfrak{j}} H'_2 \to 0$$

be an exact sequence in which the groups are Abelian, but otherwise arbitrary. A composite chain system (C, A) will be called a *combinatorial realization* of Σ' if, and only if, $\Sigma(C, A) \approx \Sigma'$.

THEOREM 2. *Σ' has a combinatorial realization.*

Assume that we have constructed a composite chain system (C, A), and homomorphisms

$$l'_{n+1} \colon Z_{n+1} \to H'_{n+1}, \qquad \mathfrak{g}_n \colon \Gamma_n \approx \Gamma'_n, \qquad k'_n \colon A_n \to \Pi'_n$$

for every $n = 1, 2, \cdots$, such that

$$(4.1) \quad \begin{matrix} \text{(a)} \\ \text{(b)} \end{matrix} \begin{cases} \mathfrak{b}_{n+1} l'_{n+1} z = \mathfrak{g}_n \beta_{n+1} z, & \mathfrak{i}_n \mathfrak{g}_n = k'_n i_n, \qquad \mathfrak{j}_n k'_n = l'_n j_n \\ l'_{n+1} Z_{n+1} = H'_{n+1}, & dC_{n+1} = l'^{-1}_n(0), \qquad \beta C_{n+1} \subset k'^{-1}_n(0), \end{cases}$$

where, as usual, $z \in Z_{n+1}$ and $i_n \colon \Gamma_n \to A_n$ is the identical map. Then it follows from (4.1b) that isomorphisms and homomorphisms,

$$\mathfrak{h}_{n+1} \colon H_{n+1} \approx H'_{n+1}, \qquad \mathfrak{f}_n \colon \Pi_n \to \Pi'_n \qquad (n = 1, 2, \cdots),$$

are defined by $\mathfrak{h} l = l'$, $\mathfrak{f} k = k'$, where H_{n+1}, Π_n are in $\Sigma = \Sigma(C, A)$. It follows from (4.1a) and (1.3) that

$$\mathfrak{b}\mathfrak{h} l z = \mathfrak{b} l' z = \mathfrak{g}\beta z = \mathfrak{g}\mathfrak{b} l z$$

$$\mathfrak{i}\mathfrak{g} = k' i = \mathfrak{f} k i = \mathfrak{f}\mathfrak{i}$$

$$\mathfrak{j}\mathfrak{f} k = \mathfrak{j} k' = l' j = \mathfrak{h} l j = \mathfrak{h}\mathfrak{j} k.$$

Therefore $\mathfrak{b}\mathfrak{h} = \mathfrak{g}\mathfrak{b}$ etc. and $(\mathfrak{h}, \mathfrak{g}, \mathfrak{f}) \colon \Sigma \to \Sigma'$ is a homomorphism. Since $\mathfrak{h}_{n+1}$, $\mathfrak{g}_n$ are isomorphisms (onto) for every n, so if $\mathfrak{f}_n$, by (7.5) on p. 435 of [17]. Therefore (C, A) is a combinatorial realization of Σ'.

We now construct (C, A) inductively, starting with $C_1 = A_0 = 0$. Let $r \geqq 1$ and assume that we have constructed the groups and homomorphisms,

$$C_r \xrightarrow{\beta} A_{r-1} \xrightarrow{j} C_{r-1} \xrightarrow{\beta} \cdots \xrightarrow{\beta} A_0,$$

likewise l'_{n+1}, k'_n, so as to satisfy (4.1) for $n = 0, \cdots, r-1$. Let Γ_r, B_r be any groups which are isomorphic to Γ'_r, $\beta_r^{-1}(0)$ and let $\mathfrak{g}_r \colon \Gamma_r \approx \Gamma'_r$, $u \colon \beta_r^{-1}(0) \approx B_r$ be any isomorphisms. We define A_r and j_r by[10]

$$A_r = \Gamma_r + B_r, \qquad j_r(\gamma + b) = u^{-1} b \qquad (\gamma \in \Gamma_r, b \in B_r).$$

Then $\Gamma_r = j_r^{-1}(0)$ and $j_r A_r = \beta_r^{-1}(0)$. Since C_r is a free Abelian group so are $\beta_r^{-1}(0)$ and B_r. Let $\{b_\lambda\}$ be a set of free generators of B_r. It follows from (4.1a),

[10] If X, Y are any (additive) groups $X + Y$ will always denote their direct sum. It is always to be assumed that $x \in X$, $y \in Y$ are identified with $(x, 0)$, $(0, y) \in X + Y$.

with $n = r - 1$, that $\mathfrak{b}_r l'_r j_r = \mathfrak{g}_{r-1}\beta_r j_r = 0$. Therefore $l'_r j_r b_\lambda = \mathfrak{j}_r x'_\lambda$, for some $x'_\lambda \in \Pi'_r$ by the exactness of Σ'. We define $k'_r : A_r \to \Pi'_r$ by

$$k'_r(\gamma + b_\lambda) = \mathfrak{i}_r\mathfrak{g}_r\gamma + x'_\lambda . \tag{4.2}$$

Then $\mathfrak{j}_r k'_r \gamma = 0 = l'_r j_r \gamma$ and $\mathfrak{j}_r k'_r b_\lambda = l'_r j_r b_\lambda$. Also $k'_r i_r = \mathfrak{i}_r \mathfrak{g}_r$. Therefore

$$\mathfrak{i}_r \mathfrak{g}_r = k'_r i_r , \qquad \mathfrak{j}_r k'_r = l'_r j_r . \tag{4.3}$$

Let $\{y_\mu\}$ be a set of elements which generate H'_{r+1}. Let Z_{r+1} be a free Abelian group with a set of free generators, $\{z_\mu\}$, in a (1–1) correspondence, $z_\mu \to y_\mu$, with $\{y_\mu\}$. Let $l'_{r+1}: Z_{r+1} \to H'_{r+1}$ be defined by $l'_{r+1} z_\mu = y_\mu$. Then $l'_{r+1} Z_{r+1} = H'_{r+1}$. Let P_{r+1} be any group such that $v: P_{r+1} \approx l_r'^{-1}(0) \subset Z_r$ and let $C_{r+1} = Z_{r+1} + P_{r+1}$. Since C_r is free Abelian so are $l_r'^{-1}(0)$, P_{r+1} and hence C_{r+1}. Let $\{p_\sigma\}$ be a set of free generators of P_{r+1}. Since $vP_{r+1} = l_r'^{-1}(0)$ it follows from (4.1a), with $n = r - 1$, that

$$\beta_r v p_\sigma = \mathfrak{g}_{r-1}^{-1}\mathfrak{b}_r l'_r v p_\sigma = 0.$$

Since $j_r A_r = \beta_r^{-1}(0)$ it follows that $vp_\sigma = j_r a_\sigma$ for some $a_\sigma \in A_r$ and from (4.3) that $\mathfrak{j}_r k'_r a_\sigma = l'_r j_r a_\sigma = l'_r v p_\sigma = 0$. Therefore there is a $\gamma'_\sigma \in \Gamma'_r$ such that $k'_r a_\sigma = \mathfrak{i}_r \gamma'_\sigma = k'_r \gamma_\sigma$, where $\gamma_\sigma = \mathfrak{g}_r^{-1}\gamma'_\sigma$. Also it follows from (4.3) and the exactness of Σ' that

$$k'_r \mathfrak{g}_r^{-1} \mathfrak{b}_{r+1} l'_{r+1} z = \mathfrak{i}_r \mathfrak{b}_{r+1} l'_{r+1} z = 0.$$

We define β_{r+1} by

$$\beta_{r+1}(z + p_\sigma) = \mathfrak{g}_r^{-1}\mathfrak{b}_{r+1} l'_{r+1} z + (a_\sigma - \gamma_\sigma).$$

Then $d_{r+1}(z + p_\sigma) = j_r \beta_{r+1}(z + p_\sigma) = j_r a_\sigma = v p_\sigma$, whence $d_{r+1}^{-1}(0) = Z_{r+1}$ and $d_{r+1} C_{r+1} = l_r'^{-1}(0)$. Also $k'_r \beta_{r+1} = 0$ and $\mathfrak{g}_r \beta_{r+1} z = \mathfrak{b}_{r+1} l'_{r+1} z$. Therefore (4.1) are satisfied when $n = r$ and the induction is complete.

ADDENDUM. *The combinatorial realization, (C, A), of Σ' may be constructed so that*

(a) $l'_n : Z_n \approx H'_n$ *if H'_n is free Abelian.*

(b) *The rank of C_n is finite if both H'_n, H'_{n-1} have finite sets of generators.*

To prove this we assume, as part of the inductive hypothesis, that these conditions are satisfied for $n \leqq r$ and also that the rank of Z_n is finite if H'_n is finitely generated. We insist that the generators $\{y_\mu\}$ of H'_{r+1} shall be free if H'_{r+1} is free Abelian, in which case $l'_{r+1}: Z_{r+1} \approx H'_{r+1}$, and finite in number if H'_{r+1} is finitely generated. In the latter case the rank of Z_{r+1} is finite. If H'_r, and hence Z_r, are finitely generated, so are $l_r'^{-1}(0)$ and P_{r+1}. Therefore the rank of C_{r+1} is finite if both H'_{r+1}, H'_r are finitely generated. This proves the addendum.

Let $\Sigma = \Sigma(C, A)$, $\Sigma' = \Sigma(C', A')$, where (C, A) and (C', A') are composite chain systems. If a given homomorphism $F: \Sigma \to \Sigma'$ is the one induced by a homomorphism $(h, f): (C, A) \to (C', A')$ we shall call (h, f) a *combinatorial realization* of F.

THEOREM 3. *Any homomorphism, $F:\Sigma \rightarrow \Sigma'$ has a combinatorial realization $(C, A) \rightarrow (C', A')$.*

Let $F = (\mathfrak{h}, \mathfrak{g}, \mathfrak{f})$. As in the proof of Theorem 2 we have

$$C_{n+1} = Z_{n+1} + P_{n+1}, \qquad A_n = \Gamma_n + B_n,$$

where $P_{n+1} \approx dC_{n+1}$, $B_n \approx jA_n$. Let $\{b_\lambda\}$, $\{z_\mu\}$, $\{p_\sigma\}$ be sets of free generators of B_n, Z_{n+1}, P_{n+1}.

Let $z'_\mu \in l_{n+1}^{-1}\mathfrak{h}_{n+1}l_{n+1}z_\mu \subset Z'_{n+1}$ and let $h^0_{n+1}:Z_{n+1} \rightarrow Z'_{n+1}$ be defined by $h^0 z_\mu = z'_\mu$ for every $n \geqq 1$. Then

$$lh^0 = \mathfrak{h}l. \tag{4.4}$$

Let $a'_\lambda \in k_n^{-1}\mathfrak{f}_n k_n b_\lambda \subset A'_n$. Then it follows from (1.3), (3.1) and (4.4), since $jA_n \subset Z_n$, that

$$\begin{aligned} lja'_\lambda &= \mathfrak{j}ka'_\lambda = \mathfrak{j}\mathfrak{f}kb_\lambda = \mathfrak{h}\mathfrak{j}kb_\lambda \\ &= \mathfrak{h}ljb_\lambda = lh^0 jb_\lambda. \end{aligned}$$

Therefore $ja'_\lambda = h^0 jb_\lambda + dc'_\lambda$, for some $c'_\lambda \in C'_{n+1}$. Let $f:A_n \rightarrow A'_n$ be defined by

$$fi = i\mathfrak{g} \qquad fb_\lambda = a'_\lambda - \beta c'_\lambda.$$

Then $kfi = ki\mathfrak{g} = \mathfrak{i}\mathfrak{g} = \mathfrak{f}\mathfrak{i} = \mathfrak{f}ki$ and $kfb_\lambda = ka'_\lambda = \mathfrak{f}kb_\lambda$. Therefore

$$\mathfrak{h}l = lh^0, \qquad i\mathfrak{g} = fi, \qquad \mathfrak{f}k = kf. \tag{4.5}$$

Since $kf = \mathfrak{f}k$ and $k\beta = 0$ we have $kf\beta p_\sigma = \mathfrak{f}k\beta p_\sigma = 0$. Therefore $f\beta p_\sigma = \beta c''_\sigma$, for some $c''_\sigma \in C'_{n+1}$. Let $h:C_{n+1} \rightarrow C'_{n+1}$ be defined by $hz = h^0 z$, $hp_\sigma = c''_\sigma$. Then $\beta hp_\sigma = f\beta p_\sigma$. Since $\beta z = \mathfrak{b}lz$ it follows from (4.5) and (4.4) that

$$\begin{aligned} f\beta z &= f\mathfrak{b}lz = \mathfrak{g}\mathfrak{b}lz = \mathfrak{b}\mathfrak{h}lz \\ &= \mathfrak{b}lhz = \beta hz. \end{aligned}$$

Therefore $f\beta = \beta h$. Also $jfi = ji\mathfrak{g} = 0 = hji$, $jfb_\lambda = ja'_\lambda - dc'_\lambda = hjb_\lambda$. Therefore $jf = hj$. Thus (h, f) is a homomorphism. Since $hz = h^0 z$ it follows from (4.5) that (h, f) is a combinatorial realization of $(\mathfrak{h}, \mathfrak{g}, \mathfrak{f})$ and the proof is complete.

CHAPTER II. THE GROUP $\Gamma(A)$

5. Definition of $\Gamma(A)$

Let A be any additive, Abelian group. We shall define $\Gamma(A)$, additively, by means of symbolic generators and relations. The symbolic generators shall be the elements of A. We emphasize the fact that an element $a \in A$ is not an element of $\Gamma(A)$. The elements of $\Gamma(A)$ are equivalence classes of words, written as sums, in the pairs $(+, a)$, $(-, a)$ for every $a \in A$. We write $(\pm, a)$ as $\pm w(a)$ and, frequently, $+w(a)$ as $w(a)$. We shall use the symbol $\equiv$ to denote equivalence between words and $=$ to indicate that two symbols, in any context, stand for the same thing. In defining $\Gamma(A)$, or any other group, by this means we always assume that the "trivial relations", $x - x \equiv 0$, are satisfied as a matter of course.

The relations for $\Gamma(A)$ are

$$(5.1)\quad \begin{matrix}\text{(a)}\\ \text{(b)}\end{matrix}\begin{cases} w(-a) \equiv w(a) \\ w(a+b+c) - w(b+c) - w(c+a) - w(a+b) \\ \qquad\qquad + w(a) + w(b) + w(c) \equiv 0, \end{cases}$$

for all elements $a, b, c \in A$. It follows from (5.1b), with $a = b = c = 0$, that

$$(5.2)\qquad w(0) \equiv 0.$$

Hence it follows from (5.1b), with $b = 0$, $a + c = d$, that

$$(5.3)\qquad w(d) - w(c) - w(d) + w(c) \equiv 0.$$

Therefore $\Gamma(A)$ is Abelian. It follows from (5.1a), (5.2) and (5.1b), with $a = b = -c$, that

$$w(a) - w(2a) + 3w(a) \equiv 0,$$

whence $w(2a) \equiv 4w(a)$. Let

$$(5.4)\qquad W(a, b) = w(a + b) - w(a) - w(b).$$

Then $W(a, b) \equiv W(b, a)$ since A and $\Gamma(A)$ are Abelian. Since $w(2a) = 4w(a)$ we have

$$(5.5)\qquad W(a, a) \equiv 2w(a).$$

Given that addition is commutative, it is easily verified that (5.1b) is equivalent to

$$(5.6)\qquad W(a, b + c) \equiv W(a, b) + W(a, c).$$

It follows from (5.4), (5.6) and induction on n that

$$(5.7)\qquad w(a_1 + \cdots + a_n) \equiv \sum_i w(a_i) + \sum_{i<j} W(a_i, a_j).$$

On taking $a_1 = \cdots = a_n$ it follows from (5.5) and (5.7) that

$$(5.8)\qquad w(na) \equiv n^2 w(a).$$

Let $\gamma(a) \in \Gamma(A)$ be the element which corresponds to $w(a)$ and $[a, b] \in \Gamma(A)$ the element corresponding to $W(a, b)$. Then

$$\gamma(a + b) = \gamma(a) + \gamma(b) + [a, b],$$

in consequence of (5.4). Therefore $-[a, b]$ is a factor set which measures the error made in supposing the map $\gamma: A \to \Gamma(A)$ to be a homomorphism. We shall deal with the generators (i.e., generating elements) $\gamma(a)$ in preference to the symbols $w(a)$.

Let g be a map[11] of the set of generators $\gamma(a)$, indexed to A by the map

[11] We admit the possibility that $g\gamma(a) \neq g\gamma(b)$ if $a \neq b$, even if $\gamma(a) = \gamma(b)$. But the map $g\gamma: A \to G$ shall be single-valued.

$a \to \gamma(a)$, into an (additive) group G. We shall say that g is *consistent* with a relation

$$\varepsilon_1 w(a_1) + \cdots + \varepsilon_n w(a_n) \equiv 0 \qquad (\varepsilon_i = \pm 1)$$

if, and only if,

$$\varepsilon_1 g\gamma(a_1) + \cdots + \varepsilon_n g\gamma(a_n) = 0.$$

If g is consistent with all the relations (5.1) it determines a homomorphism $\Gamma(A) \to G$.

We give some examples of groups $\Gamma(A)$. First let A be free Abelian and let $\{a_i\}$ be a set of free generators of A, indexed in (1–1) fashion to a set $\{i\}$.

(A) $\Gamma(A)$ *is free Abelian and is freely generated by the set of elements* $\gamma(a_i)$, $[a_j, a_k]$, *for every* $i \in \{i\}$ *and every pair* $j, k \in \{i\}$ *such that*[12] $j < k$.

Let[13] be a free Abelian group, which is freely generated by a set of elements, $\{g_i, g_{jk}\}$, indexed in (1–1) fashion to the union of $\{i\}$ and the totality of pairs (j, k) such that $j < k$. Let $\phi: G \to \Gamma(A)$ be the homomorphism which is defined by $\phi g_i = \gamma(a_i)$, $\phi g_{jk} = [a_j, a_k]$. Let $a(x) = \Sigma x_i a_i$ where $\{x_i\}$ is a set of integers, almost all[14] of which are zero. Let

$$g(x) = \sum_i x_i^2 g_i + \sum_{j<k} x_j x_k g_{jk}.$$

Then $g(-x) = g(x)$, where $-x = \{-x_i\}$. Since

$$(x_i + y_i)^2 - x_i^2 - y_i^2 = 2x_i y_i$$

$$(x_j + y_j)(x_k + y_k) - x_j x_k - y_j y_k = x_j y_k + x_k y_j$$

it follows that

$$(5.9) \quad g(x + y) - g(x) - g(y) = 2\sum_i x_i y_i g_i + \sum_{j<k} (x_j y_k + x_k y_j) g_{jk},$$

where $y = \{y_i\}$, $x + y = \{x_i + y_i\}$. Since this is bilinear in x, y and since $g(-x) = g(x)$ it follows that the correspondence $\gamma\{a(x)\} \to g(x)$ is consistent with (5.1a) and with (5.6), and hence with (5.1b). Therefore it determines a homomorphism $\phi': \Gamma(A) \to G$. Obviously $\phi'\gamma(a_i) = g_i$ and it follows from (5.4) and (5.9) that $\varphi'[a_j, a_k] = g_{jk}$ if $j < k$. Therefore $\phi'\phi = 1$. Also it follows from (5.7) and (5.8), with a_i in (5.7) replaced by $x_i a_i$, that $\phi\phi' = 1$. Therefore $\phi: G \approx \Gamma(A)$, which proves the assertion.

(B) *Let A be cyclic of (finite) order m and let a_1 be a generator of A. Then $\Gamma(A)$ is cyclic of order m or $2m$, according as m is odd or even, and is generated by $\gamma(a_1)$.*

This is a corollary of Theorem 5 below.

[12] We postulate a simple ordering of the set $\{i\}$ ($j \neq k$ and $k \nless j$ if $j < k$) as a convenient method of indicating a summation over all unordered pairs j, k ($j \neq k$). It is to be assumed, when the context requires it, that any such store of indices is simply ordered.

[13] This proof of (A) and its use in simplifying an earlier proof of Theorem 7 below were suggested by M. G. Barratt.

[14] By almost all we mean all but a finite number.

(C) *If every element in A is of finite order and also divisible by its order, then $\Gamma(A) = 0$.*

Let $a \epsilon A$ be of order m and let $a = mb$. Then

$$2\gamma(a) = [a, a] = [a, mb] = [ma, b] = 0.$$

Therefore $2\gamma(x) = 0$ for every $x \epsilon A$. In particular $2\gamma(b) = 0$. If m is even it follows that

$$\gamma(a) = \gamma(mb) = m^2\gamma(b) = 0.$$

Since $m^2\gamma(a) = \gamma(ma) = 0$ and $2\gamma(a) = 0$ we also have $\gamma(a) = 0$ if m is odd. This proves (C).

It follows from (C) that $\Gamma(R_1) = 0$ if R_1 is the group of rationals, mod. 1.

(D) *If A is the additive group of rationals, then $\Gamma(A) \approx A$.*

First let A be the additive group of any commutative ring, R. Then a homomorphism, $g:\Gamma(A) \to A$, is defined by $g\gamma(a) = a^2$. Now let R be the ring of rationals. Then it may be verified that a homomorphism, $f:A \to \Gamma(A)$, is defined by[15] $f(p/q) = pq\gamma(1/q)$, where p, q are any integers, and that $fg = 1$, $gf = 1$.

Let A be a free Abelian group and let $\{a_i\}$ be a set of free generators of A. Then the following expression for $\Gamma(A)$ is suggested by [19]. Let I_0 be the group of integers and let A^* be the group of homomorphisms $a^*:A \to I_0$, which are restricted by the condition that $a^*a_i = 0$ for almost all values of i. Then A^* is a free Abelian group, which is freely generated by $\{a_i^*\}$, where $a_i^*a_j = 1$ or 0 according as $j = i$ or $j \neq i$. We describe a homomorphism $f:A^* \to A$ as *admissible* if, and only if, $fa_i^* = 0$ for almost all values of i and as *symmetric* if, and only if,

$$a^*fb^* = b^*fa^*, \tag{5.10}$$

for every pair $a^*, b^* \varepsilon A^*$. Let

$$fa_i^* = \Sigma_j f_{ij}a_j = \Sigma_j(a_j^*fa_i^*)a_j . \tag{5.11}$$

Then (5.10) is equivalent to the condition $f_{ij} = f_{ji}$.

Let S be the additive group of all admissible, symmetric homomorphisms, $f:A^* \to A$, and let $\lambda:S \to \Gamma(A)$ be given by

$$\lambda f = \sum_{i \leqq j} (a_j^*fa_i^*)e_{ij} , \tag{5.12}$$

where $e_{ii} = \gamma(a_i)$, $e_{ij} = [a_i, a_j]$ if $i < j$. It follows from (A) and (5.11) that $\lambda^{-1}(0) = 0$. An arbitrary element $\alpha \epsilon \Gamma(A)$ is given by

$$\alpha = \sum_{i \leqq j} \gamma_{ij}e_{ij}$$

for integral values of γ_{ij}, which are zero for almost all values of i, j. Therefore $\alpha = \lambda f$, where f is given by (5.11), with $f_{ij} = \gamma_{ij}$ if $i \leqq j$. Therefore

$$\lambda:S \approx \Gamma(A). \tag{5.13}$$

[15] Since $pqk^2\gamma(1/kq) = pq\gamma(1/q)$ we need not insist that $(p, q) = 1$.

6. Induced homomorphisms of $\Gamma(A)$

Let $f:A \to A'$ be a homomorphism of A into an additive Abelian group A'. Let $\Gamma(A')$ be defined in the same way as $\Gamma(A)$ and let $\gamma(a') \in \Gamma(A')$ be defined in the same way as $\gamma(a)$. Then the correspondence $\gamma(a) \to \gamma(fa)$ obviously determines a homomorphism

$$g:\Gamma(A) \to \Gamma(A').$$

We describe g as the homomorphism *induced* by f. It is given by $g\gamma = \gamma f$. Obviously

$$g[a, b] = [fa, fb] \qquad (a, b \in A). \tag{6.1}$$

It is also obvious that $g\Gamma(A) = \Gamma(A')$ if $fA = A'$ and that $g = 1$ if $A = A'$ and $f = 1$. Let $g':\Gamma(A') \to \Gamma(A'')$ be induced by $f':A' \to A''$. Then it is obvious that $g'g:\Gamma(A) \to \Gamma(A'')$ is induced by $f'f:A \to A''$. Hence it follows that

$$g:\Gamma(A) \approx \Gamma(A')$$

if $f:A \approx A'$.

Let A admit a (multiplicitive) group, W, as a group of operators. Then so does $\Gamma(A)$, according to the rule

$$w\gamma(a) = \gamma(wa) \qquad (w \in W). \tag{6.2}$$

That is to say, $w:\Gamma(A) \to \Gamma(A)$ is the automorphism induced by $w:A \to A$. Let $f:A \to A'$ be an operator homomorphism into a group, A', which also admits W as a group of operators. Then it is easily verified that the induced homomorphism, $g:\Gamma(A) \to \Gamma(A')$, is also an operator homomorphism.

On taking A to be cyclic infinite and $A' = A$ we see that a given automorphism, $\Gamma(A) \to \Gamma(A)$ (e.g. $\gamma(a) \to -\gamma(a)$), is not necessarily induced by any endomorphism $A \to A$; also that distinct automorphisms of A (e.g. $a \to \pm a$) may induce the same automorphism of $\Gamma(A)$.

Let $g:\Gamma(A) \to \Gamma(A')$ be induced by $f:A \to A'$. Let $\{a_i\} \subset A$ be a set of generators of A and let $\{b_\lambda\} \subset f^{-1}(0)$ be a set of elements which generate $f^{-1}(0)$. Then

$$g\gamma(b_\lambda) = g[a_i, b_\lambda] = 0.$$

THEOREM 4. *Let* $fA = A'$. *Then* $g^{-1}(0)$ *is generated*[16] *by the elements*

$$\gamma(b_\lambda), \qquad [a_i, b_\lambda], \tag{6.3}$$

for all values of λ, i.

Let $\Gamma_0 \subset g^{-1}(0)$ be the sub-group generated by the elements (6.3). I say that, if $a \in A$, $b \in f^{-1}(0)$, then

$$\gamma(a + b) - \gamma(a) \in \Gamma_0 . \tag{6.4}$$

For let $b = b_{\lambda_1} + \cdots + b_{\lambda_p}$, Then it follows from (5.7) that

$$\gamma(b) = \sum_\rho \gamma(b_{\lambda_\rho}) + \sum_{\rho<\sigma} [b_{\lambda_\rho} , b_{\lambda_\sigma}].$$

[16] Cf. Theorem 6 in [20].

Each b_{λ_ρ} is a sum of generators in the set $\{a_i\}$. Therefore $[b_{\lambda_\rho}, b_{\lambda_\sigma}]$ is a sum of elements of the form $[a_i, b_{\lambda_\sigma}]$. Therefore $\gamma(b) \in \Gamma_0$. Let $a = a_{i_1} + \cdots + a_{i_q}$. Then

$$[a, b] = \sum_r \sum_\sigma [a_{i_r}, b_{\lambda_\sigma}] \in \Gamma_0 .$$

Therefore

$$\gamma(a + b) - \gamma(a) = [a, b] + \gamma(b) \in \Gamma_0 ,$$

which proves (6.4).

Let $\Gamma^* = \Gamma(A)/\Gamma_0$ and let $\alpha^* \in \Gamma^*$ be the coset containing a given element $\alpha \in \Gamma(A)$. Since $\Gamma_0 \subset g^{-1}(0)$ it follows that g induces a homomorphism,

$$g^*: \Gamma^* \to \Gamma(A'),$$

which is given by

(6.5) $$g^*\gamma(a)^* = g\gamma(a) = \gamma(fa).$$

Then $g^{*-1}(0) = g^{-1}(0)/\Gamma_0$ and we have to prove that $g^{*-1}(0) = 0$.

Let $u(a') \in f^{-1}a' \subset A$ be a "representative" of a', for each $a' \in A'$. Then

$$f\{u(fa) - a\} = 0,$$

whence

$$u(fa) = a + b(a) \qquad (a \in A),$$

where $b(a) \in f^{-1}(0)$. Therefore it follows from (6.4) that

(6.6) $$\gamma\{u(fa)\}^* = \gamma(a)^*.$$

Similarly

(6.7) $$\begin{cases} \gamma\{u(-a')\}^* = \gamma\{-u(a')\}^* = \gamma\{u(a')\}^* \\ \gamma\{u(a_1' + \cdots + a_n')\}^* = \gamma\{u(a_1') + \cdots + u(a_n')\}^*. \end{cases}$$

Let $g': \{\gamma(a')\} \to \Gamma^*$ be the correspondence which is given by

(6.8) $$g'\gamma(a') = \gamma\{u(a')\}^*.$$

Then it follows from (6.7) that g' is consistent with the relations (5.1), for $\Gamma(A')$. Therefore it determines a homomorphism $g': \Gamma(A') \to \Gamma^*$. It follows from (6.5), (6.8) and (6.6) that

$$g'g^*\gamma(a)^* = g'\gamma(fa) = \gamma\{u(fa)\}^* = \gamma(a)^*.$$

Therefore $g'g^* = 1$, whence $g^{*-1}(0) = 0$ and the theorem is proved.

Let $A' = A/B$, where $B \subset A$ is generated by $\{b_\lambda\}$, and let $g: \Gamma(A) \to \Gamma(A')$ be induced by the natural homomorphism $A \to A'$. Let A be a free Abelian group, which is freely generated by $\{a_i\}$. Then A' is defined by $\{a_i\}$, treated as symbolic generators, and the relations $b_\lambda \equiv 0$, when b_λ is expressed as a sum of the generators a_i and their negatives. It follows from (A) in §5 and from Theorem 4 that $\Gamma^* = \Gamma(A)/g^{-1}(0)$ is similarly defined by the symbolic generators $\gamma(a_i)$, $[a_j, a_k](j < k)$ and the relations

$$(6.9) \qquad \gamma(b_\lambda) \equiv 0, \qquad [a_i, b_\lambda] \equiv 0,$$

when $\gamma(b_\lambda)$, $[a_i, b_\lambda]$ are expressed in terms of $\pm\gamma(a_i)$, $\pm[a_j, a_k]$. Let us identify each element $\alpha^* \epsilon \Gamma^*$ with $g^*\alpha^* \epsilon \Gamma(A')$, where $g^*:\Gamma^* \approx \Gamma(A')$ is given by (6.5) (obviously g^* is onto). Then Theorem 4 can be restated in the form:

THEOREM 5. *Let A' be defined by symbolic generators $\{a_i\}$ and relations $\{b_\lambda \equiv 0\}$. Then $\Gamma(A')$ is defined by the set of symbolic generators $\gamma(a_i)$, $[a_j, a_k](j < k)$ and the relations* (6.9).

Let A be generated by a_1, subject to the single relation $ma_1 \equiv 0$, where $m > 0$. Since $\gamma(ma_1) = m^2\gamma(a_1)$ and $[a_1, ma_1] = m[a_1, a_1] = 2m\gamma(a_1)$ it follows from Theorem 5 that $\Gamma(A)$ is generated by $\gamma(a_1)$, subject to the relations $m^2\gamma(a_1) \equiv 0$, $2m\gamma(a_1) \equiv 0$, which reduce to the single relation $(m^2, 2m)\gamma(a_1) \equiv 0$. This proves (B) in §5 since $(m^2, 2m) = m$ or $2m$ according as m is odd or even.

Theorem 4 is not necessarily true if f is into, but not onto A'. For example, let A' be cyclic of order m^2, where m is odd. Let a_1 be a generator of A' and let $A \subset A'$ be the sub-group generated by ma_1. Then A and likewise $\Gamma(A)$ are of order m. Also $\Gamma(A')$ is of order m^2. Let $g:\Gamma(A) \to \Gamma(A')$ be the homomorphism induced by the identical map $f:A \to A'$. Since

$$\gamma(ma_1) = m^2\gamma(a_1) = 0,$$

in consequence of the relations for $\Gamma(A')$, it follows that $g\Gamma(A) = 0$, though $f^{-1}(0) = 0$.

As another example let A' be the group of rationals mod. 1 and let $A \subset A'$ by the cyclic sub-group, which is generated by $1/m$ $(m > 1)$. Let $g:\Gamma(A) \to \Gamma(A')$ be induced by the identical map $A \to A'$. In this case $\Gamma(A') = 0$, according to (C) in §5. Therefore Theorem 4 may break down even if g, but not f, is onto.

THEOREM 6. *Let A' be such that*[17] $\Gamma(A'/A_0') \neq 0$, *for each proper sub-group $A_0' \subset A'$. Then a homomorphism, $f:A \to A'$, is onto A' if the induced homomorphism, $g:\Gamma(A) \to \Gamma(A')$, is onto $\Gamma(A')$.*

Let $g\Gamma(A) = \Gamma(A')$ and let $A_0' = fA$. Then

$$f = if_0:A \to A',$$

where $f_0:A \to A_0'$ is defined by $f_0a = fa(a \epsilon A)$ and $i:A_0' \to A'$ is the identical map. Therefore

$$g = jg_0:\Gamma(A) \to \Gamma(A'),$$

where $g_0:\Gamma(A) \to \Gamma(A_0')$, $j:\Gamma(A_0') \to \Gamma(A')$ are induced by f_0, i. Since g is onto so is j. Let $g':\Gamma(A') \to \Gamma(A'/A_0')$ be the homomorphism induced by the natural map $f':A' \to A'/A_0$. Then f' is onto and so therefore are g' and

$$g'j:\Gamma(A_0') \to \Gamma(A'/A_0').$$

[17] In §8 below we shall see that this condition is always satisfied if A' is finitely generated.

But $g'j$ is induced by $f'i: A_0' \to A'/A_0'$. Since $f'iA_0' = 0$ it follows, obviously, that $g'j\Gamma(A_0') = 0$. Therefore $\Gamma(A'/A_0') = 0$ and it follows from the condition on A' that $A_0' = A'$. This proves the theorem.

7. Relation to Tensor products

Let A be a weak direct sum,[18] $\Sigma_p A_p$, where $\{A_p\}$ is any set of additive Abelian groups. Let Γ be the weak direct sum

$$\Gamma = \Sigma_p \Gamma(A_p) + \sum_{q<r} A_q \circ A_r,$$

where $A_q \circ A_r$ is the tensor product of A_q and A_r. Since $[a_q, a_r] \in \Gamma(A)$ is bilinear in $a_q \in A_q$ and $a_r \in A_r$ it follows that a homomorphism, $f: \Gamma \to \Gamma(A)$, is defined by the correspondences

$$\gamma(a_p) \to \gamma'(a_p), \; a_q \cdot a_r \to [a_q, a_r], \tag{7.1}$$

where $\gamma(a_p) \in \Gamma(A_p)$ and $\gamma'(a) \in \Gamma(A)$ means the same as $\gamma(a)$ in §5.

THEOREM 7. $f: \Gamma \approx \Gamma(A)$.

Let A_p be defined by a set of symbolic generators, a_{pi}, and relations $b_{p\lambda} \equiv 0$. We assume that each a_{pi} is distinct from each a_{qj} if $p \neq q$. Then A is defined by the combined set of generators $\{a_{pi}\}$ and the combined set of relations $\{b_{p\lambda} \equiv 0\}$. Therefore it follows from Theorem 5 that $\Gamma(A)$ is defined by the union,

$$\{S_p', S_{qr}'\},$$

of all the generators and all the relations in the sets

$$S_p' : \begin{cases} \gamma'(a_{pi}), \quad [a_{pi}, a_{pj}] & (i < j) \\ \gamma'(b_{p\lambda}) \equiv 0, \quad [a_{pi}, b_{p\lambda}] \equiv 0 \end{cases}$$

$$S_{qr}' : \begin{cases} [a_{qj}, a_{rk}] & (q < r) \\ [a_{qj}, b_{r_\mu}] \equiv 0, \; [b_{q\mu}, a_{rk}] \equiv 0. \end{cases}$$

But $\Gamma(A_p)$ is defined by S_p and $A_q \circ A_r$ by[19] S_{qr}, where S_p, S_{qr} are obtained from S_p', S_{qr}' by writing γ instead of γ' in S_p' and $x \cdot y$ instead of $[x, y]$ throughout S_{qr}' ($x = a_{qj}$ or $b_{q\mu}$, $y = a_{rk}$ or b_{r_μ}). Therefore Γ is defined by the combined system $\{S_p, S_{qr}\}$ and (7.1) transforms this system into $\{S_p', S_{qr}'\}$. This proves the theorem.

8. "A" finitely generated

Let A have a finite number of generators. Then it is a direct sum

$$A = X_1 + \cdots + X_t + Y_1 + \cdots + Y_r, \tag{8.1}$$

where X_λ is of finite order, σ_λ, and Y_μ is cyclic infinite. Moreover we may take

[18] An element in A is a set of elements $\{a_p\}$, with $a_p \in A_p$, almost all of which are zero. If $a_p = 0$ except when $p = p_1, \cdots, p_n$ we write $\{a_p\} = a_{p_1} + \cdots + a_{p_n}$.

[19] This follows from two successive applications of Theorem 6 in [20].

$\sigma_1, \cdots, \sigma_t$ to be the invariant factors of A, so that $\sigma_1 > 1$, $\sigma_\lambda \mid \sigma_{\lambda+1}$. Let this be so and let $\rho_1, \cdots, \rho_p$ be those among $\sigma_1, \cdots, \sigma_t$ which are distinct. That is to say

$$\sigma_{k_\lambda+1} = \cdots = \sigma_{k_{\lambda+1}} = \rho_\lambda \neq \rho_{\lambda+1} \qquad (k_1 = 0, k_{p+1} = t),$$

for $\lambda = 1, \cdots, p$. Let $n_\lambda = k_{\lambda+1} - k_\lambda$. Then we denote $(\sigma_1, \cdots, \sigma_t)$ by

$$(\rho_1, n_1), \cdots, (\rho_p, n_p). \tag{8.2}$$

Let $\{A_p\} = \{X_\lambda, Y_\mu\}$, in §7, and let Γ be identified with $\Gamma(A)$ by means of the isomorphism f in Theorem 7. Then it is clear that the rank of $\Gamma(A)$ is

$$r(r + 1)/2.$$

Let ρ_λ be odd. Then it also follows that ρ_λ occurs s_λ times in $\Gamma(A)$, where s_λ is calculated as follows Let

$$N_\mu = n_\mu + \cdots + n_p + r \qquad (N_{p+1} = r)$$

$$M_\lambda = n_\lambda(n_\lambda + 2N_{\lambda+1} + 1)/2.$$

Then ρ_λ occurs $n_\lambda(n_\lambda + 1)/2$ times in the summand

$$\sum_i \Gamma(X_i) + \sum_{i<j} X_i \circ X_j \qquad (k_\lambda < i, j \leqq k_{\lambda+1})$$

and $n_\lambda N_{\lambda+1}$ times in

$$\sum_i \sum_\mu X_i \circ X_\mu + \sum_i \sum_\alpha X_i \circ Y_\alpha,$$

where $\mu = k_{\lambda+1} + 1, \cdots, t$, $\alpha = 1, \cdots, r$. Therefore $s_\lambda = M_\lambda$.

In general let ρ_{h-1} be odd and ρ_h even, where $1 \leqq h \leqq p + 1$ and $h = 1$, $h = p + 1$ have the obvious meanings. If $\sigma_i = \rho_\lambda$ and $\lambda \geqq h$ then $\Gamma(X_i)$ is of order $2\rho_\lambda$. Hence it follows that ρ_λ occurs $M_\lambda - n_\lambda$ times in $\Gamma(A)$ if $\lambda = h$ or if $\lambda > h$ and $\rho_\lambda > 2\rho_{\lambda-1}$. In the latter case $2\rho_{\lambda-1}$ occurs $n_{\lambda-1}$ times. If $\rho_\lambda = 2\rho_{\lambda-1}$ and $\lambda > h$ then ρ_λ occurs $M_\lambda - n_\lambda + n_{\lambda-1}$ times. Also $2\rho_p$ occurs n_p times if $h \leqq p$. Therefore the invariant factors of $\Gamma(A)$, written in the form (8.2), are

$$(\rho_1, s_1), \cdots, (\rho_p, s_p), \tag{8.3}$$

together with

$$(2\rho_\mu, n_\mu) \tag{8.4}$$

for every μ such that $h \leqq \mu \leqq p$ and either $2\rho_\mu < \rho_{\mu+1}$ or $\mu = p$, where

$$\begin{array}{ll} \text{(a)} & \left\{\begin{array}{ll} s_l = M_l & \text{if } 1 \leqq l < h \\ s_h = M_h - n_h & \\ s_\lambda = M_\lambda - n_\lambda & \text{if } \lambda > h,\ \rho_\lambda > 2\rho_{\lambda-1} \\ s_\lambda = M_\lambda - n_\lambda + n_{\lambda-1} & \text{if } \lambda > h,\ \rho_\lambda = 2\rho_{\lambda-1}. \end{array}\right. \\ \text{(b)} & \\ \text{(c)} & \\ \text{(d)} & \end{array} \tag{8.5}$$

Notice that $\Gamma(A)$ cannot be an arbitrary group. For example its rank must be a binomial coefficient or zero. Suppose however that a given group, Γ, is known to be of the form $\Gamma(A)$, where A is finitely generated. Suppose further that the rank and invariant factors of Γ are known.

THEOREM 8. *The rank and invariant factors of A are uniquely determined by those of* Γ.

Let r' be the rank of Γ. Then the rank, r, of A is the (unique) non-negative solution of the quadratic equation

$$x^2 + x - 2r' = 0.$$

Let the invariant factors of Γ, written in the form (8.2) be

$$(\rho_1', s_1'), \cdots, (\rho_q', s_q').$$

We proceed to determine the sequence (8.2) for A. If ρ_q' is odd it follows from (8.3) that $\rho_p = \rho_q'$. Since

$$M_p = n_p(n_p + 2r + 1)/2$$

it follows from (8.5a) that n_p is the non-negative root of the quadratic

$$x^2 + (2r + 1)x - 2s_q' = 0.$$

If ρ_q' is even then $\rho_p = \rho_q'/2$, $n_p = s_q'$, according to (8.4), with $\mu = p$.

Assume that $(\rho_\lambda, n_\lambda) \cdots, (\rho_p, n_p)$ have been uniquely determined, where[20] $\lambda \leqq p$, and let $\rho_\lambda = \rho_j'$. If $j = 1$ the sequence (8.2) is determined and we number (ρ_μ, n_μ) so that $\lambda = 1$. If either $j = 2$ or if $j > 2$ and ρ_{j-2}' is odd it follows from (8.3), (8.4) and (8.5a, b) that $\rho_{\lambda-1} = \rho_{j-1}'$ and that $n_{\lambda-1}$ is the non-negative root of

$$x^2 + ax - 2s_{j-1}' = 0,$$

where $a = 2N_\lambda \pm 1$ according as ρ_{j-1}' is odd or even.

If $j > 2$ and ρ_{j-2}' is even we consider the cases

$$\text{a) } s_j' = M_\lambda - n_\lambda$$

$$\text{b) } s_j' \neq M_\lambda - n_\lambda .$$

In case (a) it follows from (8.4) and (8.5c) that $\rho_{\lambda-1} = \rho_{j-1}'/2$, $n_{\lambda-1} = s_{j-1}'$. In case (b) it follows from (8.5d) that

$$\rho_{\lambda-1} = \rho_j'/2, \qquad n_{\lambda-1} = s_j' + n_\lambda - M_\lambda .$$

Therefore $\rho_{\lambda-1}$, $n_{\lambda-1}$ are uniquely determined in each case and the theorem follows by induction on j.

Let $g:\Gamma(A) \to \Gamma(A')$ be the homomorphism induced by a homomorphism $f:A \to A'$. If A' is finitely generated so is A'/A_0', where $A_0' \subset A'$ is any subgroup. Therefore $\Gamma(A'/A_0') \neq 0$ unless $A_0' = A'$. Therefore it follows from

[20] The value of p is not determined till the induction is complete. Therefore we allow $\lambda \leqq 0$.

Theorem 6 that, if A' is finitely generated and $g\Gamma(A) = \Gamma(A')$, then $fA = A'$. We prove a kind of dual of this.

THEOREM 9. *If A is finitely generated, A' being arbitrary, then $g^{-1}(0) = 0$ implies $f^{-1}(0) = 0$.*

Since $g[a_0, a] = 0$ for any $a \epsilon A$, $a_0 \epsilon f^{-1}(0)$ this follows from:

THEOREM 10. *If A is finitely generated and $[a_0, a] = 0$ for every $a \epsilon A$, then $a_0 = 0$.*

Let A be given by (8.1) and let x_i, y_λ be generators of X_i, Y_λ. Let $[a_0, a] = 0$ for every $a \epsilon A$, where

$$a_0 = k_1x_1 + \cdots + k_tx_t + l_1y_1 + \cdots + l_ry_r .$$

First assume that $r > 0$. Then

$$[a_0, y_r] = \sum_{i=1}^{t} k_i[x_i, y_r] + \sum_{\lambda=1}^{r} l_\lambda[y_\lambda, y_r] = 0.$$

But $[x_i, y_r]$ generates the cyclic summand, $X_i \circ Y_r$, of $\Gamma(A)$, whose order is σ_i. Also $[y_\lambda, y_r]$ is a non-zero element in the free cyclic group $Y_\lambda \circ Y_r$ or $\Gamma(Y_r)$, according as $\lambda < r$ or $\lambda = r$. Therefore $k_i \equiv 0(\sigma_i)$, $l_\lambda = 0$, whence $a_0 = 0$.

Let $r = 0$. Then a similar argument, with y_r replaced by x_t, shows that $k_ix_i = 0$ if $i < t$ and that

$$k_t[x_t, x_t] = 2k_t\gamma(x_t) = 0.$$

Therefore $\sigma \mid 2k_t$, where σ is the order of $\gamma(x_t)$. But $\sigma = \sigma_t$ or $2\sigma_t$ according as σ_t is odd or even. In either case $\sigma_t \mid k_t$. Therefore $a_0 = 0$ and the theorem follows.

As a corollary of this and Theorem 8 we have:

THEOREM 11. *Let both A and A' be finitely generated. Then $f:A \approx A'$ if, and only if, $g:\Gamma(A) \approx \Gamma(A')$, where g is induced by f.*

Notice that, in consequence of Theorem 10, a finitely generated group, A, is orthogonal to itself by the pairing $(A, A) \to \Gamma(A)$, in which $(a, b) = [a, b]$.

9. Direct systems

Let $\mathfrak{A}$ be the category of all (additive) Abelian groups, with all homomorphisms as mappings. Then a functor[21] $\Gamma:\mathfrak{A} \to \mathfrak{A}$ is obviously defined by the correspondences $A \to \Gamma(A)$, $f \to \Gamma f$, where $\Gamma f:\Gamma(A) \to \Gamma(A')$ is the homomorphism induced by $f:A \to A'$. Let $\mathfrak{D} = \mathfrak{D}\mathrm{ir}$ be the category of direct systems of Abelian groups, defined as in [10], except that the groups are to be Abelian. Let

$$\Gamma_l:\mathfrak{D} \to \mathfrak{D}, \qquad L = \mathrm{Lim}_{\to}\mathfrak{D} \to \mathfrak{A}$$

be the functor defined by lifting Γ([10], §24) and the direct limit functor. Let (D, T) be a given system in $\mathfrak{D}$, with groups $T(d)$ $(d \epsilon D)$ and projections $T(d_2, d_1)$

[21] See [10]. There are obvious generalizations of Theorem 12 below to non-Abelian groups. The partial ordering in D, below, is not to be confused with the simple ordering of indices, which was introduced in §5 and which is not needed here.

$(d_1 < d_2)$. Then $\Gamma_l(D, T)$ consists of the groups $\Gamma T(d)$ and the projections $\Gamma T(d_2, d_1)$. Let

$$\lambda(d): T(d) \to L(D, T), \mu(d): \Gamma T(d) \to L\Gamma_l(D, T)$$

be the injections in (D, T) and $\Gamma_l(D, T)$. Then a homomorphism

$$\omega(D, T): L\Gamma_l(D, T) \to \Gamma L(D, T)$$

is given by

$$\omega(D, T)\{\mu(d)\gamma(t_d)\} = \gamma\{\lambda(d)t_d\}, \tag{9.1}$$

where $t_d \in T(d)$. We recall from [10] that the transformation $\omega: L\Gamma_l \to \Gamma L$, which is thus defined, is natural and that Γ is said to *commute with* L if, and only if, ω is an equivalence, meaning that $\omega(D, T)$ is an isomorphism for each system (D, T).

THEOREM 12. *The functor* Γ *commutes with* L.

Using the same notation as before we have

$$\begin{aligned}\mu(d_2)\gamma\{T(d_2, d_1)t_{d_1}\} &= \mu(d_2)\{\Gamma T(d_2, d_1)\}\gamma(t_{d_1})\\ &= \mu(d_1)\gamma(t_{d_1}).\end{aligned}$$

Therefore a single-valued map, ϕ, of the generators, $\gamma\{\lambda(d)t_d\} \in \Gamma L(D, T)$, into $L\Gamma_l(D, T)$ is defined by

$$\phi\gamma\{\lambda(d)t_d\} = \mu(d)\gamma(t_d). \tag{9.2}$$

Let $a, b, c \in L(D, T)$. Then there is a $d \in D$ such that a, b, c have representatives $r, s, t \in T(d)$. Therefore

$$\phi\gamma(-a) = \phi\gamma\{\lambda(d)(-r)\} = \mu(d)\gamma(-r) = \mu(d)\gamma(r)$$

$$\phi\gamma(a + b + c) = \phi\gamma\{\lambda(d)(r + s + t)\} = \mu(d)\gamma(r + s + t).$$

Similarly $\phi\gamma(b + c) = \mu(d)\gamma(s + t)$ etc. and it follows that ϕ is consistent with the relations (5.1). Therefore it determines a homomorphism

$$\phi: \Gamma L(D, T) \to L\Gamma_l(D, T).$$

It follows from (9.1) and (9.2) that $\phi\omega(D, T) = 1$, $\omega(D, T)\phi = 1$, which proves the theorem.

CHAPTER III. THE SEQUENCE $\Sigma(K)$

10. Definition of $\Sigma(K)$

In this chapter a *complex* will mean a pair (K, e^0), where K is a connected CW-complex and $e^0 \in K^0$ is a 0-cell, which is to be taken as base point for all the homotopy groups which we associate with K. Nevertheless we shall denote complexes by K, K' etc., remembering that, if K_i stands for (K, e_i^0) $(i = 1, 2)$ and $e_1^0 \neq e_2^0$, then $K_1 \neq K_2$. A *cellular map*, $\phi: K \to K'$, will mean one which, in addition to $\phi K^n \subset K'^n$ for every $n \geqq 0$, satisfies the condition $\phi e^0 = e'^0$, where e^0, e'^0 are the base points of K, K'.

Let K be a given complex, let $\rho_2 = \pi_2(K^2, K^1)$,

$$C_{n+1} = \pi_{n+1}(K^{n+1}, K^n), A_n = \pi_n(K^n) \qquad (n \geqq 2)$$

and let $\beta: C_{n+1} \to A_n$, $j: A_n \to C_n$ be the boundary and injection operators, where $C_2 = jA_2 \subset \rho_2$. Let $\beta: \rho_2 \to \pi_1(K^1)$ be the boundary homomorphism. Then $\beta C_2 = 1$ and, as proved in CH II, $\rho_2 = C_2 + B^*$, where [22] B^* is the image of $\beta\rho_2$ in an isomorphism, $\beta^*: \beta\rho_2 \approx B^*$, such that $\beta\beta^* = 1$. We can imbed C_2 isomorphically in ρ_2 made Abelian, which is a free $\pi_1(K)$-module. Also C_n is a free $\pi_1(K)$-module if $n > 2$. Therefore, ignoring the operators in $\pi_1(K)$, $C_n (n \geqq 2)$ is a free Abelian group. Also $j_n A_n = \beta_n^{-1}(0)$. Therefore the family of groups C_{n+1}, A_n, related by β, j with $\beta C_2 = 0$, is a composite chain system $(C, A) = (C, A)(K)$, as defined in §4 above. We define $\Sigma(K) = \Sigma(C, A)$.

Let $\Gamma_n = \Gamma_n(K)$ etc. be the groups in $\Sigma = \Sigma(K)$. It follows from CH II that there are natural isomorphisms $\pi_n \approx \pi_n(K)$, $H_n \approx H_n(\tilde{K})$ $(n \geqq 2)$, where $H_n(\tilde{K})$ is the n^{th} integral homology group of the universal covering complex, $\tilde{K}$, of K. The homomorphisms $\mathfrak{i}_n$, $\mathfrak{j}_n$ in Σ are equivalent under these isomorphisms to $i'_n \mid \Gamma_n$ where $i'_n: A_n \to \pi_n(K)$ is the injection, and to the resultant of the lifting isomorphism $\pi_n(K) \approx \pi_n(\tilde{K})$, followed by the natural homomorphism $\pi_n(\tilde{K}) \to H_n(\tilde{K})$. Also $\Gamma_n = i_n \pi_n(K^{n-1})$, where $i_n: \pi_n(K^{n-1}) \to A_n$ is the injection. Therefore $\Gamma_2 = 0$ and Σ terminates with

$$\cdots \to \Pi_3 \to H_3 \to 0 \to \Pi_2 \to H_2 \to 0.$$

An element $w \in \pi_1(K)$, operating in the usual way[23] on C_{n+1}, A_n, obviously determines an automorphism, $w: (C, A) \approx (C, A)$, which induces an automorphism, $w: \Sigma \approx \Sigma$. More generally, let Σ be any algebraic sequence, of the kind considered in §4. Then the totality of automorphisms of Σ is obviously a group $G(\Sigma)$. Let $\lambda: W \to G(\Sigma)$ be a homomorphism of a given (multiplicative) group W into $G(\Sigma)$. Let Σ', W', λ' be similarly defined. Then a *homomorphism*

$$(10.1) \qquad (F, \mathfrak{w}): (\Sigma, W, \lambda) \to (\Sigma', W', \lambda')$$

will mean a pair of homomorphisms, $F: \Sigma \to \Sigma'$, $\mathfrak{w}: W \to W'$, such that $F\lambda(w) = \lambda'\{\mathfrak{w}(w)\} F$ for each $w \in W$. Since

$$\lambda(w_0)\lambda(w) = \lambda(w_0 w) = \lambda(w_0 w w_0^{-1})\lambda(w_0)$$

it follows that $(\lambda(w_0), [w_0])$ is a homomorphism, where $[w_0]$ is the inner automorphism, $w \to w_0 w w_0^{-1}$, of W. We shall use $\lambda(K): \pi_1(K) \to G(\Sigma)$ to denote the homomorphism which describes how $\pi_1(K)$ operates on $\Sigma = \Sigma(K)$.

We shall say that homomorphisms $(F, \mathfrak{w})$, $(F^*, \mathfrak{w}^*)$, of the form (10.1), are in

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[22] In defining $\Sigma(K)$ we ignore B^* and hence lose sight of the invariant $\mathbf{k}^3(K)$ (Cf. [11], [13], [9]).

[23] I.e. through the inverses of the injection isomorphisms $\pi_1(K^n) \approx \pi_1(K)$ $(n \geqq 2)$.

the same *operator class*, $\{F, \mathfrak{w}\}$, if, and only if,

$$(F, \mathfrak{w}) = (\lambda'(w_0'), [w_0'])(F^*, \mathfrak{w}^*) = (\lambda'(w_0')F^*, [w_0']\mathfrak{w}^*),$$

for some $w_0' \in W'$. Let

$$(F', \mathfrak{w}'):(\Sigma', W', \lambda') \to (\Sigma'', W'', \lambda'') \tag{10.2}$$

be a homomorphism. Then, writing $\mathfrak{w}'(w_0') = w_0''$, we have

$$F'\lambda'(w_0') = \lambda''(w_0'')F', \qquad \mathfrak{w}'[w_0'] = [w_0'']\mathfrak{w}'.$$

Hence it follows that a single-valued product of operator classes is defined by

$$\{F', \mathfrak{w}'\} \{F, \mathfrak{w}\} = \{F'F, \mathfrak{w}'\mathfrak{w}\},$$

for all pairs of homomorphisms of the form (10.1), (10.2). It may be verified that all triples (Σ, W, λ) (the objects), together with all operator classes of homomorphisms (the mappings), constitute a category, $\mathfrak{S}^w$.

The usefulness of $\Sigma(K)$ is limited by our ignorance concerning $\pi_n(K)$ for large values of n. Therefore we shall often want to confine ourselves to a finite part of Σ. It will be convenient to start with H_q, and Σ_q will denote the part

$$\Sigma_q: \qquad H_q \to \Gamma_{q-1} \to \Pi_{q-1} \to \cdots$$

of Σ. We write $\Sigma_\infty = \Sigma$, thus defining Σ_q for $q \leqq \infty$. A homomorphism or isomorphism

$$(\mathfrak{h}, \mathfrak{g}, \mathfrak{f}):\Sigma_q \to \Sigma_q' \qquad (q < \infty)$$

will mean the same as when $q = \infty$, except that $\mathfrak{h}_{n+1}$, $\mathfrak{g}_n$, $\mathfrak{f}_n$ will only be defined for $n = 1, \cdots, q-1$.

11. The invariance of $\Sigma(K)$

Let $K = (K, e^0)$, $K' = (K', e'^0)$ be given complexes. It follows from proposition (J) in §5 of CH I, on homotopy extension, that any map $K \to K'$ is homotopic to one in which $e^0 \to e'^0$. Hence it follows from (L) in §5 of CH II that

(11.1)

a) *any map, $K \to K'$, is homotopic to a cellular map.*

b) *if $\phi_0 \simeq \phi_1:K \to K'$, where ϕ_0, ϕ_1 are cellular, then ϕ_0, ϕ_1 are related by a cellular homotopy, $\phi_t:K \to K'$* (i.e. $\phi_t K^n \subset K'^{n+1}$).

Therefore, in discussing the invariance of $\Sigma(K)$, we may confine ourselves to cellular maps and homotopies.

A cellular map, $\phi:K \to K'$, induces a family of homomorphisms[24]

$$h_{r+1}:\rho_{r+1}(K) \to \rho_{r+1}(K'), f_r:\pi_r(K^r) \to \pi_r(K'^r) \qquad (r \geqq 1), \tag{11.2}$$

such that $\beta h = f\beta$, $jf = hj$. Since $h_2 j_2 = j_2 f_2$ we have $h_2 C_2 \subset C_2' = C_2(K')$. The induced homomorphism $h_2:C_2 \to C_2'$ together with h_{n+1}, f_n for $n = 2, 3, \cdots$, obviously constitute a homomorphism

$$(h, f):(C, A) \to (C', A') = (C, A)(K').$$

[24] See [17] in CH II. $\rho_n(L) = \pi_n(L^n, L^{n-1})$ $(n \geqq 2)$, $\rho_1(L) = \pi_1(L^1)$.

This induces a homomorphism $F:\Sigma(K) \rightarrow \Sigma(K')$. Let $\mathfrak{w}:\pi_1(K) \rightarrow \pi_1(K')$ be the homomorphism, which is induced by ϕ and is given by $\mathfrak{w}\iota_1 = \iota_1 f_1$, where $\iota_1:\pi_1(L^1) \rightarrow \pi_1(L)$ is the injection ($L = K$ or K'). Then $h_{n+1}, f_n, h_2:C_2 \rightarrow C_2'$ are operator homomorphisms when $\pi_1(K)$ operates on (C', A') through $\mathfrak{w}$. Hence it follows that $(F, \mathfrak{w})$ is a homomorphism of the form (10.1) where $\Sigma = \Sigma(K)$, $W = \pi_1(K)$, $\lambda = \lambda(K)$, $\Sigma' = \Sigma(K')$, etc. We describe $(F, \mathfrak{w})$ as the homomorphism *induced* by ϕ.

Let $\phi \simeq \phi^*:K \rightarrow K'$ and let (h, f), (h^*, f^*) be the families of homomorphisms of the form (11.2), which are induced by ϕ, ϕ^*. Then[24]

$$x_0'h^* - h = d\xi + \xi d, \qquad x_0'f^* - f = \beta\xi j, \tag{11.3}$$

where $x_0' \in \pi_1(K'^1)$, which operates on C_{n+1}', A_n' through the injection $\pi_1(K'^1) \rightarrow \pi_1(K'^n)$, and $\xi:\rho(K) \rightarrow \rho(K')$ is a deformation operator as defined in §4 of CH II. The homomorphisms $\xi_3 \mid C_2$, ξ_4, ξ_5, $\cdots$ constitute a deformation operator in the sense of §3 above. Also $dC_2 = 0$ and $d\xi_3 C_2 \subset C_2'$, since $dC_3' \subset C_2'$. Therefore

$$(h, f) \simeq (w_0'h^*, w_0'f^*):(C, A) \rightarrow (C', A'),$$

in the sense of §3, where w_0' is the image of x_0' in the injection $\pi_1(K'^1) \rightarrow W'$. Hence it follows that $F = \lambda'(w_0')F^*$, where $F, F^*:\Sigma \rightarrow \Sigma'$ are induced by (h, f), (h^*, f^*). Moreover $\mathfrak{w} = [w_0']\,\mathfrak{w}^*$ in consequence of the relation

$$x_0'(f_1^*x)x_0'^{-1}(f_1x)^{-1} = \beta_2\xi_2 x,$$

which is included, additively, in (11.3). Therefore $(F, \mathfrak{w})$ and $(F^*, \mathfrak{w}^*)$ are in the same operator class, where $(F, \mathfrak{w})$, $(F^*, \mathfrak{w}^*)$ are induced by ϕ, ϕ^*. Therefore a homotopy class, $\alpha:K \rightarrow K'$, of maps induces a unique operator class, $\Sigma\alpha = \{F, \mathfrak{w}\}$, of homomorphisms.

Let $\mathfrak{K}$ be the homotopy category of all complexes (i.e. connected, CW-complexes with base points). Then it may be verified that the correspondences

$$K \rightarrow (\Sigma(K), \pi_1(K), \lambda(K)), \qquad \alpha \rightarrow \Sigma\alpha$$

determine a functor $\Sigma:\mathfrak{K} \rightarrow \mathfrak{S}^w$. We express this by saying that $(\Sigma(K), \pi_1(K), \lambda(K))$, or simply that $\Sigma(K)$ is a homotopy invariant of K.

Similarly $\Sigma_q(K)$ is a homotopy invariant of K, for any $q < \infty$. Also, for a particular value of n, the secondary modular boundary homomorphism

$$\mathfrak{b}_n(m):H_n(K, m) \rightarrow \Gamma_{n-1,m}(K),$$

is a homotopy invariant within the category of complexes such that every $(n-1)$-cycle in $\tilde{K}$ is spherical. Notice that $\mathfrak{b}_4(m)$ is defined for every complex.

The definition of $\Sigma(K)$ can be generalized as follows. Let $r \geqq 0$, let $A_n = C_n = 0$ if $n \leqq r + 1$ and if $n > r + 1$ let

$$C_{n+1} = \pi_{n+1}(K^{n-r+1}, K^{n-r}), \qquad A_n = \pi_n(K^{n-r}).$$

Let $\beta:C_{n+2} \rightarrow A_n$, $j:A_n \rightarrow C_n(C_{r+2} = jA_{r+2})$ be the boundary and injection operators. Then the groups C_{n+1}, A_n, related by β, j, constitute a system, (C, A), of the sort introduced in §1. We define $\Sigma^r(K) = \Sigma(C, A)$. Then it may be verified, in consequence of §3 and (11.1), that $\Sigma^r(K)$ is a homotopy invariant of K.

The groups which appear in $\Sigma^r(K)$ are naturally isomorphic to groups which belong to a larger class of "injected" invariants. Let $0 \leqq p \leqq q < r$, with $q > p$ if $p > 0$, let $m < n$ and let

$$\pi_r(K^n, K^q; m, p) = i\pi_r(K^m, K^p),$$

where $i:\pi_r(K^m, K^p) \to \pi_r(K^n, K^q)$ is the injection and $\pi_r(K^s, K^0) = \pi_r(K^s)$ ($s = m$ or n). Then it follows from (11.1)[25] that $\pi_r(K^n, K^q; m, p)$ is a homotopy invariant, and indeed an invariant of the n-type of K. In Chapter V below we shall see how these invariants may be defined for any arcwise connected space.

12. The sufficiency of $\Sigma(K)$

Let K, K' be given complexes, whose dimensionalities do not exceed q, where $q \leqq \infty$.

THEOREM 13. *If a map $\phi:K \to K'$ induces isomorphisms $\Sigma_q(K) \approx \Sigma_q(K')$ and $\pi_1(K) \approx \pi_1(K')$, then*[26] $\phi:K \equiv K'$.

Since the homomorphisms $H_n \approx H_n(\tilde{K})$ are natural this follows from Theorem 3 in Chapter I.

This is what we call a sufficiency[27] theorem. We shall prove the corresponding realizability theorem, subject to the restrictions $q = 4$ and $\pi_1(K) = 1$. But first we must prove a theorem concerning Γ_3.

13. Expression for $\Gamma_3(K)$

Let $u:S^3 \to S^2$ be a fixed map, which represents a generator of $\pi_3(S^2)$. Let $v:S^2 \to K^2$ be a map which represents a given element[28] $x \in \Pi_2$. Then $vu:S^3 \to K^2$ represents an element $u(x) \in \Gamma_3$. We have [29]

$$u(x + y) - u(x) - u(y) = [x, y]^*, \tag{13.1}$$

where $[x, y]^*$ is the product, or commutator (cf. [22]), of $x, y \in \Pi_2$. Also $u(-x) = u(x)$ and $[x, y]^*$ is bilinear in x, y. Therefore the map $\gamma(x) \to u(x)$ is consistent with the relations (5.1a) and (5.6), for $\Gamma(\Pi_2)$. Therefore it determines a homomorphism, $\theta:\Gamma(\Pi_2) \to \Gamma_3$, and $\theta[x, y] = [x, y]^*$. Obviously[30] θ is an operator homomorphism with respect to the operators in $\pi_1(K)$.

[25] Cf. p. 220 in CH I.

[26] $\phi:K \equiv K'$ means that ϕ is a homotopy equivalence.

[27] Cf. §14 below and §5 in [9].

[28] $\Pi_2 = \Pi_2(K)$, $\Gamma_3 = \Gamma_3(K)$ and, in the following paragraph, $\Pi_2' = \Pi_2(K)'$, $\Gamma_3' = \Gamma_3(K')$, where $\Gamma_n(L)$ etc. are the groups in $\Sigma(L)$.

[29] See (7.6) in [21]. Alternatively let $P = S_1^2 \cup S_2^2$ be formed from S^2 by pinching an equator into a point. Let $\phi: S^2 \to P$ be the identification map and $\psi:P \to K$ a map such that $\psi \mid S_1^2$, $\psi \mid S_2^2$ represent x, y, when S_1^2, S_2^2 take their orientations from S^2. Let $v = \psi\phi$. Then (13.1) follows from (11.5) in [3]. Similarly $u(-x) = u(x)$.

[30] This is obvious if the operators are defined by means of homotopies $\phi_t:S^n \to K$, in which $\phi_t p_o$ varies, where $p_o \in S^n$ is the base point. If the operators are defined by means of the covering transformations, $\tilde{K} \to \tilde{K}$, it may be deduced from (13.2) below.

Let $(\mathfrak{h}, \mathfrak{g}, \mathfrak{f}):\Sigma(K) \to \Sigma(K')$ be the homomorphism which is induced by a cellular map, $\phi:K \to K'$, into a complex K'. Since $\phi(vu) = (\phi v)u$ we have $\mathfrak{g}_3 u(x) = u(\mathfrak{f}_2 x)$. Let $\mathfrak{g}:\Gamma(\Pi_2) \to \Gamma(\Pi_2')$ be the homomorphism induced by $\mathfrak{f}_2:\Pi_2 \to \Pi_2'$ and let θ mean the same in K' as in K. Since $\theta\gamma(x) = u(x)$, $\mathfrak{g}_3 u(x) = u(\mathfrak{f}_2 x)$, $\gamma(\mathfrak{f}_2 x) = \mathfrak{g}\gamma(x)$ we have

$$\mathfrak{g}_3\theta\gamma(x) = \mathfrak{g}_3 u(x) = u(\mathfrak{f}_2 x) = \theta\gamma(\mathfrak{f}_2 x) = \theta\mathfrak{g}\gamma(x).$$

Therefore θ is natural, in the sense that

$$\mathfrak{g}_3\theta = \theta\mathfrak{g}:\Gamma(\Pi_2) \to \Gamma_3' . \tag{13.2}$$

THEOREM 14. $\theta:\Gamma(\Pi_2) \approx \Gamma_3$.

Let $\tilde{K}$ be the universal covering complex of K and let $\mathbf{p}:\tilde{K} \to K$ be the covering map. Then it follows from the standard lifting theorems that

$$\mathfrak{f}_2:\Pi_2(\tilde{K}) \approx \Pi_2 , \qquad \mathfrak{g}_3:\Gamma_3(\tilde{K}) \approx \Gamma_3 ,$$

where $\mathfrak{f}_2$, $\mathfrak{g}_3$ are induced by $\mathbf{p}$. Therefore $\mathfrak{g}:\Gamma\{\Pi_2(\tilde{K})\} \approx \Gamma\{\Pi_2)$, where $\mathfrak{g}$ is induced by $\mathfrak{f}_2$, and the theorem follows from (13.2) if it is true when K is replaced by $\tilde{K}$. therefore we may assume that $\pi_1(K) = 1$.

Let $\pi_1(K) = 1$ and let $\{a_i\}$ be a set of free generators of A_2 , which is free Abelian since $j:A_2 \approx C_2$. Let $\{e_\lambda^3\}$ be the 3-cells in K and let $c_\lambda \epsilon C_3$ be the element which is represented by a characteristic map[31] for e_λ^3 . Then $\{c_\lambda\}$ is a set of free generators of C_3 and Π_2 is defined by the generators a_i and the relations $b_\lambda \equiv 0$, where $b_\lambda = \beta c_\lambda$. By Theorem 5, $\Gamma(\Pi_2)$ is defined by the generators $\gamma(a_i)$, $[a_j , a_k]$ $(j < k)$ and the relations

$$\gamma(b_\lambda) \equiv 0, \qquad [a_i , b_\lambda] \equiv 0. \tag{13.3}$$

Let $K_0^2 = e^0 \cup \{e_i^2\}$, where $\{e_i^2\}$ is a set of 2-cells in a (1-1) correspondence, $e_i^2 \to a_i$, with $\{a_i\}$. Thus $K_0^1 = e^0$ and $\bar{e}_i^2 = e^0 \cup e_i^2$ is a 2-sphere. Moreover $\pi_2(K_0^2)$ is freely generated by the set of elements $\{a_i^0\}$, where a_i^0 is represented by a homeomorphism $\phi_i:S^2 \to \bar{e}_i^2$. Let $\psi:K_0^2 \to K^2$ be a map such that $(\psi \mid \bar{e}_i^2)\phi_i$ represents a_i and let $\psi_2:\pi_2(K_0^1) \to A_2$ be the homomorphism induced by ψ. Then $\psi_2 a_i^0 = a_i$. Therefore $\psi_2:\Pi_2(K_0^2) \approx A_2$ and it follows from Theorem 1 in CHI that $\psi_2:K_0^2 \equiv K^2$. By (D) in §5 of CH I any compact subset of K_0^2 is contained in a finite sub-complex. Therefore it follows from arguments similar to those used in the proof of Theorem 2 in [4] that $\pi_3(K_0^2)$ is freely generated by $u(a_i^0)$, $[a_j^0 , a_k^0]^*$ $(j < k)$. Therefore $\pi_3(K^2)$ is freely generated by

$$u(a_i), [a_j , a_k]^* \qquad (j < k). \tag{13.4}$$

Notice that $\theta\gamma(a_i) = u(a_i)$, $\theta[a_j , a_k] = [a_j , a_k]^*$.

Since any compact subset of K^3 is contained in a finite sub-complex it follows from the proof of Lemma 4 on p. 418 of [4] that Γ_3 is defined by the generators (13.4) and the relations $u(b_\lambda) \equiv 0$, $[a_i , b_\lambda]^* \equiv 0$. It follows from (13.1) and (5.4) and the bilinearity of $[a, b]$, $[a, b]^*$ that these relations, when expressed in terms of

[31] I.e. a map, $\phi:I^3 \to \bar{e}_\lambda^3$, such that $\phi\dot{I}^3 \subset K^2$ and $\phi \mid I^3 - \dot{I}^3$ is a homeomorphism onto e_λ^3 .

the generators (13.4), are the images under θ of the relations (13.3). Therefore $\theta:\Gamma(\Pi_2) \approx \Gamma_3$ and the theorem is proved.

14. Geometrical realizability

Let $q < \infty$ and let

$$\Sigma_q: \qquad H_q \rightarrow \Gamma_{q-1} \rightarrow \cdots \rightarrow H_2 \rightarrow 0$$

be a sequence in which the (Abelian) groups are arbitrary except that $\Gamma_2 = 0$, if $q > 2$, and

$$\theta:\Gamma(\Pi_2) \approx \Gamma_3 \tag{14.1}$$

if $q > 3$. In this case θ, like $\mathfrak{b}$, $\mathfrak{i}$, $\mathfrak{j}$ is to be a component part of Σ_q. Let Σ_q' be a sequence which also satisfies these conditions. We shall describe $(\mathfrak{h}, \mathfrak{g}, \mathfrak{f}):\Sigma_q \rightarrow \Sigma_q'$ as a *proper* homomorphism if, and only if, either $q \leqq 3$ or $q > 3$ and

$$\mathfrak{g}_3\theta = \theta\mathfrak{g}:\Gamma(\Pi_2) \rightarrow \Gamma_3', \tag{14.2}$$

where $\mathfrak{g}:\Gamma(\Pi_2) \rightarrow \Gamma(\Pi_2')$ is induced by $\mathfrak{f}_2$. We shall describe a complex K as a *geometrical realization* of Σ_q' if, and only if, $\Sigma_q(K)$ is properly isomorphic to Σ_q'.

The symbol $(C, A)_q$ will denote a composite chain system, with the groups C_r, A_{r-1} discarded if $r > q$ and $\theta:\Gamma(\Pi_2) \approx \Gamma_3$ if $q > 3$, where $\Pi_2 = A_2/\beta C_3$. A *proper* homomorphism (isomorphism), $(h, f)_q$, between two such systems, will consist of homomorphisms, (isomorphisms) $h_1, \cdots, h_q$ and $f_1, \cdots, f_{q-1}$, such that $f\beta = \beta h$, $hj = jf$ and (14.2) is satisfied if $q > 3$, where $\mathfrak{g}_3$ is the homomorphism induced by f_3. That is to say $i_3\mathfrak{g}_3 = f_3 i_3$, as in (3.4). We describe $(C, A)_q$ as a *combinatorial realization*[32] of Σ_q' if, and only if, there are homomorphisms (onto)

$$l_{n+1}':Z_{n+1} \rightarrow H_{n+1}', \ \mathfrak{g}_n:\Gamma_n \approx \Gamma_n', \ k_n':A_n \rightarrow \Pi_n' \tag{14.3}$$

for $n = 1, \cdots, q-1$, such that $k_n'^{-1}(0) = \beta C_{n+1}$, $l_n'^{-1}(0) = dC_{n+1}$,

$$\mathfrak{b}_{n+1}l_{n+1}'z = \mathfrak{g}_n\beta_{n+1}z, \qquad \mathfrak{i}_n\mathfrak{g}_n = k_n' i_n, \qquad \mathfrak{j}_n' k_n' = l_n' k_n$$

and (14.2) is satisfied if $q > 3$. If $(C, A)_q$ is a combinatorial realization of Σ_q' so, obviously, is any system $(C', A')_q$ which is properly isomorphic to $(C, A)_q$. The existence of a combinatorial realization of Σ_q' follows from the proof of Theorem 2, with $\mathfrak{g}_3$ chosen so as to satisfy (14.2) if $q > 3$.

Let $n \leqq q$ and let Σ_n' be the part of Σ_q' which begins with H_n'. We shall say that $(C, A)_n$ is *part of* $(C', A')_q$ if, and only if, $C_{r+1} = C_{r+1}'$, $A_r = A_r'$ and β_{r+1}, j_r are the same in both systems, for every $r < n$. By an *n-dimensional partial realization* of Σ_q' we shall mean a complex, K^n, of at most n dimensions, such that the part, $(C, A)_n$, of (C, A) (K^n) is a combinatorial realization of Σ_n'. Notice that a q-dimensional partial realization, K^q, is a geometrical realization of Σ_q' if, and only if, $l_q' : Z_q \approx H_q'$.

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[32] We take Σ_1' to consist of a single homomorphism, $0 \rightarrow 0$, and we admit that $(C, A)_1$ $(C_1 = A_0 = 0)$ is a combinatorial realization of Σ_1'.

Lemma 1. *Let $n < q$ and let K^n be a simply connected, n-dimensional partial realization of Σ_q'. If $\Gamma_n(K^n) \approx \Gamma_n'$ the complex $\mathbf{K}^n$ can be imbedded in an $(n+1)$-dimensional partial realization of Σ_q'.*

Let $(C, A)_n$ be part of $(C, A)(K^n)$ and let $(C, A)_n$ be extended by the construction in the proof of Theorem 2 to a combinatorial realization, $(C', A')_{n+1}$, of Σ_{n+1}'. In order to simplify the notation we take Γ_n' in A_n' to be the same as Γ_n' in Σ_n' and the isomorphism $\Gamma_n' \approx \Gamma_n'$, analogous to the one in (14.3), to be the identity. Let $\mathfrak{g}_n : \Gamma_n \approx \Gamma_n'$, where $\Gamma_n = \Gamma_n(K^n)$ and $\mathfrak{g}_n$ is defined by (14.2) if $n = 3$. Then

$$A_n = \Gamma_n + B_n, \qquad A_n' = \Gamma_n' + B_n',$$

as in the proofs of Theorems 2 and 3, where $u : jA_n \approx B_n$, $u' : jA_n' \approx B_n'$ and $ju = 1$, $ju' = 1$. Since β_n is the same in $(C', A')_{n+1}$ and in $(C, A)_n$ we have $jA_n = \beta_n^{-1}(0) = jA_n'$. Therefore an isomorphism, $f : A_n \approx A_n'$, is defined by

$$f(\gamma + b) = \mathfrak{g}_n\gamma + u'jb \qquad (\gamma \in \Gamma_n,\ b \in B_n).$$

Since $j\gamma = 0, j\mathfrak{g}_n\gamma = 0, ju' = 1$ we have $j_n f = j_n$.

Let $\{c_\lambda'\}$ be a set of free generators of C_{n+1}'. Let

$$K^{n+1} = K^n \cup \{e_\lambda^{n+1}\},$$

where the $(n+1)$-cell e_λ^{n+1} is attached to K^n by a map, $\phi_\lambda : \dot{E}_\lambda^{n+1} \to K^n$, such that $\phi_\lambda v_\lambda : \dot{I}^{n+1} \to K^n$ represents $f^{-1}\beta c_\lambda'$, where $v_\lambda : \dot{I}^{n+1} \to \dot{E}_\lambda^{n+1}$ is a homeomorphism. Then e_λ^{n+1} has a characteristic map, $\psi_\lambda : I^{n+1} \to e_\lambda^{n+1}$ which agrees with $\phi_\lambda v_\lambda$ in $\dot{I}^{n+1}$. Let $(C, A)_{n+1}$ be part of $(C, A)(K^{n+1})$ and let $c_\lambda \in C_{n+1}$ be the element which is represented by ψ_λ. Then $\{c_\lambda\}$ is a set of free generators of C_{n+1} and an isomorphism, $h : C_{n+1} \approx C_{n+1}'$, is defined by $hc_\lambda = c_\lambda'$. Moreover $\phi_\lambda v_\lambda$ represents both βc_λ and $f^{-1}\beta c_\lambda'$. Therefore $f\beta = \beta h$. Also $j_n f = j_n$, $f\gamma = \mathfrak{g}_n\gamma$ and $\mathfrak{g}_n$ satisfies (14.2) if n = 3. Therefore the identical maps of C_{r+1}, A_r $(r < n)$, together with h, f, constitute a proper isomorphism $(C, A)_{n+1} \approx (C', A')_{n+1}$. Therefore $(C, A)_{n+1}$ is a combinatorial realization of Σ_{n+1}' and the lemma is proved.

Theorem 15. *Σ_4' has a (simply connected) geometrical realization K, which is*

a) *at most 4-dimensional if H_4' is free Abelian,*

b) *a finite complex if each of H_2', H_3', H_4' is finitely generated.*

Let K^1 consist of a single 0-cell. Then K^1 is a partial realization of Σ_1'. Since $\Gamma_1(K) = 0$, $\Gamma_2(K) = 0$ and $\Gamma_3(K) \approx \Gamma\{\Pi_2(K)\}$, where K is any complex, it follows from three successive applications of Lemma 1 that Σ_4' has a 4-dimensional partial realization, K^4.

Let $(C, A)_4$ be part of (C, A) (K^4) and let l', $\mathfrak{g}$ mean the same as in (14.3). Let $z \in l_4'^{-1}(0)$. Then $\beta z = \mathfrak{g}_3^{-1}\mathfrak{b}l_4'z = 0$. Therefore $z \in j_4\pi_4(K^4)$. Let $\{z_\mu\}$ be a set of elements which generate $l_4'^{-1}(0)$ and let $a_\mu \in j_4^{-1}z_\mu$. Let $K^5 = K^4 \cup \{e_\mu^5\}$, where e_μ^5 is attached to K^4 by a map which represents a_μ, and let $\{c_\mu\}$ be the corresponding basis for $C_5 = \pi_5(K^5, K^4)$. Then $\beta c_\mu = a_\mu$ and $dc_\mu = z_\mu$. Therefore $dC_5 = l_4'^{-1}(0)$ and it follows that l_4' induces an isomorphism $\mathfrak{h}_4 : H_4(K^5) \approx H_4'$. Therefore K^5 is a full realization of Σ_4'.

If H_4' is free Abelian we may assume that $l_4'^{-1}(0) = 0$, as in the addendum to Theorem 2. In this case Σ_4' is realized by K^4. Also we may assume that, if H_2', H_3', H_4' are finitely generated, so are C_2, C_3, C_4 and hence $l_4^{-1}(0)$ and C_5. In this case K^5 is a finite complex and the theorem is proved.

We now consider the realizability of a proper homomorphism, $F_q:\Sigma_q \to \Sigma_q'$, by a map $\phi:K^q \to K'^q$, where K, K' are given complexes and $\Sigma = \Sigma(K)$, $\Sigma' = \Sigma(K')$. Let $(C, A)_q$, $(C', A')_q$ be parts of (C, A) (K), (C, A) (K'). Then it follows from the proof of Theorem 3 that F_q can be realized combinatorially, in the same way as when $q = \infty$, by a (proper) homomorphism $(h, f)_q:(C, A)_q \to (C', A')_q$. We shall describe a cellular map, $\phi:K^q - K'^q$ as a (geometrical) realization of both $(h, f)_q$ and F_q if, and only if, the homomorphisms h, f are those induced by ϕ. Notice that $h_q dC_{q+1}' \subset dC_{q+1}'$, since $l_q h_q = \mathfrak{h}_q l_q$; also that a given map, $\phi:K^q \to K'^q$, induces a homomorphism $\Sigma_q \to \Sigma_q'$ if, and only if, $h_q:C_q \to C_q'$ satisfies this condition, where h_q is induced by ϕ. This is certainly the case if $K = K^q$, for then $C_{q+1} = 0$.

Let $(h, f)_q$ be a combinatorial realization of a given proper homomorphism $F_q:\Sigma_q \to \Sigma_q'$. Let $1 \leqq n < q$, let $(C, A)_n$, $(C', A')_n$ be parts of $(C, A)_q$, $(C', A')_q$ and let $(h, f)_n$ consist of the homomorphisms $h_1, \cdots, h_n$ and $f_1, \cdots, f_{n-1}$. Let $F_n:\Sigma_n \to \Sigma_n'$ be the homomorphism which is similarly induced by F_q. Then F_n is obviously a proper homomorphism and $(h, f)_n$ is a combinatorial realization of F_n. Let $\phi_n^0:K^n \to K'^n$ be a realization of $(h, f)_n$. We assume that K, but not necessarily K', is simply connected and also that $\phi_n^0 K^1 = e'^0$, the base point in K'. Let $\mathfrak{g}_n^0$, $\mathfrak{g}_n:\Gamma_n \to \Gamma_n'$ be the homomorphisms induced by ϕ_n^0 and by f_n in $(h, f)_q$.

LEMMA 2. *If $\mathfrak{g}_n^0 = \mathfrak{g}_n$ and if $j_n A_n$ is a direct summand of C_n, then $\phi_n^0 \mid K^{n-1}$ can be extended to a realization, $K^{n+1} \to K'^{n+1}$, of $(h, f)_{n+1}$.*

First let $n = 1$ and let $\rho_2 = \pi_2(K^2, K^1)$. Then $\rho_2 = C_2 + B^*$, as in §10, where $\beta^*:\beta\rho_2 \approx B^*$ and $\beta\beta^* = 1$. Also $\beta\rho_2 = \pi_1(K^1)$, since $\pi_1(K) = 1$. If $a \in \rho_2$, $b^* \in B^*$ we have $(\beta a)b^* = a + b^* - a$, by (2.1c) in CH II, and $a + b^* - a \in B^*$ since ρ_2 is the direct sum of C_2 and B^*. Therefore B^* is invariant under the operators in $\pi_1(K^1)$. Also $\beta\rho_2$ operates identically on C_2. Therefore $h_2:C_2 \to C_2'$ can be extended to an operator homomorphism, $h^*:\rho_2 \to \pi_2(K'^2, K'^1)$, by taking $h^*B^* = 0$. Since $\beta C_2' = 1$ and $\phi_1^0 K^1 = e'^0$ we have $\beta h^*\rho_2 = f^0\beta\rho_2 = 1$, where $f^0:\pi_1(K^1) \to \pi_1(K'^1)$ is induced by ϕ_1^0. Therefore it follows from Lemma 4 in CH II that ϕ_1^0 can be extended to a realization, $K^2 \to K'^2$, of $(h, f)_2$.

Let $n > 1$ and let $f_n^0:A_n \to A_n'$ be the homomorphism induced by ϕ_n^0. If $f_n^0 = f_n$ we have $f_n^0\beta_{n+1} = \beta_{n+1}h_{n+1}$, since $(h, f)_{n+1}$ is a homomorphism. Therefore it follows from[33] Lemma 4 in CH II that ϕ_n^0 can be extended to a realization, $K^{n+1} = K'^{n+1}$, of $(h, f)_{n+1}$. Therefore the lemma will follow when we have extended $\phi_n^0 \mid K^{n-1}$ to another realization, $\phi_n:K^n \to K'^n$, of $(h, f)_n$, which induces f_n.

Since f_n^0, h_n are both induced by ϕ_n^0 we have $j_n f_n^0 = h_n j_n = j_n f_n$. Since $j_2^{-1}(0) =$

[33] If $\pi_1(K) \neq 1$ this argument fails unless h_{n+1}, f_n are operator homomorphisms associated with the homomorphism, $\pi_1(K) \to \pi_1(K')$, which is induced by ϕ_n^0.

0 it follows that $f_n^0 = f_n$ if $n = 2$. Let $n > 2$, let $\mathfrak{g}_n^0 = \mathfrak{g}_n$ and let $C_n = A^* + B^*$, where $A^* = j_n A_n$. Let $u: A^* \to A_n$ be a right inverse of j_n. Then a homomorphism, $\Delta: C_n \to A'_n$, or cochain $\Delta \in C^n(K^n, A'_n)$, is defined by

$$\Delta(a^* + b^*) = (f_n - f_n^0)ua^* \qquad (a^* \in A^*, b^* \in B^*).$$

Since $j_n(f_n^0 - f_n) = 0$ we have $j_n \Delta = 0$, whence $\Delta C_n \subset \Gamma'_n$.

Let $\phi_n: K^n \to K'^n$ be an extension of $\phi_n^0 \mid K^{n-1}$, which realizes[34] the separation cochain $d(\phi_n, \phi_n^0) = \Delta$. Let $f_n^1: A_n \to A'_n$ and $h_n^1: C_n \to C'_n$ be the homomorphisms induced by ϕ_n. Then[35]

$$h_n^1 - h_n = j_n \Delta = 0, \qquad f_n^1 - f_n^0 = \Delta j_n .$$

Therefore ϕ_n is a realization of $(h, f)_n$, because $h_n^1 = h_n$. Also

$$(f_n^1 - f_n^0)ua^* = \Delta j_n ua^* = \Delta a^* = (f_n - f_n^0)ua^*.$$

Therefore $f_n^1 ua^* = f_n ua^*$. Also $f_n^0 \gamma = \mathfrak{g}_n^0 \gamma = \mathfrak{g}_n \gamma = f_n \gamma$, if $\gamma \in \Gamma_n$, and $f_n^1 \gamma = f_n^0 \gamma$ since $\phi_n \mid K^{n-1} = \phi_n^0 \mid K^{n-1}$. Therefore $f_n^1 \gamma = f_n \gamma$. Since $j_n u = 1$ we have $A_n = \Gamma_n + uA^*$. Therefore $f_n^1 = f_n$ and the proof is complete.

THEOREM 16. *If $\pi_1(K) = 1$ and $q \leqq 4$ any proper homomorphism, $F_q: \Sigma_q(K) \to \Sigma_q(K')$, has a geometrical realization $K^q \to K'^q$.*

Let $(h, f)_q: (C, A)_q \to (C', A')_q$ be a combinatorial realization of F_q, where $(C, A)_q$, $(C', A')_q$ are parts of (C, A) (K), (C, A) (K'). Since $\Gamma_2 = 0$ the theorem follows from two successive applications of Lemma 2 if $q \leqq 3$. Let $q = 4$ and let $\phi_3^0: K^3 \to K'^3$ be a geometrical realization of $(h, f)_3$. Then ϕ_3^0 induces the homomorphism $\mathfrak{f}_2: \Pi_2 \to \Pi'_2$ in F_4, and it follows from (14.2) that $\mathfrak{g}_3^0 = \mathfrak{g}_3$, where $\mathfrak{g}_3^0$, $\mathfrak{g}_3$ are induced by ϕ_3^0, f_3. Therefore the theorem follows from another application of Lemma 2.

Let $\mathfrak{B}$ be any category. We describe two objects in $\mathfrak{B}$ as *equivalent* if, and only if, they are related by an equivalence in $\mathfrak{B}$. Let $T: \mathfrak{U} \to \mathfrak{B}$ be a given functor, where $\mathfrak{U}$ is any category. By the *sufficiency* and the *realizability* conditions, with respect to T, we mean the following,

Sufficiency: if $T\alpha$ is an equivalence, so is α, where α is a given mapping in $\mathfrak{U}$.

Realizability: a) *any object in $\mathfrak{B}$ is equivalent to the image, TU, of some object in $\mathfrak{U}$, and*

b) *any mapping. $TU \to TU'$, in $\mathfrak{B}$, is the image, $T\alpha$, of at least one mapping, $\alpha: U \to U'$, for every pair of objects, U, U', in $\mathfrak{U}$.*

Let $\mathfrak{K}_0^4$ be the homotopy category of all simply connected complexes of at most four dimensions. Then a mapping (i.e. homotopy class), $\alpha: K \to K'$, in $\mathfrak{K}_0^4$ induces a unique homomorphism $\Sigma_4\alpha: \Sigma_4(K) \to \Sigma_4(K')$, because $\pi_1(K') = 1$. Let $\mathfrak{S}_4$ be the category in which the objects are all sequences Σ_4, which satisfy the conditions $\Gamma_2 = 0$ and (14.1), and in which H_4 is free Abelian, with all proper homomorphisms as mappings. Then a functor, $\Sigma_4: K_0^4 \to \mathfrak{S}_4$ is obviously determined by the correspondences $K \to \Sigma_4(K)$, $\alpha \to \Sigma_4\alpha$.

[34] See [15]. In order to apply the existence theorem (10.5) in [15] we can take $\phi_n = \phi_n^0$ in $e_i^n - \sigma_i^n$, where $\sigma_i^n \subset e_i^n$ is an n-simplex, for each n-cell $e_i^n \in K$.

[35] See Appendix B below.

THEOREM 17. *The functor Σ_4 satisfies both the sufficiency and the realizability conditions.*

This follows from Theorems 13, 15, 16.

As pointed out in the introduction, both homotopy classes of maps $S^4 \to S^3$ induce the same homomorphism $\Sigma_4(S^4) \to \Sigma_4(S^3)$. Therefore the function $\alpha \to \Sigma_4\alpha$ is not (1-1). By taking $N = S^3 \cup S^4$, where $S^3 \cap S^4 = e^0$, we see that Σ_4 does not even induce an isomorphism of the group of equivalences $\alpha : K \equiv K$.

Sequences of the form $\Sigma_4(K)$ can be simplified algebraically by identifying Γ_3, H_3, H_2 with $\Gamma(\Pi_2)$, $\Pi_3/\mathfrak{i}\Gamma_3$, Π_2 so as to make $\theta = 1$, $\mathfrak{j}_3$ the natural homomorphism and $\mathfrak{j}_2 = 1$. When Σ_4 is thus simplified it is completely determined by Π_2 and

$$H_4 \to \Gamma(\Pi_2) \to \Pi_3 . \tag{14.4}$$

Let Π_2 be finitely generated. Then it follows from Theorem 8 that Π_2 is determined, up to an isomorphism, by $\Gamma(\Pi_2)$. Let $F_4 : \Sigma_4 \to \Sigma_4'$ be a proper homomorphism which determines an ismorphism of the sequence (14.4). Then it follows from Theorem 11 that $F_4 : \Sigma_4 \approx \Sigma_4'$ (obviously $\mathfrak{h}_3 : H_3 \approx H_3'$ if $\mathfrak{g}_3$, $\mathfrak{f}_3$ are isomorphisms, where $F_4 = (\mathfrak{h}, \mathfrak{g}, \mathfrak{f})$).

Theorem 16 is analogous to Theorems 1, 2, 3 in [5], concerning $(n$-1)-connected complexes (i.e. those with $\pi_r(K) = 0$ for $r = 1, \cdots, n - 1)$ when $n > 2$. Let $\Sigma = \Sigma(K)$, where K is $(n - 1)$-connected $(n > 2)$, and if $x \in \Pi_n$ let $u(x) \in \Gamma_{n+1}$ be defined in the same way as $u(x) \in \Gamma_3$, in §13 above, when $x \in \Pi_2$. Then $u : \Pi_n \to \Gamma_{n+1}$ is a homomorphism since $n > 2$. It is shown in [5] that $\theta : \Pi_{n,2} \approx \Gamma_{n+1}$ where $\Pi_{n,2} = \Pi_n/2\Pi_n$ and θ is induced by u. The argument leading to (13.2), with $\gamma : \Pi_2 \to \Gamma(\Pi_2)$ replaced by the natural homomorphism $\Pi_n \to \Pi_{n,2}$, shows that θ is natural. Therefore Lemmas 1 and 2 yield realizability theorems for Σ_{n+2}, with $\Gamma_r = \Pi_r = H_r = 0$ if $r < n$ and (14.1) replaced by $\theta : \Pi_{n,2} \approx \Gamma_{n+1}$, which are analogous to Theorems 15, 16.

Many of these facts have been recorded by G. W. Whitehead in [23]. In particular his results may be used to show that, if $\Sigma = \Sigma(K)$, where K is $(n - 1)$-connected, then Σ_{n+2} is internally exact $(n > 2)$ and that the sequence Σ^0_{n+2}, which is defined in §15 below, is exact if $n > 3$.

Theorem 16 is also analogous to Theorems 1, 2, 3 in [9], concerning the 3-type of an arbitrary (connected) CW-complex. In the following section we prove a theorem which leads to analogous results concerning the 4-type of a simply connected complex.

15. q-types

Let $2 \leqq q < \infty$. We recall from CH I that complexes K, K' are of the same q-type if, and only if, there are maps,

$$\phi : K^q \to K'^q, \qquad \phi' : K'^q \to K^q$$

such that $\phi'\phi \mid K^{q-1} \simeq 1$, $\phi\phi' \mid K'^{q-1} \simeq 1$. If, and only if, these conditions are satisfied, we write $\phi : K^q \equiv {}_{q-1}K'^q$. By Theorem 2 in CH I this is so if, and only if, ϕ induces isomorphisms $\pi_n(K) \approx \pi_n(K')$ for $n = 1, \cdots, q-1$. Obviously K and K^q have the same q-type. Therefore, when studying q-types, we may confine ourselves to complexes of at most q dimensions.

Let $\Sigma_q = \Sigma_q(K)$, where, to begin with, dim K may exceed q. Let

$$H_q^0 = H_q^0(K) = H_q / \mathfrak{j}\Pi_q .$$

Thus $H_q^0 \approx H_q(\tilde{K})/S_q(\tilde{K})$, where $\tilde{K}$ is the universal covering complex of K and $S_q(\tilde{K})$ consists of the "spherical" homology classes. Since $\mathfrak{j}\Pi_q = \mathfrak{b}_q^{-1}(0)$ an isomorphism

$$\mathfrak{b}_q^0 : H_q^0 \approx \mathfrak{b}_q H_q = \mathfrak{i}_q^{-1}(0)$$

is induced by $\mathfrak{b}_q$. Let $\Sigma_q^0 = \Sigma_q^0(K)$ be the exact sequence

$$\Sigma_q^0: \qquad 0 \rightarrow H_q^0 \xrightarrow{\mathfrak{b}^0} \Gamma_{q-1} \xrightarrow{\mathfrak{i}} \cdots \xrightarrow{\mathfrak{j}} H_2 \rightarrow 0.$$

Since $l_q Z_q = H_q$, $l_q j_q A_q = \mathfrak{j}_q \Pi_q$, $H_q^0(K^q) = Z_q / j_q A_q$ it follows that l_q induces an isomorphism $H_q^0(K^q) \approx H_q^0$, by means of which we identify $\Sigma_q^0(K^q)$ with $\Sigma_q^0(K)$. From the purely algebraic point of view Σ_q^0 may be regarded as a sequence Σ, in which every group preceding H_q is zero. Therefore we need not redefine the terms homomorphism etc.

Let $\Sigma_q' = \Sigma_q(K')$ and let $(\mathfrak{h}, \mathfrak{g}, \mathfrak{f}) : \Sigma_q \rightarrow \Sigma_q'$ be any homomorphism. Since $\mathfrak{b}_q \mathfrak{h}_q = \mathfrak{g}_{q-1} \mathfrak{b}_q$ it follows that

$$\mathfrak{h}_q \mathfrak{b}_q^{-1}(0) \subset \mathfrak{b}_q^{-1}(0) = \mathfrak{j}_q \Pi_q' .$$

Therefore $\mathfrak{h}_q$ induces a homomorphism

$$\mathfrak{h}_q^0 : H_q^0 \rightarrow H_q'^0 .$$

Obviously $\mathfrak{b}_q^0 \mathfrak{h}_q^0 = \mathfrak{g}_{q-1} \mathfrak{b}_q$. Therefore a homomorphism, $F_q^0 : \Sigma_q^0 \rightarrow \Sigma_q'^0$, which consists of $\mathfrak{h}_q^0$ and of $\mathfrak{g}_n$, $\mathfrak{f}_n$, $\mathfrak{h}_n$ for $n = 2, \cdots, q-1$, is *induced* by $(\mathfrak{h}, \mathfrak{g}, \mathfrak{f})$. If $(\mathfrak{h}, \mathfrak{g}, \mathfrak{f})$ is induced by a map $\phi : K^q \rightarrow K'^q$ we shall say that ϕ *induces*, or *realizes*, F_q^0.

THEOREM 18. a) *If $\phi_1 : \pi_1(K) \approx \pi_1(K')$ and $F_q^0 : \Sigma_q^0 \approx \Sigma_q'^0$, where ϕ_1 and F_q^0 are induced by $\phi : K^q \rightarrow K'^q$, then $\phi : K^q \equiv {}_{q-1}K'^q$.*

b) *Any proper homomorphism. $F_4^0 : \Sigma_4^0 \rightarrow \Sigma_4'^0$ has a realization $K^4 \rightarrow K'^4$.*

Part (a) follows from Theorem 2 in CH I.

Let $\{z_\mu\}$ be a set of free generators of $H_4(K^4)$. Let

$$z_\mu' \in \mathfrak{b}_q^{-1} \mathfrak{g}_3 \mathfrak{b}_4 z_\mu \subset H_4(K'^4),$$

where $\mathfrak{g}_3 : \Gamma_3 \rightarrow \Gamma_3'$ is in F_4^0, and let $\mathfrak{h}_4 : H_4(K^4) \rightarrow H_4(K'^4)$ be the homomorphism which is defined by $\mathfrak{h}_4 z_\mu = z_\mu'$. Then $\mathfrak{b}_4 \mathfrak{h}_4 = \mathfrak{g}_3 \mathfrak{b}_4$ and it follows that $\mathfrak{h}_4$ induces a homomorphism $\mathfrak{h}_4^* : H_4^0(K^4) \rightarrow H_4^0(K'^4)$, such that $\mathfrak{b}_4^0 \mathfrak{h}_4^* = \mathfrak{g}_3 \mathfrak{b}_4^0 = \mathfrak{b}_4^0 \mathfrak{h}_4$. Since $(\mathfrak{b}_4^0)^{-1}(0) = 0$ it follows that $\mathfrak{h}_4^* = \mathfrak{h}_4^0$. On replacing $\mathfrak{h}_4^0$ by $\mathfrak{h}_4$ we have a proper homomorphism, $F_4 : \Sigma_4 \to \Sigma'_4$, which induces F^0_4. Part (b) now follows from Theorem 16.

An algebraic sequence Σ^0_4 is a special kind of Σ_4, namely one such that $\mathfrak{b}_4^{-1}(0) = 0$. Therefore Theorem 15 applies unchanged to the geometrical realization of Σ^0_4. Notice, however, that any sequence Σ^0_4 is realized by a complex K^4, since $\Sigma^0_4(K) = \Sigma^0_4(K^4)$, even if $\dim K > 4$.

Notice that, by analogy with (14.4), Σ^0_4 may be replaced by a pair of arbitrary Abelian groups, Π_2, Π_3, and a homomorphism $\mathfrak{i} : \Gamma(\Pi_2) \to \Pi_3$, which may be arbitrary. Therefore the "algebraic 4-type" of a simply connected complex is a comparatively simple affair. The algebraic $(n + 2)$-type of an $(n - 1)$-connected complex $(n > 2)$ may be similarly defined, with $\Gamma(\Pi_2)$ replaced by $\Pi_{n,2}$.

Chapter IV. The Pontrjagin squares

16. The main theorem

We give the following definition of a net of finite, simplicial complexes, which differs slightly from the one in [18]. Let $\{K(d)\}$ be a set of such complexes, which is indexed to a directed set D. Instead of taking a projection, $K(d_2) \to K(d_1)$, to be a single map, where $d_1 < d_2$, we take it to be a homotopy class of simplicial maps. We then assume that, if $d_1 < d_2$, there is a single projection,

$$K(d_1, d_2) : K(d_2) \to K(d_1),$$

such that $K(d, d) = 1$ and $K(d_1, d_2)K(d_2, d_3) = K(d_1, d_3)$ if $d_1 < d_2 < d_3$. Let (D, K) denote this net and let (D', K') be a similar net. By a *homomorphism*,

$$(R^*, \rho) : (D, K) \to (D', K'), \tag{16.1}$$

we shall mean an order preserving map, $R^* : D' \to D$, together with a family of homotopy classes,

$$\rho(d') : K(R^*d') \to K'(d'),$$

where $\rho(d')$ is defined for every $d' \in D'$ and

$$\rho(d'_1)K(R^*d'_1, R^*d'_2) = K'(d'_1, d'_2)\rho(d'_2) \qquad (d'_1 \lesssim d'_2).$$

It is easily verified that all nets, with all homomorphisms as mappings, is a category $\mathfrak{X}$. We shall sometimes denote nets by X, X' etc.

Let $\mathfrak{A}$ and $\mathfrak{D}$ mean the same as in §9 above and let $(\mathfrak{X}, \mathfrak{A})$ be the Cartesian product of $\mathfrak{X}$ and $\mathfrak{A}$, in which the mappings, (ξ, α), are pairs of homomorphisms, $\xi : X \to X'$, $\alpha : A \to A'$. Let $(\mathfrak{K}_\sigma, \mathfrak{A})$ be similarly defined, where $\mathfrak{K}_\sigma$ is the homotopy category of finite, simplicial complexes. Let $P : (\mathfrak{K}_\sigma, \mathfrak{A}) \to \mathfrak{A}$ be a functor, which is contravariant in $\mathfrak{K}_\sigma$ and covariant in $\mathfrak{A}$. Let $(\rho, \alpha) : (K, A) \to (K', A')$ be any mapping in $(\mathfrak{K}_\sigma, \mathfrak{A})$. Then $P(\rho, \alpha)$ is a homomorphism

$$P(\rho, \alpha) : P(K', A) \to P(K, A').$$

Let (D, K) be a given net and let $P\{K, A\}$ denote the family of groups

$$P(K(d), A),$$

for every $d \in D$. Then it follows without difficulty that $(D, P\{K, A\})$, with the homomorphisms[36]

$$(16.2) \qquad T(d_2, d_1) = PK(d_1, d_2): P(K(d_1), A) \to P(K(d_2), A),$$

is a direct system of groups; also that a "lifted" functor,

$$P_l: (\mathfrak{X}, \mathfrak{A}) \to \mathfrak{D},$$

is defined by

$$(16.3) \qquad \begin{cases} P_l((D, K), A) = (D, P\{K, A\}) \\ P_l((R^*, \rho), \alpha) = (R^*, P\{\rho, \alpha\}), \end{cases}$$

where (R^*, ρ) means the same as in (16.1) and $P\{\rho, \alpha\}$ denotes the family of homomorphisms

$$P(\rho(d'), \alpha): P(K'(d'), A) \to P(K(R^*d'), A').$$

Therefore LP_l is a functor,

$$LP_l: (\mathfrak{X}, \mathfrak{A}) \to \mathfrak{A},$$

where $L: \mathfrak{D} \to \mathfrak{A}$ is the direct limit functor.

Let $\tau: P \to Q$ be a natural transformation, where $Q: (\mathfrak{K}_\sigma, \mathfrak{A}) \to \mathfrak{A}$ is a functor of the same variance as P. Let $Q_l: (\mathfrak{X}, \mathfrak{A}) \to \mathfrak{D}$ be defined in the same way as P_l. Let (X, A) mean the same as before and let $\tau\{K, A\}$ denote the family of homomorphisms

$$\tau(K(d), A): P(K(d), A) \to Q(K(d), A).$$

Since τ is natural, and since P, Q are contravariant in $\mathfrak{K}_\sigma$, we have

$$\tau(K(d_2), A)PK(d_1, d_2) = QK(d_1, d_2)\tau(K(d_1), A),$$

or

$$\tau(K(d_2), A)T(d_2, d_1) = U(d_2, d_1)\tau(K(d_1), A),$$

where $T(d_2, d_1)$ is given by (16.2) and $U(d_2, d_1) = QK(d_1, d_2)$. Therefore a homomorphism,

$$\tau_l(X, A): P_l(X, A) \to Q_l(X, A),$$

is defined by $\tau_l(X, A) = (1, \tau\{K, A\})$.

Let (R^*, ρ) and α mean the same as in (16.3) and let $S = P$ or Q. Then

$$S(\rho(d'), \alpha)$$

is a homomorphism

$$S(\rho(d'), \alpha): S(K'(d'), A) \to S(K(R^*d'), A').$$

Since τ is natural we have

$$\tau(K(R^*d'), A')P(\rho(d'), \alpha) = Q(\rho(d'), \alpha)\tau(K'(d'), A).$$

[36] $PK(d_1, d_2)$ stands for $P(K(d_1, d_2), 1)$.

Therefore, writing $\xi = (R^*, \rho)$, it follows from (16.3) that

$$\begin{aligned} \tau_l(X, A')P_l(\xi, \alpha) &= (1, \tau\{K, A'\})(R^*, P\{\rho, \alpha\}) \\ &= (R^*, \tau\{K, A'\}P\{\rho, \alpha\}) \\ &= (R^*, Q\{\rho, \alpha\}\tau\{K', A\}) \\ &= (R^*, Q\{\rho, \alpha\})(1, \tau\{K', A\}) \\ &= Q_l(\xi, \alpha)\tau_l(X', A). \end{aligned} \tag{16.4}$$

Therefore τ_l is natural. Let $L\tau_l: LP_l \to LQ_l$ be the transformation which is defined by

$$(L\tau_l)(X, A') = L(\tau_l(X, A').$$

On applying the functor L to both sides of (16.4) we see that $L\tau_l$ is natural.[37]

The r^{th} cohomology functor $H^r: (\mathfrak{K}_\sigma, \mathfrak{A}) \to \mathfrak{A}$ is contravariant in $\mathfrak{K}_\sigma$ and covariant in $\mathfrak{A}$. We define H_l^r in the same way as P_l and write

$$H^r = LH_l^r: (\mathfrak{X}, \mathfrak{A}) \to \mathfrak{A}.$$

Then H^r is the Čech cohomology functor. We define the cup-product,

$$\mathbf{y} \cup \mathbf{z} \in H^{2n}(X, \Gamma(A)),$$

of elements $\mathbf{y}, \mathbf{z} \in H^n(X, A)$, by means of the pairing $(a, b) = [a, b] \in \Gamma(A)$, where

$$a, b \in A.$$

It is obvious that a covariant functor

$$\Gamma: (\mathfrak{X}, \mathfrak{A}) \to (\mathfrak{X}, \mathfrak{A})$$

is defined by $\Gamma(X, A) = (X, \Gamma(A))$, $\Gamma(\xi, \alpha) = (\xi, \Gamma\alpha)$.
Thus we have functors

$$\Gamma H^r, H^r\Gamma: (\mathfrak{X}, \mathfrak{A}) \to \mathfrak{A}, \tag{16.5}$$

which are contravariant in $\mathfrak{X}$ and covariant in $\mathfrak{A}$.

THEOREM 19. *Let n be even. Then there is a natural transformation,*[38]

$$\eta: \Gamma H^n \to H^{2n}\Gamma, \tag{16.6}$$

such that

$$\eta(X, A)[\mathbf{y}, \mathbf{z}] = \mathbf{y} \cup \mathbf{z}. \tag{16.7}$$

for every pair $\mathbf{y}, \mathbf{z} \in H^n(X, A)$.

Assume that the analogous theorem has been proved for the category $(\mathfrak{K}_\sigma, \mathfrak{A})$

[37] Cf. the concluding remarks in §9 of [10].

[38] We do not assert that η is uniquely determined by naturalness and (16.7). When this theorem is quoted it is to be understood that η is the transformation which is defined in the course of the proof.

and let $\tau:\Gamma H^n \to H^{2n}\Gamma$ be a natural transformation which satisfies (16.7), where ΓH^r, $H^r\Gamma:(\mathfrak{K}_\sigma, \mathfrak{A}) \to \mathfrak{A}$ are defined in the same way as ΓH^r, $H^r\Gamma$ in (16.5). It follows from (16.3) and the definitions of $\Gamma_l:\mathfrak{D} \to \mathfrak{D}$ and $\Gamma:(\mathfrak{K}_\sigma, \mathfrak{A}) \to (\mathfrak{K}_\sigma, \mathfrak{A})$ that

$$(\Gamma H^r)_l = \Gamma_l H_l^r, \qquad (H^r\Gamma)_l = H_l^r\Gamma.$$

Therefore

$$L\tau_l:L\Gamma_l H_l^n \to LH_l^{2n}\Gamma$$

is a natural transformation. It follows from Theorem 12 that $\omega^{-1}:\Gamma L \to L\Gamma_l$ is a natural equivalence, where ω is defined by (9.1). Therefore[37]

$$\omega^{-1}H_l^n:\Gamma LH_l^n \to L\Gamma_l H_l^n$$

is a natural transformation. Therefore, writing $LH_l^r = H^r$, it follows that

$$\eta = (L\tau_l)(\omega^{-1}H_l^n):\Gamma H^n \to H^{2n}\Gamma$$

is a natural transformation.

We now verify (16.7). Let $X = (D, K)$. Then it follows from (16.3) that

$$H_l^r(X, A) = (D, H^r\{K, A\}), \qquad \Gamma_l H_l^r(X, A) = (D, \Gamma H^r\{K, A\}).$$

Let

$$\lambda(d):H^n(K(d), A) \to H^n(X, A)$$
$$\mu(d):\Gamma H^n(K(d), A) \to L\Gamma_l H_l^n(X, A)$$
$$\nu(d):H^{2n}(K(d), A) \to H^{2n}(X, A)$$

be the injections. Let $(1, \rho):\Gamma_l H_l^n(X, A) \to H_l^{2n}(X, A)$ be a homomorphism of the direct system $\Gamma_l H_l^n(X, A)$ into the direct system $H_l^{2n}(X, A)$. Then

$$L(1, \rho):L\Gamma_l H_l^n(X, A) \to H^{2n}(X, A)$$

is given by

$$L(1, \rho)\mu(d)g(d) = \nu(d)\rho(d)g(d),$$

where $g(d) \in \Gamma H^n(K(d), A)$.

Let

$$\mathbf{y} = \lambda(d)y(d), \qquad \mathbf{z} = \lambda(d)z(d),$$

where $y(d)$, $z(d) \in H^n(K(d), A)$. Since

$$[y(d), z(d)] = \gamma(y(d) + z(d)) - \gamma(y(d)) - \gamma(z(d))$$

it follows from (9.1) that

$$\omega(H_l^n(X, A))\mu(d)[y(d), z(d)] = [\mathbf{y}, \mathbf{z}].$$

Therefore

$$(\omega^{-1}H_l^n)(X, A)[\mathbf{y}, \mathbf{z}] = \omega(H_l^n(X, A))^{-1}[\mathbf{y}, \mathbf{z}]$$
$$= \mu(d)[y(d), z(d)].$$

By hypothesis $\tau(K(d), A)[y(d), z(d)] = y(d) \cup z(d)$. Therefore

$$\begin{aligned}\eta(X, A)[\mathbf{y}, \mathbf{z}] &= (L\tau\iota)(X, A)\{(\omega^{-1}H_\iota^n)(X, A)[\mathbf{y}, \mathbf{z}]\} \\ &= L(1, \tau\{K, A\})\mu(d)[y(d), z(d)] \\ &= \nu(d)\tau(K(d),A)[y(d), z(d)] \\ &= \nu(d)(y(d) \cup z(d)) \\ &= \mathbf{y} \cup \mathbf{z}.\end{aligned}$$

Therefore it only remains to prove the theorem for the category $(\mathfrak{K}_\sigma, \mathfrak{A})$.

Let K be a finite simplicial complex, let $C^r = C^r(K)$ be the group of integral, r-dimensional cochains in K and let $c_1, \cdots, c_q$ be a canonical basis for C^n. Then

$$\delta c_i = \sigma_i d_i \qquad (i = 1, \cdots, q; \sigma_i \mid \sigma_{i+1}), \tag{16.8}$$

where $\sigma_i d_i = 0$ if $i > t$ and $(d_1, \cdots, d_t)$ is part of a canonical basis for C^{n+1}. We recall from [3] the definition of the *Pontrjagin Square*

$$\mathfrak{p}c = c \cup c + c \cup_1 \delta c \qquad (c \in C^n). \tag{16.9}$$

The cochain group, $C^n(A)$, is the tensor product,

$$C^n(A) = A \circ C,$$

and the group of cocycles, $Z^n(A) \subset C^n(A)$, consists of those, and only those, cochains,

$$x = a_1 \cdot c_1 + \cdots + a_q \cdot c_q,$$

such that $\sigma_1 a_2 = \cdots = \sigma_q a_q = 0$, where $\sigma_i = 0$ if $i > t$. The cup-product of cocycles,

$$y = a_1 \cdot c_1 + \cdots + a_q \cdot c_q, \qquad z = b_1 \cdot c_1 + \cdots + b_q \cdot c_q \tag{16.10}$$

is defined by

$$y \cup z = \textstyle\sum_i \sum_j [a_i, b_j] \cdot c_i \cup c_j.$$

If $i < j$ then $\sigma_i \mid \sigma_j$ and, since n is even,

$$c_i \cup c_j \backsim c_j \cup c_i \qquad \text{mod. } \sigma_i.$$

Also $\sigma_i a_i = \sigma_i b_i = 0$. Therefore

$$y \cup z \backsim \sum_i [a_i, b_i] \cdot c_i \cup c_i + \sum_{i<j} ([a_i, b_j] + [a_j, b_i]) \cdot c_i \cup c_j. \tag{16.11}$$

I say that a homomorphism

$$v: \Gamma(Z^n(A)) \to Z^{2n}(\Gamma(A)) \tag{16.12}$$

is defined by

$$v\gamma(y) = \Sigma_i \gamma(a_i) \cdot \mathfrak{p}c_i + \sum_{i<j} [a_i, a_j] \cdot c_i \cup c_j, \tag{16.13}$$

where $y \in Z^n(A)$ is given by (16.10). For since $\sigma_i a_i = 0$ we have

$$\sigma_i^2\gamma(a_i) = \gamma(\sigma_i a_i) = 0$$
$$2\sigma_i\gamma(a_i) = \sigma_i[a_i, a_i] = [\sigma_i a_i, a_i] = 0.$$

Therefore $(\sigma_i^2, 2\sigma_i)\gamma(a_i) = 0$. That is to say $\sigma_i\gamma(a_i) = 0$ if σ_i is odd and $2\sigma_i\gamma(a_i) = 0$ if σ_i is even. Obviously $\delta \mathfrak{p} c_i \equiv 0$, mod. σ_i, and it is proved in [3] that $\delta\mathfrak{p}c_i \equiv 0$, mod. $2\sigma_i$, if σ_i is even. Therefore, and since $\sigma_i[a_i, a_j] = \sigma_j[a_i, a_j] = 0$, it follows that $\delta v\gamma(y) = 0$. That is to say, $v\gamma(y) \in Z^{2n}(\Gamma(A))$.

Obviously $v\gamma(-y) = v\gamma(y)$. Therefore v is consistent with (2.1a). Let $z \in Z^n(A)$ be given by (16.10). Then

$$y + z = (a_1 + b_1)\cdot c_1 + \cdots + (a_q + b_q)\cdot c_q .$$

Since $\gamma(a_i + b_i) - \gamma(a_i) - \gamma(b_i) = [a_i, b_i]$ and

$$[a_i + b_i, a_j + b_j] - [a_i, a_j] - [b_i, b_j] = [a_i, b_j] + [a_j, b_i]$$

we have

$$v[y, z] = v\gamma(y + z) - v\gamma(y) - v\gamma(z)$$
$$(16.14) \qquad = \sum_i [a_i, b_i]\cdot \mathfrak{p}c_i + \sum_{i<j} ([a_i, b_j] + [a_j, b_i])\cdot c_i \cup c_j .$$

The right hand side of (16.14) is bilinear with respect to $(a_1, \cdots, a_q)$ and $(b_1, \cdots, b_q)$. Therefore v is consistent with (5.6). Therefore (16.12) is a homomorphism.

Since $\sigma_i[a_i, b_i] = [\sigma_i a_i, b_i] = 0$ it follows from (16.8) and (16.9) that

$$[a_i, b_i]\cdot\mathfrak{p}c_i = [a_i, b_i]\cdot c_i \cup c_i + \sigma_i[a_i, b_i]\cdot c_i \cup_1 d_i$$
$$= [a_i, b_i]\cdot c_i \cup c_i .$$

Therefore it follows from (16.14) and (16.11) that

$$(16.15) \qquad v[y, z] \backsim y \cup z.$$

Let $y = \delta w$ where $w \in C^{n-1}(A)$. Let

$$w = \sum_{\lambda=1}^{s} a_\lambda \cdot \bar{c}_\lambda ,$$

where $(\bar{c}_1, \cdots, \bar{c}_s)$ is part of a canonical basis for C^{n-1} and $\delta\bar{c}_\lambda = \tau_\lambda c_{t+\lambda}$. Then

$$y = \textstyle\sum_\lambda a_\lambda \cdot \delta\bar{c}_\lambda = \sum_\lambda \tau_\lambda\, a_\lambda \cdot c_{t+\lambda} .$$

If $\delta c = 0$ we have $\mathfrak{p}c = c \cup c$. Therefore

$$\gamma(\tau_\lambda a_\lambda)\cdot\mathfrak{p}c_{t+\lambda} = \gamma(a_\lambda)\cdot\tau_\lambda^2(c_{t+\lambda} \cup c_{t+\lambda})$$
$$= \gamma(a_\lambda)\cdot\delta\bar{c}_\lambda \cup \delta\bar{c}_\lambda$$
$$= \delta\{\gamma(a_\lambda)\cdot\bar{c}_\lambda \cup \delta\bar{c}_\lambda\}$$

$$[\tau_\lambda a_\lambda, \tau_\mu a_\mu]\cdot c_{t+\lambda} \cup c_{t+\lambda} = [a_\lambda, a_\mu]\cdot\delta\bar{c}_\lambda \cup \delta\bar{c}_\mu = \delta([a_\lambda, a_\mu]\cdot\bar{c}_\lambda \cup \delta\bar{c}_\mu).$$

Therefore it follows from (16.13) that $v\gamma(\delta w) \backsim 0$. Also it follows from (16.15) that

$$v[\delta w, z] \backsim (\delta w) \cup z \backsim 0.$$

Therefore it follows from Theorem 4 in §6 above that the kernel of the homomorphism

$$\Gamma(Z^n(A)) \to \Gamma(H^n(K, A)),$$

which is induced by the natural homomorphism, $Z^n(A) \to H^n(K, A)$, is carried by v into the group of coboundaries in $Z^{2n}(\Gamma(A))$. Therefore v induces a homomorphism

$$\tau(K, A): \Gamma(H^n(K, A)) \to H^{2n}(K, \Gamma(A)).$$

Also it follows from (16.15) that

$$\tau(K, A)[y^*, z^*] = y^* \cup z^*,$$

where $y^*, z^* \epsilon H^n(K, A)$.

Let $(\rho, \alpha): (K, A) \to (K', A')$ be any mapping in the category $(\mathfrak{K}_\sigma, \mathfrak{A})$. Then (ρ, α) is the resultant of $(1, \alpha)$, followed by $(\rho, 1)$, where each 1 denotes the appropriate identity. It is obvious that not only τ, but even v is natural with respect to the homomorphisms $\alpha: A \to A'$. It remains to prove that τ is natural with respect to maps $K \to K'$.

Let

$$f^n: Z^n(K', A) \to Z^n(K, A)$$
$$f^{2n}: Z^{2n}(K', \Gamma(A)) \to Z^{2n}(K, \Gamma(A))$$

be the homomorphisms induced by a simplicial map $\phi: K \to K'$. Let

$$g^n: \Gamma(Z^n(K', A)) \to \Gamma(Z^n(K, A))$$

be the homomorphism induced by f^n and let

$$v': \Gamma(Z^n(K', A)) \to Z^{2n}(K', \Gamma(A))$$

be defined in the same way as v, by means of a canonical basis,[39] $(c_1', \cdots, c_{q'}')$, for $C^n(K)$. We have to prove that

$$f^{2n}v'\gamma(y') \backsim v'g^n\gamma(y') = v'\gamma(f^n y'),$$

for any $y' \epsilon Z^n(K', A)$.

Let $\delta c_i' = \sigma_i' d_i' (i = 1, \cdots, t')$, $\delta c_j' = 0 (j = t' + 1, \cdots, q')$, where $(d_1', \cdots, d_{t'}')$ is part of a basis for $C^{n+1}(K')$. Then $Z^n(K', A)$ is generated by cocycles of the form $a \cdot c_i'$, where $\sigma_i' a = 0$ and $\sigma_i' = 0$ if $i > t'$. Therefore it follows from Theorem 5 that $\Gamma(Z^n(K', A))$ is generated by the elements $\gamma(a \cdot c_i')$, where $\sigma_i' a = 0$, together with $[y', z']$, for every pair $y', z' \epsilon Z^n(K', A)$. Since

$$f^{2n}v'[y', z'] \backsim f^{2n}(y' \cup z') \backsim f^n y' \cup f^n z'$$
$$\backsim v[f^n y', f^n z'] = vg^n[y', z']$$

[39] Possibly $K' = K$. In this case the following argument shows that τ is independent of the choice of the canonical basis $(c_1, \cdots, c_q)$.

it only remains to prove that, if $\sigma_i' a = 0$, then

$$f^{2n} v' \gamma(a \cdot c_i') \backsim v\gamma(f^n(a \cdot c_i')) = v\gamma(a \cdot \phi^n c_i'), \tag{16.16}$$

where $\phi^r : C^r(K') \to C^r(K)$ is the (integral) cochain mapping induced by ϕ.

Let $\phi^n c_i' = n_1 c_1 + \cdots + n_q c_q$ for a fixed, but arbitrary value of i. Let $\sigma_i' a = 0$. Then it follows from one of the preceding arguments that

$$(\sigma_i'^2, 2\sigma_i')\gamma(a) = 0. \tag{16.17}$$

If σ_i' is odd, then

$$\begin{aligned} \phi^{2n} \mathfrak{p} c_i' &\equiv \phi^{2n}(c_i' \cup c_i') && \text{mod. } \sigma_i' \\ &\backsim \phi^n c_i' \cup \phi^n c_i' \\ &\equiv \mathfrak{p}\phi^n c_i' . \end{aligned}$$

If σ_i' is even then

$$\phi^{2n} \mathfrak{p} c_i' \backsim \mathfrak{p}\phi^n c_i' \qquad \text{mod. } 2\sigma_i' , \tag{16.18}$$

as shown in [3]. In either case

$$\phi^{2n} \mathfrak{p} c_i' \backsim \mathfrak{p}\phi^n c_i' = \mathfrak{p}(n_1 c_1 + \cdots + n_q c_q) \qquad \text{mod. } (\sigma_i'^2, 2\sigma_i'). \tag{16.19}$$

Since $\delta c_i' \equiv 0$, mod. σ_i', and since $\delta c_j = \sigma_j d_j$, where $(d_1, \cdots, d_t)$ is part of a basis for $C^{n+1}(K)$, it follows that $\sigma_i' \mid n_j \sigma_j$. Therefore $n_j c_j$ is a cocycle, mod. σ_i', for each $j = 1, \cdots, q$. Let σ_i' be even. Then it follows from (16.17,) (16.19) and (4.7) in [3] that

$$\begin{aligned} f^{2n} v' \gamma(a \cdot c_i') &= f^{2n}(\gamma(a) \cdot \mathfrak{p} c_i') = \gamma(a) \cdot \phi^{2n} \mathfrak{p} c_i' \\ &\backsim \gamma(a) \cdot \mathfrak{p}(n_1 c_1 + \cdots + n_q c_q) \\ &\backsim \gamma(a) \cdot (\Sigma_i \mathfrak{p}(n_i c_i) + 2 \sum_{i<j} n_i c_i \cup n_j c_j) \\ &= \sum_i n_i^2 \gamma(a) \cdot \mathfrak{p} c_i + \sum_{i<j} [a, a] \cdot n_i c_i \cup n_j c_j \\ &= \sum_i \gamma(n_i a) \cdot \mathfrak{p} c_i + \sum_{i<j} [n_i a, n_j a] \cdot c_i \cup c_j \\ &= v\gamma(n_1 a \cdot c_1 + \cdots + n_q a \cdot c_q) \\ &= v\gamma(a \cdot \phi^n c_i'). \end{aligned}$$

If σ_i' is odd we have the same result on replacing each Pontrjagin square, $\mathfrak{p}c$, by $c \cup c$. This proves (16.16) and hence the theorem.

Let n be even, let $\mathbf{y} \in H^n(X, A)$ and let

$$\mathfrak{p}\mathbf{y} = \eta(X, A)\gamma(\mathbf{y}).$$

We call $\mathfrak{p}\mathbf{y}$ the *Pontrjagin square* of $\mathbf{y}$. It follows from (16.7) that

$$\mathfrak{p}(\mathbf{y} + \mathbf{z}) = \mathfrak{p}\mathbf{y} + \mathfrak{p}\mathbf{z} + \mathbf{y} \cup \mathbf{z}. \tag{16.20}$$

Thus $-\mathbf{y} \cup \mathbf{z}$ appears as a factor set, which measures the error made in supposing

$$\mathfrak{p}: H^n(X, A) \to H^{2n}(X, \Gamma(A))$$

to be a homomorphism. We also have $\mathfrak{p}(r\mathbf{y}) = r^2\mathfrak{p}\mathbf{y}$, where r is any integer. Therefore (16.20), with $\mathbf{y} = \mathbf{z}$, gives

$$2\mathfrak{p}\mathbf{y} = \mathbf{y} \cup \mathbf{y}. \tag{16.21}$$

Let $(\xi, \alpha): (X, A) \to (X', A')$ be any mapping in the category $(\mathfrak{X}, \mathfrak{A})$. Since η is natural we have

$$\begin{aligned} \mathfrak{p}H^n(\xi, \alpha)\mathbf{y} &= \eta(X, A')\gamma(H^n(\xi, \alpha)\mathbf{y}) \\ &= \eta(X, A')\Gamma H^n(\xi, \alpha)\gamma(\mathbf{y}) \\ &= H^{2n}\Gamma(\xi, \alpha)\eta(X', A)\gamma(\mathbf{y}) \\ &= H^{2n}(\xi, \Gamma\alpha)\mathfrak{p}\mathbf{y} \end{aligned}$$

or, writing $f = H^n(\xi, \alpha)$, $g = H^{2n}(\xi, \Gamma\alpha)$,

$$\mathfrak{p}f = g\mathfrak{p}: H^n(X', A) \to H^{2n}(X, \Gamma(A')). \tag{16.22}$$

Let $|X|$ be an arbitrary topological space and let D be the directed set, which consists of all finite coverings of $|X|$ by open[40] sets. Let $K(d)$ be the nerve of the covering d. Then (D, K) is a net. Let (D', K') be similarly defined in terms of a space $|X'|$. Then a map $\phi: |X| \to |X'|$ induces the homomorphism.

$$(R^*, \rho): (D, K) \to (D', K'),$$

in which $R^* d'$ is the covering $\{\phi^{-1}U'\}$, where $d' = \{U'\}$, and $\rho(d')$ is determined by the transformation, $\phi^{-1}U' \to U'$, of the vertices of $K(R^* d')$ into those of $K'(d')$. Therefore η, and likewise $\mathfrak{p}$, are topological invariants of $|X|$.

Let K be a finite cell complex, which need not be a polyhedron. We take

$$\begin{cases} C_r(K) = H_r(K^r, K^{r-1}) \\ C^r(K) = \operatorname{Hom}\{C_r(K), I_0\} \end{cases} \tag{16.23}$$

to be the groups of r-dimensional, integral chains and cochains in K, where I_0 is the group of integers. By Theorem 13 in CH I there is a finite, simplicial complex, L, which is of the same homotopy type as K. Let $\phi: K \to L$, $\psi: L \to K$ be cellular maps such that $\psi\phi \simeq 1$, $\phi\psi \simeq 1$. Let

$$\phi^r: C^r(L) \to C^r(K), \qquad \psi^r: C^r(K) \to C^r(L)$$

be the cochain equivalences induced by ϕ, ψ. We define

$$c \cup c' = \phi^{2n}(\psi^n c \cup \psi^n c'), \qquad \mathfrak{p}c = \phi^{2n}\mathfrak{p}\psi^n c, \tag{16.24}$$

where c, c' are any elements of $C^n(K)$. If c, c' are cocycles mod. σ, so are $\psi^n c$, $\psi^n c'$ and

$$\psi^{2n}(c \cup c') = \psi^{2n}\phi^{2n}(\psi^n c \cup \psi^n c') \backsim \psi^n c \cup \psi^n c', \qquad \text{mod. } \sigma.$$

[40] We could equally well take closed sets

Similarly $\psi^{2n}\mathfrak{p}r \smile \mathfrak{p}\psi^{n}c$, mod. 2σ, if σ is even. Notice that this relation is analogous to (16.18).

Let X, Y be the nets which are defined by means of all the finite, open coverings of the spaces, K, L. Since K, L are compacta it follows from the Čech cohomology theory that $\phi:K \to L$ induces isomorphisms $H^r(Y, G) \approx H^r(X, G)$, for every $r \geqq 0$ and every coefficient group G. Also the cohomology group $H^r(L, G)$, which is calculated in terms of cochains in $C^r(L)$, may be identified with $H^r(Y, G)$. When this is done $\eta(Y, A)$ becomes the homomorphism, $\tau(L, A)$, which is defined by means of (16.13). It follows from the final arguments in the proof of Theorem 18 that $H^r(X, G)$ and $\eta(X, G)$ may be similarly identified with $H^r(K, G)$ and $\tau(K, G)$, which are defined in terms of $C^r(K)$, when cup-products and Pontrjagin squares of cochains are given by (16.24).

If K has a simplicial sub-division we take this to be L and, as in [3], we take $\phi:K \to L$ to be the identity and $\psi:L \to K$ to be such that $\psi L_0 \subset K_0$, for every subcomplex $K_0 \subset K$, where L_0 is the subcomplex of L, which covers K_0.

17. Secondary boundary operators

Let K be a finite cell complex, let $C_r(K)$, $C^r(K)$ be defined by (16.23) and let cup-products and Pontrjagin squares of integral cochains be defined by (16.24). Let $(c_1, \cdots, c_q)$ be a canonical basis for $C^n(K)$, with $\delta c_i = \sigma_i d_i$, where $(d_1, \cdots, d_t)$ is part of a basis for $C^{n+1}(K)$ and $\sigma_i d_i = 0$ if $i > t$. Let $m \geqq 0$ and let

$$H^n(A) = H^n(K, A), \qquad H_n(m) = H_n(K, I_m),$$

where I_m is the group of integers reduced mod.m. Let

$$y^* \epsilon H^n(A), \qquad z_* \epsilon H_n(m), \qquad a(m) \epsilon A_m = A/mA$$

be the cohomology, homology and residue classes of $y \epsilon Z^n(A)$, $z \epsilon Z_n(K, I_m)$, $a \epsilon A$. Then a homomorphism[41]

$$u_m = u_m^n : H^n(A) \to \text{Hom }\{H_n(m), A\}$$

is defined by

$$(u_m y^*)z_* = (c_1 z)a_1(m) + \cdots + (c_q z)a_q(m), \tag{17.1}$$

where $y = a_1 \cdot c_1 + \cdots a_q \cdot c_q$, with $\sigma_i a_i = 0$. If K is without $(n - 1)$-dimensional torsion, then

$$u_0 : H^n(A) \approx \text{Hom }(H_n, A),$$

where $H_n = H_n(0)$.

Let K be simply connected. We make the natural identification $\Pi_2(K) = H_2$ and we also identify $\Gamma(\Pi_2(K))$ with $\Gamma_3 = \Gamma_3(K)$ by means of θ, in Theorem 14. Also K has no 1-dimensional torsion and we identify each $y^* \epsilon H^2(A)$ with

[41] Cf. (12.2) in [14].

$u_0 y^*$. Moreover we take $A = H_2$, so that $H^2(H_2)$ is the additive group of the ring of endomorphisms of H_2. Then we have maps

$$\mathrm{Hom}\,(H_2, H_2) \xrightarrow[\mathfrak{p}]{} H^4(\Gamma_3) \xrightarrow[u_m]{} \mathrm{Hom}\,\{H_4(m), \Gamma_{3,m}\}.$$

Since $\pi_1(K) = 1$ the group $C_n(K)$ $(n \geqq 3)$ may be identified with C_n in the system $(C, A)(K)$. Let $\mathfrak{b}(m) : H_4(m) \to \Gamma_{3,m}$ be the secondary modular boundary homomorphism. Let $1 \in H^2(H_2)$ be the identity $1 : H_2 \to H_2$.

THEOREM 20. $\mathfrak{b}(m) = u_m \mathfrak{p}(1)$.

First let K be any finite cell complex, which need not be simply connected, and let the notations be the same as in (17.1). Let $a \in A$ and let $a \cdot c \in Z^n(A)$, where $c = n_1 c_1 + \cdots + n_q c_q$. Then

$$\begin{aligned} \{u_m(a\cdot c)^*\} z_* &= \left\{ u_m \left(\sum_{i=1}^{q} n_i a \cdot c_i \right)^* \right\} z_* \\ &= \sum_{i=1}^{q} (c_i z)(n_i a)(m) \\ &= \sum_{i=1}^{q} (n_i c_i z) a(m) \\ &= (cz) a(m). \end{aligned} \tag{17.2}$$

Let $\phi : K \to K'$ be a cellular map into a finite cell complex, K', and let

$$\alpha : A \to A'$$

be a given homomorphism. Let $H'_n(m) = H'_n(K', I_m)$ and let

$$f : H_n(m) \to H'_n(m), \qquad \bar{\alpha} : A_m \to A'_m \tag{17.3}$$

be the homomorphisms induced by ϕ and α. Consider the diagram

$$\begin{array}{ccccc} H^n(K', A') & \xrightarrow{\phi^*} & H^n(K, A') & \xleftarrow{\alpha_*} & H^n(K, A) \\ u_m \downarrow & & \downarrow u_m & & \downarrow u_m \\ \{H'_n(m), A'_m\} & \xrightarrow{f^*} & \{H_n(m), A'_m\} & \xleftarrow{\bar{\alpha}_*} & \{H_n(m), A_m\} \end{array}$$

in which $\{H, A\}$ denotes Hom (H, A) and $\phi^*, f^*, \alpha_*, \bar{\alpha}_*$ are induced by $\phi, f, \alpha, \bar{\alpha}$. Thus

$$\alpha_*(a\cdot c)^* = (\alpha a \cdot c)^*, \qquad \bar{\alpha}_* h = \bar{\alpha} h, \qquad f^* h' = h' f,$$

where $(a \cdot c) \in Z^n(K, A)$, $h \in \{H_n(m), A_m\}$, $h' \in \{H'_n(m), A'_m\}$. Let $(c'_1, \cdots, c'_{q'})$ be the canonical basis for $C^n(K')$, in terms of which u_m, operating on $H^n(K', A)$, is defined and let f, ϕ^* etc. also denote the corresponding maps of chains and cochains. Let

$$y' = a'_1 \cdot c'_1 + \cdots + a'_{q'} \cdot c'_{q'} \in Z^n(K', A').$$

Then $\phi^* y' = a'_1 \cdot \phi^* c'_1 + \cdots + a'_{q'} \phi^* c'_{q'}$. Let $y = a_1 \cdot c_1 + \cdots a_q \cdot c_q$. Then it follows from (17.2) that

$$\begin{aligned}\{u_m(\phi^*y'^*)\}z_* &= \{u_m(\phi^*y')^*\}z_* \\ &= \Sigma_i\{(\phi^*c_i')z\}a_i'(m) \\ &= \Sigma_i(c_i' fz)a_i'(m) \\ &= (u_m y'^*)fz_* \\ &= \{f^*(u_m y'^*)\}z_* \\ \{u_m(\alpha_* y^*)\}z_* &= \Sigma_i(c_i z)(\alpha a_i)(m) \\ &= \Sigma_i(c_i z)\bar{\alpha}a_i(m) \\ &= \bar{\alpha}\{(u_m y^*)z_*\} \\ &= \{\bar{\alpha}_*(u_m y^*)\}z_* .\end{aligned}$$

Therefore

$$u_m\phi^* = f^*u_m, \qquad u_m\alpha_* = \bar{\alpha}_*u_m . \tag{17.3}$$

Now let K, K' be simply connected and, as before, let

$$\Pi_2 = H_2, \qquad \Gamma(H_2) = \Gamma_3, \qquad H^2(H_2) = \{H_2, H_2\},$$

with the analogous identifications in K'. Let $\phi:K \to K'$ and $\alpha:A \to A'$ mean the same as before and, returning to the notation used in Chapter I, let

$$\begin{aligned}&\mathfrak{h}:H_2 \to H_2' = H_2'(0), &&\mathfrak{h}(m):H_4(m) \to H_4'(m)\\ &\mathfrak{g}:\Gamma_3 \to \Gamma_3' = \Gamma(H_2'), &&\mathfrak{g}(m):\Gamma_{3,m} \to \Gamma_{3,m}'\end{aligned}$$

be the homomorphisms induced by ϕ. It follows from (17.3) and (16.22) that the diagram

$$\begin{array}{ccccc} \{H_2', H_2'\} & \xrightarrow[\mathfrak{h}^*]{} & \{H_2, H_2'\} & \xleftarrow[\mathfrak{h}_*]{} & \{H_2, H_2\} \\ \mathfrak{p}\downarrow & & \downarrow\mathfrak{p} & & \downarrow\mathfrak{p} \\ H^4(K', \Gamma_3') & \xrightarrow{\phi^*} & H^4(K, \Gamma_3') & \xleftarrow{\mathfrak{g}_*} & H^4(K, \Gamma_3) \\ u_m\downarrow & & \downarrow u_m & & \downarrow u_m \\ \{H_4'(m), \Gamma_{3,m}'\} & \xrightarrow{\mathfrak{h}(m)^*} & \{H_4(m), \Gamma_{3,m}'\} & \xleftarrow{\mathfrak{g}(m)_*} & \{H_4(m), \Gamma_{3,m}\} \end{array}$$

is commutative, where $\mathfrak{h}_*e = \mathfrak{h}e:H_2 \to H_2'$, $\mathfrak{h}^*e' = e'\mathfrak{h}$ for every $e \in \{H_2, H_2\}$ $e' \in \{H_2', H_2'\}$ and the two bottom layers are the same as before, with $n = 4$, $A = \Gamma_3$ and $\alpha = \mathfrak{g}$. Since $\mathfrak{h}^*(1) = \mathfrak{h} = \mathfrak{h}_*(1)$ it follows that

$$\begin{aligned}\{u_m\mathfrak{p}(1)\}\mathfrak{h}(m) &= \mathfrak{h}(m)^*u_m\mathfrak{p}(1) = u_m\mathfrak{p}\mathfrak{h}^*(1) \\ &= u_m\mathfrak{p}\mathfrak{h}_*(1) = \mathfrak{g}(m)_*u_m\mathfrak{p}(1) \\ &= \mathfrak{g}(m)u_m\mathfrak{p}(1).\end{aligned}$$

We also have $\mathfrak{b}(m)\mathfrak{h}(m) = \mathfrak{g}(m)\mathfrak{b}(m)$, according to (3.6).

Therefore

$$\{\mathfrak{b}(m) - u_m\mathfrak{p}(1)\}\mathfrak{h}(m) = \mathfrak{g}(m)\{\mathfrak{b}(m) - u_m\mathfrak{p}(1)\}. \tag{17.4}$$

Let K' be any simply connected complex, let $K = K'^4$ and let $\phi: K \to K'$ be the identical map. Then $\mathfrak{h}(m)$ is onto and it follows from (17.4) that Theorem 19 is true of K' if it is true of K. Therefore we need only consider complexes of at most four dimensions.

Let K' be any complex of the same homotopy type as K and let $\phi: K \to K'$ be a homotopy equivalence. Then $\mathfrak{g}(m)$ is an isomorphism and it follows from (17.4) that Theorem 19 is true of K if it is true of K'. Therefore we may replace K by any complex of the same homotopy type. Therefore we may take K to be a reduced, 4-dimensional complex, as defined in [3].

Let K be a reduced, 4-dimensional complex. Then

$$K^2 = e^0 \cup e_1^2 \cup \cdots \cup e_q^2, \qquad K^3 = K^2 \cup e_1^3 \cup \cdots \cup e_t^3 \cup e_{t+1}^3 \cup \cdots \cup e_{t+l}^3,$$

where e_i^3 $(i = 1, \cdots, t)$ is attached to K^2 by a map, $S_i^2 \to \bar{e}_i^2$, of degree σ_i, and $\bar{e}_{t+\lambda}^3$ $(\lambda = 1, \cdots, l)$ is a 3-sphere attached to e^0. Obviously $Z_2(K) = C_2(K)$. Moreover we may assume that $\sigma_i \mid \sigma_{i+1}$ in which case $(c^1 \cdots, c^q)$ is a canonical basis for $C_2(K)$, where c^i is represented by a homeomorphism $S^2 \to \bar{e}_i^2$. Let $z \in Z_2(K)$ and let $(c_1, \cdots, c_q)$ be the basis for $C^2(K)$, which is dual to $(c^1, \cdots, c^q)$. Then

$$z_* = (c_1z)c_*^1 + \cdots + (c_qz)c_*^q.$$

Therefore it follows from (17.1), with $m = 0, n = 2, H^2(H_2) = \{H_2, H_2\}, u_0 = 1$ and $a_i = c_*^i$, that

$$(c_*^1 \cdot c_1 + \cdots + c_*^q \cdot c_q)^* z_* = z_*.$$

Therefore $(c_*^1 \cdot c_1 + \cdots + c_*^q \cdot c_q)^* = 1$.

Since K is reduced we have $\mathfrak{p}r = c \cup c$ for any $c \in C^2(K)$, according to (10.1) in [3]. Therefore it follows from (16.13) that

$$\begin{aligned} \mathfrak{p}(1) &= (\Sigma_i e^{ii} \cdot \mathfrak{p}c_i + \sum_{i<j} e^{ij} \cdot c_i \cup c_j)^* \\ &= (\sum_{i \leq j} e^{ij} \cdot c_i \cup c_j)^* \end{aligned}$$

where $e^{ii} = \gamma(c_*^i)$, $e^{ij} = [c_*^i, c_*^j]$. Also it follows from (14.1) in [3] and (17.2) above, with $n = 4$, $A = \Gamma_3$ that

$$\begin{aligned} \mathfrak{b}(m)z_* &= \sum_{i \leq j} \{(c_i \cup c_j)z\} e^{ij}(m) \\ &= \{u_m \sum_{i \leq j} (e^{ij} \cdot c_i \cup c_j)^*\} z_* \\ &= \{u_m \mathfrak{p}(1)\} z_*. \end{aligned} \tag{17.5}$$

Therefore $\mathfrak{b}(m) = u_m\mathfrak{p}(1)$ and the proof is complete.

Let K be without 2-dimensional torsion, so that H^2 and H_2 are free Abelian groups of the same rank. Subject to this condition, G. Hirsch ([19]) has given a very elegant expression for the kernel, G, of the natural homomorphism

$$\pi_3(K) \to H_3 .$$

Let S be the group of symmetric homomorphisms, $f:H^2 \to H_2$, which is defined at the end of §5 above, with $A = H_2$, $A^* = H^2$. Let $z_* \in H_4$ and let $f_z:H^2 \to H_2$ be given by

$$f_z c^* = c^* \cap z_* \qquad (c^* \in H^2).$$

Then $f_z \in S$ and a homomorphism, $\mu:H_4 \to S$, is defined by $\mu z_* = f_z$. Hirsch's theorem states that $G \approx S/\mu H_4$. We give an alternative proof of this.

Let $c_1^*, \cdots, c_q^*$ be a basis for H^2. Then it follows from (5.13) and (5.12) that $\lambda:S \approx \Gamma_3$, where

$$\lambda f = \sum_{i \leqq j} (c_i^* f c_j^*) e^{ij},$$

and from (17.5), with $m = 0$, that

$$\begin{aligned} \mathfrak{b} z_* &= \sum_{j \leqq i} \{c_i^*(c_j^* \cap z_*)\} e^{ij} = \sum_{j \leqq i} (c_i^* f_z c_j^*) e^{ij} \\ &= \lambda f_z = \lambda\mu z_* \end{aligned}$$

Therefore $\mathfrak{b} = \lambda\mu$ and it follows that λ induces an isomorphism

$$S/\mu H_4 \approx \Gamma_3/\mathfrak{b} H_4 .$$

Therefore Hirsch's theorem follows from the exactness of $\Sigma(K)$.

Let us discard the condition that $\pi_1(K) = 1$ but let K be without $(n-1)$-dimensional torsion for some $n \geqq 2$. We take $A = H_n = H_n(0)$ and use u_0^n to identify $H^n(H_n)$ with the additive group of the ring of endomorphisms of H_n. It follows from (17.1), with $m = 0$ and $a_i = z_*^i$, that $H^n(H_n)$ operates on H_n according to the rule

$$e z_* = (c_1 z) z_*^1 + \cdots + (c_q z) z_*^q ,$$

where $e = (z_*^1 \cdot c_1 + \cdots + z_*^q \cdot c_q)^*$. Let $\bar{e}':H^n(H_n) \to H^n(H_n)$ be the endomorphism, which is induced by a given endomorphism $e':H_n \to H_n$. Then

$$\begin{aligned} (\bar{e}' e) z_* &= (e' z_*^1 \cdot c_1 + \cdots + e' z_*^q \cdot c_q)^* z_* \\ &= (c_1 z) e' z_*^1 + \cdots + (c_q z) e' z_*^q \\ &= e'(e z_*). \end{aligned}$$

Therefore $\bar{e}' e = e' e$. Let $g(e)$ be the endomorphism of $H^{2n}\{\Gamma(H_n)\}$ which is induced by $\Gamma e:\Gamma(H_n) \to \Gamma(H_n)$. Then it follows from (16.22), with $f = \bar{e}$, $g = g(e)$, that

$$\mathfrak{p} e = \mathfrak{p}\bar{e}(1) = g(e)\mathfrak{p}(1).$$

Therefore $\mathfrak{p}:H^n(H_n) \to H^{2n}\{\Gamma(H_n)\}$ is determined by the correspondence

$$e \to g(e)$$

together with $\mathfrak{p}(1)$.

18. The calculation of $\Sigma_4(K)$

We return to the sequence

$$\Sigma_4: \quad H_4 \xrightarrow{\mathfrak{b}} \Gamma_3 \xrightarrow{\mathfrak{i}} \Pi_3 \xrightarrow{\mathfrak{j}} H_3 \to 0.$$

We make the same identifications, $\Pi_2 = H_2$ and $\Gamma(H_2) = \Gamma_3$ as before. We also identify each $\bar{\gamma} \in \Gamma_3/\mathfrak{b}H_4$ with $\bar{\mathfrak{i}}\bar{\gamma} \in \mathfrak{i}_3\Gamma_3 = G$, say, where

$$\bar{\mathfrak{i}}: \Gamma_3/\mathfrak{b}H_4 \approx G$$

is the isomorphism induced by $\mathfrak{i}_3$. Then $\mathfrak{i}_3$ becomes the natural homomorphism $\Gamma_3 \to G$.

The group Π_3 is an extension of H_3 by G. Let Π_3' be an equivalent extension. Then there is a homomorphism, $\mathfrak{j}_3': \Pi_3' \to H_3$, and an isomorphism, $\mathfrak{f}: \Pi_3 \approx \Pi_3'$, such that

$$\mathfrak{f}g = g, \qquad \mathfrak{j}_3'\mathfrak{f} = \mathfrak{j}_3, \qquad (g \in G),$$

whence $\mathfrak{j}_3'^{-1}(0) = \mathfrak{j}_3'^{-1}(0) = G$. Let Σ_4' be the sequence which is obtained from Σ_4 on replacing $\mathfrak{j}_3: \Pi_3 \to H_3$ by $\mathfrak{j}_3': \Pi_3' \to H_3$. Then $F: \Sigma_4 \approx \Sigma_4'$ is a proper isomorphism, where F consists of $\mathfrak{f}: \Pi_3 \approx \Pi_3'$ and the identical automorphisms of the other groups. Therefore Σ_4 is determined, up to a proper isomorphism, by the groups H_2, H_3, H_4, the homomorphism $\mathfrak{b}: H_4 \to \Gamma(H_2)$ and the cohomology class in $H^2(H_3, G)$ which determines the equivalence class of the extension Π_3.

Now let $\pi_1(K) = 1$ and let K be a finite complex. Then $H_3 = T + B$, where T is the torsion group and B is free Abelian. Let $T_1, \cdots, T_p$ be cyclic summands of T, whose orders, $\tau_1, \cdots, \tau_p$, are the coefficients of 3-dimensional torsion. Since B is free there is a homomorphism, $\mathfrak{j}^*: B \to \Pi_3$, such that $\mathfrak{j}_3\mathfrak{j}^*b = b$ for each $b \in B$. Therefore Π_3 is the direct sum

$$\Pi_3 = \Pi_3^0 + \mathfrak{j}^*B,$$

where $\Pi_3^0 = \mathfrak{j}_3^{-1}T$. Thus Π_3^0 is an extension of T by G and the equivalence class of Π_3 is obviously determined by that of Π_3^0.

Let $(c^1, \cdots, c^p, y^1, \cdots, y^r)$ be a canonical basis for C_4 such that $dc^\lambda = \tau_\lambda z^\lambda$, $dy^\mu = 0$ where $l_3z^\lambda \in H_3$ generates T_λ. let $a^\lambda \in j_3^{-1}z^\lambda$, $x^\lambda = k_3a^\lambda \in \Pi_3$. Then

$$\mathfrak{j}_3x^\lambda = \mathfrak{j}_3k_3a^\lambda = l_3j_3a^\lambda = l_3z^\lambda.$$

Therefore $x^\lambda \in \Pi_3^0$ and x^λ is a representative of l_3z^λ.

Since $j_3(\beta c^\lambda - \tau_\lambda a^\lambda) = dc^\lambda - \tau_\lambda z^\lambda = 0$ we have

$$\beta c^\lambda - \tau_\lambda a^\lambda = \gamma^\lambda \in \Gamma(H_2).$$

Let $g^\lambda = -\mathfrak{i}_3\gamma^\lambda \in G$. Then $\tau_\lambda x^\lambda = k_3\tau_\lambda a^\lambda = -k_3\gamma^\lambda = g^\lambda$. Therefore the equivalence class of Π_3^0 is uniquely determined[42] by $g^1(\tau_1), \cdots, g^p(\tau_p)$, where $g^\lambda(\tau_\lambda) \in G_{\tau_\lambda}$ is the residue class containing g^λ.

[42] See §16 of [12].

It follows from the definition of $\mathfrak{b}(m)$ that

$$\mathfrak{b}(\tau_\lambda)c_*^\lambda = \gamma^\lambda(\tau_\lambda) \in \Gamma(H_2)_{\tau_\lambda}$$

where c_*^λ is the homology class, mod. τ_λ, which contains c^λ. Therefore

$$g^\lambda(\tau_\lambda) = -\mathfrak{i}_3(\tau_\lambda)\mathfrak{b}(\tau_\lambda)c_*^\lambda,$$

where $\mathfrak{i}_3(\tau_\lambda):\Gamma(H_2)_{\tau_\lambda} \to G_{\tau_\lambda}$ is the homomorphism induced by $\mathfrak{i}_3$. Therefore the equivalence class of Π_3^0 is determined by $\mathfrak{b}$, which determines G, and by

$$\mathfrak{b}(\tau_1), \cdots, \mathfrak{b}(\tau_p),$$

which determine $g^1(\tau_1), \cdots, g^p(\tau_p)$.

Since $\mathfrak{b}(m) = u_m\mathfrak{p}(1)$ the sequence $\Sigma_4(K)$ is determined, up to a proper isomorphism, by the groups

$$H_2, H_3, H_4(m), \qquad H^4\{\Gamma(H_2)\} \quad (m = 0, \tau_1, \cdots, \tau_p),$$

the element $\mathfrak{p}(1) \in H^4\{\Gamma(H_2)\}$ and the family of homomorphisms u_m. If K is a (finite) simplicial complex all these items can, theoretically, be calculated by finite constructions.[43]

Let $n > 2$, let $K = K^{n+2}$ be $(n-1)$-connected and let us make the natural identifications

$$\Gamma_{n+1} = \Pi_n/2\Pi_n = H_n(2)$$

so that $\theta = 1$ at the end of §14 above. Then Theorem 19 has an analogue, namely Theorem 4 in [5], which states that $\mathfrak{b}_{n+2}(2)$ is the dual of the Steenrod homomorphism ([24])

$$Sq_{n-2}:H^n(2) \to H^{n+2}(2).$$

Further light is thrown on the calculation of $\Sigma_{n+2}(K)$ in forthcoming papers by S. C. Chang and P. J. Hilton. Chang defines certain numerical invariants, called "secondary torsions", which can be calculated constructively if K is given as a simplicial complex. An analysis which is similar to, but rather simpler than the one above, shows that $\Sigma_{n+2}(K)$ is determined, up to a "proper" isomorphism, by the Betti numbers and torsions of K, together with the secondary torsions defined by Chang.

Chapter V. The sequence of a general space

19. The Complex K(X)

Let P be any (geometric) simplicial complex, which may be infinite but which has the weak topology. That is to say, every (closed) simplex in P has its natural topology and a sub-set of P is closed provided it meets each simplex in a closed sub-set of the latter. By a *local ordering* of the vertices of P we shall mean an ordering, $\mathfrak{o}(\sigma^n)$, of the vertices of each simplex, $\sigma^n \in P$, such that, if σ^{n-1} is a face of σ^n, then $\mathfrak{o}(\sigma^{n-1})$ is the ordering induced by $\mathfrak{o}(\sigma^n)$. Let such an ordering

[43] This does not mean that we have found a finite algorithm for deciding whether or no $\Sigma_4(K) \approx \Sigma_4(K')$. Some of the difficulties in this question, even when K, K' have no torsion, are indicated on p. 88 of [3].

be given. Let the simplexes of P be divided into equivalences classes by an equivalence relation, $\equiv$, such that

a) $\sigma_1^r \equiv \sigma_2^s$ *implies* $r = s$,

(19.1) b) *if* $\sigma^n \equiv \tau^n$, *where* $\sigma^n = v_0 \cdots v_n$, $\tau^n = w_0 \cdots w_n$ *and the vertices* v_i, w_i *are written in their correct order, then* $v_{i_0} \cdots v_{i_r} \equiv w_{i_0} \cdots w_{i_r}$ *for each sub-set* $0 \leqq i_0 < i_2 < \cdots i_r \leqq n$.

Let $h(\tau^n, \sigma^n)\colon \sigma^n \to \tau^n$ be the order preserving barycentric map (onto) for every pair of simplexes $\sigma^n, \tau^n \in P$. Let p_1, p_2 be points in P. We write $p_2 \equiv p_2$ if, and only if, there are equivalent simplexes, $\sigma_1^n, \sigma_2^n \in P$, such that $p_i \in \sigma_i^n - \dot{\sigma}_i^n$ and $p_2 = h(\sigma_2^n, \sigma_1^n)p_1$. Obviously $p_1 \equiv p_2$ is an equivalence relation. Let K be the space whose points are these equivalence classes of points in P and which has the identification topology determined by the map $\mathbf{k}\colon P \to K$, where $\mathbf{k}p$ is the class containing p.

LEMMA 3. *K is a* CW-*complex, whose cells are the sets* $\mathbf{k}(\sigma^n - \dot{\sigma}^n)$, *for each simplex*, σ^n, *of* P.

First assume that the following supplementary conditions are satisfied:

a) $v \not\equiv v'$ *if* v, v' *are distinct vertices of the same simplex of* P

(19.2) b) *if* $\sigma^n = v_0 \cdots v_n$, $\tau^n = w_0 \cdots w_n$ *and if* $v_i \equiv w_{j_i}$ *for each* $i = 0, \cdots,$ n, *then* $\sigma^n \equiv \tau^n$.

Let α be the cardinal number of the aggregate of classes $\mathbf{k}v$, for each vertex, $v \in P$. Let $\mathbf{k}v \to e(\mathbf{k}v)$ be a $(1 - 1)$ correspondence between the aggregate $\{\mathbf{k}v\}$ and a set of basis vectors, $\{e\}$, in a non-topologized vector space, A, of rank α (Cf. [25]). Then a simplicial complex, L, with $\{e\}$ as the aggregate of its vertices, is defined as follows. Let $e_0, \cdots, e_n$ be any finite sub-set of $\{e\}$. Then the rectilinear simplex, $e_0 \cdots e_n \subset A$, is a simplex of L if, and only if, there is a simplex $v_0 \cdots v_n \in P$, such that $e_i = e(\mathbf{k}v_i)$ $(i = 0, \cdots, n)$. We give L the weak topology.

Let $\sigma^n = v_0 \cdots v_n$ be a given simplex of P. Then it follows from the definition of L that $e(\mathbf{k}v_0) \cdots e(\mathbf{k}v_n)$ is a simplex in L. Therefore a simplicial map, $\mathbf{l}\colon P \to L$, is defined by $\mathbf{l}v = e(\mathbf{k}v)$, for each vertex $v \in P$. Since P has the weak topology it follows that $\mathbf{l}$ is continuous. Notice that, in consequence of (19.2a), the map $\mathbf{l} \mid \sigma^n$ is non-degenerate. Let $p \equiv q$, where p, q are points in P, and let

$$\sigma^n = v_0 \cdots v_n, \qquad \tau^n = w_0 \cdots w_n$$

be the simplexes of P, whose interiors contain p, q. Then $\sigma^n \equiv \tau^n$ and it follows from (19.1b) that $\mathbf{k}v_i = \mathbf{k}w_i$, for each $i = 0, \cdots, n$. Therefore $\mathbf{l}\sigma^n = \mathbf{l}\tau^n$. Also $q = h(\tau^n, \sigma^n)p$ and the map $h(\tau^n, \sigma^n)$, likewise $\mathbf{l} \mid \sigma^n$ and $\mathbf{l} \mid \tau^n$ are barycentric. Therefore $\mathbf{l}p = \mathbf{l}q$. Therefore the map $\mathbf{lk}^{-1}\colon K \to L$ is single-valued. Since K has the identification topology determined by $\mathbf{k}$ the map $\mathbf{lk}^{-1}$ is continuous.[44]

Similarly it follows from (19.2) that $\mathbf{kl}^{-1}\colon L \to K$ is single-valued. It is obviously continuous in each simplex of L, and hence throughout L, since L has

[44] See §5 of [6].

the weak topology clearly $(\mathbf{l}\mathbf{k}^{-1})\mathbf{k}\mathbf{l}^{-1} = 1$, $(\mathbf{k}\mathbf{l}^{-1})\mathbf{l}\mathbf{k}^{-1} = 1$. Therefore $\mathbf{k}\mathbf{l}^{-1}$ is a homeomorphism onto K. Therefore K is a simplicial complex, with the weak topology, whose cells are the interiors of the simplexes $\mathbf{k}\mathbf{l}^{-1}(\mathbf{l}\sigma^n) = \mathbf{k}\sigma^n$, for each simplex, $\sigma^n \in P$. This proves the lemma, subject to the conditions (19.2).

Now assume that (19.2a) is satisfied and let P' be the derived complex of P, in which each new vertex is placed at the centroid of its simplex. We define a local ordering in P' by placing the centroid of σ^n after the centroid of σ^m if $m < n$. The equivalence relation between the simplexes of P induces a similar relation between those of P', in such a way that the equivalence classes of points are unaltered. Also it may be verified that the equivalence relation in P' satisfies both (19.2a) and (19.2b). Therefore $\mathbf{k}P'$ is a simplicial complex and $K = \mathbf{k}P$ is a "block complex", in which the blocks are the sub-complexes of $\mathbf{k}P'$, which cover the sets $\mathbf{k}\sigma^n$. It may be verified that K is a CW-complex, with the combinatorial structure described in the lemma.

Finally let P be general. Then the induced equivalence relation between the simplexes of the derived complex, P', obviously satisfies (19.2a). Therefore the induced equivalence relation between the simplexes of the second derived complex of P satisfies both (19.2a) and (19.2b). Again it is easily verified that K is a CW-complex, with the structure described in the lemma. This completes the proof.

We now proceed to the definition of $K(X)$. Let v_0 be the origin and v_i the point $(t_1, t_2, \cdots)$ in Hilbert space, R^∞, where $t_i = 1$ and $t_j = 0$ if $j \neq i$. Let $v_0, v_1, \cdots$ be ordered so that $v_\lambda < v_{\lambda+1}$ and let $\Delta^n \subset R^\infty$ be the rectilinear simplex $v_0 \cdots v_n$. Let $f: \Delta^n \to X$ be a given map and let (f, Δ^n) be the rectilinear n-simplex, whose points are the pairs (f, r), for every point $r \in \Delta^n$, and whose topology and affine geometry are such that the map $r \to (f, r)$ is a barycentric homeomorphism. If σ^i is any face of Δ^n we shall denote the corresponding face of (f, Δ^n) by (f, σ^i). We emphasize the fact that $(f, r) \neq (f', r)$ if $f, f': \Delta^n \to X$ are different maps, even if $fr = f'r$. Also $(f, r) \neq (g, r)$ if $r \in \Delta^{n-1}$ and $g = f \mid \Delta^{n-1}$. Therefore no two of the simplexes (f, Δ^n), (g, Δ^m) have a point in common.

Let $P(X)$ be the union of all the (disjoint) simplicial complexes (f, Δ^n), for every $n \geqq 0$ and every map $f: \Delta^n \to X$. We give $P(X)$ the weak topology, which makes each (f, Δ^n) both open and closed in $P(X)$. The simplexes of $P(X)$ are the simplexes (f, σ^r), where σ^r is any face of Δ^n. The ordering $v_0, v_1, \cdots, v_n$, for each $n > 0$, and the maps $r \to (f, r)$ determine a local ordering in $P(X)$. Let (f, σ^i) and (g, τ^j) be faces of (f, Δ^m) and (g, Δ^n). We define $(f, \sigma^i) \equiv (g, \tau^j)$ if, and only if, $i = j$ and[45]

$$f \mid \sigma^i = (g \mid \tau^i)h(\tau^i, \sigma^i).$$

It is easily verified that this is an equivalence relation and it obviously satisfies (19.1). Therefore a CW-complex, $K(X) = \mathbf{k}P(X)$, is defined as in Lemma 3. Notice that $K(X)$ is uniquely determined by X. Notice also that $fr = f'r'$ if $\mathbf{k}(f, r) = \mathbf{k}(f', r')$, though the converse is not necessarily true.

[45] $h(\tau^i, \sigma^i)$ will always mean the order preserving, barycentric map, $\sigma^i \to \tau^i$, onto τ^i.

Let $S(X)$ be the abstract singular complex of X, as defined in [16]. Any n-cell, $s^n \epsilon S(X)$, has a unique representative map,[46] $f(s^n):\Delta^n \to X$. It is obvious that the correspondence $s^n \to \mathbf{k}\{f(s^n), \Delta^n\}$ determines an isomorphic chain mapping of $S(X)$ onto $K(X)$, when the latter is treated as an abstract complex.

Let $\phi:X \to Y$ be a given map into a space Y. Then a map, $K\phi:K(X) \to K(Y)$, is obviously defined by

$$(K\phi)\mathbf{k}(f, r) = \mathbf{k}(\phi f, r),$$

where $\mathbf{k}:P(Y) \to K(Y)$ is defined in the same way as $\mathbf{k}:P(X) \to K(X)$. Moreover the correspondences $X \to K(X)$ and $\phi \to K\phi$ determine a functor $\mathfrak{T} \to \mathfrak{T}_k$, where $\mathfrak{T}$ and $\mathfrak{T}_k$ are the topological categories of all topological spaces and all CW-complexes.[47]

20. The maps κ and λ_ϕ

It is obvious that a (single-valued and continuous) map, $\kappa:K(X) \to X$, is defined by $\kappa\mathbf{k}(f, r) = fr$. Let $\phi:X \to Y$ be a given map. Then

$$\phi\kappa\mathbf{k}(f, r) = \phi fr = \kappa\{(K\phi)\mathbf{k}(f, r)\},$$

where $\kappa:K(Y) \to Y$ is defined in the same way as $\kappa:K(X) \to X$. Therefore κ is natural in the sense that

$$\phi\kappa = \kappa K\phi. \tag{20.1}$$

Let Q be any simplicial complex with the weak topology and with a local ordering of its vertices. Let $\phi:Q \to X$ be a given map. Then a map, $\lambda_\phi:Q \to K(X)$, is defined by

$$\lambda_\phi q = \mathbf{k}\{(\phi \mid \sigma^n)h(\sigma^n, \Delta^n), h(\Delta^n, \sigma^n)q\} \tag{20.2}$$

for each point $q \epsilon Q$, where σ^n is any simplex of Q, which contains q. Obviously $\kappa\lambda_\phi = \phi$. Moreover, if Q_0 is a subcomplex of Q, with the local ordering induced by the one in Q, and if $\phi_0 = \phi \mid Q_0$, then

$$\lambda_{\phi_0} = \lambda_\phi \mid Q_0. \tag{20.3}$$

Let $\theta:Q \to L$ be an isomorphism of Q onto a simplicial complex, L, with the weak topology. Let L have the local ordering which makes θ order preserving in each simplex of Q. Let $\psi:L \to X$ be a given map. Then it may be verified that

$$\lambda_{\psi\theta} = \lambda_\psi\theta:Q \to K(X). \tag{20.4}$$

[46] Strictly speaking the unique representative of s^n is the pair $(f(s^n), \mathfrak{o}_n)$, where $\mathfrak{o}_n$ is our fixed ordering of the vertices $v_0, \cdots, v_n$. Eilenberg has communicated to me a simplified definition of $S(X)$, to be used in a forthcoming book with N. E. Steenrod, in which a cell of $S(X)$ is simply a map $f:\Delta^n \to X$ and its faces are the cells $(f \mid \sigma^i)h(\sigma^i, \Delta^i)$, for each face, σ^i, of Δ^n.

[47] Notice that $K\phi$ maps each cell of $K(X)$ homeomorphically onto a cell of $K(Y)$. Therefore $\mathfrak{T} \to \mathfrak{T}_k$ maps $\mathfrak{T}$ into the category of complexes in which the mappings are of this restricted sort.

Let Q^* be any simplicial subdivision of Q and let a local ordering be defined in Q^*, which is independent of the one in Q. Let $\phi^*: Q^* \to X$ be a given map and let $\lambda^*_{\phi^*}$ be defined in the same way as λ_ϕ, but in terms of Q^* and the local ordering in Q^*.

LEMMA 4. *If $\phi \simeq \phi^*$ then $\lambda_\phi \simeq \lambda^*_{\phi^*}$.*

Let $L = Q \times I$, let θ_0, θ_1^* be the maps of Q into $Q \times 0$, $Q \times 1$, which are given by $\theta_0 q = (q, 0)$, $\theta_1^* q = (q, 1)$ and let Q_0, Q_1^* be the triangulations of $Q \times 0$, $Q \times 1$ which make θ_0, θ_1^* isomorphisms. Then L is a polyhedral complex, which consists of the simplexes in Q_0, Q_1^* and the convex prisims $\sigma^n \times I$, for every simplex, $\sigma^n \in Q$. Let $v(\sigma^n) = (q, \frac{1}{2})$, where q is the centroid of σ^n, and let L' be the triangulation of L, which is defined inductively by starring each $\sigma^n \times I$ from $v(\sigma^n)$ as centre, taking $\sigma^m \times I$ before $\sigma^n \times I$ if $m < n$. We define a local ordering in L' by giving Q_0, Q_1^* the local orderings which make θ_0, θ_1^* order preserving and placing $v(\sigma^n)$ after all the vertices of L', which lie on the boundary of $\sigma^n \times I$.

Let $\psi: L' \to X$ be a map such that $\psi(q, 0) = \phi q$, $\psi(q, 1) = \phi^* q$. Let $\lambda_\psi: L' \to K(X)$ be defined in the same way as λ_ϕ. Then it follows from (20.3) that λ_ψ determines a homotopy $\lambda_{\psi_0}\theta_0 \simeq \lambda^*_{\psi_1}\theta_1^*$, where $\psi_0 = \psi \mid Q_0$, $\psi_1 = \psi \mid Q_1^*$. Since $\phi = \psi_0\theta_0$, $\phi^* = \psi_1\theta_1^*$ the lemma follows from (20.4).

Let $\mu, \mu': Q \to K(X)$ be given maps.

THEOREM 21. *If $\kappa\mu \simeq \kappa\mu'$, then $\mu \simeq \mu'$.*

We first show that, in the presence of Lemma 4, this is equivalent to

$$\mu \simeq \lambda_{\kappa\mu}. \tag{20.5}$$

For $\kappa\mu = \kappa\lambda_{\kappa\mu}$. Therefore (20.5) follows from Theorem 21 with $\mu' = \lambda_{\kappa\mu}$. Conversely, if $\kappa\mu \simeq \kappa\mu'$ it follows from (20.5) and Lemma 4 that $\mu \simeq \lambda_{\kappa\mu} \simeq \lambda_{\kappa\mu'} \simeq \mu'$. Therefore Theorem 21 is equivalent to (20.5). We shall prove (20.5).

Let $K = K(X)$. Though K is not a simplicial complex we shall describe $\mu: Q \to K$ as *simplicial* if, and only if, it can be defined as follows. Let a barycentric map

$$\theta(\sigma^n): \sigma^n \to P(X),$$

onto a simplex of $P(X)$, be defined for each simplex $\sigma^n \in Q$ in such a way that the (simplicial) map $\mu: Q \to K$ is single-valued, where $\mu q = \mathbf{k}\theta(\sigma^n)q$ if $q \in \sigma^n$. Since $\mu \mid \sigma^n$ is continuous and Q has the weak topology it follows that μ is continuous.

Let the simplex $\theta(\sigma^n)\sigma^n$ be $d(\sigma^n)$-dimensional, let $j = d(\sigma^n)$ and let

$$\mu(\sigma^n) = h(\Delta^j, \theta(\sigma^n)\sigma^n)\theta'(\sigma^n): \sigma^n \to \Delta^j,$$

where $\theta'(\sigma^n): \sigma^n \to \theta(\sigma^n)\sigma^n$ is the map induced by $\theta(\sigma^n)$. Let

$$\theta(\sigma^n)\sigma^n = (g(\sigma^n), \sigma_0^j) \subset P(X),$$

where $\sigma_0^j \subset \Delta^k$, for some $k \geqq j$, and let

$$f(\sigma^n) = (g(\sigma^n) \mid \sigma_0^j)h(\sigma_0^j, \Delta^j): \Delta^j \to X.$$

Let σ^m be any face of σ^n and let $\mu(\sigma^n)\sigma^m = \sigma_1^i \subset \Delta^j$. Since $\mu: Q \to K$ is single-valued it may be verified that $i = d(\sigma^m)$ and that

$$(20.6)\qquad \begin{cases} \text{a)}\ \mu(\sigma^n) \mid \sigma^m = \iota h(\sigma_1^i, \Delta^i)\mu(\sigma^m) \\ \text{b)}\ f(\sigma^n) \mid \sigma_1^i = f(\sigma^m)h(\Delta^i, \sigma_1^i), \end{cases}$$

where $\iota:\sigma_1^i \to \Delta^j$ is the identical map. Also $(f(\sigma^n), \Delta^j) \equiv (g(\sigma^n), \sigma_0^j)$, whence

$$(20.7)\qquad \mu q = \mathbf{k}(f(\sigma^n), \mu(\sigma^n)q) \qquad (q \in \sigma^n).$$

Conversely, given $\mu(\sigma^n):\sigma^n \to \Delta^j$, $f(\sigma^n):\Delta^j \to X$, satisfying (20.6), for each $\sigma^n \in Q$, a simplicial map $\mu:Q \to K$ is defined[48] by (20.7).

We shall describe the simplicial map μ as non-degenerate if, and only if, $d(\sigma^n) = n$ for each $\sigma^n \in Q$. Let this be so and let $\mu(\sigma^n)$, $f(\sigma^n)$ mean the same as in (20.6). Then we can order the vertices of each simplex $\sigma^n \in Q$ so that $\mu(\sigma^n)$ preserves order. It follows from (20.6a) that we thus define a local ordering in Q and from Lemma 4, with $Q^* = Q$, $\phi^* = \phi = \kappa\mu$, that we lose no generality in assuming this to be the one by means of which $\lambda_{\kappa\mu}$ is defined. Then

$$\mu(\sigma^n) = h(\Delta^n, \sigma^n), \qquad \kappa\mu \mid \sigma^n = f(\sigma^n)\mu(\sigma^n),$$

in consequence of (20.7). Therefore it follows from (20.2) and (20.7) that $\mu = \lambda_{\kappa\mu}$. Therefore the theorem will follow when we have proved that a given map, $Q \to K$, is homotopic to a non-degenerate, simplicial map $Q^* \to K$, where Q^* is a simplicial sub-division of Q.

Let P'' be the second derived complex of $P(X)$. Then $K'' = \mathbf{k}P''$ is a simplicial sub-division of K, as shown in the proof of Lemma 3. Let $\delta_t:P'' \to P(X)$ be the canonical homotopy in which $\delta_t v = (1 - t)v + t\delta_1 v$, where v is any vertex of P'', $\delta_t v$ is treated as a vector and $\delta_1 v$ is the last vertex of the simplex of $P(X)$, which contains v in its interior. Obviously $\delta_t p \equiv \delta_t p'$ if $p \equiv p'$. Therefore[44] a homotopy, $\rho_t:K'' \to K$, is defined by $\rho_t\mathbf{k} = \mathbf{k}\delta_t$. Clearly $\rho_1:K'' \to K$ is simplicial.

Let $\mu_0:Q \to K$ be a given map. By Theorem 36 on p. 320 of [7] we have $\mu_0 \simeq \mu_1$, where $\mu_1:Q \to K$ is simplicial with respect to K'' and some simplicial sub-division, Q^*, of Q. Then $\mu_0 \simeq \rho_1\mu_1$. The resultant of simplicial maps, $Q \to L \to K$, is obviously simplicial, where L is any simplicial complex with the weak topology. Therefore $\rho_1\mu_1$ is simplicial and it follows that we lose no generality by assuming that the given map $\mu:Q \to K$ is simplicial.

Let μ be simplicial and let $\mu(\sigma^n)$, $f(\sigma^n)$ mean the same as in (20.6). Let

$$\mu^*(\sigma^n) = h(\Delta^n, \sigma^n), \qquad f^*(\sigma^n) = f(\sigma^n)\mu(\sigma^n)h(\sigma^n, \Delta^n).$$

Let $\sigma_1^m = \mu^*(\sigma^n)\sigma^m$ where σ^m is any face of σ^n. Then

$$\begin{aligned}\mu^*(\sigma^n) \mid \sigma^m &= h(\Delta^n, \sigma^n) \mid \sigma^m = \iota h(\sigma_1^m, \sigma^m) \\ &= \iota h(\sigma_1^m, \Delta^m)\mu^*(\sigma^m),\end{aligned}$$

where $\iota:\sigma_1^m \to \Delta^n$ is the identical map. Let $\sigma_1^i = \mu(\sigma^n)\sigma^m \subset \Delta^j$.

[48] When μ is thus defined it is to be understood that $\mu(\sigma^n)\sigma^n = \Delta^j (j = d(\sigma^n))$ and that $i = d(\sigma^m)$ in (20.6).

Since $h(\sigma^n, \Delta^n)\sigma_1^m = \sigma^m$ we have

$$\begin{aligned} f^*(\sigma^n) \mid \sigma_1^m &= f(\sigma^n)\mu(\sigma^n)h(\sigma^n, \Delta^n) \mid \sigma_1^m \\ &= \{f(\sigma^n)\mu(\sigma^n) \mid \sigma^m\}h(\sigma^m, \sigma_1^m) \\ &= \{f(\sigma^n) \mid \sigma_1^i\}\mu' h(\sigma^m, \sigma_1^m), \end{aligned}$$

where $\mu':\sigma^m \to \sigma_1^i$ is the map induced by $\mu(\sigma^n)$. It follows from (20.6a) that

$$\mu' = h(\sigma_1^i, \Delta^i)\mu(\sigma^m).$$

Hence, and from (20.6b), we have

$$\begin{aligned} f^*(\sigma^n) \mid \sigma_1^m &= f(\sigma^m)h(\Delta^i, \sigma_1^i)h(\sigma_1^i, \Delta^i)\mu(\sigma^m)h(\sigma^m, \sigma_1^m) \\ &= f(\sigma^m)\mu(\sigma^m)h(\sigma^m, \sigma_1^m) \\ &= f(\sigma^m)\mu(\sigma^m)h(\sigma^m, \Delta^m)h(\Delta^m, \sigma_1^m) \\ &= f^*(\sigma^m)h(\Delta^m, \sigma_1^m). \end{aligned}$$

Therefore the families of maps $\mu^*(\sigma^n), f^*(\sigma^n)$ satisfy (20.6) and a non-degenerate, simplicial map, $\mu^*:Q \to K$, is defined by

$$\mu^* q = \mathbf{k}(f^*(\sigma^n), \mu^*(\sigma^n)q) \qquad (q \in \sigma^n).$$

Finally we prove that $\mu \simeq \mu^*$. Let $u_0, \cdots, u_n$ be the vertices of σ^n, written in their correct order. Let $j = d(\sigma^n)$ and let

$$\nu_0(\sigma^n), \nu_1(\sigma^n):\sigma^n \to \Delta^{j+n+1}$$

be the barycentric maps which are given by

$$\nu_0(\sigma^n)u_\alpha = \mu(\sigma^n)u_\alpha, \quad \nu_1(\sigma^n)u_\alpha = v_{j+1+\alpha} \qquad (\alpha = 0, \cdots, n).$$

Let $\nu_t(\sigma^n)q = (1 - t)\nu_0(\sigma^n)q + t\nu_1(\sigma^n)q$, for each $q \in \sigma^n$, where $0 \leqq t \leqq 1$ and $\nu_t(\sigma^n)q$ is treated as a vector in Δ^{j+n+1}. Let $\rho(\sigma^n):\Delta^{j+n+1} \to \Delta^j$ be the barycentric retraction which is given by

$$\rho(\sigma^n) \mid \Delta^j = 1, \quad \rho(\sigma^n)v_{j+1+\alpha} = \mu(\sigma^n)u_\alpha \qquad (\alpha = 0, \cdots, n)$$

and let $F(\sigma^n) = f(\sigma^n)\rho(\sigma^n):\Delta^{j+n+1} \to X$. Let

$$\theta_t(\sigma^n):\sigma^n \to P(X)$$

be the homotopy which is given by

(20.8) $$\theta_t(\sigma^n)q = (F(\sigma^n), \nu_t(\sigma^n)q) \qquad (q \in \sigma^n).$$

I say that a homotopy, $\mu_t:Q \to K$, of $\mu_0 = \mu$ into $\mu_1 = \mu^*$ is given by $\mu_t q = \mathbf{k}\theta_t(\sigma^n)q$. This will follow when we have proved that

(20.9) $$\mathbf{k}\theta_i(\sigma^n) = \mu_i \mid \sigma^n \qquad (i = 0, 1)$$

and, since Q has the weak topology and $\mathbf{k}\theta_t(\sigma^n)$ is continuous throughout σ^n, for each $\sigma^n \in Q$, that μ_t is single-valued. The fact that μ_t is single-valued will follow when we have proved that

$$\mathbf{k}\theta_t(\sigma^m) = \mathbf{k}\theta_t(\sigma^n) \mid \sigma^m \tag{20.10}$$

where σ^m is any face of σ^n.

Since $\nu_0(\sigma^n)q = \mu(\sigma^n)q$ $(q \in \sigma^n)$ and $F(\sigma^n) \mid \Delta^j = f(\sigma^n)$ it follows from (20.7) and (20.8) that $\mathbf{k}\theta_0(\sigma^n) = \mu \mid \sigma^n$. Let $\Delta_1^n = v_{j+1} \cdots v_{j+n+1}$. Then $\mu(\sigma^n)h(\sigma^n, \Delta_1^n)$ and $h(\Delta_1^n, \sigma^n)$ are the maps induced by $\rho(\sigma^n)$ and $\nu_1(\sigma^n)$. Therefore

$$\begin{aligned} F(\sigma^n) \mid \Delta_1^n &= f(\sigma^n)\mu(\sigma^n)h(\sigma^n, \Delta_1^n) \\ &= f(\sigma^n)\mu(\sigma^n)h(\sigma^n, \Delta^n)h(\Delta^n, \Delta_1^n) \\ &= f^*(\sigma^n)h(\Delta^n, \Delta_1^n) \\ \nu_1(\sigma^n)q &= h(\Delta_1^n, \sigma^n)q \qquad\qquad (q \in \sigma^n) \\ &= h(\Delta_1^n, \Delta^n)h(\Delta^n, \sigma^n)q \\ &= h(\Delta_1^n, \Delta^n)\mu^*(\sigma^n)q. \end{aligned}$$

Hence it follows that $\mathbf{k}\theta_1(\sigma^n) = \mu^* \mid \sigma^n$ and we have proved (20.9).

Let

$$\sigma^m = u_{r_0} \cdots u_{r_m}, \qquad \sigma_1^i = v_{s_0} \cdots v_{s_i} \qquad (r_\alpha < r_{\alpha+1}\,;\, s_\beta < s_{\beta+1})$$

$$v_{s_0} \cdots v_{s_i} v_{j+1+r_0} \cdots v_{j+1+r_m} = \sigma_1^{i+m+1}.$$

Then $\rho(\sigma^n)\sigma_1^{i+m+1} = \mu(\sigma^n)\sigma^m = \sigma_1^i$. Let $\rho' : \sigma_1^{i+m+1} \to \sigma_1^i$ be the map induced by $\rho(\sigma^n)$. Then it follows from (20.6a) that

$$\begin{aligned} &\rho(\sigma^m)\, h(\Delta^{i+m+1}, \sigma_1^{i+m+1})v_{j+1+r_\alpha} \\ &= \rho(\sigma^m)v_{i+1+\alpha} = \mu(\sigma^m)u_{r_\alpha} \\ &= h(\Delta^i, \sigma_1^i)\mu(\sigma^n)u_{r_\alpha} = h(\Delta^i, \sigma_1^i)\rho' v_{j+1+r_\alpha}. \end{aligned}$$

Therefore

$$h(\Delta^i, \sigma_1^i)\rho' = \rho(\sigma^m)h(\Delta^{i+m+1}, \sigma_1^{i+m+1})$$

and it follows from (20.6b) that

$$\begin{aligned} F(\sigma^n) \mid \sigma_1^{i+m+1} &= \{f(\sigma^n)\rho(\sigma^n)\} \mid \sigma_1^{i+m+1} \\ &= \{f(\sigma^n) \mid \sigma_1^i\}\rho' \\ &= f(\sigma^m)h(\Delta^i, \sigma_1^i)\rho' \\ &= f(\sigma^m)\rho(\sigma^m)h(\Delta^{i+m+1}, \sigma_1^{i+m+1}) \\ &= F(\sigma^m)h\,(\Delta^{i+m+1}, \sigma_1^{i+m+1}). \end{aligned}$$

Therefore $(F(\sigma^n), \sigma_1^{i+m+1}) \equiv (F(\sigma^m), \Delta^{i+m+1})$. Also

$$h(\sigma_1^i, \Delta^i)v_\beta = v_{s_\beta} = h(\sigma_1^{i+m+1}, \Delta^{i+m+1})v_\beta \qquad (\beta \leqq i).$$

Therefore

$$\begin{aligned} \nu_0(\sigma^n)u_{r_\alpha} &= \mu(\sigma^n)u_{r_\alpha} = h(\sigma_1^i, \Delta^i)\mu(\sigma^m)u_{r_\alpha} \\ &= h(\sigma_1^{i+m+1}, \Delta^{i+m+1})\nu_0(\sigma^m)u_{r_\alpha} \\ \nu_1(\sigma^n)u_{r_\alpha} &= v_{j+1+r_\alpha} = h(\sigma_1^{i+m+1}, \Delta^{i+m+1})\nu_1(\sigma^m)u_{r_\alpha}. \end{aligned}$$

Therefore $\nu_t(\sigma^n)q = h(\sigma_1^{i+m+1}, \Delta^{i+m+1})\nu_t(\sigma^m)q$ if $q \epsilon \sigma^m$. This proves (20.10) and hence the theorem.

Let $\phi \simeq \phi':X \to Y$, where Y is any space. Then it follows from (20.1) that

$$\kappa K\phi = \phi\kappa \simeq \phi'\kappa = \kappa K\phi'.$$

Therefore $K\phi \simeq K\phi'$, by Theorem 21. Therefore $\{K\phi\}$ is a single-valued function of $\{\phi\}$, where $\{\psi\}$ denotes the homotopy class of a map ψ. It may be verified that the correspondences $K \to K(X)$, $\{\phi\} \to \{K\phi\}$ determine a functor of the homotopy category of all spaces into the homotopy category of CW-complexes.

21. The sequence $\Sigma(X)$.

There is a unique map, $f:\Delta^0 \to X$, such that $f\Delta^0$ is a given point in X. Therefore $\kappa \mid K^0(X)$ is a (1–1) map onto X, whence $\kappa K(X) = X$. Since $K(X)$ is locally contractible, according to (M) in §5 of Ch I, each of its components is arcwise connected. Therefore each component of $K(X)$ is mapped by κ into a single arc-component of X. Let $K_0(X)$, $K_1(X)$ be given components of $K(X)$ and let $e_i^0 \epsilon K_i^0(X)$, $x_i = \kappa e_i^0$ $(i = 0, 1)$. If x_0, x_1, are in the same arc component of X there is a map, $f:\Delta^1 \to X$, such that $fv_i = x_i$. Then $\mathbf{k}(f, \Delta^1)$ is a 1-cell in $K(X)$, whose extremities are e_0^0, e_1^0. Therefore $K_0(X) = K_1(X)$ and κ maps precisely one component of $K(X)$ onto a given arc-component of X.

Let X be arcwise connected and let a 0-cell $e^0 \epsilon K^0(X)$ and the point $x_0 = \kappa e^0$ be chosen as base points in $K(X)$ and X.

Theorem 22. *$\kappa_n:\pi_n\{K(X)\} \approx \pi_n(X)$ for every $n = 1, 2, \cdots$, where κ_n is induced by κ.*

Let $\phi:(\dot{\Delta}^{n+1}, \Delta^0) \to (X, x_0)$ be a map which represents a given element $a \epsilon \pi_n(X)$. Since $\phi\Delta^0 = x_0$ we have $e^0 = \mathbf{k}(\phi \mid \Delta^0, \Delta^0)$ and $\lambda_\phi\Delta^0 = e^0$. Therefore $\lambda_\phi:\dot{\Delta}^{n+1} \to K(X)$ represents an element $a_0 \epsilon \pi_n\{K(X)\}$. Since $\phi = \kappa\lambda_\phi$ we have $a = \kappa_n a_0$. Therefore κ_n is onto.

Let $\mu:\dot{\Delta}^{n+1} \to K(X)$ be a map which represents a given element $a_0 \epsilon \kappa_n^{-1}(0)$ and let μ' be the constant map $\dot{\Delta}^{n+1} \to e^0$. Since $\kappa_n a_0 = 0$ we have $\kappa\mu \simeq \kappa\mu'$. Therefore $\mu \simeq \mu'$ by Theorem 21. Therefore $a_0 = 0$ and Theorem 22 is proved.

It follows from (20.1) that the isomorphisms κ_n are natural with respect to the homomorphisms induced by maps $\phi:X \to Y$ and $K\phi$, where Y is any arcwise connected space.

We recall from CH I that X is *dominated* by a CW-complex, L, if, and only if, there are maps $\phi:X \to L$, $\psi:L \to X$, such that $\psi\phi \simeq 1$.

Theorem 23. *If X is dominated by a* CW-*complex then* $\kappa:K(X) \equiv X$.

This follows from Theorem 22 and Theorem 1 in CH I.

Let X be itself a CW-complex. Then it follows from Theorem 23 that κ induces a proper isomorphism

$$\Sigma_q(X) \approx \Sigma_q\{K(X)\} \qquad (q \leqq \infty),$$

which is natural in consequence of (20.1). If X is an arbitrary, arcwise connected space we choose base points $e^0 \in K^0(X)$, $\kappa e^0 \in X$ and define $\Sigma_q(X)$ as $\Sigma_q\{K(X)\}$.

The complex $K(X)$ can also be used to extend the domain of definition of other invariants from CW-complexes to arbitrary spaces. For example the n-type of X may be defined as the n-type of $K(X)$. The same applies to the injected groups discussed at the end of §11.

Appendix A. On spaces dominated by complexes

We have seen, in Theorem 23, that any arcwise connected space, X, which is dominated by a CW-complex, is of the same homotopy type as some CW-complex[49] K. Let $\lambda: X \equiv K$ and let $\kappa: K \to X$ be a homotopy inverse of λ. Let $\lambda X \subset K_0$ where K_0 is a sub-complex of K, let $\lambda_0: X \to K_0$ be the map induced by λ and let $\kappa_0 = \kappa \mid K_0$. Then $\kappa_0\lambda_0 = \kappa\lambda \simeq 1$. Therefore X is dominated by K_0. If X is compact, so is λX. Therefore $\lambda X \subset K_0$, where K_0 is a finite sub-complex of K, by (D) in §5 of Ch I. Therefore X is dominated by a finite CW-complex.

Theorem 24. *An arcwise connected space, X, which is dominated by a* CW-*complex with a countable aggregate of cells, is of the same homotopy type as some locally finite polyhedron.*

By Theorem 23, $\kappa: K \equiv X$, where $K = K(X)$. Let $\lambda: X \to K$ be a homotopy inverse of κ. Let $\phi: X \to L$, $\psi: L \to X$ be such that $\psi\phi \simeq 1$, where L is a countable CW-complex. Then

$$\lambda \simeq \lambda\psi\phi = \mu\phi: X \to K,$$

where $\mu = \lambda\psi: L \to K$. Let e^n be any cell of L. Since $\mu\bar{e}^n$ is compact it is contained in a finite sub-complex of K. Since L is countable it follows that μL and hence $\mu\phi X$, is contained in a countable sub-complex, $K_0 \subset K$. Since $\mu\phi \simeq \lambda$ we have $\mu\phi: X \equiv K$ and we may replace λ by $\mu\phi$. Thus we assume to begin with that $\lambda X \subset K_0$.

Let $\rho_t: K \to K$ be a homotopy of $\rho_0 = \lambda\kappa$ into $\rho_1 = 1$. Then there is a countable sub-complex, $K_1 \subset K$, such that $\rho_t K_0 \subset K_1$, for the same reason that $\mu L \subset K_0$. By repeating this argument we define a sequence of countable sub-complexes $K_0, K_1, \cdots$, such that $\rho_t K_n \subset K_{n+1}$. The union of the complexes K_n is a countable sub-complex, K^*, such that $\lambda X \subset K^*$ and $\rho_t K^* \subset K^*$. Therefore $\lambda^*\kappa^* \simeq 1$ (in K^*) and $\kappa^*\lambda^* = \kappa\lambda \simeq 1$, where $\lambda^*: X \to K^*$ is the map induced by λ and $\kappa^* = \kappa \mid K^*$. Therefore $\lambda^*: X \equiv K^*$. But $K^* \equiv P$, where P is a locally finite polyhedron, by Theorem 13 in CH I. This proves Theorem 24.

It follows from Theorem 24, and the remarks which precede it, that any compact space which is dominated by a CW-complex, and in particular any ANR compactum, is of the same homotopy type as some locally finite poly-

[49] This may be proved more directly by a construction of the sort used in [8].

hedron. We leave open the question whether or no it is of the same homotopy type as a polyhedron of finite dimensionality.

Appendix B. On separation cochains

Let X, X' and $Y \subset X$, $Y' \subset X'$ be given topological spaces and let

$$A_n = \pi_n(X, y_0), \qquad C_n = \pi_n(X, Y, y_0) \qquad\qquad (n \geqq 2)$$
$$A'_n = \pi_n(X', y'_0), \qquad C'_n = \pi_n(X', Y', y'_0),$$

where y_0, y'_0 are base points in Y, Y'. Let

$$j: A_n \to C_n, \qquad j': A'_n \to C'_n,$$

be the injections and let

$$f, f^0: A_n \to A'_n, \qquad h, h^0: C_n \to C'_n$$

be the homomorphisms induced by given maps

$$\phi, \phi^0: (X, Y, y_0) \to (X', Y', y'_0).$$

Let $\phi \mid Y = \phi^0 \mid Y$. Then ϕ, ϕ^0 determine a *separation homomorphism*,

$$\Delta = \Delta(\phi, \phi^0): C_n \to A'_n,$$

which is defined in the same way as Eilenberg's separation co-chain, except that attention must be paid to the base points. The purpose of this section is to prove that

$$\text{(B1)} \qquad \begin{array}{l} \text{a)} \\ \text{b)} \end{array} \begin{cases} h - h^0 = j'\Delta \\ f - f^0 = \Delta j. \end{cases}$$

We recall the definition of Δ. Let E^n_1, E^n_2 be "Northern" and "Southern" hemispheres on an n-sphere, S^n, and let S^{n-1} be the "equatorial" $(n-1)$-sphere. Let $I^n \subset R^n$ be the n-cube, which is given by $0 \leqq t_1, \cdots, t_n \leqq 1$, where $t_1, \cdots, t_n$ are Cartesian coordinates for R^n. Let $\theta_i: E^n_i \to I^n$ be fixed homomorphisms (onto), such that $\theta_1 \mid S^{n-1} = \theta_2 \mid S^{n-1}$. Let E^n_i be oriented by means of the map θ_i^{-1} $(i = 1, 2)$ and let S^n take its orientation from E^n_1. Thus, taking orientation into account,

$$\text{(B2)} \qquad \dot{E}^n_1 = \dot{E}^n_2 = S^{n-1}, \qquad S^n = E^n_1 - E^n_2 .$$

We shall use maps of I^n and of the (oriented) n-elements E^n_1, E^n_2 to represent elements of homotopy groups, both absolute and relative. We shall also use maps of S^n to represent elements of absolute homotopy groups. Let $p_0 = (\frac{1}{2}, 0, \cdots, 0) \in \dot{I}^n$. It will be convenient to take p_0 and $\theta_i^{-1} p_0$ as base points in I^n and S^n.

Let $\lambda: (I^n, \dot{I}^n, p_0) \to (X, Y, y_0)$ be a map representing a given element $c \in C_n$. Then $\Delta c \in A'_n$ is the element represented by $\lambda(\phi, \phi^0): S^n \to X'$, where

$$\begin{aligned} \lambda(\phi, \phi^0)q &= \phi\lambda\theta_1 q && \text{if } q \in E^n_1 \\ &= \phi^0\lambda\theta_2 q && \text{if } q \in E^n_2 . \end{aligned}$$

Obviously Δc is unaltered by a homotopy of the form

$$\lambda_t:(I^n, \dot{I}^n, p_0) \rightarrow (X, Y, y_0).$$

Therefore it does not depend on the choice of the representative map λ.

Any pair of elements $c, c' \in C_n$ may be represented by maps, $\lambda, \lambda':I^n \rightarrow Y$, such that

$$\lambda(t_1, \cdots, t_n) = y_0 \qquad \text{if } t_1 \leqq \tfrac{1}{2}$$

$$\lambda'(t_1, \cdots, t_n) = y_0 \qquad \text{if } t_1 \geqq \tfrac{1}{2},$$

and $c + c'$ by $\lambda^*:I^n \rightarrow Y$, where

$$\begin{aligned} \lambda^*(t_1, \cdots, t_n) &= \lambda(t_1, \cdots, t_n) && \text{if } t_1 \geqq \tfrac{1}{2} \\ &= \lambda'(t_1, \cdots, t_n) && \text{if } t_1 \leqq \tfrac{1}{2}. \end{aligned}$$

Then $\lambda(\phi, \phi^0)E_3^n = y_0'$, $\lambda'(\phi, \phi^0)E_4^n = y_0'$, where E_3^n, $E_4^n \subset S^n$ are "Western" and "Eastern" hemispheres, and

$$\begin{aligned} \lambda^*(\phi, \phi^0) &= \lambda(\phi, \phi^0) && \text{in } E_4^n \\ &= \lambda'(\phi, \phi^0) && \text{in } E_3^n. \end{aligned}$$

Hence it follows that $\Delta(c + c') = \Delta c + \Delta c'$. Therefore Δ is a homomorphism.

Let $b_0 \in \pi_n(S^n)$ and $b_i \in \pi_n(S^n, S^{n-1})$ be the elements which are represented by the identical maps $S^n \rightarrow S^n$ and $E_i^n \rightarrow S^n$ $(i = 1, 2)$. Then it follows from (B2) that

$$b_1 - b_2 = j^*b_0,$$

where $j^*:\pi_n(S^n) \rightarrow \pi_n(S^n, S^{n-1})$ is the injection. On carrying this relation into X' by means of the homomorphism

$$\pi_n(S^n, S^{n-1}) \rightarrow \pi_n(X', Y'),$$

which is induced by $\lambda(\phi, \phi^0)$, we have (B1a).

Let $\lambda:(I^n, \dot{I}^n) \rightarrow (X, y_0)$ be a map which represents a given element $a \in A_n$. Then $\lambda(\phi, \phi^0)S^{n-1} = y_0$ and fa, f^0a are represented by the maps

$$\lambda(\phi, \phi^0) \mid E_1^n, \qquad \lambda(\phi, \phi^0) \mid E_2^n.$$

Since $S^n = E_1^n - E_2^n$ the map $\lambda(\phi, \phi^0)$ represents $fa - f^0a$. But λ also represents ja and $\lambda(\phi, \phi^0)$ represents Δja. This proves (B1b).

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