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The Coset Poset and Probabilistic Zeta Function of a Finite Group[1]

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We investigate the topological properties of the poset of proper cosets xH in a finite group G. Of particular interest is the reduced Euler characteristic, which is closely related to the value at -1 of the probabilistic zeta function of G. Our main result gives divisibility properties of this reduced Euler characteristic. © 2000 Academic Press

1. INTRODUCTION

For a finite group G and a non-negative integer s, let $P(G, s)$ be the probability that a randomly chosen ordered s-tuple from G generates G. Hall [17] gave an explicit formula for $P(G, s)$, exhibiting the latter as a finite Dirichlet series $\sum_n a_n n^{-s}$, with $a_n \in \mathbb{Z}$ and $a_n = 0$ unless n divides $|G|$. For example,

$$P(A_5, s) = 1 - \frac{5}{5^s} - \frac{6}{6^s} - \frac{10}{10^s} + \frac{20}{20^s} + \frac{60}{30^s} - \frac{60}{60^s}.$$

In view of Hall's formula, we can speak of $P(G, s)$ for an arbitrary complex number s. The reciprocal of this function of s is sometimes called the *zeta function* of G; see [6, 21].

The present paper arose from an attempt to understand the value of the zeta function at $s = -1$. More precisely, I wanted to explain some

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surprising divisibility properties of $P(G,-1)$, which is an integer, that I observed empirically. For example,

$$P(A_5,-1) = 1 - 25 - 36 - 100 + 400 + 1800 - 3600 = -1560,$$

which is divisible by $60 = |A_5|$. Similarly, $P(A_6,-1)$ is divisible by $|A_6|$, while $P(A_7,-1)$ is divisible by $|A_7|/3$.

The main theorem of this paper is a general divisibility result of this sort. The theorem specifies, for each prime p, a power p^a that divides $P(G,-1)$; the exponent a is defined in terms of the p-local structure of G. See Section 4 for the precise statement, which is somewhat technical. See Section 6 for some easily stated special cases.

Perhaps more interesting than the result itself is the nature of the proof, which is topological. The starting point is an observation of S. Bouc [private communication], giving a topological interpretation of $P(G,-1)$. Consider the *coset poset* $\mathscr{C}(G)$, consisting of proper cosets xH ($H < G$, $x \in G$), ordered by inclusion. Recall that we can apply topological concepts to a poset $\mathscr{P}$ by using the simplicial complex $\Delta(\mathscr{P})$ whose simplices are the finite chains in $\mathscr{P}$. In particular, we can speak of the Euler characteristic $\chi(\mathscr{P}) := \chi(\Delta(\mathscr{P}))$ and the reduced Euler characteristic $\tilde{\chi}(\mathscr{P}) := \chi(\mathscr{P}) - 1$. Bouc's observation, then, is that

$$P(G,-1) = -\tilde{\chi}(\mathscr{C}(G)). \tag{1}$$

This makes it possible to study divisibility properties of $P(G,-1)$ by using group actions on $\mathscr{C}(G)$ and proving the contractibility of certain fixed-point sets. This technique goes back to [10] and was studied further in [11, 22]. The group we use is G, acting by conjugation, or $(G \times G) \rtimes \mathbb{Z}_2$, acting by translation and inversion.

The remainder of the paper is organized as follows. In Section 2 we recall Hall's formula for $P(G,s)$ and a generalization of it due to Gaschütz [15]. In Section 3 we prove Bouc's result (1). We state and prove our main theorem in Section 4. The statement involves subgroups $N^*(P)$ for any p-subgroup $P \leq G$. We make some remarks about $N^*(P)$ in Section 5, in order to facilitate applications of the theorem. We are then able to give, in Section 6, several special cases and examples. The proof of the theorem, as we have already noted, makes use of an action of G or $(G \times G) \rtimes \mathbb{Z}_2$ on $\mathscr{C}(G)$. There is a bigger group that acts on $\mathscr{C}(G)$, called the biholomorph of G. In Section 7 we indicate briefly how the use of this bigger group can sometimes yield sharper results.

Having studied the Euler characteristic of the coset poset, one naturally wants to go further and study its homotopy type. Our results here are meager, but we show in Section 8 that $\mathscr{C}(G)$ has the homotopy type of a bouquet of spheres if G is solvable. The spheres all have the same dimension (which may be less than the dimension of the coset poset). And the number of spheres is $\left|P(G,-1)\right|$. Our results and methods here are closely related to those of Gaschütz [16] and Bouc [7].

Given that $P(G,-1)$ can be computed in terms of the coset poset, it is natural to ask whether the function $P(G,s)$ itself can be obtained from the coset poset. We show that this is the case in Section 9. More precisely, we define an analogue of $P(G,s)$ for an arbitrary finite lattice; when the lattice is taken to be the *coset lattice* of G (obtained by adjoining a largest and smallest element to the coset poset), we recover $P(G,s)$.

This paper raises more questions than it answers. Several such questions are stated in Sections 2.3 and 8.4.

Serge Bouc was extremely helpful; he told me about the topological interpretation (1) of $P(G,-1)$, without which this work could not have been done, and he made numerous other suggestions. Nigel Boston gave me the computation of $P(A_7,-1)$, so that I could stop trying to prove my original guess that $P(G,-1)$ is divisible by $|G|$ if G is simple. Jacques Thévenaz made several helpful comments and suggestions. John Shareshian provided answers to two questions I had asked in a preliminary version of this paper. Finally, I benefited greatly from the interest, questions, and encouragement of Keith Dennis.

2. HALL'S FORMULA AND GASCHÜTZ'S GENERALIZATION

2.1. *Hall's Formula*

Hall [17] did not actually discuss the probabilistic function $P(G,s)$, but rather the closely related "Eulerian function" $\phi(G,s)$. This is defined, for a non-negative integer s, to be the number of ordered s-tuples $(x_1,\ldots,x_s)$ such that $G=\langle x_1,\ldots,x_s\rangle$. Thus $P(G,s)=\phi(G,s)/\,|G|^s$. Hall introduced Möbius inversion on posets in order to compute $\phi(G,s)$. The resulting formula is

$$\phi(G,s)=\sum_{H\le G}\mu(H,G)\;|H|^s\,, \tag{2}$$

where μ is the Möbius function of the lattice of subgroups of G. We recall the proof:

For any subgroup $K\le G$, we have

$$\sum_{H\le K}\phi(H,s)=|K|^s\,,$$

since every s-tuple generates some subgroup. Möbius inversion now yields

$$\phi(K,s)=\sum_{H\le K}\mu(H,K)\;|H|^s\,;$$

setting $K=G$, we obtain (2).

We can now divide both sides of (2) by $|G|^s$ to obtain the expression for $P(G,s)$ as a finite Dirichlet series mentioned in Section 1:

$$P(G,s) = \sum_{H\le G} \frac{\mu(H,G)}{|G:H|^s}. \tag{3}$$

In particular,

$$P(G,-1) = \sum_{H\le G} \mu(H,G)\,|G:H|\,. \tag{4}$$

2.2. *A Relative Version*

Let N be a normal subgroup of G and let $\overline{G} = G/N$. Fix a non-negative integer s such that $\overline{G}$ admits s generators, let $y = (y_1, \dots, y_s)$ be a generating s-tuple of $\overline{G}$, and let $P(G,N,s)$ be the probability that a random lift of y to an s-tuple in G generates G; thus $P(G,N,s) = \phi(G,N,s)/\,|N|^s$, where $\phi(G,N,s)$ is the number of generating s-tuples of G lying over y. Gaschütz [15] proved that this number is independent of the choice of y, so we are justified in omitting y from the notation. More precisely, Gaschütz gave a formula for $\phi(G,N,s)$, generalizing Hall's formula (2), in which y plays no role. Although he avoided explicit mention of the Möbius function, his formula can be written as

$$\phi(G,N,s) = \sum_{\substack{H\le G\\ HN=G}} \mu(H,G)\,|H\cap N|^s\,. \tag{5}$$

This reduces to (2) if $N = G$. Dividing (5) by $|N|^s$, we obtain

$$P(G,N,s) = \sum_{HN=G} \frac{\mu(H,G)}{|G:H|^s} \tag{6}$$

since $|N : H\cap N| = |G:H|$ if $HN = G$.

We recall the proof of (5): For any $K \le G$ let $\alpha(K)$ be the number of lifts of y to K and let $\beta(K)$ be the number of such lifts that generate K. Then $\alpha(K) = \sum_{H\le K}\beta(H)$, hence $\beta(K) = \sum_{H\le K}\mu(H,K)\alpha(H)$. Equation (5) is now obtained by setting $K = G$ and noting that

$$\alpha(H) = \begin{cases} |H\cap N|^s & \text{if } HN = G\\ 0 & \text{otherwise.}\end{cases}$$

Note further that $\phi(G,s) = \phi(\overline{G},s)\phi(G,N,s)$ [16, Satz 1], since there are $\phi(\overline{G},s)$ generating s-tuples of $\overline{G}$, each of which lifts to $\phi(G,N,s)$ generating s-tuples of G. Hence

$$P(G,s) = P(\overline{G},s)P(G,N,s). \tag{7}$$

We know this initially for sufficiently large positive integers s, but it remains valid as an identity in the ring of Dirichlet series [16, p. 475]. In particular, we can set $s = -1$ to get

$$P(G, -1) = P(\overline{G}, -1)P(G, N, -1), \tag{8}$$

with

$$P(G, N, -1) = \sum_{HN=G} \mu(H, G)\ |G : H|\,. \tag{9}$$

2.3. *Consequences*

We call attention to three corollaries of the results of Section 2.2. First, there is a striking consequence of the fact that $\phi(G, N, s)$ is independent of the generating s-tuple y of $\overline{G}$. Although this was stated by Gaschütz [15] in 1955, it does not seem to be as widely known as it ought to be:

COROLLARY 1. *Let G be a finite group and let $\overline{G}$ be an arbitrary quotient. Let s be an integer such that G admits s generators. Then every generating s-tuple of $\overline{G}$ lifts to a generating s-tuple of G.*

Proof. By hypothesis there is some generating s-tuple y of $\overline{G}$ that lifts to a generating s-tuple of G, so $\phi(G, N, s) > 0$. The result now follows from the fact that $\phi(G, N, s)$ counts the number of lifts of an arbitrary y. ∎

The assumption that G is finite is crucial here; the corollary is false, for instance, if $G = \mathbb{Z}$.

Next, we specialize to the case where N is a p-group for some prime p. Then $|G : H| = |N : H \cap N|$ is a nontrivial power of p if $HN = G$, $H < G$. So the sum in (9) is congruent to $1 \mod p$. Equation (8) therefore yields:

COROLLARY 2. *If the normal subgroup N is a p-group, then the p-part of $P(G, -1)$ is the same as that of $P(\overline{G}, -1)$.*

Finally, we specialize further to the case of a minimal normal subgroup.

COROLLARY 3. *Let N be a minimal normal subgroup of G that is abelian (hence an elementary abelian p-group for some prime p). Then*

$$P(G, s) = P(\overline{G}, s)\left(1 - \frac{c}{|N|^s}\right), \tag{10}$$

where c is the number of complements of N in G (possibly 0).

Proof. Any $H < G$ such that $HN = G$ must map isomorphically onto $\overline{G}$, i.e., must be a complement of N. ($H \cap N$ is a proper $\overline{G}$-submodule of N, hence it is trivial.) In particular there cannot be any inclusion relations among such subgroups H, so they are all maximal and have $\mu(H, G) = -1$. The result now follows from (6) and (7). ∎

By repeated use of (10), Gaschütz [16] obtains an Euler product expansion for the Dirichlet series $P(G, s)$ if G is solvable. See also Bouc [7]. Very briefly, if

$$1 = N_0 < N_1 < \cdots < N_k = G$$

is a chief series, then there is a factor of the form $1 - c_i q_i^{-s}$ for each index $i = 1, \ldots, k$ such that N_i/N_{i-1} has a complement in G/N_{i-1}. Here c_i is the number of complements, which is worked out explicitly by Gaschütz, and $q_i = \left|N_i : N_{i-1}\right|$ is a prime power. Bouc [private communication] has asked whether there is a converse to this result:

Question 1. *If G is a finite group such that $P(G, s)$ has an Euler product expansion with factors of the form $1 - c_i q_i^{-s}$, is G solvable*?

Remark. Bouc [8, 9] has associated to any finite group G a polynomial $\tilde{P}(G)$ in infinitely many variables X_S, one for each finite simple group S. One recovers Hall's function $\phi(G, s)$ by making the substitution $X_S = |S|^s$. Question 1 is motivated, in part, by the fact that G is solvable if and only if $\tilde{P}(G)$ has a particular type of factorization. Bouc's polynomial also has the property that it is irreducible if (and only if) G is simple. Thus one might be tempted to ask whether $P(G, s)$ is irreducible in the ring of finite Dirichlet series if G is simple. But this is already known to be false for the simple group of order 168; see Boston [6, p. 161].

2.4. *Direct Products*

Bouc [8, 9] has given a detailed analysis of the behavior of his polynomial $\tilde{P}(G)$, hence also $\phi(G, s)$ and $P(G, s)$, with respect to direct products. We confine ourselves here to recording one easy observation, for later reference.

We say that two groups G and H are *coprime* if they have no nontrivial isomorphic quotients. Equivalently, G and H are coprime if no proper subgroup $K \leq G \times H$ surjects onto both factors. This is a special case of the analysis of subgroups of a direct product given in [26, Chap. 2, (4.19)] or [27, Lemma 1.1], but we recall the proof: If $\theta\colon G/G' \to H/H'$ is an isomorphism between nontrivial finite quotients, then $K := \{(x, y) \in G \times H : \theta(xG') = yH'\}$ is a proper subgroup that surjects onto both factors. Conversely, if $K \leq G \times H$ surjects onto both factors, then there are canonical isomorphisms

$$G/G' \xleftarrow{\cong} K/(G' \times H') \xrightarrow{\cong} H/H',$$

where $G' = \{x \in G : (x, 1) \in K\}$ and $H' = \{y \in H : (1, y) \in K\}$; the quotients are nontrivial if K is a proper subgroup.

PROPOSITION 1. *Let G and H be coprime finite groups. Then*

$$P(G \times H, s) = P(G, s)P(H, s). \tag{11}$$

In particular,

$$P(G \times H, -1) = P(G, -1)P(H, -1). \tag{12}$$

Proof. Using the second definition of "coprime," one sees that an s-tuple from $G \times H$ generates $G \times H$ if and only if its projections onto the factors generate G and H. Equation (11) follows at once for positive integers s. As in Section 2.2, the equation then holds as an identity in the ring of finite Dirichlet series, whence (12). ∎

We will give a topological explanation of (12) in Section 8.3; see Proposition 12.

3. TOPOLOGICAL INTERPRETATION OF $P(G, -1)$

We prove here Bouc's formula (1) and a relative version of it. Recall that $\mu(H, G)$ can be computed by counting chains of subgroups

$$H = H_0 < H_1 < \cdots < H_l = G,$$

where a chain of length l is counted with the sign $(-1)^l$; see Hall [17, (2.21)]. It follows that the term $\mu(H, G)\ |G : H|$ in (4) is a similar signed count of chains of cosets

$$C_0 < C_1 < \cdots < C_l = G,$$

where $C_0 = xH$ for some $x \in G$. (Use the fact that $xH \subseteq yK$ if and only if $H \leq K$ and $yK = xK$.) The sum in (4) is therefore the signed count of chains of cosets

$$C_0 < C_1 < \cdots < C_l = G$$

with C_0 arbitrary. Such chains are in 1–1 correspondence with chains

$$C_0 < C_1 < \cdots < C_{l-1}$$

of *proper* cosets, i.e., with simplices in the simplicial complex $\Delta(\mathscr{C}(G))$, but we are counting them with the opposite of the usual sign. This proves (1). Note that we get the *reduced* Euler characteristic because we have counted the empty simplex; this is the case $l = 0$ above.

The proof of (1) extends with virtually no change to the relative situation considered in Section 2.2. Starting from (9), one obtains

$$P(G, N, -1) = -\tilde{\chi}(\mathscr{C}(G, N)), \tag{13}$$

where $\mathscr{C}(G, N)$ is the poset of proper cosets xH such that $HN = G$. Thus Eq. (8) can be written as

$$\tilde{\chi}(\mathscr{C}(G)) = -\tilde{\chi}(\mathscr{C}(\overline{G}))\tilde{\chi}(\mathscr{C}(G, N)). \tag{14}$$

We will give a topological explanation of this in Section 8.1; see Proposition 10.

4. THE MAIN THEOREM

4.1. *Statement of the Theorem*

Fix a prime p. For any p-group $P \le G$, let $N^*(P)$ be the subgroup generated by the elements $x \in G$ such that P and P^x generate a p-group, where $P^x = x^{-1}Px$. (See Section 5.1 for other descriptions of $N^*(P)$.) Roughly speaking, we will show that we get good divisibility results for $P(G, -1)$ if there are lots of p-subgroups P with $N^*(P) < G$. We should therefore think of these P as the "good" p-subgroups. Note that if P is "bad" (i.e., $N^*(P) = G$), then so is any subgroup of P. The number q in the following theorem is a measure of how bad a p-subgroup can be.

MAIN THEOREM. *Let G be a finite group, p a prime, and q the maximal order of a p-subgroup P such that $N^*(P) = G$. Then $\tilde{\chi}(\mathscr{C}(G))$, hence also $P(G, -1)$, is divisible by $|G|_p/q$, where $|G|_p$ is the p-part of the order of G. If $p = 2$ and G is not a 2-group, this can be improved by a factor of 2: $\tilde{\chi}(\mathscr{C}(G))$ is divisible by $2|G|_2/q$.*

Remark. If p is odd, the theorem is vacuous unless p divides $|G|$. If $p = 2$, however, the theorem implies that $\tilde{\chi}(\mathscr{C}(G))$ is divisible by 2 if G is not a 2-group, even if $|G|$ is odd.

We will give two proofs of the first assertion of the theorem. The first proof is slightly more straightforward, but the second one is more easily extended to give the second assertion, i.e., the improvement when $p = 2$.

4.2. *Proof of the First Assertion*; *Method* 1

The conjugation action of G on itself permutes the cosets xH and hence induces an action of G on $\mathscr{C} = \mathscr{C}(G)$.

LEMMA 1. *An element $y \in G$ stabilizes the coset xH if and only if y and y^x are in the normalizer $N(H)$ and $y^x \equiv y \bmod H$.*

Proof. We have $y(xH)y^{-1} = yxy^{-1}(yHy^{-1})$. This equals xH if and only if (a) $y \in N(H)$ and (b) $x^{-1}yxy^{-1} \in H$. Note that (b) can also be written as (c) $y^x y^{-1} \in H$. The lemma now follows at once since (a) and (c) imply $y^x \in Hy \subseteq N(H)$. ∎

LEMMA 2. *Let P be a p-subgroup of G. Then any coset fixed by P meets $N^*(P)$.*

Proof. If xH is fixed by P, then Lemma 1 implies that P and P^x normalize H and that $P^xH = PH$. Call this group K. Then K has a Sylow p-subgroup containing P and a K-conjugate of P^x, hence an H-conjugate of P^x. Thus there is an $h \in H$ such that $\langle P, P^{xh} \rangle$ is a p-group, i.e., $xh \in N^*(P)$. ∎

PROPOSITION 2. *Let P be a p-subgroup of G such that $N^*(P) < G$. Then the fixed-point set $\mathscr{C}^P$ is contractible.*

Proof. Since $P \leq N^*(P) < G$, $N^*(P)$ is an element of $\mathscr{C}^P$. Lemma 2 now gives us a conical contraction $xH \supseteq xH \cap N^*(P) \subseteq N^*(P)$ of $\mathscr{C}^P$ (see [22, Section 1.5]); we have used here the fact that an intersection of two cosets, if nonempty, is again a coset. ∎

We can now prove that $\tilde{\chi}(\mathscr{C})$ is divisible by $|G|_p/q$, where q is the maximal order of a p-subgroup P such that $N^*(P) = G$. The method of proof is spelled out in Brown-Thévenaz [11, Section 2] and Quillen [22, Section 4], so we will be brief. Let $\Delta = \Delta(\mathscr{C})$, and consider the action of S on Δ, where S is a Sylow p-subgroup of G. Let

$$\Delta' = \bigcup_{\substack{P \leq S \\ |P| > q}} \Delta^P.$$

The family of subcomplexes Δ^P is closed under intersection, and each is contractible by Proposition 2; hence the union Δ' is contractible. So if we compute $\chi(\mathscr{C}) = \chi(\Delta)$ by counting simplices, the simplices in Δ' contribute 1. On the other hand, the stabilizer in S of any simplex not in Δ' has order dividing q, so each S-orbit of such simplices contributes a multiple of $|S|/q = |G|_p/q$ to $\chi(\mathscr{C})$. Hence $\chi(\mathscr{C}) \equiv 1 \bmod |G|_p/q$.

4.3. *Proof of the First Assertion*; *Method* 2

Although we defined $\mathscr{C}(G)$ to be the set of proper *left* cosets, it is also the set of proper *right* cosets, since $Hx = xH^x$. So G acts on $\mathscr{C}$ by both left and right translation. We therefore get a (left) action of $G \times G$ on $\mathscr{C}$, with (y, z) acting by $xH \mapsto yxHz^{-1}$. (The conjugation action that we used above is obtained by restricting this action to the diagonal.) We have the following analogue of Lemma 1, whose proof is left to the reader:

LEMMA 3. *An element $(y, z) \in G \times G$ stabilizes the coset xH if and only if y^x and z are in the normalizer $N(H)$ and $y^x \equiv z$* mod *H.*

Given p-subgroups $P, Q \leq G$, let $N^*(P, Q)$ be the group generated by the elements $x \in G$ such that $\langle P^x, Q \rangle$ is a p-group. If $P = Q$, this reduces to $N^*(P)$. The analogue of Lemma 2 is:

LEMMA 4. *Let R be a p-subgroup of $G \times G$. Let P (resp. Q) be the projection of R on the first (resp. second) factor. Then any coset xH fixed by R meets $N^*(P, Q)$.*

Proof. If xH is fixed by R, then Lemma 3 implies that P^x and Q normalize H and that $P^x H = QH$. Call this group K. Then K has a Sylow p-subgroup containing Q and a K-conjugate of P^x, hence an H-conjugate of P^x. Thus there is an $h \in H$ such that $\langle P^{xh}, Q \rangle$ is a p-group, i.e., $xh \in N^*(P, Q)$. ∎

Recall that q is the maximal order of a p-subgroup of G such that $N^*(P) = G$.

PROPOSITION 3. *Let S be a Sylow p-subgroup of G, let R be a subgroup of $S \times S$, and let P, Q be the projections of R on the factors as in Lemma 4.*

1. *If $N^*(P, Q) < G$, then $\mathscr{C}^R$ is contractible.*
2. *If $N^*(P, Q) = G$, then $|R| \leq q\,|S|$; hence $|S \times S : R| \geq |S|\,/q$.*

Proof. If $N^*(P, Q) < G$, then $N^*(P, Q)$ is an element of $\mathscr{C}^R$, and Lemma 4 gives us a conical contraction of $\mathscr{C}^R$, whence 1. To prove 2, note first that $N^*(P, Q) \leq N^*(P \cap Q)$. So if $N^*(P, Q) = G$, then $|P \cap Q| \leq q$. Hence

$$|R| \leq |P| \cdot |Q| = |P \cap Q| \cdot |PQ| \leq q\,|S|\,.$$

∎

The proof of the first assertion of the main theorem is now completed exactly as in Section 4.2, by letting $S \times S$ act on Δ and considering

$$\Delta' = \bigcup_{\substack{R \leq S \times S \\ |R| > q|S|}} \Delta^R.$$

4.4. *Proof of the Second Assertion*

Assume now that $p = 2$ and that G is not a 2-group. We continue to denote by S a fixed Sylow 2-subgroup of G and by q the maximal order of a 2-subgroup P such that $N^*(P) = G$. Since the elements of $\mathscr{C}$ are both left cosets and right cosets, the inversion permutation $x \mapsto x^{-1}$ of G induces an automorphism of $\mathscr{C}$. The action of $G \times G$ on $\mathscr{C}$ therefore extends to an action of $(G \times G) \rtimes \mathbb{Z}_2$, with the generator τ of $\mathbb{Z}_2$ acting on $\mathscr{C}$ by inversion and on $G \times G$ by interchanging the factors. In particular, we have an action of the 2-group $(S \times S) \rtimes \mathbb{Z}_2$ on $\mathscr{C}$.

The second assertion of the main theorem follows from the next proposition, via the same methods we have been using:

PROPOSITION 4. *Let $T \leq (S \times S) \rtimes \mathbb{Z}_2$ be a subgroup such that $|T| > q\,|S|$. Then $\mathscr{C}^T$ is contractible.*

Proof. Let $R = T \cap (S \times S)$ and let P and Q be the projections of R on the factors. If $R = T$, we are done by Proposition 3. So we may assume that T is generated by R and an element $w = (y, z)\tau$ that normalizes R. Now $R = R^w$ has projections Q^z, P^y, so we must have $P = Q^z$ and $Q = P^y$. Replacing T by its conjugate $T^{(1,y^{-1})}$, we are reduced to the case where $Q = P$.

If $N^*(P) < G$, we get a conical contraction of $\mathscr{C}^T$ as in the proof of Proposition 3. (Note that $N^*(P)$, if proper, is in $\mathscr{C}^T$; in fact, it is a subgroup containing S and hence is fixed by the whole group $(S \times S) \rtimes \mathbb{Z}_2$.) So we may assume that $N^*(P) = G$. Then $|P| \leq q$, and we have

$$|T| = 2\,|R| \leq 2\,|P|^2 \leq 2q^2 \leq 2q\,|S|. \tag{15}$$

On the other hand, $|T| \geq 2q\,|S|$ by hypothesis, so the inequalities in (15) must be equalities. Thus $R = P \times P$ and $|P| = q = |S|$, whence $T = (S \times S) \rtimes \mathbb{Z}_2$. In this case, I claim that $\mathscr{C}^T$ is the set of proper subgroups $H \geq S$. Since G is not a 2-group, this implies that $\mathscr{C}^T$ has a smallest element and is therefore contractible.

To prove the claim, suppose xH is fixed by T. Since xH is fixed by $S \times S$, we have $S, S^x \leq H$. And since xH is fixed by τ, we have $xH = Hx^{-1} = x^{-1}H^{x^{-1}}$, so $x \in N(H)$ and $x^2 \in H$. It follows that $x \in H$, for otherwise H would have index 2 in $\langle H, x\rangle$, contradicting the fact that H contains a Sylow 2-subgroup of G. Thus $xH = H$, and the claim follows. ∎

5. REMARKS ON $N^*(P)$

5.1. *Alternate Definitions*

In order to apply the main theorem, one needs to be able to compute $N^*(P)$ for a p-subgroup $P \leq G$. Recall that $N^*(P)$ is defined to be the group generated by the set

$$C(P) = \{x \in G : \langle P, P^x\rangle \text{ is a } p\text{-group}\}.$$

We begin by giving two reformulations of this definition, the first of which explains the notation $N^*(P)$.

PROPOSITION 5. *Let P be a p-subgroup of G.*

(a) *$N^*(P)$ is generated by the set*

$$A(P) = \bigcup_H N(H),$$

where H ranges over the p-subgroups of G containing P.

(b) *Fix a Sylow p-subgroup S of G containing P. Then $N^*(P)$ is generated by the set*

$$B(P) = \{x \in G : P^x \leq S\}.$$

Proof. Trivially $B(P) \subseteq C(P)$, so $\langle B(P)\rangle \leq N^*(P)$. For the opposite inclusion, suppose $x \in C(P)$. Then there is an element $y \in G$ such that $\langle P, P^x\rangle^y \leq S$. We now have $y \in B(P)$ and $xy \in B(P)$, so $x \in \langle B(P)\rangle$. This proves (b). For (a), we trivially have $A(P) \subseteq C(P)$, so $\langle A(P)\rangle \leq N^*(P)$. To prove the opposite inclusion, we use Alperin's fusion theorem [1]; see also Barker [4] for a short proof of Alperin's theorem. Suppose $x \in C(P)$. Then there is a Sylow subgroup S containing P and P^x. According to the fusion theorem, the conjugation map $c_x : P \to P^x$ can be factored as a composite of conjugation isomorphisms

$$P = P_0 \to P_1 \to \cdots \to P_n = P^x$$

with the following property: For each $i = 1, \ldots, n$, there is a subgroup H_i, with $P_{i-1}, P_i \leq H_i \leq S$, such that the map $P_{i-1} \to P_i$ is conjugation by an element $x_i \in N(H_i)$. Hence $x = x_0 x_1 \cdots x_n$ with $x_0 \in N(P) \subseteq A(P)$, so it suffices to show $x_i \in \langle A(P)\rangle$ for $i = 1, \ldots, n$. Let $y_i = x_1 \cdots x_i$, so that $P_i = P^{y_i}$. Assume inductively that $x_1, \ldots, x_{i-1} \in \langle A(P)\rangle$. Since $P^{y_{i-1}} \leq H_i$, we then have $x_i \in A(P^{y_{i-1}}) = A(P)^{y_{i-1}} \subseteq \langle A(P)\rangle$. ∎

5.2. *Examples*

EXAMPLE 1. If S is a Sylow p-subgroup of G, then $N^*(S) = N(S)$.

The next two examples will make use of Burnside's fusion theorem [12, Sect. 123], which says the following. Let S be a Sylow subgroup of G; if P and Q are normal subgroups of S that are conjugate in G, then P and Q are conjugate in $N(S)$. We recall the proof: If $Q = P^x$, then S and S^x are Sylow subgroups of $N(Q)$, and hence $S = S^{xy}$ for some $y \in N(Q)$; then $Q = P^{xy}$ with $xy \in N(S)$.

EXAMPLE 2. Suppose $|P| = |G|_p / p$, and let S be a Sylow p-subgroup containing P. Then $N^*(P) = \langle N(P), N(S)\rangle$. We can see this, for instance, using $B(P)$. Suppose x is an element of G such that $P^x \leq S$. Then P and P^x are normal in S, so we can apply Burnside's theorem to find $y \in N(S)$ such that $xy \in N(P)$; hence $x \in \langle N(P), N(S)\rangle$.

EXAMPLE 3. If S is abelian, then a similar use of Burnside's theorem shows that $N^*(P) = \langle N(P), N(S)\rangle$ for any $P \leq S$. (More generally, this holds if every subgroup of S is normal.) If, in addition, the normalizer of S equals its centralizer, then $N(S) \leq N(P)$ and $N^*(P) = N(P)$.

EXAMPLE 4. Suppose that G has a cyclic or generalized quaternion Sylow p-subgroup i.e., that G has p-rank 1. If $P \leq G$ is a subgroup of order p, then $N^*(P) = N(P)$. In fact, if $\langle P, P^x\rangle$ is a p-group, then P is its unique subgroup of order p, so $P^x = P$ and $x \in N(P)$.

EXAMPLE 5. Suppose any two distinct Sylow p-subgroups have trivial intersection. Then for any $P \leq G$ of order p, $N^*(P) = N(S)$, where S is the unique Sylow subgroup containing P. To see this, note that, for any $x \in G$, S^x is the unique Sylow subgroup containing P^x. So if $\langle P, P^x\rangle$ is a p-group, we must have $S^x = S$, i.e., $x \in N(S)$.

EXAMPLE 6. Suppose $p = 2$ and the Sylow 2-subgroups are dihedral. If $P \leq G$ is cyclic of order at least 4, then $N^*(P) = N(P)$, by an argument similar to that of Example 4. Similarly, if $P \leq G$ is dihedral of order at least 8 and P' is its cyclic subgroup of order 4, then $N^*(P) \leq N(P')$.

5.3. *Connection with Connectivity of p-group Posets*

Let p^e be a power of p that divides $|G|$, and let $\mathcal{S}_{p,e}(G)$ be the poset of p-subgroups of G of order $> p^e$. For example, $\mathcal{S}_{p,0}(G)$ is the poset $\mathcal{S}_p(G)$ of nontrivial p-subgroups introduced by Brown [10] and studied further by Quillen [22].

PROPOSITION 6. *Suppose $\mathcal{S}_{p,e}(G)$ is disconnected. Then $N^*(P) < G$ for every p-subgroup P such that $|P| > p^e$, i.e., the number q in the main theorem satisfies $q \leq p^e$.*

Proof. Let $\mathcal{S} = \mathcal{S}_{p,e}(G)$. Given $P \in \mathcal{S}$ and $x \in G$ such that $\langle P, P^x\rangle$ is a p-group, P and P^x are in the same connected component of $\mathcal{S}$ via the path $P \leq \langle P, P^x\rangle \geq P^x$. Hence $N^*(P)$, acting on $\mathcal{S}$ by conjugation, stabilizes the component containing P. Since G is transitive on the maximal elements of $\mathcal{S}$, it is transitive on components, and we conclude that $N^*(P) < G$. ∎

Remark. The estimate $q \leq p^e$ is not sharp. In other words, it is not true in general that q is the smallest power p^e such that $\mathcal{S}_{p,e}(G)$ is disconnected. There are easy counterexamples involving direct products. John Shareshian [private communication] has provided a more interesting counterexample, with $G = SL(3, 3)$ and $p = 3$. Shareshian has shown that $q = 3$ but that $\mathcal{S}_{3,1}(G)$ is connected; the smallest power 3^e such that $\mathcal{S}_{3,e}(G)$ is disconnected is 3^2 in this case.

In case $e = 0$, the condition that $\mathcal{S}_p(G)$ be disconnected has been extensively studied. Finite groups with this property are said to have a *strongly p-embedded subgroup*. There are several characterizations of such groups; see [2, Sect. 46] or [22, Sect. 5]. And there is also a classification of them; see [3, (6.2)] for the list. The proofs in [22, Sect. 5] extend easily to the case of general e and yield:

PROPOSITION 7. *Let $\mathcal{S} = \mathcal{S}_{p,e}(G)$, where p^e is a proper divisor of $|G|$. The following conditions on a subgroup $M \leq G$ are equivalent*:

1. *M contains the stabilizer of a component of $\mathcal{S}$.*
2. *For some Sylow p-subgroup S of G, M contains $N(P)$ for all $P \in \mathcal{S}$ such that $P \leq S$.*
3. *$p^{e+1} \mid |M|$, and M contains $N(P)$ for all $P \in \mathcal{S}$ such that $P \leq M$.*
4. *M contains $N(S)$ for some Sylow p-subgroup S, and for each $P \in \mathcal{S}$ with $P \leq M$, M contains all p-subgroups $\geq P$.*
5. *$p^{e+1} \mid |M|$, and for $x \notin M$, $p^{e+1} \nmid |M \cap M^x|$.*

COROLLARY. *The stabilizer of the component of $\mathcal{S}$ containing a Sylow p-subgroup S is generated by the groups $N(P)$ with $P \in \mathcal{S}$, $P \leq S$.*

6. SPECIAL CASES AND EXAMPLES

6.1. *Special Cases of the Theorem*

Fix a prime p dividing $|G|$. Assume, for simplicity, that $O_p(G) = 1$, i.e., that G has no nontrivial normal p-subgroups. (If this fails, we can replace G by $G/O_p(G)$; see Section 2.3, Corollary 2.) Let S be a Sylow p-subgroup of G.

1. $P(G, -1)$ is divisible by p; for $p = 2$, $P(G, -1)$ is divisible by 4. This follows from the fact that the number q in the main theorem satisfies $q < |S|$ (Section 5.2, Example 1).

2. If $N^*(P) < G$ for every subgroup P of order p, then $P(G, -1)$ is divisible by $|G|_p$ (or $2|G|_2$ if $p = 2$). This holds, in particular, if S is cyclic or generalized quaternion, or if S is abelian and its normalizer equals its centralizer, or if G has a strongly p-embedded subgroup.

Here we have $q = 1$, since the hypothesis implies that $N^*(P) < G$ for every nontrivial p-subgroup P. This explains the first sentence. For the second sentence, see Section 5.2 (Examples 3 and 4) and Section 5.3 (Proposition 6). Note that the case of a strongly p-embedded subgroup includes the case where the Sylow subgroups have trivial intersection; here we have an alternate proof that $N^*(P) < G$ via Example 5 of Section 5.2.

3. If $p = 2$ and S is dihedral, then $P(G, -1)$ is divisible by $|G|_2/2$. If $N^*(V) < G$ for every subgroup V isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, then $P(G, -1)$ is divisible by $|G|_2$. This holds, in particular, if S is dihedral of order 8 and is self-normalizing.

This follows from Section 5.2, Example 6. For the case where $|S| = 8$, see also Example 2.

6.2. *Example*: $G = A_7$

The order of A_7 is $2^3 \cdot 3^2 \cdot 5 \cdot 7$, so statement 1 in Section 6.1 gives divisibility of $P(A_7, -1)$ by $2^2 \cdot 3 \cdot 5 \cdot 7$. One can check that the Sylow 2-subgroup is dihedral of order 8 and is self-normalizing, so statement 3 gives us another factor of 2, i.e., $P(A_7, -1)$ is divisible by $2^3 \cdot 3 \cdot 5 \cdot 7 = |A_7|/3$. Finally, one can check that $N^*(P) = A_7$ if P is the group of order 3 generated by a 3-cycle. The main theorem therefore does not allow us to do any better at the prime 3. Moreover, direct computation shows that $P(A_7, -1)/|A_7| = -1640/3$, so one cannot do better at the prime 3.

Remark. Further computations of $P(G, -1)/|G|$ for simple groups G can be found in Table I. These computations were done with the aid of the computer algebra system GAP [14], using programs kindly provided by S. Bouc.

6.3. *Example*: $G = PSL(2, l)$

Let G be the simple group $PSL(2, l)$, where l is a prime ≥ 5. The subgroup structure of G is well known; see, for instance, Burnside [12, Chap. XX], Dickson [13, Chap. XII], or Huppert [18, Sect. II.8]. This makes it easy to apply the results of Section 6.1.

We have $|G| = 2lmn$, where $m = (l-1)/2$ and $n = (l+1)/2 = m+1$. The odd Sylow subgroups of G are all cyclic, so $P(G, -1)$ is divisible by $|G|_p$ for each odd prime p (Section 6.1, statement 2). The only possible difficulty, then, is at the prime 2, where the Sylow subgroup S is dihedral. Its order is the largest power 2^a such that $l \equiv \pm 1 \bmod 2^a$. Equivalently, its order is $2b$, where b is the 2-part of m or n, whichever is even.

If $b = 2$, then statement 1 in Section 6.1 gives divisibility of $P(G, -1)$ by $2b = 4$, hence by $|G|$. If $b = 4$, then one checks from the list of subgroups of G that S is self-normalizing, so we can apply statement 3 and we again get divisibility of $P(G, -1)$ by $|G|$. On the other hand, if $b \geq 8$, i.e., if $l \equiv \pm 1 \bmod 16$, then the main theorem only gives divisibility of $P(G, -1)$ by $|G|/2$. Indeed, if $V \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $N^*(V)$ contains $N(V)$, which is a maximal subgroup isomorphic to S_4, as well as a dihedral group of order at least 16; hence $N^*(V) = G$. Thus the main theorem only gives divisibility of

TABLE I
$P(G, -1)$ for Some Simple Groups G

G	$\lvert G\rvert$	$P(G, -1)/\lvert G\rvert$
A_5	$2^2 \cdot 3 \cdot 5$	-26
A_6	$2^3 \cdot 3^2 \cdot 5$	265
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	$-1640/3$
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$30401/2^2 \cdot 3$
A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	$-760573/2^2 \cdot 3^2$
$PSL(2, 7)$	$2^3 \cdot 3 \cdot 7$	-17
$PSL(2, 8)$	$2^3 \cdot 3^2 \cdot 7$	-246
$PSL(2, 11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	202
$PSL(3, 3)$	$2^4 \cdot 3^3 \cdot 13$	$-3515/2 \cdot 3$
$PSL(3, 4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$-81017/2$
$PSL(3, 5)$	$2^5 \cdot 3 \cdot 5^3 \cdot 31$	$-16197/2^2 \cdot 5$
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	-1756
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	$753371/2^3 \cdot 3$
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	$1474753/2^3$
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	$2617621/3$
$Sz(8)$	$2^6 \cdot 5 \cdot 7 \cdot 13$	-17838
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	5540
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	$40704809/2^3 \cdot 3 \cdot 5$

$P(G, -1)$ by $|G|/2$ in this case. Nevertheless, explicit computation, based on the calculation of $P(G, s)$ by Hall [17], shows that $P(G, -1)$ is divisible by $|G|$. We will give a topological explanation of this in Section 7.

7. THE BIHOLOMORPH

Our proof of the main theorem made use of the action of $(G \times G) \rtimes \mathbb{Z}_2$ on $\mathscr{C}(G)$. We discuss here a bigger group that acts, namely the biholomorph of G, which can sometimes be used to improve the divisibility result in the main theorem.

Let $S(G)$ be the group of permutations of the underlying set of G. Let $L, R \leq S(G)$ be the groups of left and right translations. They are isomorphic copies of G that commute with one another and are normalized by $\mathrm{Aut}(G)$. The product $LR\,\mathrm{Aut}(G) \leq S(G)$ is called the *holomorph* of G, and denoted $\mathrm{Hol}(G)$. It can also be described as the normalizer in $S(G)$ of L (or of R), and it is isomorphic to $G \rtimes \mathrm{Aut}(G)$; see Burnside [12, Sect. 64]. Yet another description is that $\mathrm{Hol}(G)$ consists of all $h \in S(G)$ that preserve the ternary relation $(x, y, z) \mapsto xy^{-1}z$, i.e., that satisfy $h(xy^{-1}z) = h(x)h(y)^{-1}h(z)$; see Mac Lane [20, Sect. IV.1, Exercise 1]. If G is centerless, then we can identify LR with $G \times G$, so that $\mathrm{Hol}(G) = (G \times G)\,\mathrm{Aut}(G)$; the intersection $(G \times G) \cap \mathrm{Aut}(G)$ is G, embedded in $G \times G$ as the diagonal subgroup and in $\mathrm{Aut}(G)$ as the group of inner automorphisms.

The *biholomorph* $\mathrm{Bi}(G)$ is obtained from $\mathrm{Hol}(G)$ by adjoining the inversion permutation $\tau\colon x \mapsto x^{-1}$, which normalizes $\mathrm{Hol}(G)$. If G is nonabelian, then $\mathrm{Bi}(G) = \mathrm{Hol}(G) \rtimes \mathbb{Z}_2$. If G is centerless, then

$$\mathrm{Bi}(G) = (G \times G)\,\mathrm{Aut}(G)\langle\tau\rangle \cong (G \rtimes \mathrm{Aut}(G)) \rtimes \mathbb{Z}_2.$$

All elements of $\mathrm{Bi}(G)$ map cosets to cosets, hence we have an action of $\mathrm{Bi}(G)$ on $\mathscr{C}(G)$. In principle, then, we should be able to improve the main theorem by using this action, provided we can prove the contractibility of enough fixed-point sets.

To illustrate this, we continue the discussion begun in Section 6.3 of the group $G = PSL(2, l)$, $l \equiv \pm 1 \bmod 16$. We have $\mathrm{Aut}(G) = PGL(2, l)$, which contains G with index 2. Thus $\mathrm{Bi}(G) = (G \times G)\,\mathrm{Aut}(G)\langle\tau\rangle$ contains $(G \times G) \rtimes \mathbb{Z}_2$ with index 2. If S is a Sylow 2-subgroup of G and $T > S$ is a Sylow 2-subgroup of $PGL(2, l)$, then $\mathrm{Bi}(G)$ has a Sylow 2-subgroup $U = (S \times S)T\langle\tau\rangle$. Consider the action of U on $\mathscr{C}(G)$.

Using the methods of Section 4.4, one checks that the "bad" subgroups of U (those whose fixed-point sets in $\mathscr{C}(G)$ are not known to be contractible) are no bigger than the bad subgroups of $(S \times S) \rtimes \mathbb{Z}_2$. Their index in U, however, is twice as big as the corresponding index in $(S \times S) \rtimes \mathbb{Z}_2$, so we obtain an improvement of the main theorem by a factor of 2. Roughly speaking, then, the outer automorphism of order 2 explains why $P(G, -1)$ is divisible by $|G|$, and not just $|G|/2$. Further details are left to the interested reader.

8. THE HOMOTOPY TYPE OF THE COSET POSET

We assume in this section that the reader is familiar with standard terminology and results from the topological theory of posets, as in Quillen [22, Sect. 1] or Björner [5, Sects. 9 and 10].

8.1. *Quotients*

The main result of this subsection is Proposition 10, which arose from a question asked by S. Bouc. We begin with two easy special cases, whose proofs are more elementary than the proof in the general case. Let G be a finite group, N a normal subgroup, and $\overline{G}$ the quotient group G/N.

PROPOSITION 8. *If N is contained in the Frattini subgroup of G, then the quotient map $q: G \twoheadrightarrow \overline{G}$ induces a homotopy equivalence $\mathscr{C}(G) \to \mathscr{C}(\overline{G})$.*

Proof. The hypothesis implies that $q(C)$ is a proper coset in $\overline{G}$ for any $C \in \mathscr{C}(G)$, so we do indeed get a map $\mathscr{C}(G) \to \mathscr{C}(\overline{G})$. It is a homotopy equivalence, with homotopy inverse given by q^{-1}, since $C \subseteq q^{-1}(q(C))$ for $C \in \mathscr{C}(G)$ and $D = q(q^{-1}(D))$ for $D \in \mathscr{C}(\overline{G})$. ∎

Note that, for general N, the same proof gives a homotopy equivalence $\mathscr{C}_0 \to \mathscr{C}(\overline{G})$, where $\mathscr{C}_0$ is the set of cosets in G such that $q(C) \neq \overline{G}$. So one way to analyze $\mathscr{C}(G)$ is to start with $\mathscr{C}_0$, which is $\mathscr{C}(\overline{G})$ up to homotopy, and then examine the effect of adjoining the proper cosets C such that $q(C) = \overline{G}$. Here is a case where that analysis is particularly easy:

PROPOSITION 9. *Suppose N is a minimal normal subgroup and is abelian. Let c be the number of complements of N in G. Then $\Delta(\mathscr{C}(G))$ is homotopy equivalent to the join of $\Delta(\mathscr{C}(\overline{G}))$ with a discrete set of $c\,|N|$ points.*

Proof. If $C = xH$ is a proper coset such that $q(C) = \overline{G}$, then H is a complement of N, as we noted in Section 2.3, proof of Corollary 3. So the effect of adjoining C to $\mathscr{C}_0$ is to cone off an isomorphic copy of $\mathscr{C}(\overline{G})$. Thus we get $\Delta(\mathscr{C})$ from $\Delta(\mathscr{C}_0)$ by coning off $c\,|N|$ copies of $\Delta(\mathscr{C}(\overline{G}))$. Moreover, the inclusion of each of these copies into $\Delta(\mathscr{C}_0)$ is a homotopy equivalence, since q maps the set of strict predecessors of xH isomorphically onto $\mathscr{C}(\overline{G})$. Up to homotopy, then, we are starting with $\Delta(\mathscr{C}(\overline{G}))$ and coning it off $c\,|N|$ times, i.e., we are joining $\Delta(\mathscr{C}(\overline{G}))$ to a set of $c\,|N|$ points. ∎

Note that we might have $c = 0$. This is the case if and only if N is contained in the Frattini subgroup of G, in which case we are in the situation of Proposition 8.

Finally, we return to an arbitrary quotient and prove a general result that includes the previous two as special cases. Recall from Section 3 that $\mathscr{C}(G, N)$ denotes the set of proper cosets C such that $q(C) = \overline{G}$, i.e., $\mathscr{C}(G, N)$ is the set-theoretic complement of $\mathscr{C}_0$ in $\mathscr{C}(G)$. Recall also that the *join* $X * Y$ of two posets X, Y is the disjoint union of X and Y, with the ordering that induces the given orderings on X and Y and satisfies $x < y$ for all $x \in X$ and $y \in Y$. There is an order-preserving map $j: \mathscr{C}(G) \to \mathscr{C}(\overline{G}) * \mathscr{C}(G, N)$ such that j is the identity on $\mathscr{C}(G, N)$ and j maps $\mathscr{C}_0$ to $\mathscr{C}(\overline{G})$ via q.

PROPOSITION 10. *The map $j: \mathscr{C}(G) \to \mathscr{C}(\overline{G}) * \mathscr{C}(G, N)$ is a homotopy equivalence.*

Proof. We use Quillen's "Theorem A" (see [22, Theorem 1.6, 5, Theorem 10.5]). Thus we must show, for each $D \in \mathscr{C}(\overline{G}) * \mathscr{C}(G, N)$, that the fiber $\mathscr{F} := \{C \in \mathscr{C}(G) : j(C) \leq D\}$ is a contractible subposet of $\mathscr{C}(G)$.

This is trivial if $D \in \mathscr{C}(\overline{G})$, since $\mathscr{F}$ then has a largest element $q^{-1}(D)$; so we may assume $D \in \mathscr{C}(G, N)$, and, in fact, we may assume that D is a subgroup H such that $q(H) = \overline{G}$.

Then $\mathscr{F} = \mathscr{C}_0 \cup \mathscr{C}_1$, where $\mathscr{C}_1$ is the set of cosets contained in H. The intersection $\mathscr{C}_0' = \mathscr{C}_0 \cap \mathscr{C}_1$ consists of the cosets in H that do not surject onto $\overline{G}$. The proof that q induces a homotopy equivalence $\mathscr{C}_0 \to \mathscr{C}(\overline{G})$ applies equally well to the surjection $H \twoheadrightarrow \overline{G}$, so q also induces a homotopy equivalence $\mathscr{C}_0' \to \mathscr{C}(\overline{G})$. Thus the inclusion $\mathscr{C}_0' \hookrightarrow \mathscr{C}_0$ is a homotopy equivalence. On the other hand, $\mathscr{C}_1$ has a largest element and so is contractible. Passing now to associated simplicial complexes, one checks that $\Delta(\mathscr{F}) = \Delta(\mathscr{C}_0) \cup_{\Delta(\mathscr{C}_0')} \Delta(\mathscr{C}_1)$, with $\Delta(\mathscr{C}_0')$ a strong deformation retract of $\Delta(\mathscr{C}_0)$. Thus $\Delta(\mathscr{F})$ admits a strong deformation retraction onto $\Delta(\mathscr{C}_1)$, hence $\mathscr{F}$ is indeed contractible. ∎

Remark. Proposition 10 provides a topological explanation for Eq. (14).

8.2. *Solvable Groups*

Using Proposition 9 and an obvious induction argument, we obtain:

PROPOSITION 11. *Let G be a finite solvable group and let*

$$1 = N_0 < N_1 < \cdots < N_k = G$$

be a chief series. Then $\mathscr{C}(G)$ has the homotopy type of a bouquet of $(d-1)$-spheres, where d is the number of indices $i = 1, \ldots, k$ such that N_i/N_{i-1} has a complement in G/N_{i-1}. The number of spheres is $(-1)^{d-1}\tilde{\chi}(\mathscr{C}(G)) = (-1)^d P(G, -1)$.

The number d can also be described as the number of factors in the Euler product expansion of $P(G, s)$; see Section 2.3.

It would be interesting to understand the nonzero homology group of $\mathscr{C}(G)$ as a representation of the holomorph or biholomorph of G. Consider, for example, the simplest case, where G is an elementary abelian p-group for some prime p, i.e., G is the additive group of a vector space $V = \mathbb{F}_p^r$. Then the holomorph is the same as the biholomorph and is the affine group $A = V \rtimes GL(V)$. The coset poset consists of proper affine subspaces of V, and it is homotopy equivalent to a bouquet of $(p^r - 1)(p^{r-1} - 1) \cdots (p - 1)$ spheres of dimension $r - 1$. The resulting homology representation was studied by Solomon [24, 25] and Lusztig [19]; its restriction to $GL(V)$ is the Gelfand–Graev representation.

8.3. *Further Results*

We give here a few easy observations about the homotopy type of the coset poset.

First we consider the direct product of two coprime groups (see Section 2.4). The analysis of the coset poset for such a product follows closely Quillen's study of products in [22, Proposition 2.6], so we will be brief. We call a coset $C \subseteq G \times H$ *saturating* if it surjects onto both factors.

LEMMA 5. *For any groups G, H, let $\mathscr{C}_0(G \times H)$ be the set of non-saturating cosets in $G \times H$. Then $\mathscr{C}_0(G \times H)$ is homotopy equivalent to the join $\mathscr{C}(G) * \mathscr{C}(H)$.*

Proof. For any group G, let $\mathscr{C}^+(G)$ be the set of all cosets in G, i.e., $\mathscr{C}^+(G) = \mathscr{C}(G) \cup \{G\}$. Let $\mathscr{C}_{00}(G \times H)$ be the set of proper cosets of the form $C_1 \times C_2$ with $C_1 \in \mathscr{C}^+(G)$ and $C_2 \in \mathscr{C}^+(H)$. Then $\mathscr{C}_{00}(G \times H)$ can be identified with $\mathscr{C}^+(G) \times \mathscr{C}^+(H) - \{(G, H)\}$, whose geometric realization is homeomorphic to that of $\mathscr{C}(G) * \mathscr{C}(H)$; see [22, Proposition 1.9]. On the other hand, we can deform $\mathscr{C}_0(G \times H)$ onto $\mathscr{C}_{00}(G \times H)$ via $C \subseteq p(C) \times q(C)$ for $C \in \mathscr{C}_0(G)$, where $p: G \times H \to G$ and $q: G \times H \to H$ are the projections. ∎

If G and H are coprime, there are no saturating proper cosets, so $\mathscr{C}_0(G \times H) = \mathscr{C}(G \times H)$. Consequently:

PROPOSITION 12. *If G and H are coprime, then $\mathscr{C}(G \times H)$ is homotopy equivalent to $\mathscr{C}(G) * \mathscr{C}(H)$. In particular,*

$$\tilde{\chi}(\mathscr{C}(G \times H)) = -\tilde{\chi}(\mathscr{C}(G))\tilde{\chi}(\mathscr{C}(H)). \tag{16}$$

This provides a topological explanation for Eq. (12).

One can still get interesting results about $\mathscr{C}(G \times H)$ if G and H are not coprime, by using Lemma 5 as a starting point; one must then examine the effect of adjoining the saturating cosets to $\mathscr{C}_0(G \times H)$. If H is simple, for example, every saturating proper subgroup $K \leq G \times H$ is the graph of a surjection $G \twoheadrightarrow H$. In particular, every such K is maximal and isomorphic to G, so the adjunction of a coset zK simply cones off a copy of $\mathscr{C}(G)$. Moreover, this copy of $\mathscr{C}(G)$, consisting of the strict predecessors of zK, is null-homotopic in $\mathscr{C}_0(G \times H)$; in fact, it maps to the contractible poset $\mathscr{C}(G) \times \mathscr{C}^+(H)$ under the homotopy equivalence $\mathscr{C}_0(G \times H) \to \mathscr{C}_{00}(G \times H)$ constructed in the proof of Lemma 5. Thus the adjunction is equivalent, up to homotopy, to wedging on a copy of the suspension of $\mathscr{C}(G)$. This proves:

PROPOSITION 13. *If H is simple, then $\mathscr{C}(G \times H)$ is homotopy equivalent to the wedge of $\mathscr{C}(G) * \mathscr{C}(H)$ and $m\,|H|$ copies of the suspension of $\mathscr{C}(G)$, where m is the number of surjections $G \twoheadrightarrow H$.*

Finally, we consider the connectedness of $\mathscr{C}(G)$.

PROPOSITION 14. *$\mathscr{C}(G)$ is connected unless G is cyclic of prime power order.*

Proof. If G is not cyclic, then we can connect any coset xH to the singleton coset $\{1\}$ by the path $xH \supseteq \{x\} \subseteq \langle x\rangle \supseteq \{1\}$, so $\mathscr{C}(G)$ is connected. If G is cyclic but not of prime power order, a variant of this argument, which is left to the reader, again shows that $\mathscr{C}(G)$ is connected. Alternatively, we can appeal to Proposition 11 or Proposition 12, each of which implies that $\mathscr{C}(G)$ has the homotopy type of a bouquet of $(d-1)$-spheres, where d is the number of distinct prime divisors of $|G|$; so $\mathscr{C}(G)$ is connected if $d \geq 2$. ∎

8.4. *Questions*

Aside from the results in the previous subsections, I know practically nothing about the homotopy type of $\mathscr{C}(G)$. For example, one can ask the following, in the spirit of Bouc's question in Section 2.3:

QUESTION 2. *Can one characterize finite solvable groups in terms of the combinatorial topology of the coset poset*?

What I have in mind here is an analogue of Shareshian's theorem [23] that a finite group is solvable if and only if its subgroup lattice is nonpure shellable. For example, one might hope for some sort of converse to Proposition 11. But it is not clear what form such a converse might take. Indeed, Shareshian [private communication] has shown that the coset poset of the simple group A_5 is homotopy equivalent to a bouquet of 2-spheres. His proof has led him to ask whether there is a connection, at least for simple groups, between the homotopy properties of $\mathscr{C}(G)$ and those of the subgroup lattice of G. At present there are not enough examples to enable one to formulate the question more precisely.

Here are two questions that are even more basic:

QUESTION 3. For which finite groups G is $\mathscr{C}(G)$ simply-connected?

QUESTION 4. Are there any finite groups G for which $\mathscr{C}(G)$ is contractible?

Finally, I remark that it would be interesting to investigate the p-local analogue of $\mathscr{C}(G)$, consisting of cosets xP with P a p-group.

9. $P(G, s)$ DEPENDS ONLY ON THE COSET POSET

In this section we show how $P(G, s)$, not just $P(G, -1)$, can be computed from the coset poset $\mathscr{C}(G)$. To this end, we define an analogue of $P(G, s)$ for an arbitrary finite lattice. Taking the lattice to be the coset lattice of G (defined below) and making a change of variable, we recover $P(G, s)$.

Let $\widehat{L}$ be a finite lattice with smallest element $\hat{0}$ and largest element $\hat{1}$. Let L be the proper part of $\widehat{L}$, i.e., $L = \widehat{L} - \{\hat{0}, \hat{1}\}$. Assume $L \neq \emptyset$, and let $A \subseteq L$ be its set of minimal elements, i.e., the set of atoms of $\widehat{L}$. For any strictly positive integer s, we say that an s-tuple of atoms $(x_1, \ldots, x_s)$ is *generating* if $x_1 \vee \cdots \vee x_s = \hat{1}$, where $\vee$ denotes the join (least upper bound). Equivalently, $(x_1, \ldots, x_s)$ is generating if $\{x_1, \ldots, x_s\}$ has no upper bound in L. We can now define functions $\phi(L, s)$ and $P(L, s)$ by asking how many generating s-tuples there are or what is the probability that a random s-tuple is generating. Arguing as in the derivation of Hall's formula (2), we obtain

$$\phi(L, s) = \sum_{x \in \widehat{L}} \mu(x, \hat{1})\ |x|^s\,, \tag{17}$$

where $|x|$ is the number of atoms $\leq x$ and μ is the Möbius function of $\widehat{L}$. (Start with the equation $|y|^s = \sum_{x \leq y} \alpha(x)$, where $\alpha(x)$ is the number of s-tuples of atoms whose join is x.) We can omit the term corresponding to $x = \hat{0}$ in (17), so that the right-hand side becomes a Dirichlet polynomial. Notice that $|\hat{1}|$ is the total number of atoms, so we can divide by $|\hat{1}|^s$ to get a probability. Setting $|\hat{1} : x| = |\hat{1}| / |x|$, we obtain

$$P(L, s) = \sum_{x > \hat{0}} \frac{\mu(x, \hat{1})}{|\hat{1} : x|^s}. \tag{18}$$

As before, the right-hand side can be used to define the left-hand side for an arbitrary complex number s. In particular, we can set $s = 0$ and obtain an integer

$$P(L, 0) = \sum_{x > \hat{0}} \mu(x, \hat{1}) = -\mu(\hat{0}, \hat{1}) = -\tilde{\chi}(L). \tag{19}$$

To recover $P(G, s)$, let $\widehat{L}$ be the *coset lattice* of G, consisting of all cosets (including G itself) and the empty set. Thus the proper part L is the coset poset $\mathscr{C}(G)$. The atoms are the one-element cosets and hence can be identified with the elements of G. The meet (greatest lower bound) in $\widehat{L}$ is given by set-theoretic intersection. The join of two cosets $x_1 H_1, x_2 H_2$ is given by

$$x_1 H_1 \vee x_2 H_2 = x_1 H = x_2 H,$$

where $H = \langle x_1^{-1} x_2, H_1, H_2 \rangle$, and similarly for more than two cosets. Applying this to one-element cosets (identified with group elements), we conclude that an $(s+1)$-tuple $(x_0, x_1, \ldots, x_s)$ is generating in the coset lattice if and only if the s-tuple $(x_0^{-1} x_1, x_0^{-1} x_2, \ldots, x_0^{-1} x_s)$ generates the group G. Hence

$$\phi(\mathscr{C}(G), s+1) = |G|\, \phi(G, s)$$

and

$$P(\mathscr{C}(G), s+1) = P(G, s).$$

The formulas (17), (18), and (19) in this case reduce to results about $\phi(G, s)$ and $P(G, s)$ that we have already seen in Sections 2 and 3.

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