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Connectivity of the coset poset and the subgroup poset of a group

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Abstract. We study the connectivity of the coset poset and the subgroup poset of a group, focusing in particular on simple connectivity. The coset poset was recently introduced by K. S. Brown in connection with the probabilistic zeta function of a group. We take Brown's study of the homotopy type of the coset poset further, and in particular generalize his results on direct products and classify direct products with simply connected coset posets.

The homotopy type of the subgroup poset $L(G)$ has been examined previously by Kratzer, Thévenaz, and Shareshian. We generalize some results of Kratzer and Thévenaz, and determine $\pi_1(L(G))$ in nearly all cases.

1 Introduction

One may apply topological concepts to any poset (partially ordered set) P by means of the simplicial complex $\Delta(P)$ consisting of all finite chains in P. The basic topological theory of posets is described in [2] and in [22], and the topology of posets arising from groups has been studied extensively (see [1], [6], [16], [21], [26], [31]). We will often use P to denote both $\Delta(P)$ and its geometric realization $|\Delta(P)| = |P|$.

The coset poset $\mathscr{C}(G)$ of a finite group G (the poset of all left cosets of all proper subgroups of G, ordered by inclusion) was introduced by Brown [6] in connection with the probabilistic zeta function $P(G,s)$. (The choice of left cosets over right cosets is irrelevant since each left coset is a right coset ($xH = (xHx^{-1})x$) and vice-versa.) Brown showed that $P(G,-1) = -\tilde{\chi}(\Delta(\mathscr{C}(G)))$, the reduced Euler characteristic of $\mathscr{C}(G)$, and used this relationship to prove certain divisibility results about $P(G,-1)$. The connection between $P(G,-1)$ and $\tilde{\chi}(\mathscr{C}(G))$ motivates the study of the homotopy type of $\mathscr{C}(G)$. In this paper we study the connectivity of $\mathscr{C}(G)$ and further Brown's study of the homotopy type of $\mathscr{C}(G)$ in terms of normal subgroups and quotients. Brown has asked, in [6, Question 3], 'For which finite groups G is $\mathscr{C}(G)$ simply con-

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nected?' One of the main goals of this paper is to study this question (for both finite and infinite groups). Our main results on the problem are as follows.

(1) If G is not a 2-generated group (i.e. G is not generated by two elements), then $\mathscr{C}(G)$ is simply connected (Corollary 2.5).

(2) If $G = H \times K$ with H, K non-trivial, then $\pi_1(\mathscr{C}(G)) \neq 1$ if and only if both H and K are cyclic of prime-power order (Theorem 3.6).

(3) If $G = H \rtimes K$ with H, K non-trivial and if K is not finite cyclic or H is not K-cyclic (we say that H is K-cyclic if there is an element $h \in H$ which is not contained in any proper K-invariant subgroup of H), then $\mathscr{C}(G)$ is simply connected (Proposition 3.5).

(4) If the subgroup poset $L(G)$ is disconnected, or if G has a maximal subgroup isomorphic to $\mathbb{Z}/p^n$ (with p prime), then $\mathscr{C}(G)$ is not simply connected (Propositions 3.11 and 6.2).

(5) The simple groups A_5, $\mathrm{PSL}_2(\mathbb{F}_7)$ and A_6 have simply connected coset posets (Propositions 4.9, 4.14 and 4.15).

Further results appear in Propositions 3.5, 3.8, and 3.10.

Brown gave the following description of the homotopy type of $\mathscr{C}(G)$ for any finite solvable group G in [6, Proposition 11]. (Recall that a *chief series* in a group G is a maximal chain in the lattice of normal subgroups of G.)

Theorem 1.1 (Brown). *Let G be a finite solvable group, let $\{1\} = N_0 \lhd \cdots \lhd N_k = G$ be a chief series for G and let c_i be the number of complements of N_i/N_{i-1} in G/N_{i-1} $(1 \leqslant i \leqslant k)$. Then $\mathscr{C}(G)$ is homotopy-equivalent to a bouquet of spheres, each of dimension $d-1$, where d is the number of indices i such that $c_i \neq 0$. The number of spheres is given by*

$$\left|\prod_{i=1}^{k}\left(c_i\frac{|N_i|}{|N_{i-1}|}-1\right)\right|.$$

The number of spheres may be calculated by induction, using [6, Corollary 3]. We remark that when G is solvable, the number d of complemented factors in a chief series for G is bounded below by $|\pi(G)|$, the number of distinct primes dividing $o(G)$. Indeed, any solvable group G may be built up by a series of extensions with kernel of prime-power order. In this process, each time we add a new prime the kernel P and quotient Q of the extension are relatively prime, and hence $H^2(Q,P) = 0$ by [5, Chapter 3, Proposition 10.1] so that the extension splits. In light of the above theorem, this shows that if G is a finite solvable group, $\mathscr{C}(G)$ is k-connected whenever $|\pi(G)| \geqslant k+2$.

The paper is structured as follows. In Section 2, we introduce the notion of an *atomized* poset, which generalizes the coset poset of a group and the subgroup poset of a torsion group. The main theorem of this section gives conditions under which such posets are k-connected. The main results on the simple connectivity of $\mathscr{C}(G)$ appear in Section 3, along with generalizations of some results from [6]. In Section 4, we prove that the first three non-abelian simple groups have simply connected coset posets.

In Section 5 we discuss the connectivity of $L(G)$, the subgroup poset. This is the poset of all proper non-trivial subgroups of G, ordered by inclusion; it has been studied previously in [16], [18], [31]. Using the homotopy complementation formula of Björner and Walker [3], we generalize results from [16] and determine $\pi_1(L(G))$ for nearly all finite groups.

The final section of the paper discusses a relationship between $\mathscr{C}(G)$ and $L(G)$ for certain groups G. Kratzer and Thévenaz [16] have proven an analogue of Theorem 1.1 for the subgroup poset (Theorem 5.1). The striking similarity between these two results motivates our discussion.

We now make some remarks on notation and conventions. The k-skeleton of a simplicial complex Δ is denoted by $\Delta^{\leqslant k}$. In a poset P, a chain $p_0 < p_1 < \cdots < p_n$ is said to have length n. A wedge-sum of empty spaces, or a wedge-sum over an empty index set, is defined to be a point. For any space X, we set $X * \emptyset = X$, where $*$ denotes the join. We say that X is (-1)-connected if and only if $X \neq \emptyset$, and any space is (-2)-connected. In addition, X is called 0-connected if and only if it has exactly one path component, and X is k-connected (for $k \geqslant 1$) if and only if it is 0-connected and $\pi_n(X) = 1$ for $1 \leqslant n \leqslant k$ (so that the empty space is k-connected only for $k = -2$). Finally, $H_n(\Delta)$ will denote the (simplicial) homology of Δ with coefficients in $\mathbb{Z}$, and $\tilde{H}_n(\Delta)$ will denote the reduced simplicial homology of Δ (again over $\mathbb{Z}$).

2 Connectivity of atomized posets

In this section we introduce *atomized* posets. For any group G, $\mathscr{C}(G)$ is atomized, and if G is a torsion group then $L(G)$ is atomized as well. The main result of this section is Theorem 2.3, which gives conditions under which an atomized poset is k-connected.

Definition 2.1. We call a poset P *atomized* if every element of P lies above some minimal element and every finite set of minimal elements with an upper bound has a join.

We call the minimal elements of P *atoms*, and denote the set of atoms of P by $\mathscr{A}(P)$. If $S \subset \mathscr{A}(P)$ is finite, we say that S *generates* its join (or that S generates P if $P_{\geqslant S}$ is empty), and we write $\langle S \rangle$ for the object generated by S.

The proper part of any finite length lattice is atomized, but the converse is not true. (Consider, for example, the five-element poset $\{\hat{0} < a, b < c, d\}$.) The subgroup poset of $\mathbb{Z}$ has no minimal elements and so it is not atomized, and in fact if a group G has an element of infinite order then $L(G)$ is not atomized.

In the coset poset of any group or the subgroup poset of any torsion group, the definition of generation coincides with the standard group-theoretic definitions, where the coset generated by elements $x_1, x_2, \ldots \in G$ is $x_1\langle x_1^{-1}x_2, x_1^{-1}x_3, \ldots \rangle$.

The following lemma shows that, up to homotopy, we can replace any atomized poset P with a smaller simplicial complex $\mathscr{M}(P)$. This complex has many fewer vertices but much higher dimension; it will play an important role in our analysis of the coset poset.

Lemma 2.2. *Let P be an atomized poset and let $\mathcal{M}(P)$ denote the simplicial complex with vertex set $\mathcal{A}(P)$ and with a simplex for each finite set $S \subset \mathcal{A}(P)$ with $\langle S \rangle \neq P$. Then $\Delta(P) \simeq \mathcal{M}(P)$.*

Proof. We use the Nerve Theorem [2, (10.6)]. Consider the cones $P_{\geqslant x}$ with $x \in \mathcal{A}(P)$. Since P is atomized, $\bigcup_{x \in \mathcal{A}(P)} \Delta(P_{\geqslant x}) = \Delta(P)$. If $S \subset \mathcal{A}(P)$ is finite and $\langle S \rangle \neq P$, then $\bigcap_{s \in S} P_{\geqslant s} = P_{\geqslant \langle S \rangle} \simeq *$. So each finite intersection is either empty or contractible, and the Nerve Theorem tells us that $\Delta(P) \simeq \mathcal{M}(P)$, as $\mathcal{M}(P)$ is the nerve of this cover. □

We call $\mathcal{M}(P)$ the minimal cover of P. When $P = \mathcal{C}(G)$ for some group G, we denote $\mathcal{M}(\mathcal{C}(G))$ by $\mathcal{M}(G)$. This complex has vertex set G and a simplex for each finite subset of G contained in a proper coset.

Theorem 2.3. *Let P be an atomized poset such that no k atoms generate P. Then P is $(k-2)$-connected. In particular, for any group G,*

(a) *if G is not k-generated, then $\mathcal{C}(G)$ is $(k-1)$-connected;*

(b) *if G is a torsion group in which any k elements of prime order generate a proper subgroup, then $L(G)$ is $(k-2)$-connected.*

Proof. By Lemma 2.2, it suffices to check that $\mathcal{M}(P)$ is $(k-2)$-connected. Since no k atoms generate P, any k atoms form a simplex in $\mathcal{M}(P)$. So $\mathcal{M}(P)^{\leqslant k-1}$ is the $(k-1)$-skeleton of the full simplex with vertex set $\mathcal{A}(P)$, and thus $\mathcal{M}(P)$ is $(k-2)$-connected. (If $\mathcal{A}(P)$ is infinite the 'full simplex' on the set $\mathcal{A}(P)$ is the simplicial complex whose simplices are all the finite subsets of $\mathcal{A}(P)$.) □

Remark 2.4. Brown has asked in [6] whether there exist (finite) groups G with $\mathcal{C}(G)$ contractible. Theorem 2.3 shows that if G is not finitely generated, then $\mathcal{C}(G)$ is contractible.

Theorem 2.3 specializes to the following result.

Corollary 2.5. *If G is not 2-generated (respectively P is an atomized poset not generated by three atoms), then $\mathcal{C}(G)$ (respectively P) is simply connected.*

Corollary 2.5 does not characterize finite groups with simply connected coset posets. In fact, A_5 affords a counter-example (see Proposition 4.9). Also, one can prove Corollary 2.5 without the use of $\mathcal{M}(P)$. Any loop in $|\Delta(P)|$ is easily seen to be homotopic to an edge cycle

$$(a_1 \leqslant \langle a_1, a_2 \rangle \geqslant a_2 \leqslant \langle a_2, a_3 \rangle \geqslant \cdots \geqslant a_n \leqslant \langle a_n, a_1 \rangle \geqslant a_1),$$

where each a_i is an atom. One can successively shorten this cycle by replacing a segment $(a_i \leqslant \cdots \geqslant a_{i+2})$ by $(a_i \leqslant \langle a_i, a_{i+2} \rangle \geqslant a_{i+2})$, since both lie in the cone under $\langle a_i, a_{i+1}, a_{i+2} \rangle$ and hence are homotopic. Details are left to the reader.

3 Connectivity of the coset poset

This section contains the main results on the connectivity of $\mathscr{C}(G)$. We begin by considering the coset poset of a (non-simple) group G in terms of an extension $1 \to N \to G \to G/N \to 1$.

Definition 3.1. For any semi-direct product $G = H \rtimes K$ (with H and K arbitrary groups), let $f : G \to H$ be the function $f(h,k) = h$, and let $\pi : G \to K$ be the map $\pi(h,k) = k$. We call a coset $gT \in \mathscr{C}(G)$ *saturating* if $\pi(T) = K$ and the only K-invariant subgroup of H that contains $f(T)$ is H itself (a subgroup $I \leqslant H$ is called K-invariant if $K \leqslant N_G(I)$).

The direct product $P \times Q$ of posets P and Q is the Cartesian product of P and Q, together with the ordering defined by $(p,q) \leqslant (p',q')$ if and only if $p \leqslant p'$ and $q \leqslant q'$. The join $P * Q$ of P and Q is the disjoint union of P and Q, together with the ordering that induces the original orderings on P and Q and satisfies $p < q$ for all $p \in P$, $q \in Q$. There are canonical homeomorphisms $|P \times Q| \cong |P| \times |Q|$ and $|P * Q| \cong |P| * |Q|$, so long as one takes the associated compactly generated topology on the right-hand side when P and Q are not locally countable (see [33]).

Lemma 3.2. *Let $G = H \rtimes K$, with H and K non-trivial groups. Let $\mathscr{C}_0(G)$ be the poset of all non-saturating cosets and let $\mathscr{C}_K(H)$ denote the poset of all cosets of proper K-invariant subgroups of H. Then $\mathscr{C}_0(G)$ is homotopy-equivalent to $\mathscr{C}_K(H) * \mathscr{C}(K)$.*

Proof. Let $\mathscr{C}^+(K)$ denote the set of all cosets in K (including K itself) and let $\mathscr{C}_K^+(H) = \mathscr{C}_K(H) \cup \{H\}$. If $\mathscr{C}_{00}(H \rtimes K)$ denotes the set of all proper cosets of the form $(x,y)(I \rtimes J)$, with I a K-invariant subgroup of H and $J \leqslant K$, then the map

$$(x,y)(I \rtimes J) \mapsto (xI, yJ)$$

is easily checked to be a well-defined poset isomorphism

$$\mathscr{C}_{00}(H \rtimes K) \xrightarrow{\cong} \mathscr{C}_K^+(H) \times \mathscr{C}^+(K) - \{(H,K)\}.$$

The latter is homeomorphic to $\mathscr{C}_K(H) * \mathscr{C}(K)$ (see [21, Proposition 1.9] or the proof of [16, Proposition 2.5]). Finally, we have an increasing poset map Φ from $\mathscr{C}_0(H \rtimes K)$ onto $\mathscr{C}_{00}(H \rtimes K)$ given by

$$\Phi((x,y)T) = (x,y)(\hat{f}(T) \rtimes \pi(T))$$

where $\hat{f}(T)$ is the smallest K-invariant subgroup containing $f(T)$ (i.e. the intersection of all invariant subgroups containing $f(T)$). This map is a homotopy equivalence by [2, Corollary 10.12]. □

In the case where the action of K on H is trivial, f becomes the quotient map $H \times K \twoheadrightarrow H$ and all subgroups of H are K-invariant. In this case $\mathscr{C}_0(H \times K)$ is the poset of all cosets which do not surject onto both factors, and we obtain the following result from [6, Section 8].

Proposition 3.3 (Brown). *For any finite groups H and K, $\mathscr{C}_0(H \times K) \simeq \mathscr{C}(H) * \mathscr{C}(K)$. If H and K have no non-trivial isomorphic quotients then $\mathscr{C}(H \times K) \simeq \mathscr{C}(H) * \mathscr{C}(K)$.*

The condition on quotients implies that there are no saturating subgroups (and hence no saturating cosets); see [6, Section 2.4].

We will now use Lemma 3.2 to show that most semi-direct products have simply connected coset posets. First we need the following simple lemma, which appears (for finite groups) as [6, Proposition 14]. The result extends, with the same proof, to infinite groups (the argument for $\mathbb{Z}$ requires a simple modification).

Lemma 3.4. *Let G be a non-trivial group. Then $\mathscr{C}(G)$ is connected unless G is cyclic of prime-power order.*

Given a semi-direct product $G = H \rtimes K$, recall that H is K-cyclic if H has an element h that is not contained in any proper K-invariant subgroup of H.

Proposition 3.5. *Let $G = H \rtimes K$ with H and K non-trivial groups. If K is not finite cyclic, or if H is not K-cyclic, then $\mathscr{C}(G)$ is simply connected. Furthermore, if G is a torsion group and K is not cyclic of prime-power order, then $\mathscr{C}(G)$ is simply connected.*

Proof. By Lemma 3.2, we have $\mathscr{C}_0(G) \simeq \mathscr{C}_K(H) * \mathscr{C}(K)$. We claim that in each of the above cases, $\mathscr{C}_0(G)$ is simply connected. The join of a connected space and a non-empty space is always simply connected (see [19]), and both $\mathscr{C}_K(H)$ and $\mathscr{C}(K)$ are always non-empty (since $\{1\} \in \mathscr{C}_K(H)$) and so it suffices to show that one or the other is connected. If H is not K-cyclic, then for every element $h \in H$ there is a K-invariant subgroup $I_h < H$ containing h, and thus we have a path $hT \geqslant \{h\} \leqslant I_h \geqslant \{1\}$ joining any coset $hT \in \mathscr{C}_K(H)$ to the trivial subgroup. In the other cases, Lemma 3.4 shows that $\mathscr{C}(K)$ is connected. Thus $\pi_1(\mathscr{C}_0(G)) = 1$ in each case.

We now show that every loop in $\mathscr{C}(G)$ is null-homotopic. As mentioned after Corollary 2.5 it suffices to consider edge cycles of the form

$$C = (\{x_1\}, x_1T_1, \{x_2\}, x_2T_2, \dots, \{x_n\}, x_nT_n, \{x_1\}),$$

where each T_i is cyclic. If H is not K-cyclic, then every cyclic subgroup of G lies in $\mathscr{C}_0(G)$ and hence C lies in $|\mathscr{C}_0(G)|$ and must be null-homotopic.

Next assume that K is not a finite cyclic group. If none of the vertices of C are saturating cosets, we are done, so assume that some coset x_iT_i saturates. Since T_i is cyclic and K is not finite cyclic, we must have $T_i \cong K \cong \mathbb{Z}$. So no subgroup of T_i surjects onto K, and hence no subcoset of x_iT_i saturates. Now

$$\Delta(\mathscr{C}_0(G) \cup \{x_iT_i\}) = \Delta(\mathscr{C}_0(G)) \cup \Delta(\mathscr{C}(G)_{\leqslant x_iT_i}),$$

and $\mathscr{C}(G)_{\leqslant x_iT_i}$ is contractible since it has a maximal element. So we have written $|\mathscr{C}_0(G) \cup \{x_iT_i\}|$ as the union of two simply connected spaces whose intersection is $|\mathscr{C}(G)_{<x_iT_i}| \cong |\mathscr{C}(T_i)|$. By Lemma 3.4, this intersection is connected, and the Van Kampen theorem now shows that $\mathscr{C}_0(G) \cup \{x_iT_i\}$ is simply connected. Repeating the

process eventually shows that C lies in a simply connected poset, and hence is null-homotopic.

If K is not cyclic of prime-power order and G is a torsion group, then each T_i is finite. Thus there are finitely many cosets in the set

$$S = \{xT : xT \subset x_i T_i \text{ for some } i\} - \mathscr{C}_0(G).$$

If xT is a minimum element of S, then

$$\Delta(\mathscr{C}_0(G) \cup \{xT\}) = \Delta(\mathscr{C}_0(G)) \cup \Delta(\mathscr{C}(G)_{\leqslant xT}),$$

and we proceed as above, noting that the intersection of these two pieces is the order complex of $\mathscr{C}(G)_{<xT} \cong \mathscr{C}(T)$, which is connected (because T surjects onto K, which is not cyclic of prime-power order). This process may be repeated until we have added all of S, and hence the poset $S \cup \mathscr{C}_0(G)$ is simply connected. The cycle C lies in this poset and is thus null-homotopic. □

Theorem 3.6. *Let H and K be non-trivial groups. Then $\pi_1(\mathscr{C}(H \times K)) \neq 1$ if and only if both H and K are cyclic of prime-power order.*

Proof. It is easy to check that if both groups are cyclic of prime-power order then there are just two complemented factors in any chief series for $H \times K$ (in the sense of Theorem 1.1) and the desired result follows from that theorem. In the other direction, we apply Proposition 3.5. □

The question of simple connectivity for the coset poset of a finite (non-trivial) semi-direct product is now reduced to the case of products $H \rtimes \mathbb{Z}/p^n$, where p is prime and H is $(\mathbb{Z}/p^n)$-cyclic. When H is solvable, Theorem 1.1 applies, and so we are most interested in the case where H is a non-solvable group. The simplest example, then, is $S_5 \cong A_5 \rtimes \mathbb{Z}/2$. In this case the coset poset is still simply connected. For the proof we will need the following result of Brown [6, Proposition 10], which we note extends (with the same proof) to infinite groups.

Proposition 3.7 (Brown). *For any group G and normal subgroup N, there is a homotopy equivalence $\mathscr{C}(G) \simeq \mathscr{C}(G/N) * \mathscr{C}(G,N)$, where the latter poset is the collection of all cosets $xH \in \mathscr{C}(G)$ which surject onto G/N under the quotient map.*

Proposition 3.8. *For $n > 3$, the coset poset of S_n is simply connected.*

Proof. We have $A_n \lhd S_n$ with $S_n/A_n \cong \mathbb{Z}/2$, and so Proposition 3.7 gives

$$\mathscr{C}(S_n) \simeq \mathscr{C}(\mathbb{Z}/2) * \mathscr{C}(S_n, A_n) \cong \operatorname{Susp}(\mathscr{C}(S_n, A_n)).$$

Therefore it will suffice to show that $\mathscr{C}(S_n, A_n)$ is connected.

Since A_n has index 2 in S_n, the elements of $\mathscr{C}(S_n, A_n)$ are exactly the cosets xH where H is not contained in A_n. Letting S denote the set of proper subgroups of S_n not contained in A_n, we have $\mathscr{C}(S_n, A_n) = \bigcup_{x \in S_n} \{xH : H \in S\}$.

First we show that for each x the poset $\{xH : H \in S\}$ is connected. Each such poset is isomorphic to S, and so it suffices to consider this case. Consider the action of S_n on the set $\{1, \dots, n\}$. We have $\mathrm{Stab}(i) \cong S_{n-1}$ for each i, and $\mathrm{Stab}(i) \cap \mathrm{Stab}(j) \cong S_{n-2}$ for all i, j. These groups are all in S (since $n > 3$), and hence for each pair i, j there is a path in S between $\mathrm{Stab}(i)$ and $\mathrm{Stab}(j)$.

We now show that for each $K \in S$ there exists i such that we have a path (in S) from K to $\mathrm{Stab}(i)$. Since K is not contained in A_n, there is an element $k \in K$ such that $k \notin A_n$. First assume that k is not an n-cycle. Then if the orbits of $\langle k \rangle$ have orders $a_1, \dots, a_m$, there is a path of the form

$$K \geqslant \langle k \rangle \leqslant S_{a_1} \times \cdots \times S_{a_m} \geqslant S_{a_1} \leqslant S_{n-1}$$

connecting K to the stabilizer of some point. If k is an n-cycle, we consider two cases, depending on the parity of $n/2$; note that n must be even since the n-cycle k is not in A_n. If $n/2$ is odd, then $k^{n/2} \notin A_n$ and replacing k by $k^{n/2}$ reduces to the above case. When $n/2$ is even, assume without loss of generality that $k = (1\ 2 \dots n)$. It is easy to check that (for $m = n/2$) the element $t = (m\ \ m+2)(m-1\ \ m+3)\dots(2\ \ 2m)$ lies in the normalizer of $\langle (1\ 2 \dots n) \rangle$. This element fixes 1, and hence we have

$$K \geqslant \langle k \rangle \leqslant N(\langle k \rangle) \geqslant \langle t \rangle \leqslant S_{n-1}.$$

To finish the proof, we must find a path in $\mathscr{C}(S_n, A_n)$ from $C_x = \{xH : H \in S\}$ to $C_y = \{yH : H \in S\}$ for each pair $x, y \in S_n$. If $x^{-1}y \notin A_n$ then $x\langle x^{-1}y \rangle$ lies in $C_x \cap C_y$ and we are done. If $x^{-1}y \in A_n$, then either $x, y \in A_n$ or $x, y \notin A_n$. In the first case, take $z \notin A_n$. Then $x^{-1}z, z^{-1}y \notin A_n$ and hence $C_x \cap C_z \neq \varnothing$ and $C_z \cap C_y \neq \varnothing$. Since C_z is connected, this yields a path joining C_x to C_y. If $x, y \notin A_n$, then choosing $z \in A_n$ we may complete the proof in a similar manner. □

Remark 3.9. Similarly, $\mathscr{C}(\mathbb{Z})$ and $\mathscr{C}(\mathbb{Z} \rtimes \mathbb{Z}/2)$, where $\mathbb{Z}/2$ acts by inversion, are simply connected (again the normal subgroup of index 2 plays a crucial role).

We note a simple consequence of Proposition 3.7 for a general group extension. For the proof, one uses the fact that if X is k-connected then $X * Y$ is k-connected for any Y; see [19].

Proposition 3.10. *Let G be a group with quotient $\bar{G}$, and suppose that $\mathscr{C}(\bar{G})$ is k-connected. Then $\mathscr{C}(G)$ is k-connected as well.*

We conclude this section by discussing several cases in which $\mathscr{C}(G)$ is not simply connected. First, recall that $\mathscr{C}(G)$ is connected unless G is cyclic of prime-power order. Also, Theorem 3.6 tells us that $\pi_1(\mathbb{Z}/p^n \times \mathbb{Z}/q^m) \neq 1$ when p, q are prime and $m, n \neq 0$. Our next result gives a general condition under which $\mathscr{C}(G)$ is not simply connected. Recall that $\tilde{H}_n(X)$ denotes reduced simplicial homology with coefficients in $\mathbb{Z}$.

Proposition 3.11. *Let G be a non-cyclic group with a cyclic maximal subgroup M of prime-power order. Then $H_1(\mathscr{C}(G)) \neq 1$.*

Proof. Let $\{x_i\}_{i\in I}$ be a set of left coset representatives for M. Let $X = \Delta(\mathscr{C}(G))$, $Y = \Delta(\mathscr{C}(G) - \{x_iM\}_{i\in I})$, and $Z = \Delta(\bigcup_{i\in I} \mathscr{C}(G)_{\leqslant x_iM})$. Then we have $X = Y \cup Z$, yielding a Mayer–Vietoris sequence

$$\cdots \to \tilde{H}_1(X) \xrightarrow{\partial} \tilde{H}_0(Y \cap Z) \xrightarrow{h} \tilde{H}_0(Y) \oplus \tilde{H}_0(Z) \to \tilde{H}_0(X).$$

Now $\operatorname{Im}\partial = \operatorname{Ker} h$ contains all homology classes of the form $[xM' - yM']$ where $M' < M$ is the unique maximal subgroup of M and $xM = yM$. (The image of $[xM' - yM']$ in $\tilde{H}_0(Y)$ is trivial because, as we will show, Y is connected, and its image in Z is trivial because the simplicial boundary of $(yM', xM) + (xM, xM')$ is $xM' - yM'$.) If $xM' \neq yM'$, then xM' and yM' lie in different connected components of $Y \cap Z$ and hence $[xM' - yM'] \neq 0$ in $\tilde{H}_0(Y \cap Z)$. So $\tilde{H}_1(X) \neq 1$.

To complete the proof we must show that Y is connected. Consider a coset xH, where $H \neq M$. If $\langle x \rangle \neq M$, then we have a path

$$xH \geqslant \{x\} \leqslant \langle x \rangle \geqslant \{1\}$$

connecting xH to the identity. If $\langle x \rangle = M$ then choose some $g \in G$, $g \notin M$. We now have a path

$$xH \geqslant \{x\} \leqslant x\langle g \rangle \geqslant \{xg\} \leqslant \langle xg \rangle \geqslant \{1\}$$

in Y connecting xH to the identity. □

The class of finite groups to which Proposition 3.11 applies is rather small. The p-groups with a cyclic maximal subgroup have been classified (see [5]), and there are only a few types. Any other finite group with a maximal subgroup $M \cong \mathbb{Z}/p^n$ (p prime) is either a semi-direct product $A \rtimes \mathbb{Z}/p^n$, where A is elementary abelian, or of the form $\mathbb{Z}/p^n \rtimes \mathbb{Z}/q$ where q is prime. This can be shown as follows. If $M \lhd G$, then clearly $G \cong \mathbb{Z}/p^n \rtimes \mathbb{Z}/q$. Suppose that $M \ntriangleleft G$. Herstein [13] proved (by elegant and elementary methods) that a group with an abelian maximal subgroup is solvable. Now M is a Sylow p-subgroup of G and we have $N_G(M) = M$, so that M lies in the center of its normalizer and must have a complement (by Burnside's theorem [23, p. 289]). Hence $G = T \rtimes M$ for some $T < G$, and maximality of M implies that T is a minimal normal subgroup. Finally, a minimal normal subgroup of a solvable group is elementary abelian (by Lemma 5.12). The simplest interesting example of such a group is $A_4 \cong (\mathbb{Z}/2)^2 \rtimes \mathbb{Z}/3$, and other examples may be constructed by letting a generator of the multiplicative group $\mathbb{F}_{p^n}^{\times}$ (with p a prime) act by multiplication on the additive group of $\mathbb{F}_{p^n}$ (of course one needs the highly restrictive assumption that $p^n - 1$ is a prime power).

We note that there exist infinite groups with maximal subgroups isomorphic to $\mathbb{Z}/p\mathbb{Z}$, for p a large enough prime. Specifically there are the Tarski Monsters, in which every proper non-trivial subgroup has order p; see [20, Chapter 9]. (For such a group, $\mathscr{C}(G)$ is 1-dimensional and it is easy to construct loops of any length explicitly.) There is another family of infinite groups for which $\mathscr{C}(G)$ is not simply connected, as discussed after Proposition 6.2. We close this section with an open question.

Question 3.12. Does there exist a finite non-solvable group G with $\mathscr{C}(G)$ not simply connected?

4 2-transitive covers and simple groups

In this section, we will examine the coset posets of the finite simple groups A_5, $\mathrm{PSL}_2(\mathbb{F}_7)$, and A_6 utilizing the notion of a 2-transitive cover. Some computational details are omitted, and a complete presentation of the results in the first two cases (and all the necessary background information on $\mathrm{PSL}_2(\mathbb{F}_7)$) may be found in [22]. First we examine the notion of a 2-transitive cover.

4.1 2-transitive covers.

Definition 4.1. Let G be a group. We call a collection of subgroups $S \subset L(G)$ a *cover* of G if every element of G lies in some subgroup $H \in S$. We call S *2-transitive* if for each $H \in S$, the action of G on the left cosets of H is 2-transitive.

In addition, we say that a 2-transitive cover S is *n-regular* if for each $H \in S$ there is an element $g \in G$ with $o(g) = n$ whose action on G/H is non-trivial. (This is equivalent to requiring that no subgroup $H \in S$ contains all elements of G of order n.)

Remark 4.2. A group G has a 2-transitive cover if and only if for every element $g \in G$ there is a 2-transitive action of G in which g fixes a point.

If G is a simple group, then any cover is automatically n-regular (for any $n > 1$ such that G contains elements of order n) because every action of G is faithful.

We need the following standard result (see [29]).

Lemma 4.3. *Let Δ be a simplicial complex, and let $T \subset \Delta^{\leqslant 1}$ be a maximal tree. Then $\pi_1(\Delta)$ has a presentation with a generator for each (ordered) edge (u,v) with $\{u,v\} \in \Delta^{\leqslant 1}$, and with the following relations*:

(i) $(u,v) = 1$ *if* $\{u,v\} \in T$;

(ii) $(u,v)(v,u) = 1$ *if* $\{u,v\} \in \Delta^{\leqslant 1}$;

(iii) $(u,v)(v,w)(w,u) = 1$ *if* $\{u,v,w\} \in \Delta^{\leqslant 2}$.

When $\Delta = \mathscr{M}(G)$ for a non-cyclic group G, we always take T to be the collection of edges $\{1,g\}$ $(g \in G)$. (Since G is non-cyclic, all such edges exist.) The resulting presentation for $\pi_1(\mathscr{M}(G)) \cong \pi_1(\mathscr{C}(G))$ is the *standard presentation*.

Definition 4.4. Let G be a non-cyclic group. We say that $\mathcal{M}(G)$ is *n-locally simply connected* if G contains elements of order n, and each generator (g,h) (in the standard presentation for $\pi_1(\mathcal{M}(G))$) with $o(g) = n$ is trivial.

Proposition 4.5. *Let G be a non-cyclic group containing elements of order n. If $\mathcal{M}(G)$ is n-locally simply connected and G admits an n-regular 2-transitive cover, then $\mathcal{M}(G)$ (and hence $\mathcal{C}(G)$) is simply connected.*

Proof. Let $\{g_1, g_2\}$ be any edge of $\mathcal{M}(G)$. We must show that the corresponding generators in the standard presentation for $\pi_1(\mathcal{M}(G))$ are trivial. When $\langle g_1, g_2\rangle \neq G$, this is obvious, and so assume that $\langle g_1, g_2\rangle = G$. It will suffice to show that we can find an element z of order n and a subgroup $H \in L(G)$ such that $g_1 \equiv g_2 \equiv z \pmod{H}$. (Then the set $\{g_1, g_2, z\}$ forms a 2-simplex in $\mathcal{M}(G)$ and we have $(g_1, g_2) = (g_1, z)(z, g_2) = 1$ since $\mathcal{M}(G)$ is n-locally simply connected.)

Let S be an n-regular 2-transitive cover of G. Then there exists $H \in S$ with $g_1^{-1}g_2 \in H$, and there is an element $z \in G$ with $o(z) = n$ which acts non-trivially on the left cosets of H. Let $H, x_1H, \ldots, x_kH$ denote these cosets. Since z acts non-trivially on G/H and G acts 2-transitively, for each $i \leqslant k$ some conjugate of z sends H to x_iH, so that there is an element of order n in $\{g \in G : gH = x_iH\} = x_iH$. So we have found an element $z^\alpha \in g_1H = g_2H$, as desired; note that $g_i \notin H$ since $\langle g_1, g_2\rangle = G$. □

We now examine the extent to which the above ideas may be applied to finite simple groups. All 2-transitive actions of finite simple groups are known: see [8], [9], [11]. A finite group cannot be the union of a single conjugacy class of proper subgroups, and so any group with a 2-transitive cover has at least two distinct 2-transitive actions. The finite simple groups with multiple 2-transitive actions are listed below. This table is derived from [9, p. 197]; note that only an extension of $\mathrm{PSL}_2(\mathbb{F}_8)$ acts 2-transitively on 28 points; see [10].

Group	Degree	No. of actions (resp.)
$A_5 \cong \mathrm{PSL}_2(\mathbb{F}_5)$	5, 6	1, 1
$\mathrm{PSL}_2(\mathbb{F}_7) \cong \mathrm{PSL}_3(\mathbb{F}_2)$	7, 8	2, 1
$A_6 \cong \mathrm{PSL}_2(\mathbb{F}_9)$	6, 10	2, 1
$\mathrm{PSL}_2(\mathbb{F}_{11})$	11, 12	2, 1
A_7	7, 15	1, 2
M_{11}	11, 12	1, 1
$A_8 \cong \mathrm{PSL}_4(\mathbb{F}_2)$	8, 15	1, 2
M_{12}	12	2
HS	176	2
$\mathrm{PSL}_n(\mathbb{F}_q)$, $n \geqslant 3$	$\frac{q^d-1}{q-1}$	2
$\mathrm{Sp}_{2n}(\mathbb{F}_2)$, $n \geqslant 3$	$2^{n-1}(2^n \pm 1)$	1, 1

We begin by considering the two infinite families, $\mathrm{Sp}_{2n}(\mathbb{F}_2)$ and $\mathrm{PSL}_n(\mathbb{F}_q)$ $(n \geqslant 3)$. In the first case, the stabilizers for the above two actions do form a cover. For a proof, see [25, Lemma 4.1]. These actions can be described in terms of a set of quadratic forms [11], and the cited proof explicitly constructs fixed points.

The groups $\mathrm{PSL}_n(\mathbb{F}_q)$ for $n \geqslant 3$ do not admit 2-transitive covers. The 2-transitive actions above are on lines and hyperplanes, respectively, in $\mathbb{F}_q^n$, and since the stabilizers in a single action cannot form a cover, it will suffice to show that if an element $g \in \mathrm{PSL}_n(\mathbb{F}_q)$ does not fix any line, then it also does not fix any hyperplane. To prove this, let $A \in \mathrm{SL}_n(\mathbb{F}_q)$ represent g, and suppose that A fixes the hyperplane spanned by $v_1, \dots, v_{n-1}$. Then in a basis of the form $v_1, \dots, v_{n-1}, v_n$, the matrix of A has as its last row the vector $(0, \dots, 0, \lambda)$ for some $\lambda \in \mathbb{F}_q^*$, and thus $x - \lambda$ divides the characteristic polynomial of A. So A has an eigenvector, i.e. it fixes a line.

Remark 4.6. Except when $n = 6$, $p = 2$, Zsigmondy's Theorem [4, p. 508] shows that there is a prime dividing $o(\mathrm{PSL}_n(\mathbb{F}_q))$ but not dividing the order of the stabilizers in the above actions.

Although not isomorphic as actions, the actions of $\mathrm{PSL}_n(\mathbb{F}_q)$ on lines and hyperplanes do have the same permutation character. In fact, a simple duality argument shows that for arbitrary elements $A \in \mathrm{GL}_n(\mathbb{F}_q)$, the number of lines fixed by A equals the number of hyperplanes fixed by A. The main point is that after choosing a basis for $(\mathbb{F}_q)^n$, the adjoint of A is represented in the dual basis by the transpose A^t, which is conjugate to A (since they have the same characteristic polynomial). (This argument shows that the stabilizers of points and of hyperplanes are interchanged by the transpose-inverse automorphism of $\mathrm{GL}_n(\mathbb{F}_q)$, which descends to an automorphism of $\mathrm{PSL}_n(\mathbb{F}_q)$. Saxl [24] has shown that no finite simple group is the union of two conjugacy classes interchanged by an automorphism.)

Each of the remaining groups appears in the Atlas [10]. It is well known (see [27]) that a 2-transitive permutation character τ always has the form $1 + \chi$ where χ is an irreducible character. After identifying the irreducible characters corresponding to our 2-transitive actions in the character table, it is a simple matter to check whether the stabilizers form a cover (an element g has a fixed point if and only if $\tau(g) \neq 0$, i.e. if and only if $\chi(g) \neq -1$). In each case, the 'maximal subgroups' section of the entry for the group in question lists the stabilizers together with the decomposition of the permutation representation into irreducible characters.

In summary, one obtains the following result:

Proposition 4.7. *The only finite simple groups admitting* 2*-transitive covers are* A_5, $\mathrm{PSL}_2(\mathbb{F}_7)$, A_6, M_{11}, *and* $\mathrm{Sp}_{2n}(\mathbb{F}_2)$ *for* $n \geqslant 3$.

Remark 4.8. Proposition 4.7 depends heavily on the correctness of the information in [10]. We have made no attempt to verify this information, beyond checking that none of the known errors (see [15]) affect the problem at hand.

4.2 The coset poset of A_5.

We will establish the following result.

Proposition 4.9. *The coset poset of* $A_5 \cong \mathrm{PSL}_2(\mathbb{F}_5)$ *has the homotopy type of a bouquet of* 1560 *two-dimensional spheres.*

It can be checked that there are 1018 proper cosets in A_5, and hence $\mathscr{C}(A_5)$ has 1018 vertices. Evidently $\mathscr{C}(A_5)$ is far too large to admit direct analysis.

Shareshian has given (in an unpublished manuscript) a proof of this result using the theory of shellability. The proof below is somewhat simpler than Shareshian's argument.

Following Shareshian, we will show that $\mathscr{C} = \mathscr{C}(A_5)$ has the homotopy type of a two-dimensional complex. For this portion of the proof we work directly with $\mathscr{C}$. We will show that $\mathscr{C}$ is simply connected by applying Proposition 4.5. To show that $\mathscr{C}(A_5)$ has the homotopy type of a bouquet of 2-spheres, we appeal to the general fact that a k-dimensional complex which is $(k-1)$-connected is homotopy-equivalent to a bouquet of k-spheres. The number of spheres in the bouquet can be calculated from the Euler characteristic $\chi(\mathscr{C}(A_5))$, computed in [6].

Claim 4.10. *Let* $\mathscr{C}^-$ *denote the poset* $\mathscr{C}$ *with all cosets of all copies of* $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *removed. Then* $\Delta(\mathscr{C}^-)$ *is two-dimensional and* $\mathscr{C}^- \simeq \mathscr{C}$.

Proof (Shareshian). Quillen's Theorem A (see [21, Theorem 1.6] or [2, Theorem 10.5]) shows that the inclusion $\mathscr{C}^- \hookrightarrow \mathscr{C}$ is a homotopy equivalence (there is a unique subgroup lying above any copy of $\mathbb{Z}/2 \times \mathbb{Z}/2$, namely a copy of A_4), and so it remains to check that $\mathscr{C}^-$ is two-dimensional. This follows easily from the fact that each maximal subgroup of A_5 is isomorphic to $A_4, D_{10} \cong Z/5 \rtimes \mathbb{Z}/2$, or S_3. □

Claim 4.11. *The coset poset of* A_5 *is simply connected.*

Sketch of proof. It suffices to check that $\mathscr{M}(A_5)$ is 2-locally simply connected. This may be proved by the following method. For every pair of elements $t, x \in A_5$ with $o(t) = 2$, there is an element $z \in A_5$ such that $\langle z, t\rangle, \langle z, x\rangle, \langle z^{-1}t, z^{-1}x\rangle \neq A_5$. (Up to automorphism, there are only a few cases to check, because when $\langle t, x\rangle \neq A_5$ we may set $z = 1$.) The generators corresponding to the edge $\{t, x\}$ are now trivial, because $\{1, x, z\}$, $\{1, t, z\}$, and $\{t, x, z\}$ are simplices in $\mathscr{M}(A_5)$. □

4.3 The coset poset of $\mathrm{PSL}_2(\mathbb{F}_7)$. We now consider the simple group $G = \mathrm{PSL}_2(\mathbb{F}_7)$, and show that $\mathscr{C}(G)$ is simply connected. Other facts about the homotopy type of $\mathscr{C}(G)$ are discussed at the end of the section.

For basic facts about the groups $\mathrm{PSL}_2(\mathbb{F}_p)$, we refer to [7] (see also [22] and [30]). We write the elements of $\mathrm{PSL}_2(\mathbb{F}_p)$ as 'Möbius transformations'

$$f : \mathbb{F}_p \cup \{\infty\} \to \mathbb{F}_p \cup \{\infty\}, \quad x \mapsto \frac{ax+b}{cx+d},$$

where $a, b, c, d \in \mathbb{F}_p$, $\det(f) = ad - bc = 1$, and ∞ is dealt with in the usual manner. The action of $\mathrm{PSL}_2(\mathbb{F}_p)$ on $\mathbb{F}_p \cup \{\infty\}$ is 2-transitive. For a proof of the following result, see [7] or [22].

Lemma 4.12. *Any maximal subgroup of G is either the stabilizer of a point in $\mathbb{F}_7 \cup \{\infty\}$ (and is isomorphic to $\mathbb{Z}/7 \rtimes \mathbb{Z}/3$) or is isomorphic to S_4. The two conjugacy classes of subgroups isomorphic to S_4 are interchanged by the transpose-inverse automorphism of* $\mathrm{GL}_3(\mathbb{F}_2) \cong G$.

The following lemma will help to minimize the amount of computation in the proof of simple connectivity.

Lemma 4.13. *Let*

$$\alpha = \frac{ax+b}{cx+d}$$

be a non-trivial element of $\mathrm{PSL}_2(\mathbb{F}_7)$. *Define the trace-squared of α to be* $\mathrm{tr}^2(\alpha) = (a+d)^2$. (*Being the square of the trace of each representative of α in* $\mathrm{SL}_2(\mathbb{F}_7)$, *it is well defined.*) *The order and trace-squared of α are related as follows*:

$$o(\alpha) = \begin{cases} 2, & \mathrm{tr}^2(\alpha) = 0 \\ 3, & \mathrm{tr}^2(\alpha) = 1 \\ 4, & \mathrm{tr}^2(\alpha) = 2 \\ 7, & \mathrm{tr}^2(\alpha) = 4. \end{cases}$$

Proof. The group $G = \mathrm{PSL}_2(\mathbb{F}_7)$ contains just one conjugacy class of elements of order n for $n = 2, 3, 4$, and tr^2 is preserved under conjugation. So in these cases the result follows from checking, for example, that the elements

$$\frac{-1}{x}, \quad \frac{x-1}{x} \quad \text{and} \quad \frac{4x-1}{x}$$

have orders 2, 3 and 4, respectively.

The elements of order 7 in G are exactly those with one fixed point in $\mathbb{F}_7 \cup \infty$. Thus if

$$\alpha = \frac{ax+b}{cx+d} \in G - \mathrm{Stab}(\infty)$$

and $o(\alpha) = 7$, then the equation

$$\frac{ax+b}{cx+d} = x$$

has exactly one solution in $\mathbb{F}_7$ and hence its discriminant $(d-a)^2 + 4bc = \mathrm{tr}^2(\alpha) - 4$ is zero. So $\mathrm{tr}^2(\alpha) = 4$ in this case, and any element of order 7 in $\mathrm{Stab}(\infty)$ is conjugate to an element of order 7 outside of $\mathrm{Stab}(\infty)$. □

Proposition 4.14. *The coset poset of* $\mathrm{PSL}_2(\mathbb{F}_7)$ *is simply connected.*

Proof. As in the case of $A_5 \cong \mathrm{PSL}_2(\mathbb{F}_5)$, we show that $\mathcal{M}(G)$ is 2-locally simply connected, i.e. we check that each generator (g,h) with $o(g) = 2$ in the standard presentation for $\pi_1(\mathcal{M}(G))$ is trivial.

Recall that $(g,h) = 1$ if $\langle g,h\rangle \neq G$. An automorphism class of generating pairs of a group H is a set of the form $\{(\phi(x),\phi(y)) : \phi \in \mathrm{Aut}(H) \text{ and } \langle x,y\rangle = H\}$. To show that each generator (g,h) with $o(g) = 2$ is trivial, it suffices to check one representative from each automorphism class of generating pairs. Letting $\Phi_{a,b}$ denote the number of automorphism classes of generating pairs of G in which all representatives (x,y) satisfies $o(x) = a$ and $o(y) = b$, Möbius inversion allows one to calculate $\Phi_{a,b}$ using the Möbius function of G (as calculated in [12]). For these computations it is also necessary to know that $\mathrm{Aut}(G) \cong \mathrm{PGL}_2(\mathbb{F}_7)$ has order 336. The method of calculation is described in [12, §§1, 3]. (One can avoid Möbius inversion as indicated in the discussion following the proof below.)

Case 1. $o(h) = 3$. Möbius inversion shows that $\Phi_{2,3} = 1$, and the pair (g,h), where

$$g = \frac{x-2}{x-1}, \quad h = \frac{4x}{2},$$

represents the unique automorphism class of generators because $o(gh) = 7$ and no proper subgroup contains elements of orders 2 and 7. Letting

$$z = \frac{3x-2}{-2x-3}$$

we have $o(z) = 2$ and $g(-1) = h(-1) = z(-1) = -2$, so that

$$\langle z^{-1}g, z^{-1}h\rangle \leqslant \mathrm{Stab}(-1).$$

Now two elements of order 2 cannot generate a simple group, so that $\langle g,z\rangle \neq G$, and since $o(hz) = 3$, the pair (h,z) does not fall into the unique automorphism class of generators with orders 2 and 3. Thus $\{z,g,h\}$ is a 2-simplex in $\mathcal{M}(G)$ and we have the relations $(g,h) = (g,z)(z,h) = 1$.

Case 2. $o(h) = 4$. Möbius inversion shows that $\Phi_{2,4}(G) = 1$. Define

$$g = \frac{-1}{x}, \quad h = \frac{4x+1}{-x} \quad \text{and} \quad z = \frac{-2}{4x}.$$

The argument is now similar to Case 1 (note that $g(0) = h(0) = z(0) = \infty$).

Case 3. $o(h) = 7$. In this case, there are three automorphism classes of generating pairs. This does not require Möbius inversion: simply note that any pair of elements with these orders generates G, and that there are 21 elements of order 2 and 48 of order 7. Since $\mathrm{Aut}(G)$ acts without fixed points on these ordered pairs and $|\mathrm{Aut}(G)| = 336$, we have $\Phi_{2,7}(G) = (21 \cdot 48)/336 = 3$.

Let

$$h = x + 1 \quad \text{and} \quad g = \frac{b}{cx},$$

(so that $o(h) = 7$ and $o(g) = 2$). We have

$$hg = \frac{b}{cx} + 1 = \frac{cx + b}{cx}$$

and so $\mathrm{tr}^2(hg) = c^2$, which implies that the pairs (g_1, h), (g_2, h) and (g_3, h), where

$$g_1 = \frac{-1}{x}, \quad g_2 = \frac{2}{3x} \quad \text{and} \quad g_3 = \frac{3}{2x},$$

represent the three generating automorphism classes.

Next assume that there exist elements $z_i \in \mathrm{Stab}(\infty)$ such that $o(z_i) = 3$ and $\{h, g_i, z_i\}$ forms a simplex in $\mathscr{M}(G)$ for $i = 1, 2, 3$. Then we have $\langle h, z_i \rangle \leqslant \mathrm{Stab}(\infty)$ and $(g_i, z_i) = 1$ because $o(g_i) = 2$ and $o(z_i) = 3$. So $(g_i, h) = (g_i, z_i)(z_i, h) = 1$. We will now find such elements z_i. Consider the equations $h(x) = g_i(x)$, i.e.

$$x + 1 = \frac{b}{-b^{-1}x}$$

where $b = -1, 2, 3$. Equivalently (since ∞ can never be a solution), we want to solve the equation $x^2 + x + b^2 = 0$, and by examining the discriminant we see that solutions $x_i \in \mathbb{F}_7$ exist when $b = -1$ or 3, but not when $b = 2$. For $i = 1, 3$ there is an element $z_i \in \mathrm{Stab}(\infty)$ with $o(z_i) = 3$ and $z_i(x_i) = g_i(x_i) = h(x_i)$, namely

$$z_i = \frac{2x + (2x_i + 4)}{4}.$$

Thus $(g_1, h) = (g_3, h) = 1$.

Finally, we must show that $(g_2, h) = 1$ where

$$g_2 = \frac{2}{3x}.$$

Letting

$$z_2 = \frac{2x}{4},$$

we have $o(z_2) = 3$ and $z_2 \in \mathrm{Stab}(\infty)$. So it suffices to show that these three elements lie in a proper coset, i.e. that $\langle z_2^{-1}g_2, z_2^{-1}h\rangle \neq G$. We have $o(z_2^{-1}g_2) = 2$, $o(z_2^{-1}h) = 3$ and $o(z_2^{-1}g_2z_2^{-1}h) = 4$, and so these elements do not lie in the unique automorphism class found at the start of the proof. $\square$

We now give a more elementary proof that $\Phi_{2,3}(G) = \Phi_{2,4}(G) = 1$. Choose $k_1, k_2 \in G$ with $o(k_1) = 3$, $o(k_2) = 4$. Letting I denote the set of involutions in G, we have $|I| = 21$. Let $I_{k_i} = \{t \in I \mid \langle k_i, t\rangle = G\}$. Since G has unique conjugacy classes of elements of orders 3 and 4, it will suffice to show that $C_{\mathrm{Aut}(G)}(k_i)$ acts transitively on I_{k_i}.

The above actions are semi-regular, i.e. no non-trivial elements act with fixed points (since an automorphism fixing a generating set is trivial). Therefore we have $|C_{\mathrm{Aut}(G)}(k_1)| = 6$ and $|C_{\mathrm{Aut}(G)}(k_2)| = 8$, since the group $\mathrm{Aut}(G)$ has order 336 and acts transitively on the 56 elements of order 3 and on the 42 elements of order 4. When a group H acts semi-regularly on a set, each orbit has size $|H|$, so that $|I_{k_1}|$ is either 6, 12 or 18 and $|I_{k_2}|$ is either 8 or 16. Now k_1 lies in two distinct subgroups of G isomorphic to S_4 and since S_4 contains nine involutions, we have more than nine elements in $I - I_{k_1}$. So $|I_{k_1}| = 6$ and the action is transitive. For k_2, the nine involutions in an S_4 containing k_2 force $|I_{k_2}| = 8$. (The fact that each of the 56 elements of order 3 in G lies in exactly two subgroups isomorphic to S_4 follows from a standard counting argument, using the fact that there are 14 copies of S_4 in G; generally $\mathrm{PSL}_2(\mathbb{F}_p)$ contains $p(p^2-1)/24$ copies of S_4 whenever $p \equiv \pm 1 \pmod 8$; see [7], [22].)

We will now show that $\mathscr{C}(G)$ has the homotopy type of a three-dimensional complex and that $H_2(\mathscr{C}(G)) \neq 0$. Any chain of length four (recall that a chain of length four has five vertices) lies under a subgroup isomorphic to S_4, and in fact must contain a coset xH where $H \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ or $\mathbb{Z}/4$. But cosets of copies of $\mathbb{Z}/2 \times \mathbb{Z}/2$ and $\mathbb{Z}/4$ may be removed from $\mathscr{C}(G)$ without changing the homotopy type (by applying Quillen's Theorem A to the inclusion map). Since the reduced Euler characteristic of $\mathscr{C}(G)$ is $17 \cdot 168$ from [6, Table I] and the only even dimension in which $\mathscr{C}(G)$ has homology is dimension 2, it now follows that $H_2(\mathscr{C}(G))$ has rank at least $17 \cdot 168$.

This method does not seem to show that $\mathscr{C}(G)$ has the homotopy type of a two-dimensional complex. In fact, we expect (see Question 6.1) that since $H_2(L(G))$ is non-zero, $H_3(\mathscr{C}(G))$ is non-zero as well. (One shows that $H_2(L(G)) \neq 0$ as follows. We have $\tilde{\chi}(L(G)) = \mu_G(\{1\}) = 0$ from [12]. The above argument shows that $L(G)$ has the homotopy type of a two-dimensional complex, and hence

$$\operatorname{rank} H_1(L(G)) = \operatorname{rank} H_2(L(G)).$$

Shareshian [28, Lemma 3.11] has shown that $H_1(L(G)) \neq 0$; his argument shows that $\operatorname{rank} H_1(L(G)) \geqslant 21$.)

4.4 The coset poset of A_6. We will now sketch a proof that $\mathscr{M}(A_6)$ is 2-locally simply connected, and hence $\mathscr{C}(A_6)$ is simply connected.

Proposition 4.15. *The coset poset of A_6 is simply connected.*

Sketch of proof. Let $\Phi_{2,n}$ denote the number of automorphism classes of generating pairs (t,x) with $o(t) = 2$ and $o(x) = n$. Möbius inversion (see [12] for the inversion formula) shows that $\Phi_{2,3} = 0$, $\Phi_{2,4} = 1$ and $\Phi_{2,5} = 2$. (Note that $o(\mathrm{Aut}(A_6)) = 360 \cdot 4$ due to the outer automorphism of S_6.)

The pair $t = (15)(34)$, $x = (1234)(56)$ generates A_6, because $o(tx) = 5$, and no maximal subgroup of A_6 contains elements of orders 4 and 5; see [10]. Letting $z = (152)$, we see that $o(z^{-1}t) = 2$, $o(z^{-1}x) = 3$ and hence

$$\langle z^{-1}t, z^{-1}x\rangle \neq A_6 \quad (\Phi_{2,3} = 0).$$

Thus in the standard presentation, $(x,t) = (x,z)(z,t) = (x,z)$. Letting $w = (12)(34)$ we have $o(wx) = o(wz) = 2$ so that $\langle wx, wz\rangle \neq A_6$, and also $\langle w, x\rangle, \langle w, z\rangle \neq A_6$ (the former lies in a copy of S_4). Hence $(x,z) = 1$.

Next let $x = (12345)$, $t_1 = (16)(23)$, $t_2 = (16)(24)$. These represent distinct automorphism classes of pairs, since $o(xt_1) \neq o(xt_2)$. Also, each pair generates A_6 (note that $o(x^2t_1) = o(xt_2) = 4$). Letting $z_1 = (45)(23)$, $z_2 = (24)(15)$ one checks that $\langle z_i, t_i\rangle, \langle z_i, x\rangle, \langle z_i x, z_i t_i\rangle \neq A_6$, completing the proof. □

We now give a direct calculuation of the numbers $\Phi_{2,n}$, similar to the calculation after the proof of Proposition 4.14. To check that $\Phi_{2,3} = 0$, it suffices to check that $\langle (123), t\rangle \neq A_6$ for all involutions t (since all elements of order 3 in A_6 are conjugate in $\mathrm{Aut}(A_6)$). If $\langle (123), t\rangle = A_6$ then t cannot fix 4, 5, or 6, and so up to inner automorphisms of S_6 we can assume that $t = (34)(56)$. Then $\langle (123), t\rangle$ lies in the copy of S_4 corresponding to $\{1,2,3,4\}$.

Now let I be the set of involutions in A_6, so that $|I| = 45$, and let

$$I_x = \{t \in I \mid \langle x, t\rangle = A_6\}$$

for any $x \in A_6$. Choose $x_4, x_5 \in A_6$ with $o(x_4) = 4$, $o(x_5) = 5$. Since S_6 has a unique conjugacy classes of elements of orders 4 and 5, $\Phi_{2,n}$ is the number of orbits of $C_{\mathrm{Aut}(A_6)}(x_n)$ on I_{x_n}. Additionally, $|C_{\mathrm{Aut}(A_6)}(x_n)|$ is just $|\mathrm{Aut}(A_6)| = 1440$ divided by the number of elements of order n, so that $|C_{\mathrm{Aut}(A_6)}(x_4)| = 16$ and $|C_{\mathrm{Aut}(A_6)}(x_5)| = 10$.

When $n = 4$, x_4 lies in two subgroups of the form $(\mathbb{Z}/3)^2 \rtimes \mathbb{Z}/4$. (This follows from a simple counting argument, using the fact that these subgroups are maximal and form a single conjugacy class; see [10].) Each of these subgroups contains nine involutions and their intersection has order at most 12, and hence contains at most three involutions. Hence $|I_{x_4}| \leqslant 45 - 15 = 30$, and since $C_{\mathrm{Aut}(A_6)}(x_4)$ has order 16 and acts semi-regularly on I_{x_4}, the action must be transitive. When $n = 5$, x_5 lies in two copies of A_5; this follows as above, since these subgroups are maximal and fall into 2 conjugacy classes. Since A_5 contains 15 involutions, we have $|I_{x_5}| < 45 - 15 = 30$. Since $|C_{\mathrm{Aut}(A_6)}(x_5)| = 10$, it has at most two orbits on I_{x_5}, and the classes exhibited in the proof of Proposition 4.15 show that $\Phi_{2,5} = 2$.

5 Connectivity of the subgroup poset

In this section we will consider the connectivity of the subgroup poset, focusing in particular on $\pi_1(L(G))$ for finite groups G. Some of the results discussed here could have been proven using the methods from earlier sections, but the present approach, based on the homotopy complementation formula of Björner and Walker [3], is more powerful.

Kratzer and Thévenaz [16] have proved the following theorem for finite solvable groups, which is strikingly similar to Theorem 1.1. As mentioned in [31], this result follows easily by induction from the homotopy complementation formula (Lemma 5.4).

Theorem 5.1 (Kratzer–Thévenaz). *Let G be a finite solvable group, and let*

$$1 = N_0 \lhd N_1 \lhd \cdots \lhd N_d = G$$

be a chief series for G. Let c_i denote the number of complements of N_i/N_{i-1} in G/N_{i-1}. Then $L(G)$ is homotopy-equivalent to a bouquet of $c_1 c_2 \dots c_d$ spheres of dimension $d-2$.

The number d in this theorem is at least $|\pi(G)|$, the number of distinct primes dividing $o(G)$ (this is immediate, since each factor N_i/N_{i-1} is of prime-power order). Note that if $c_i = 0$ for some i, then $L(G)$ is in fact contractible. In fact, more is true.

Lemma 5.2. *Let G be a group with a normal subgroup N. If N does not have a complement in G, then $L(G)$ is contractible.*

Proof. This is an immediate consequence of [3, Theorem 3.2]. □

We will say that a group G is complemented if each normal subgroup $N < G$ has a complement. The above result then reduces the study of the homotopy type of the subgroup poset to the case of complemented groups.

Definition 5.3. For any finite group G, let $d(G)$ denote the length of a chief series for G, i.e. the rank of the lattice of normal subgroups of G.

From Theorem 5.1 we see that if G is a non-trivial finite, solvable, complemented group, then $L(G)$ is k-connected if and only if $d(G) \geqslant k+3$. Our next goal is to establish the 'if' portion of this statement for any finite group; it is clearly true for any non-complemented group.

We now state the homotopy complementation formula of [3], specialized to the case of the proper part of a bounded lattice. Let $\bar{L}$ be a bounded lattice, and let $L = \bar{L} - \{\hat{0}, \hat{1}\}$ be its proper part. We say that elements p, q in a poset P are *complements* if $p \wedge q = \hat{0}$ and $p \vee q = \hat{1}$, and we denote the set of complements of $p \in P$ by $p^{\perp}$. A subset $A \subset P$ is an *antichain* if no two distinct elements of A are comparable. We shall use the conventions on wedge-sums and joins stated in the Introduction.

Lemma 5.4 (Björner–Walker). *Let L be the proper part of a bounded lattice. If there is an element $x \in L$ such that $x^{\perp}$ is an antichain, then L is homotopy-equivalent to*

$$\bigvee_{y \in x^{\perp}} \mathrm{Susp}(L_{<y} * L_{>y}).$$

The theorem of Björner and Walker is more general than this. In the present case their proof becomes surprisingly simple.

For G finite and N abelian, the following result appears as [16, Corollaire 4.8].

Proposition 5.5. *Let G be a group with a non-trivial proper normal subgroup N. Then $L(G)$ is homotopy-equivalent to $\bigvee_{H \in N^{\perp}} \mathrm{Susp}(L(G/N) * L_H(N))$, where $L_H(N)$ denotes the poset of H-invariant subgroups in $L(N)$.*

Proof. Note that when $N \lhd G$, the lattice-theoretic and group-theoretic notions of complement coincide. It is easy to check that the complements of N form an antichain in $L(G)$; when G is finite, they all have the same order. The result now follows easily from Lemma 5.4 and the observation that any subgroup T containing a complement $H \in N^{\perp}$ has the form $T = I \rtimes H$ where $I = T \cap N$ is H-invariant, and that $H \cong G/N$ implies that $L(G)_{<H} \cong L(G/N)$. □

Note that the proposition implies that if G is non-simple then $\pi_1(L(G))$ is free.

The following corollary generalizing [16, Proposition 4.4] will be used in the last section.

Corollary 5.6. *If H and K have no isomorphic quotients other than the trivial group, then $L(H \times K) \simeq \mathrm{Susp}(L(H) * L(K))$.*

Proof. If $H' \neq H$ is a complement to K in G and $q_1 : H \times K \to K$ is the projection map, then the condition on quotients implies that $I \leqslant I \cdot q_1(H') \geqslant q_1(H')$ is a conical contraction of $L_{H'}(K)$. The corollary now follows from Proposition 5.5. □

Our next result strengthens and generalizes part of [16, Proposition 4.2]. We use the conventions on connectivity established in the Introduction.

Lemma 5.7. *Let G be a group with a non-trivial proper normal subgroup N. If $L(G/N)$ is k-connected then $L(G)$ is $(k+1)$-connected. In particular, if $L(G/N)$ is contractible then so is $L(G)$.*

Proof. If $L(G/N)$ is k-connected, then its join with any space X is k-connected and Susp $(L(G/N) * X)$ is $(k+1)$-connected; see [19]. Thus Proposition 5.5 shows that $L(G)$ is $(k+1)$-connected. □

Theorem 5.8. *For any non-trivial finite group G, the poset $L(G)$ is $(d(G) - 3)$-connected.*

Proof. This follows immediately from Lemma 5.7 by induction. □

In order to improve on these results, it is necessary to understand which groups have path-connected subgroup posets. For non-simple finite groups, the following result was proven by Lucido [18] using the classification of the finite simple groups. When G is not simple, the result follows easily from Proposition 5.5 and the discussion after Proposition 3.11.

Proposition 5.9 (Lucido). *If G is a finite group, then $L(G)$ is disconnected if and only if $G \cong A \rtimes \mathbb{Z}/p$, where $A \neq \{1\}$ is elementary abelian, p is prime and $L_{\mathbb{Z}/p}(A)$ is empty.*

Definition 5.10. Write $\mathscr{F}$ for the collection of finite groups of the form $A \rtimes \mathbb{Z}/p$, with A elementary abelian, p prime and $L_{\mathbb{Z}/p}(A) = \emptyset$, and $\mathscr{F}'$ for the collection of groups in $\mathscr{F}$ with $A \neq 1$.

The following result is immediate from Proposition 5.9 and Lemma 5.7.

Proposition 5.11. *If a finite group G has a proper, non-trivial quotient $\overline{G}$ which is not in $\mathscr{F}$, then $L(G)$ is simply connected.*

Of course, one can deduce results about higher connectivity as well.

We finish by examining those finite groups G for which we have yet to determine whether or not $L(G)$ is simply connected. Our results do not apply to non-abelian simple groups, but Shareshian [28, Proposition 3.14] has shown that if G is a minimal simple group (i.e. if G is a finite non-abelian simple group all of whose proper subgroups are solvable) then $H_1(L(G)) \neq 0$.

Any non-simple (complemented) finite groups a for which we have not determined simple connectivity of $L(G)$ may be written as a (non-trivial) semi-direct product $H \rtimes K$, where $K \in \mathscr{F}$. We break these groups down into two cases. First we consider groups in which $K \in \mathscr{F}'$, and then we consider groups in which every proper, non-trivial quotient has prime order. The following lemma, a simple consequence of [14, Theorem 7.8], will be useful.

Lemma 5.12. *Each minimal normal subgroup of a finite group is a direct power of a simple group. In particular, if G is a finite group whose only characteristic subgroups are 1 and G, then G is a direct power of a simple group (since G is a minimal normal subgroup of $G \rtimes \mathrm{Aut}(G)$).*

If S is a non-abelian simple group then the n simple direct factors of the direct power S^n are permuted by $\mathrm{Aut}(S^n)$.

Let $G = H \rtimes K$ with $K = A \rtimes \mathbb{Z}/p \in \mathscr{F}'$. We may also assume that G is not in $\mathscr{F}$. If H has a complement K' such that $L_{K'}(H) = \emptyset$, then the wedge decomposition for $L(G)$ (see Proposition 5.5) contains the factor

$$\mathrm{Susp}(L_{K'}(H) * L(K')) = \mathrm{Susp}(L(K')),$$

and since $K' \cong K$ we see that $L(K')$ is disconnected and $\pi_1(L(G)) \neq 1$.

On the other hand, if $L_{K'}(H) \neq \emptyset$ for each $K' \in H^{\perp}$, then every term in the wedge decomposition for $L(G)$ is simply connected, and so $L(G)$ is simply connected. So in this case (with $K \in \mathscr{F}'$), we have $\pi_1(L(G)) \neq 1$ if and only if some complement of H a is maximal subgroup of G.

Remark 5.13. It is not difficult to check that, if k is the number of complements $K' \in H^{\perp}$ with $L_{K'}(H) = \emptyset$, then $\pi_1(L(G))$ is a free group on $k(1 + o(A))$ generators. The fact that $\pi_1(L(G))$ is free follows from Proposition 5.5, which shows that $L(G)$ is a wedge of suspensions. To compute the number of generators, note that if $K' \in H^{\perp}$ and $L_{K'}(H) \neq \emptyset$, then the corresponding term $\mathrm{Susp}(L_{K'}(H) * L(K'))$ is simply connected. On the other hand, if $L_{K'}(H) = \emptyset$, i.e. if K' is maximal in G, then $\pi_1(\mathrm{Susp}(L_{K'}(H) * L(K'))) = \pi_1(\mathrm{Susp}(L(K')))$. This is the free group on $n-1$ generators, where n is the number of connected components in $K' \cong K$. As shown in [18], this is exactly $1 + o(A)$.

Finally, we consider finite, non-simple (complemented) groups G in which every proper, non-trivial quotient has prime order. Up to isomorphism, any such group has the form $G = H \rtimes \mathbb{Z}/p$ with H a minimal normal subgroup. Lemma 5.12 shows that $H \cong S^n$ with S simple. Hence if $G \notin \mathscr{F}'$, then $G = S \rtimes \mathbb{Z}/p$ or $G = S^p \rtimes \mathbb{Z}/p$, with $\mathbb{Z}/p$ permuting the simple direct factors of S^p transitively. We now show that in the latter case $\pi_1(L(G)) = 1$.

Proposition 5.14. *Let S be a finite, non-abelian simple group and let p be a prime. If $G = S^p \rtimes \mathbb{Z}/p$, with $\mathbb{Z}/p$ acting regularly on the simple direct factors of S^p, then $L(G)$ is simply connected.*

Proof. Since $L(\mathbb{Z}/p)$ is empty, Proposition 5.5 shows that

$$L(G) \simeq \bigvee_{K \in (S^p)^{\perp}} \mathrm{Susp}(L_K(S^p)).$$

Hence $\pi_1(L(G)) = 1$ if and only if $L_K(S^p)$ is connected for all $K \in (S^p)^{\perp}$. Let $K \in (S^p)^{\perp}$. Let $K = \langle k \rangle$ and let ϕ be the automorphism of S^p induced by k. Let $S_1, \dots, S_p$ be the simple direct factors of S^p. We may assume without loss of generality that $\phi(S_i) = S_{i+1}$ for $1 \leqslant i \leqslant p-1$ and $\phi(S_p) = S_1$.

Let $D_K \in L_K(S^p)$ denote the subgroup consisting of all elements fixed by ϕ (and hence by K). Note that $D_K \neq \{1\}$ by Thompson's theorem on fixed-point free automorphisms [32]. For each $I \in L_K(S^p)$ we will construct a path (in $L_K(S^p)$) from I to D_K. Let $f_i : S^p \to S_i$ denote the ith projection map. It is not hard to check that $I \in L_K(S^p)$ implies that $\hat{I} = f_1(I) \times \cdots \times f_p(I) \in L_K(S^p)$ (assuming that $\hat{I} \neq S^p$). Now since I is non-trivial and K-invariant, there is a non-trivial element $s = (s_1, 1, \dots, 1) \in f_1(I) < \hat{I}$ and since $\hat{I}$ is also K-invariant, $\phi^i(s) \in \hat{I}$ for each i. Moreover, the product $s' = \prod_{i=0}^{p-1} \phi^i(s) \in \hat{I}$ is non-trivial and invariant under ϕ. Thus when $\hat{I} \neq S^p$ we have a path $I \leqslant \hat{I} \geqslant \langle s' \rangle < D_K$.

If $\hat{I} = S^p$, then I surjects onto S_i for each i and in particular I is not nilpotent. The automorphism ϕ either fixes I pointwise or induces an automorphism of I of prime order. In the latter case, Thompson's theorem implies that ϕ fixes some non-trivial element $i \in I$. Hence we have a path $I \geqslant \langle i \rangle \leqslant D_K$ in $L_K(S^p)$, so that $L_K(S^p)$ is connected and the proof is complete. □

In summary, we have

Theorem 5.15. *Let G be a finite group which is neither simple nor a semi-direct product $S \rtimes \mathbb{Z}/p$ (with S simple and p prime), and assume further that $L(G)$ is 0-connected (i.e. $G \notin \mathscr{F}$). Then $L(G)$ is simply connected unless $G \cong H \rtimes K$ with $K = A \rtimes \mathbb{Z}/p \in \mathscr{F}'$ and K maximal in G. In this case $\pi_1(L(G))$ is a free group on $k(1 + o(A))$ generators, where k is the number of complements of H which are maximal in G.*

In particular, if $G = H \times K$ with H, K non-trivial, then $L(G)$ is simply connected unless $H \in \mathscr{F}$ and K has prime order (or vice versa).

6 The homology of $\mathscr{C}(G)$ and $L(G)$

We end by discussing a relationship between the homology of the coset poset and the homology of the subgroup poset which exists at least for certain groups. This discussion is motivated by the striking similarity between Theorems 1.1 and 5.1.

Question 6.1. If G is a group, then is it true that for any $n > 0$

$$\operatorname{rank} \tilde{H}_{n+1}(\mathscr{C}(G)) \geqslant \operatorname{rank} \tilde{H}_n(L(G))? \tag{1}$$

All finite solvable groups satisfy (1). This follows from Theorems 1.1 and 5.1. It is easy to check that the number of spheres in the coset poset is greater than the number in the subgroup poset. In light of Lemma 5.2, any non-complemented group satisfies (1) trivially, so that we may restrict attention to non-abelian simple groups and non-trivial semi-direct products $H \rtimes K$ with K simple. We provide an affirmative answer to Question 6.1 in the case $n = 0$. This result is closely related to [26, Theorem 2.5].

Proposition 6.2. *For any group G, there is a surjection*

$$\tilde{H}_1(\mathscr{C}(G)) \twoheadrightarrow \tilde{H}_0(L(G)).$$

Proof. If G is cyclic then $L(G)$ is connected, unless $G \cong \mathbb{Z}/pq$ with p, q prime, and the result is trivial. When $G \cong \mathbb{Z}/pq$, the result follows from Theorem 1.1.

We now assume that G is not cyclic. Let $X = \Delta(\mathscr{C}(G))$, $Y = \Delta(\mathscr{C}(G)_{\geqslant \{1\}})$, and $Z = \Delta(\mathscr{C}(G) - \{1\})$. Then $Y \simeq *$ and $Y \cap Z \cong \Delta(L(G))$, and so assuming that Z is connected the result follows from the Mayer–Vietoris sequence for the decomposition $X = Y \cup Z$. To show that $\mathscr{C}(G) - \{1\}$ is connected, choose $x \in G - 1$. Then any vertex yH of $\mathscr{C}(G) - \{1\}$ is connected to $\{x\}$ via the path

$$yH \geqslant \{y\} \leqslant x\langle x^{-1}y \rangle \geqslant \{x\}.$$

(We have $yH \neq \{1\}$ and so we may assume that $y \neq 1$.) □

Remark 6.3. Proposition 6.2 shows that when $L(G)$ is disconnected, $\mathscr{C}(G)$ is not simply connected. For finite groups, this result is eclipsed by Proposition 3.11 (see Proposition 5.9). There is, though, an interesting class of infinite groups to which Proposition 6.2 applies. In [20, Chapter 9], it is shown that there is a continuum of non-isomorphic infinite groups G all of whose non-trivial proper subgroups are infinite cyclic. Furthermore, in each of these groups, any two maximal subgroups intersect trivially (and any subgroup lies in a maximal subgroup) so that $L(G)$ is disconnected in every case.

Our next goal is to show that if $p \equiv \pm 3 \pmod 8$ and $p \not\equiv \pm 1 \pmod 5$ then the simple group $\mathrm{PSL}_2(\mathbb{F}_p)$ satisfies (1). First, we have the following result due to Shareshian [28, Lemma 3.8].

Lemma 6.4 (Shareshian). *Let p be an odd prime and let G be a simple group isomorphic to one of the following*:

(i) $\mathrm{PSL}_2(\mathbb{F}_p)$ *with* $p \equiv \pm 3 \pmod 8$ *and* $p \not\equiv \pm 1 \pmod 5$;

(ii) $\mathrm{PSL}_2(\mathbb{F}_{2^p})$;

(iii) $\mathrm{PSL}_2(\mathbb{F}_{3^p})$;

(iv) $\mathrm{Sz}(2^p)$.

Then $L(G)$ *has the homotopy type of a wedge of* $o(G)$ *circles.*

The proof of the next result is analogous to Shareshian's proof of Lemma 6.4.

Lemma 6.5. *If* $p \equiv \pm 3 \pmod 8$ *and* $p \not\equiv \pm 1 \pmod 5$ *then* $\mathscr{C}(\mathrm{PSL}_2(\mathbb{F}_p))$ *has the homotopy type of a two-dimensional complex.*

Sketch of proof. For $p = 5$, this is just Claim 4.10. Let $G = \mathrm{PSL}_2(\mathbb{F}_p)$. We begin by removing from $\mathscr{C}(G)$ all cosets xH which are not intersections of maximal cosets, i.e. we remove all cosets xH for which H is not an intersection of maximal subgroups. The resulting poset $\mathscr{C}_0$ is homotopy-equivalent to $\mathscr{C}(G)$ by [28, Corollary 2.5]. Similarly, the poset L_0 consisting of all subgroups in $L(G)$ which are intersections of maximal subgroups is homotopy-equivalent to $L(G)$.

Now any maximal chain of length k in the coset poset corresponds to a chain of length $k-1$ in the subgroup poset (we simply take all underlying subgroups, except for the identity). By [28, Lemma 3.4], any chain in L_0 has length at most two, and hence any chain in $\mathscr{C}_0$ has length at most three. Shareshian's argument in the proof of Lemma 6.4 that all 2-simplices in $\Delta(L_0)$ can be removed without changing the homotopy type also shows that all 3-simplices may be removed from C_0 without changing the homotopy type: when removed in the correct order, each corresponds to an 'elementary collapse'. □

The following computation of the Euler characteristic of $\mathscr{C}(\mathrm{PSL}_2(\mathbb{F}_p))$ was communicated to us by K. S. Brown.

Lemma 6.6. *If* $p \equiv \pm 3 \pmod 8$ *and* $p \equiv \pm 2 \pmod 5$, *then the Euler characteristic of* $\mathscr{C}(\mathrm{PSL}_2(\mathbb{F}_p))$ *is* $o(\mathrm{PSL}_2(\mathbb{F}_p))\big(\frac{1}{12}p(p-1)(p+1) - p - 4\big) + 1$.

Proof. For any finite group G we have $\chi(\mathscr{C}(G)) = -P(G,-1) + 1$, where $P(G,s)$ is the probabilistic zeta function of G (see [6]). Möbius inversion allows one to compute $P(G,-1)$ from the Möbius function of G; see [6, Section 2.1]. For $G = \mathrm{PSL}_2(\mathbb{F}_p)$, the Möbius function was calculated by Hall [12], and one easily derives the above result. □

Proposition 6.7. *If* $p \equiv \pm 3 \pmod 8$ *and* $p \not\equiv \pm 1 \pmod 5$ *then* $G = \mathrm{PSL}_2(\mathbb{F}_p)$ *satisfies* (1).

Proof. By Lemmas 6.4 and 6.5, it suffices to check that $H_2(\mathscr{C}(G))$ has rank at least $o(G)$, and for $p = 5$ this follows from Proposition 4.9. Assume that $p > 5$. Since the Euler characteristic of $\mathscr{C}(G)$ is simply $\operatorname{rank} H_2(\mathscr{C}(G)) - \operatorname{rank} H_1(\mathscr{C}(G))$, Lemma 6.6 shows that $H_2(\mathscr{C}(G))$ has rank at least

$$o(G)\left(\frac{p}{12}(p-1)(p+1) - p - 4\right) + 1.$$

The conditions of the proposition force $p \geqslant 11$, so that

$$\frac{p}{12}(p-1)(p+1) - p - 4 \geqslant 95$$

(the left-hand side being an increasing function of p). Thus $\operatorname{rank} H_2(\mathscr{C}(G)) \geqslant o(G)$, as desired. □

At least two other simple groups satisfy (1), namely $\mathrm{PSL}_2(\mathbb{F}_8)$ and $\mathrm{Sz}(8)$. The proof is analogous to that given above, using [6, Table I] for the computation of $P(G,-1)$ and hence $\chi(\mathscr{C}(G))$. Presumably Question 6.1 can be answered for all groups listed in Lemma 6.4.

We now show that certain direct products satisfy (1). In particular, given any finite collection of non-isomorphic non-abelian simple groups satisfying (1), their direct product Π also satisfies (1), and if G is a finite solvable group then $G \times \Pi$ satisfies (1).

Proposition 6.8. *If* H, K *are finite groups satisfying* (1) *and* H, K *have no non-trivial isomorphic quotients, then* $G \times H$ *also satisfies* (1).

Proof. Let $G = H \times K$. Recall that Proposition 3.3 and Corollary 5.6 show that $\mathscr{C}(G) \simeq \mathscr{C}(H) * \mathscr{C}(K)$ and $L(G) \simeq \mathrm{Susp}(L(H) * L(K))$. We need only consider the case in which G is not solvable, and in light of Proposition 6.2 we need only check condition (1) for $n \geqslant 1$.

If $L(H)$, $L(K)$ are both empty then G is solvable and we are done. If $L(H)$ is empty but $L(K)$ is not, then $L(G) \simeq \mathrm{Susp}\, L(K)$ and so $\tilde{H}_i(L(G)) \cong \tilde{H}_{i-1}(L(K))$ for

$i \geqslant 1$. Letting $\tilde{\beta}_i(X)$ denote the rank of the ith (reduced) homology group of the space X for $i \geqslant 0$, we have (for $n \geqslant 1$)

$$\begin{aligned}\tilde{\beta}_n(L(G)) = \tilde{\beta}_{n-1}(L(K)) &\leqslant \tilde{\beta}_n(\mathscr{C}(K)) \\ &\leqslant \sum_{i+j=n} \tilde{\beta}_i(\mathscr{C}(H)) \cdot \tilde{\beta}_j(\mathscr{C}(K)) = \tilde{\beta}_{n+1}(\mathscr{C}(G)),\end{aligned}$$

the last equality following from [19, Lemma 2.1]. If $L(K)$ is empty and $L(H)$ is not, the situation is symmetric.

Now assume that $L(H)$, $L(K)$ are each non-empty. Then for any $n \geqslant 1$ we have

$$\tilde{\beta}_{n+1}(\mathscr{C}(G)) = \sum_{i+j=n} \tilde{\beta}_i(\mathscr{C}(H)) \cdot \tilde{\beta}_j(\mathscr{C}(K))$$

and

$$\tilde{\beta}_n(L(G)) = \tilde{\beta}_{n-1}(L(H) * L(K)) = \sum_{k+l=n-2} \tilde{\beta}_k(L(H)) \cdot \tilde{\beta}_l(L(K))$$

by [19, Lemma 2.1] (note that $L(G)$ is simply connected so that there is no problem when $n = 1$). By assumption

$$\tilde{\beta}_{i-1}(L(H)) \leqslant \tilde{\beta}_i(\mathscr{C}(H)) \quad \text{and} \quad \tilde{\beta}_{i-1}(L(K)) \leqslant \tilde{\beta}_i(\mathscr{C}(K)),$$

and so for each m ($0 \leqslant m \leqslant n-2$) we have

$$\tilde{\beta}_m(L(H)) \cdot \tilde{\beta}_{n-2-m}(L(K)) \leqslant \tilde{\beta}_{m+1}(\mathscr{C}(H)) \cdot \tilde{\beta}_{n-(m+1)}(\mathscr{C}(K)).$$

Thus $\tilde{\beta}_n(L(G)) \leqslant \tilde{\beta}_{n+1}(\mathscr{C}(G))$ as desired. □

Thévenaz [31] has 'found' the spheres in the subgroup poset of a solvable group, that is, he has given a proof of Theorem 5.1 by analyzing a certain collection of spherical subposets of $L(G)$. It would be interesting to explore similar ideas in the coset poset. In particular, such a proof of Theorem 1.1 might allow one to construct explicitly an injection from $H_n(L(G))$ into $H_{n+1}(\mathscr{C}(G))$ (for G finite and solvable), and could shed further light on Question 6.1.

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