Available online at www.sciencedirect.com
ScienceDirect
Journal of Combinatorial Theory, Series A 114 (2007) 733–746
Journal of Combinatorial Theory
Series A
www.elsevier.com/locate/jcta

Shelling the coset poset

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Received 28 July 2005
Available online 22 November 2006
Communicated by Anders Björner

Abstract

It is shown that the coset lattice of a finite group has shellable order complex if and only if the group is complemented. Furthermore, the coset lattice is shown to have a Cohen–Macaulay order complex in exactly the same conditions. The group theoretical tools used are relatively elementary, and avoid the classification of finite simple groups and of minimal finite simple groups.

Keywords: Shelling; Lexicographic shelling; Coset poset; Coset lattice; Subgroup lattice; Complemented group

1. Introduction

We start by recalling the definition of a shelling. All posets, lattices, simplicial complexes, and groups in this paper are finite.

Definition 1.1. If Δ is a simplicial complex, then a *shelling* of Δ is an ordering $F_1, \ldots, F_n$ of the facets (maximal faces) of Δ such that $F_k \cap (\bigcup_{i=1}^{k-1} F_i)$ is a non-empty union of facets of F_k for $k \geqslant 2$. If Δ has a shelling, we say it is *shellable*.

We will use this definition in the context of a poset P by recalling the *order complex* $|P|$ to be the simplicial complex with vertex set P and faces chains in P. We say that a poset P is shellable if $|P|$ is. Recall also that P is *graded*, and $|P|$ is *pure*, if all maximal chains in P have the same length.

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0097-3165/$ – see front matter
doi:10.1016/j.jcta.2006.08.010

The idea of a shelling (and the property of shellability) were first formally introduced by Bruggesser and Mani in [8], though similar ideas had been assumed implicitly since the beginning of the 20th century. See Chapter 8 of [24] for a development of some of the history and basic results on shellability. Since its introduction, it has been studied extensively by combinatorialists. Particularly, in the 1980s and 1990s Björner and Wachs wrote several papers [2,4–6] developing the theory of shellability for posets. Especially important to those interested in group theory are [5] and [6], as they extend the older definition of shellability (which only applied to graded posets) to apply to any poset. This extension makes Theorem 1.3 much more interesting!

We henceforth assume that a reader has seen the basic definitions and results of, say, [5], although we try to state clearly what we are using.

Recall that the subgroup lattice (denoted $L(G)$) is the lattice of all subgroups of a group G. Shellings of subgroup lattices have been of interest for quite some time. In fact, one of the main results of Björner's first paper on shelling posets [2] was to show that supersolvable groups have shellable subgroup lattices. (Recall a supersolvable group is a group having chief series with every factor of prime order.) As mentioned before, at that time, shellability was a property that applied only to graded posets. Under this definition, Björner had the shellable subgroup lattices completely characterized, if we recall the following theorem of Iwasawa:

Theorem 1.2. (Iwasawa [13]) *Let G be a finite group. Then $L(G)$ is graded if and only if G is supersolvable.*

Of course, when Björner and Wachs updated the definition of a shelling to allow non-graded posets in [5,6], shellable subgroup lattices were no longer characterized. This gap was soon filled by Shareshian:

Theorem 1.3. (Shareshian [19]) *Let G be a finite group. Then the subgroup lattice $L(G)$ is shellable if and only if G is solvable.*

A nice summary article on shellability and group theory was written by Welker in [23]. This article is now somewhat out of date, and it has some errors, but it is very useful as an overview of the topic. The reader should be warned, however, that at the time it was written shellability was still considered to apply only to graded posets.

Shareshian's result is surprising and pretty, and it would be nice to find something similar for other lattices on groups. In this paper, we consider cosets. The *coset poset* $\mathfrak{C}(G)$ (poetically named by K. Brown in [7]) is the poset of all cosets of proper subgroups of G, ordered by inclusion. The *coset lattice* $\hat{\mathfrak{C}}(G)$ is $\mathfrak{C}(G) \cup \{G, \emptyset\}$, likewise ordered by inclusion. The meet operation is intersection, while $H_1x_1 \vee H_2x_2 = \langle H_1, H_2, x_1x_2^{-1}\rangle x_2$. Clearly, $\mathfrak{C}(G)$ is shellable if and only if $\hat{\mathfrak{C}}(G)$ is, so we study the two interchangeably. If $\mathfrak{C}(G)$ is shellable, we will call G *coset-shellable*.

The history of the coset poset is discussed in the last chapter of [18]. Most results proved have been either negative results, or else so similar to the situation in the subgroup lattice as to be uninteresting. More recently, K. Brown rediscovered the coset poset, and studied its homotopy type while proving some divisibility results on the so-called probabilistic zeta function [7]. In particular, he showed that if G is a solvable group, then $|\mathfrak{C}(G)|$ has the homotopy type of a bouquet of spheres, all of the same dimension.

In Section 2 we show that there are finite groups G which have a shellable subgroup lattice, but a non-shellable coset lattice. In particular, we show that for $\mathfrak{C}(G)$ to be shellable, G must be supersolvable, and every Sylow subgroup of G must be elementary abelian. Our main tool is the above-mentioned result of Brown, together with the fact that a pure shellable complex is homotopic to a bouquet of top-dimensional spheres. In Section 3 we use linear algebra to construct an invariant on subgroups of such groups. Finally, in Section 4 we use this invariant to construct a so-called *EL*-shelling, and to finish the proof of our main theorem:

Theorem 1.4 *(Main Theorem). If G is a finite group, then $\mathfrak{C}(G)$ is shellable if and only if G is supersolvable with all Sylow subgroups elementary abelian.*

Our theorem is even more interesting when we connect it with a paper of P. Hall [12]. We recall that a group G is *complemented* if for every subgroup $H \subseteq G$, there is a *complement* K which satisfies (i) $K \cap H = 1$ and (ii) $HK = KH = G$. Hall proved the equivalence of the first three properties in the following restatement of our theorem:

Theorem 1.5 *(Restatement of Main Theorem). If G is a finite group, then the following are equivalent*:

(1) *G is supersolvable with all Sylow subgroups elementary abelian,*
(2) *G is complemented,*
(3) *G is a subgroup of the direct product of groups of square free order,*
(4) *G is coset-shellable,*
(5) *$\mathfrak{C}(G)$ is homotopy Cohen–Macaulay,*
(6) *$\mathfrak{C}(G)$ is sequentially Cohen–Macaulay over some field, and*
(7) *$\mathfrak{C}(G)$ is Cohen–Macaulay over some field.*

Parts (5)–(7) are discussed in Section 2.3, where we define the three used versions of the Cohen–Macaulay property, and give further references.

Notice that, contrary to the situation of Shareshian's Theorem, $\mathfrak{C}(G)$ is shellable if and only if it is pure and shellable. Thus, non-pure shellability only comes in the negative direction of our proof. Of course, now that it has been defined, one cannot ignore it!

Complemented groups have also been called *completely factorizable* groups, and have been studied by other people, see, for example, [1] or [15]. Ramras has further examined the homotopy type of the coset poset in [16].

2. Coset posets that are not shellable

2.1. *p*-Groups

It is often easier to show that something is not shellable, than to show that it is. So we start our search for shellings of the coset lattice by finding groups for which $\mathfrak{C}(G)$ is certainly *not* shellable. The following lemma will be very useful in this endeavor.

Lemma 2.1. *If P is a shellable poset, then every interval in P is also shellable.* (*Thus, if G is coset-shellable, then so is every subgroup $H \subseteq G$.*)

Proof. Since every interval in a poset P is a "link" (for more information, see the beginning of Section 2.3), the first part follows immediately from Proposition 10.14 in [6].

Fig. 1. The coset poset of $\mathbb{Z}_4$ is not connected, so not shellable.

For the second part, we note that the interval $[\emptyset, H]$ in $\hat{\mathfrak{C}}(G)$ is isomorphic to $\hat{\mathfrak{C}}(H)$. □

Corollary 2.2. *If G is a finite coset-shellable group, then G is solvable.*

Proof. Note that the interval $[1, G]$ in $\hat{\mathfrak{C}}(G)$ is isomorphic to the subgroup lattice of G. Apply Lemma 2.1 and Shareshian's Theorem (Theorem 1.3). □

A proof of Corollary 2.2 that does not rely on Shareshian's Theorem will also be given, in Section 2.2.

At first glance, one might hope that perhaps all solvable groups have a shellable coset poset. Soon enough, however, one considers the coset-poset of $\mathbb{Z}_4$, pictured in Fig. 1. We see that $\mathfrak{C}(\mathbb{Z}_4)$ is not even connected, and connectivity is an easy consequence of the definition of shellability as long as all facets have dimension at least 1.

A similar situation holds for an arbitrary prime p: $\mathbb{Z}_{p^2}$ has only one non-trivial proper subgroup, so $\mathfrak{C}(\mathbb{Z}_{p^2})$ falls into p connected components, and, in particular, is not shellable. Hence, no group G with a subgroup isomorphic with $\mathbb{Z}_{p^2}$ can be coset-shellable. Can we eliminate any other p-groups from the possibility of coset-shellability? In fact we can. We recall the following theorem of K. Brown:

Theorem 2.3. (Proposition 11 from [7]) *Let G be a finite solvable group with a chief series $1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_k = G$. Then $\mathfrak{C}(G)$ has the homotopy type of a bouquet of $(d-1)$-spheres, where d is the number of indices $i = 1, \ldots, k$ such that N_i/N_{i-1} has a complement in G/N_{i-1}.*

It follows from the proof in [7] that for $G \neq 1$ the number of spheres is, in fact, non-zero. (It is

$$\prod_{i=1}^{k} \bigl(1 - c_i |N_i/N_{i-1}|\bigr),$$

where c_i is the number of complements N_i/N_{i-1} has in G/N_{i-1}; also related is [7, Corollary 3].)

Proposition 2.4.

(1) *If H is a finite p-group which is coset-shellable, then H is elementary abelian.*
(2) *If G is a finite group which is coset-shellable, then all Sylow subgroups of G are elementary abelian.*

Proof. (1) If H is a finite p-group, then $L(H)$ is graded (by, for example, Iwasawa's Theorem, Theorem 1.2), hence $\mathfrak{C}(H)$ is also graded. But it is well known (see, for example, [5]) that a graded, shellable poset P has homotopy type of a bouquet of r-spheres, where r is the length of a maximal chain in P. By the above theorem, we see that every chief factor of H must be complemented, and hence H has trivial Frattini subgroup $\Phi(H)$ (otherwise any minimal normal subgroup contained in $\Phi(H)$ is an uncomplemented chief factor).

But for a finite p-group, $\Phi(H) = H'H^p$ (see, for example, [17, 5.3.2]), so H is abelian of exponent p, that is, elementary abelian.

(2) Apply Lemma 2.1 to the interval $[\emptyset, H]$ in $\hat{\mathfrak{C}}(G)$, where H is a Sylow subgroup of G. □

2.2. *Non-supersolvable groups*

We now have that for a finite group G to be coset-shellable, G must be solvable with elementary abelian Sylow subgroups. A little more holds: G must in fact be supersolvable. To prove this, it suffices by Lemma 2.1 and the discussion of the previous section to restrict ourselves to groups G such that:

(1) G is not supersolvable,
(2) all proper subgroups of G are supersolvable, and
(3) all Sylow subgroups of G are elementary abelian.

A closely related idea is that of minimal non-complemented groups, which are non-complemented groups with every proper subgroup complemented. A complete characterization of such groups is given in [15], although we do not use their characterization.

In light of Shareshian's Theorem and Corollary 2.2, it might seem at first glance that a stronger version of Condition 1 would be to require G to be solvable but not supersolvable. The following result of Doerk, however, shows that this would be redundant.

Lemma 2.5. (Doerk [9, Hilfssatz C]) *If every maximal subgroup of G is supersolvable, then G is solvable.*

We notice also that this frees our characterization of groups that are not coset-shellable from Shareshian's Theorem, which relies on Thompson's classification of minimal finite simple groups. We will also need this for Section 2.3.

For any normal subgroup N in G, let $q: G \to G/N$ be the quotient map. Then we take $\mathfrak{C}_0(G)$ to be the subposet of $\mathfrak{C}(G)$ of all Hx such that $q(Hx) \neq G/N$. Thus, $\mathfrak{C}_0(G)$ is obtained from $\mathfrak{C}(G)$ by removing cosets Kx when $KN = G$. We will use the following proposition to show that, for G satisfying Conditions 1–3, $\mathfrak{C}(G)$ has the wrong homotopy type to be shellable.

Theorem 2.6. (K. Brown [7], Proposition 8 and following discussion) *The quotient map $q: G \to G/N$ induces a homotopy equivalence $\mathfrak{C}_0(G) \to \mathfrak{C}(G/N)$.*

The following lemma from group theory will be useful.

Lemma 2.7. *Let G be a solvable group, with H a proper subgroup. Then*

(1) *If N is an abelian normal subgroup of G with $NH = G$, then H is maximal in G if and only if $N/N \cap H$ is a chief factor for G.*
(2) *H is maximal if and only if it is a complement to a chief factor N_{i+1}/N_i, i.e., if and only if $HN_{i+1} = G$ and $H \cap N_{i+1} = N_i$.*

Part (1) may be found in [17, Theorem 5.4.2]. Part (2) follows from part (1) by taking N_{i+1} to be minimal such that $HN_{i+1} = G$.

We use Lemma 2.7 in proving the following:

Lemma 2.8. *Let G be a group satisfying Conditions* 1–3 *above. Let n be the length of a longest chain in $\mathfrak{C}(G)$. Then G has a minimal normal subgroup N, of non-prime order, over which $|\mathfrak{C}_0(G)|$ is the subcomplex of $|\mathfrak{C}(G)|$ generated by all chains of length n.*

Proof. Our proof goes in five steps:

(1) Every chief factor N_{i+1}/N_i of G is complemented in G/N_i.

We apply a theorem of Gaschütz, proved in [14, Theorem 3.3.2], which says that a normal abelian p-subgroup N has a complement in G if and only if N has a complement in a Sylow p-subgroup P containing N. Let N_{i+1}/N_i be a chief factor of G. Then since G has all Sylow subgroups elementary abelian, G/N_i has all Sylow subgroups elementary abelian. But an elementary abelian group is a complemented group (see Theorem 1.5), hence N_{i+1}/N_i has a complement in any Sylow subgroup P/N_i containing it, and so by Gaschütz we get that N_{i+1}/N_i has a complement in G/N_i.

(2) A chief factor N_{i+1}/N_i is of non-prime order only if $N_i = 1$.

Suppose otherwise, that N_{i+1}/N_i is of non-prime order with $N_i \neq 1$, so that G/N_i is solvable but not supersolvable. Then N_i/N_{i-1} has a complement by part (1), so there is a group K with $G/N_i \cong K/N_{i-1}$. Since all subgroups of G are supersolvable, we see that G/N_i is supersolvable, a contradiction.

(3) There exists a minimal normal subgroup $N \subseteq G$ of non-prime order, and N is a complement to any maximal subgroup K of non-prime index.

Since G is not supersolvable, there is some factor of non-prime order in any chief series of G, and by part (2) it must be of the form $N_1/1$. We notice that in the situation of the second part of Lemma 2.7, we have $[G : H] = [N_{i+1} : N_i]$. Since $N_1/1$ is the only factor of non-prime order, we get the desired result.

(4) A maximal chain C has length less than n if and only if the top element of C is a coset Kx of some complement K of N.

Suppose Hx is the top element of a chain C in $\mathfrak{C}(G)$. By Iwasawa's Theorem and the supersolvability of H, we have that every maximal chain in the interval $(\emptyset, Hx]$ has length $n + 1 - a$, where a is the number of primes with multiplicity dividing $[G : H]$. We see that C has length n if and only if $a = 1$, so C is of length less than n if and only if H is of non-prime index in G if and only if H is a complement of N.

(5) In the situation of part (4), $C \setminus \{Kx\}$ can be extended to a chain of length n.

Let K_1x be the coset immediately under Kx in C. Then since K is supersolvable, $[K : K_1]$ is a prime. Then $[G : NK_1] = [K : K_1]$ is also a prime. Moreover, NK_1 is supersolvable, so if $|N| = p^a$, then there is a chain $K_1 = H_0 < H_1 < \cdots < H_a = NK_1$ between K_1 and NK_1. The desired chain then follows C up to K_1x, and ends at the top with $K_1x < H_1x < \cdots < H_ax = NK_1x$.

We have shown that $|\mathfrak{C}_0(G)|$ is obtained from $|\mathfrak{C}(G)|$ by removing the facets of dimension less than n, thus that $|\mathfrak{C}_0(G)|$ is the subcomplex of $|\mathfrak{C}(G)|$ generated by all n-faces. □

We notice in passing that the argument in part (2) actually shows that a complement K of N has $\mathrm{Core}_G K = 1$, thus that a group G satisfying Conditions 1–3 is *primitive*. Such groups have highly restricted structure, see Chapter A.15 of [10] for an overview.

We relate the preceding lemma to the following result from Björner and Wachs:

Lemma 2.9. (Björner and Wachs [5, Theorem 2.9]) *If Δ is shellable, then the subcomplex generated by all faces of dimensions between r and s is also shellable, for all $r \leqslant s$.*

We are now ready to prove our goal for this section.

Theorem 2.10. *If G is not supersolvable, then G is not coset-shellable.*

Proof. By the preceding discussion, it suffices to consider G solvable with every subgroup a complemented group. Let N be the minimal normal subgroup of order p^a constructed in Lemma 2.8. Then the resulting $|\mathfrak{C}_0(G)|$ is the subcomplex of $|\mathfrak{C}(G)|$ generated by the faces of dimension n. Theorem 2.6 gives us that $|\mathfrak{C}_0(G)| \simeq |\mathfrak{C}(G/N)|$.

If $\mathfrak{C}_0(G)$ were shellable, then $|\mathfrak{C}_0(G)|$ would have the homotopy type of a bouquet of n-spheres, as discussed in the proof of Proposition 2.4. But Theorem 2.3 and the comment following give that $|\mathfrak{C}_0(G)| \simeq |\mathfrak{C}(G/N)|$ is homotopic to a non-empty bouquet of $(n-a)$-spheres. Thus, $\mathfrak{C}_0(G)$ is not shellable, and by Lemma 2.9 we see that $\mathfrak{C}(G)$ is not shellable. □

We have now proved that (4) ⇒ (1)–(3) in our Restatement of the Main Theorem, Theorem 1.5. The following subsection, which the rest of the paper does not depend on, deals with (5)–(7). A reader who is unfamiliar with the Cohen–Macaulay property for simplicial complexes may, if desired, skip directly to Section 3.

2.3. *Cohen–Macaulay coset lattices*

In fact, we have proven slightly more in Section 2.2. Two properties that are closely related to shellability are that of being Cohen–Macaulay and (generalized to non-pure complexes) that of being sequentially Cohen–Macaulay. Recall that the *link* of a face F_0 in a simplicial complex Δ is $\mathrm{lk}_\Delta F_0 = \{F \in \Delta\colon\ F \cup F_0 \in \Delta,\ F \cap F_0 = \emptyset\}$. Links in the order complexes of posets are closely related to intervals. More specifically, if C is a maximal chain containing x and y, and C' is C with all z such that $x < z < y$ removed, then it is easy to see that $\mathrm{lk}_P C'$ is the order complex of the interval (x, y). In general, the link of a chain in a bounded poset is the so-called "join" of intervals.

Let k be a field. A simplicial complex Δ is *Cohen–Macaulay over k* if for every face $F \in \Delta$, $\tilde{H}_i(\mathrm{lk}_\Delta F, k) = 0$ for $i < \dim \mathrm{lk}_\Delta F$, i.e., if every link has the homology of a wedge of top-dimensional spheres. It will come as little surprise after the preceding discussion of links in posets that one can prove the following fact: a poset P is Cohen–Macaulay if and only if every interval (x, y) in P has the homological wedge of spheres property (see [3] for a proof of this and further discussion of links and joins). The complex Δ is *homotopy Cohen–Macaulay* if every such link is homotopic to (rather than merely having the homology of) a wedge of top-dimensional spheres. Since a graded shellable poset has the homotopy type of a wedge of top-dimensional spheres, and since every interval in a shellable poset is shellable, we see that (the order complex of) a graded shellable poset is homotopy Cohen–Macaulay. A homotopy Cohen–Macaulay complex is Cohen–Macaulay over any field.

There is an extension of the Cohen–Macaulay property to non-pure complexes. The *pure i-skeleton* of a simplicial complex Δ is the subcomplex generated by all faces of dimension i. We say that Δ is *sequentially Cohen–Macaulay* if its pure i-skeleton is Cohen–Macaulay for all i. A pure, sequentially Cohen–Macaulay complex is obviously Cohen–Macaulay.

A reference for background on Cohen–Macaulay complexes is [20]. Useful properties of sequentially Cohen–Macaulay complexes are given in [22]. We recall some facts presented in the latter.

Lemma 2.11. *Let Δ be a simplicial complex, P a poset*:

(1) *If Δ is shellable, then Δ is sequentially Cohen–Macaulay* [22, Corollary 1.6].
(2) *If P is sequentially Cohen–Macaulay, then all intervals in P are also sequentially Cohen–Macaulay* [22, Theorem 1.5].

Then in the previous two sections we have actually shown

Proposition 2.12. *If $\mathfrak{C}(G)$ is sequentially Cohen–Macaulay, then G is a complemented group.*

Proof. The proof of Proposition 2.4 shows that if P is a p-group, but not elementary abelian, then $\mathfrak{C}(P)$ has the homotopy type of a wedge of spheres of the wrong dimension. Hence the homology does not vanish below the top dimension, and $\mathfrak{C}(P)$ is not (sequentially) Cohen–Macaulay. Lemma 2.11 part (2) then gives that all Sylow subgroups of a group G with $\mathfrak{C}(G)$ sequentially Cohen–Macaulay must be elementary abelian.

Similarly, in the proof of Theorem 2.10 we show that $\mathfrak{C}_0(G)$ is not Cohen–Macaulay. By Lemma 2.8 we have that $\mathfrak{C}_0(G)$ is the pure n-skeleton of $\mathfrak{C}(G)$, and then the definition gives that $\mathfrak{C}(G)$ is not sequentially Cohen–Macaulay unless G is supersolvable. □

Proposition 2.12 and the fact that complemented groups are supersolvable then give that (6) and (7) are equivalent, and that both imply (1)–(3) in our Restatement of the Main Theorem. Then (4) ⇒ (5) ⇒ (6) is clear from the definition of homotopy Cohen–Macaulay, and it remains only to prove (1)–(3) ⇒ (4). This will be the subject of Section 4.

3. Some linear algebra

We now take a brief break from shellings and homotopy type to do some linear algebra. First, we introduce some notation. Fix a vector space V with (ordered) basis $\mathfrak{B} = \{e_1, \ldots, e_n\}$, and consider a subspace $U \subseteq V$. Let $\{g_1, \ldots, g_k\}$ be a set of generators for U. Then we can write the coordinates of the g_i's as row vectors $[g_i]_{\mathfrak{B}}$, put these in a matrix $\begin{bmatrix} g_1 \\ \vdots \\ g_k \end{bmatrix}_{\mathfrak{B}}$, and reduce to reduced row echelon form M. Denote the set of pivot columns for M (i.e., the columns with a leading 1 in some row of M) as $I_{V,\mathfrak{B}}(U)$, or just $I(U)$ if the choice of V and $\mathfrak{B}$ is clear.

Lemma 3.1. *$I(U)$ is an invariant for the subspace U of V with respect to $\mathfrak{B}$.*

Proof. We need only show that $I(U)$ does not depend on the choice of generators for U. Suppose generators $\{h_i\}$ give row reduced matrix M_h and generators $\{g_i\}$ give row reduced matrix M_g. But then the row reduced matrix of $\{h_i\} \cup \{g_i\}$ must be both M_h and M_g by uniqueness of reduced row echelon form, hence $M_g = M_h$. In particular, the pivot columns are the same. □

We mention some elementary properties of our invariant.

Proposition 3.2. *Fix V and $\mathfrak{B}$ as above, and let U_1, U_2 be subspaces of V. Then*

(1) $|I(U_1)| = \dim U_1$.
(2) *If $U_1 \subseteq U_2$, then $I(U_1) \subseteq I(U_2)$.*

Proof. From a first course in linear algebra, the number of pivots of a matrix is the dimension of the row space, and adding rows to the matrix adds pivots, but does not change the ones we had before. □

We will need the following lemma in our application of $I(U)$ to the next section. Briefly, part (2) will correspond with having a unique lexicographically first path in intervals of $\mathfrak{C}(G)$.

Lemma 3.3. *Fix V and $\mathfrak{B}$ as above, and let $U_1 \subseteq U_2$ be subspaces of V. Then*

(1) *If k is the largest number in $I(U_2) \setminus I(U_1)$, then there is a unique subspace $W_\uparrow$ such that $U_1 \subseteq W_\uparrow \subseteq U_2$ and $I(W_\uparrow) = I(U_1) \cup \{k\}$.*
(2) *If j is the smallest number in $I(U_2) \setminus I(U_1)$, then there is a unique subspace $W_\downarrow$ such that $U_1 \subseteq W_\downarrow \subseteq U_2$ and $I(W_\downarrow) = I(U_2) \setminus \{j\}$.*

Proof. (1) It is immediate from the definition of $I(U_2)$ that there is some $g \in U_2$ with a 1 in the kth coordinate, and 0's in all preceding coordinates when written as a vector with respect to $\mathfrak{B}$. Suppose g_1 and g_2 both have this property. Then $g_1 - g_2$ is 0 in all coordinates up to and including k, hence $g_1 - g_2 \in U_1$. It follows that the desired $W_\uparrow = \langle U_1, g\rangle$ is unique.

(2) First, such a subspace exists. Suppose $U_2 = \langle g_1, \dots, g_n\rangle$, where the g_i's are row reduced as in the definition of $I(U)$. Reorder so that $g_1, \dots, g_l$ are the generators (rows) with pivots in $I(U_2) \setminus I(U_1)$, ordered from least to greatest (where $l = |I(U_2) \setminus I(U_1)|$). Then $U_2 = \langle U_1, g_1, \dots, g_l\rangle$, and $W_\downarrow = \langle U_1, g_2, \dots, g_l\rangle$ is a space with the desired properties.

Suppose W is another such space. Represent $W = \langle U_1, h_2, \dots, h_l\rangle$ in the same way as we did for U_2 in the preceding paragraph. Let $W_0 = \langle g_2, \dots, g_l, h_2, \dots, h_l\rangle$. Then the h_i's and g_i's are all zero in coordinates up to and including j, so $j \notin I(W_0)$. Also, $W_0 \subseteq U_2$ so $I(W_0) \subseteq I(U_2)$. But the g_i's and h_i's were row reduced with respect to U_1, so are zero in all pivots of U_1, so $I(U_1) \cap I(W_0) = \emptyset$. To summarize, $I(W_0) \subseteq I(U_2) \setminus (\{j\} \cup I(U_1))$. But since W_0 is at least $(l-1)$-dimensional (as the g_i's are linearly independent), we get that this is actually an equality. Thus, $\langle g_2, \dots, g_l\rangle = \langle h_2, \dots, h_l\rangle = W_0$, and we have that $W = W_\downarrow$. □

This ends our excursion into linear algebra. We are now ready to apply the results of this section.

4. Shelling the coset poset

To show that the coset poset of a finite complemented group G is shellable, we actually exhibit a co*EL*-labeling. First, let us recall the definition of an *EL*-labeling.

A *cover relation* is a pair $x \leftarrow y$ in a poset P such that $x \lneqq y$ and such that there is no z with $x \lneqq z \lneqq y$. In this situation, we say that y *covers* x. We recall that the usual picture one draws of a poset P is the *Hasse diagram*, where we arrange vertices corresponding with the elements of

P such that x is below y if $x < y$, and draw an edge between x and y if $x \leftarrow y$. We say that a poset is *bounded* if it has a unique top and bottom, that is, unique upper and lower bounds.

Let λ be a labeling of the cover relations (equivalently, of the edges of the Hasse diagram) of P with elements of some poset L—for us, L will always be the integers. Then λ is an *EL-labeling* if for every interval $[x, y]$ of P we have (i) there is a unique (strictly) increasing maximal chain on $[x, y]$, and (ii) this maximal chain is first among maximal chains on $[x, y]$ with respect to the lexicographic ordering. If λ is an *EL*-labeling of the dual of P, then we say λ is a co*EL-labeling*.

Björner first introduced *EL*-labelings in [2], and showed that if a bounded poset P has an *EL*-labeling, then P is shellable. For this reason, posets with an *EL*-labeling (or co*EL*-labeling) are often called *EL-shellable* (or co*EL-shellable*). As we mentioned before, we will use the invariants $I(U)$ discussed in the previous section to construct a co*EL*-labeling of $\hat{\mathfrak{C}}(G)$.

Let $G = G_1 \times \cdots \times G_r$ be the direct product of square free groups $\{G_i\}$, and identify each group G_i with its inclusion in G. Fix p a prime. Let H be a subgroup of G, with H^* a Sylow p-subgroup of H. Let G^* be a Sylow p-subgroup of G, with H^* contained in G^*. By the normality of G_i and an order argument, we get that $G^* \cap G_i$ is either isomorphic to $\mathbb{Z}_p$ or 1 (depending on whether $p \mid |G_i|$). Let e_i be a generator of $G^* \cap G_i$ when this intersection is non-trivial. Let $\mathfrak{B}$ be an ordered basis of such generators e_i, taken from each non-trivial $G^* \cap G_i$. Think of the elementary abelian subgroup G^* as a vector space over $\mathbb{Z}_p$, and define $I^p(H)$ to be $I_{G^*,\mathfrak{B}}(H^*)$.

Lemma 4.1. *$I^p(H)$ is well defined.*

Proof. We need to check that $I^p(H)$ is independent of the choice of H^*, G^*, and $\mathfrak{B}$. Recall that $I_{G^*,\mathfrak{B}}(H^*)$, as defined in Section 3, is the set of pivots of the matrix with rows generating H^*.

Notice that an element $g = e_1^{\alpha_1} \cdots e_r^{\alpha_r} \in G^*$ has $\alpha_j = 0$ for $j < i$ if and only if $g \in G_i G_{i+1} \cdots G_r$. Now the pivot associated with e_i is in $I_{G^*,\mathfrak{B}}(H^*)$ if and only if there is an element in the matrix of row-reduced generators for H^* with first non-zero position i, that is, of the form $h = e_i^{\alpha_i} \cdots e_r^{\alpha_r}$ with $\alpha_i \neq 0$. We see that this happens if and only if $h \in G_i \cdots G_r \setminus G_{i+1} \cdots G_r$. We also notice that a set of generators is (weakly) row-reduced if and only if the first non-zero positions are strictly increasing. Thus, the set of pivots for H^* is determined by the subgroups of the form $G_j \cdots G_r$, and as $G_j \cdots G_r$ does not depend on G^* or $\mathfrak{B}$, it follows immediately that $I^p(H)$ is independent of them.

It remains to check that $I^p(H)$ is independent of the choice of H^*. Any alternate choice differs only by conjugation by some element $x \in H$. But a set of generators $\{h_1, \ldots, h_k\}$ for H^* has the same matrix representation with respect to $\mathfrak{B}$ as their conjugates $\{x^{-1}h_1x, \ldots, x^{-1}h_kx\}$ has with respect to $x^{-1}\mathfrak{B}x$. In particular, the set of pivots is unchanged. □

We need a couple more lemmas.

Lemma 4.2. *Let M be a maximal subgroup of a supersolvable group G. If $G = HM$, then $Hx \cap M$ is a maximal coset of Hx.*

Proof. Since $G = HM$, we can write $Hx = Hm$ for some $m \in M$. So $Hx \cap M = (H \cap M)m \neq \emptyset$. Also, $|G| = |HM| = \frac{|H||M|}{|H \cap M|} = [H : H \cap M] \cdot \frac{|G|}{[G:M]}$. Since $[G : M]$ is prime, it follows that $[H : H \cap M] = [G : M]$ is also prime, hence that $H \cap M$ is maximal in H. □

The following lemma (due to G. Zappa) is proved, for example, in [17, 5.4.8].

Lemma 4.3. *Let G be a finite supersolvable group. Then G has a chief series $1 = N_0 \subseteq \cdots \subseteq N_k = G$ with $[N_1 : N_0] \geqslant [N_2 : N_1] \geqslant \cdots \geqslant [N_k : N_{k-1}]$.*

In particular, if p is the largest prime dividing $|G|$ and q is the smallest; then G has a normal Sylow p-subgroup and a normal Hall q'-subgroup.

Corollary 4.4. *Let G be a finite supersolvable group. If p is the smallest prime dividing $[H_0 : H_n]$ for subgroups $H_n \subseteq H_0$ of G, then there is a unique subgroup H_1 with $H_n \subseteq H_1 \subseteq H_0$ and such that p does not divide $\frac{[H_0:H_n]}{[H_0:H_1]}$.*

Proof. Let $\pi = \{q\colon q \leqslant p,\ q \mid |H_0|\}$, and K be a Hall π'-subgroup of H_0. Then $K \lhd H_0$ by the lemma, hence KH_n is a subgroup of H_0 with the desired properties. □

We are now ready to prove the Main Theorem. The high level idea is to use the changes in the invariants $I^p(H)$ to label cover relations. Unfortunately, that gives us a lot of identically labeled chains. So we pick out some distinguished cover relations, and change their labels to have a unique increasing chain. The details follow.

Theorem 4.5. *If G is supersolvable with all subgroups elementary abelian, then $\hat{\mathfrak{C}}(G)$ is coEL-shellable, and so G is coset-shellable.*

Proof. We recall by the theorem of P. Hall restated in Theorem 1.5 that $G \subseteq G_1 \times \cdots \times G_r$ where each G_i is of square free order. If $\hat{\mathfrak{C}}(G_1 \times \cdots \times G_r)$ is co*EL*-shellable, then it follows immediately from the definition that the interval $[\emptyset, G] \cong \hat{\mathfrak{C}}(G)$ is as well. So we can assume without loss of generality that $G = G_1 \times \cdots \times G_r$, the direct product of groups of square free order.

For each i, and each p dividing $|G_i|$, pick $M^*_{p,i}$ to be a maximal subgroup of index p (a Hall p'-subgroup) in G_i. Such $M^*_{p,i}$'s exist because G_i is solvable, and it is a well-known characterization of solvable groups that they have Hall p'-subgroups for each prime p (see [17, Chapter 9] for more background). Then set $M_{p,i}$ to be $M^*_{p,i} \times \prod_{j \neq i} G_j$. Fix $l(p, j)$ to be an order preserving map into the positive integers of the lexicographic ordering on the pairs (p, j) for all p dividing $|G|$ and $j = 1, \ldots, r$. We will use $M_{p,j}$ to pick out the distinguished edges mentioned above. Most edges will be labeled with $l(p, j)$ for an appropriate p and j, while these distinguished edges will be labeled with the negative of $l(p, j)$.

More precisely, suppose $H_1x \subseteq H_0x$ is a cover relation in $\hat{\mathfrak{C}}(G)$. Since G is supersolvable, $[H_0 : H_1] = p$ for some prime p, hence the Sylow p-subgroups of H_1 have dimension (as vector spaces over $\mathbb{Z}_p$) one lower than those of H_0. It follows that $I^p(H_0) = I^p(H_1) \cup \{j\}$ for some j. Then label the edge $H_0x \to H_1x$ as

$$\lambda(H_0x \to H_1x) = \begin{cases} -l(p, j) & \text{if } H_1x = H_0x \cap M_{p,j}, \\ l(p, j) & \text{otherwise.} \end{cases}$$

Finally, label $\lambda(x \to \emptyset) = 0$. We will show that λ is a co*EL*-labeling.

Intervals in $\hat{\mathfrak{C}}(G)$ all have either the form $[\emptyset, H_0x]$, or $[H_nx, H_0x]$. We consider these types of intervals separately, and show there is a unique increasing chain which is lexicographically first.

On $[\emptyset, H_0x]$, we notice from Proposition 3.2 that every maximal chain from H_0x down to $\emptyset$ has 0 on the last edge, and $\pm l(p, j)$ (over all pairs (p, j) such that $j \in I^p(H_0)$) on the preceding edges. In fact, for each such pair (p, j), exactly one of $+l(p, j)$ or $-l(p, j)$ occurs exactly once on any maximal chain. Finally, since $j \in I^p(H_0)$, there is an element of order p in H_0 of the form $e_j e_{j+1} \cdots e_r$, where each $e_i \in G_i$ and $e_j \neq 0$. As $G_i \subseteq M_{p,j}$ for $i \neq j$, we see that $H_0 M_{p,j} = H_0 M_{p,j} M_{p,j} \subseteq (\langle e_i \rangle \prod_{i \neq j} G_i) M_{p,j} = \langle e_i \rangle M_{p,j} = G$.

Then since 0 is the last edge, the only possible increasing chain is the one with labels $-l(p, j)$ in increasing order. By Lemma 4.2 there is such a chain, it is clearly unique and lexicographically first.

For $[H_n x, H_0 x]$, the situation is only slightly more complicated. Let a pair (p, j) be called *admissible* for the given interval if p divides $[H_0 : H_n]$ and $j \in I^p(H_0) \setminus I^p(H_n)$. If $l(p, j)$ is minimal among admissible (p, j), then there is a unique $H_1 x$ of index p in $H_0 x$ with $H_0 x \to H_1 x$ labeled $\pm l(p, j)$ by Corollary 4.4 and Lemma 3.3. Moreover, any chain on $[H_n x, H_0 x]$ has exactly one edge with label $\pm l(p, j)$ for each admissible (p, j).

Suppose C is an increasing chain on $[H_n x, H_0 x]$. Suppose $H_i x \to H_{i+1} x$ in C is labeled $+l(p, j)$. Then $l(p, j)$ is minimal among (p, j) admissible for $[H_n x, H_i x]$ since the chain is increasing. Thus $H_i x \to H_{i+1} x$ is the unique edge down from $H_i x$ labeled with $\pm l(p, j)$, and since the label was positive we see that $H_n x \nsubseteq M_{p,j}$. It follows that the unique increasing chain on $[H_n x, H_0 x]$ is the lexicographically first one labeled with $-l(p, j)$ in increasing order for (p, j) such that $H_n x \subseteq M_{p,j}$, followed by $+l(p, j)$ for all other admissible (p, j). □

5. Examples

At first glance, the labeling constructed in Theorem 4.5 might seem to come "from left field." It is helpful to work out what happens for the case where G is a group of square free order. In this case, many of the complications we faced in the proof disappear. For example, we do not have to worry about I^p, since if $[H_0 : H_1] = p$, then $I^p(H_1) = \emptyset$ and $I^p(H_0) = \{1\}$. Similarly, we can just take $l(p, j) = p$, since the only possible value of j is 1. The only $M_{p,j}$'s we have are $M_{p,1}$, which we can denote as M_p.

Thus we see that for any H_0, H_1 with $[H_0 : H_1] = p$ we get

$$\lambda(H_0 x \to H_1 x) = \begin{cases} -p & \text{if } H_1 x = H_0 x \cap M_p, \\ p & \text{otherwise} \end{cases}$$

and $\lambda(x \to \emptyset) = 0$. An example for $\mathbb{Z}_6$ is worked out in Fig. 2. An exercise for the reader might be to work out the labeling for S_3.

On the opposite extreme, it is not so hard to understand the co*EL*-shelling on $\mathbb{Z}_p^n$—it is just the change in I^p, with $l(p, j)$ becoming j. We will not say anything more about this, but an example for $\mathbb{Z}_2^2$ is worked out in Fig. 3.

6. Consequences and conclusion

A (co)*EL*-labeling of a lattice L tells us a lot about the homotopy type of $L \setminus \{0, 1\}$. In particular, the *falling chains* (for our purposes, weakly decreasing maximal chains) in an *EL*-labeling give a basis for the non-trivial homology/cohomology group. See [5, Section 5] for a discussion of this in a more general setting. Our co*EL*-labeling for $\mathfrak{C}(G)$ (where G is a complemented group) thus helps us understand the cohomology of the order complex in a very concrete way.

In showing the shellability of a solvable group's subgroup lattice, Shareshian [19] produces a so-called "coatom ordering." Unfortunately, while the existence of a coatom ordering implies the existence of something with similar properties to a co*EL*-labeling (a "co*CL*-labeling"),

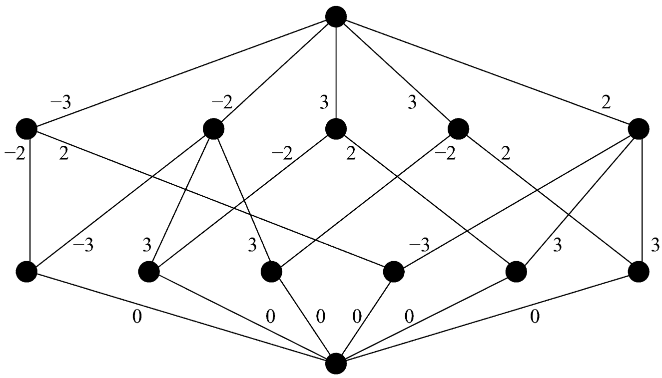

Fig. 2. The coEL-labeling of $\mathfrak{C}(\mathbb{Z}_6)$. The leftmost two maximal cosets are $M_3 = M_{3,1}$ and $M_2 = M_{2,1}$, respectively.

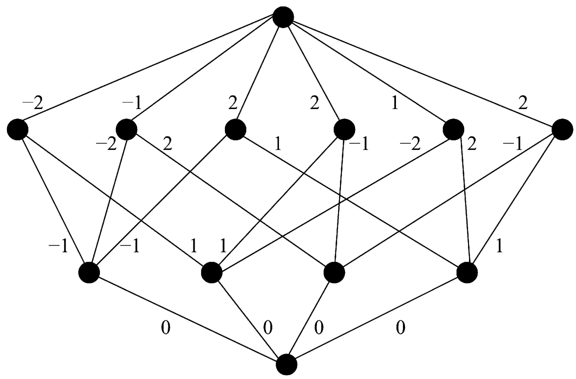

Fig. 3. The coEL-labeling of $\mathfrak{C}(\mathbb{Z}_2^2)$. The leftmost two maximal cosets are $M_{2,2}$ and $M_{2,1}$, respectively.

Shareshian is not able to exhibit such a labeling. Such a labeling would be interesting, as it could presumably be used to give an alternative proof and/or expand upon a result of Thévenaz [21, Theorem 1.4]. Perhaps techniques like we use here could be used on the chief series for a solvable group (where every factor is an elementary abelian p-group) to produce a (co)EL-labeling in the subgroup lattice.

Acknowledgments

I thank my advisor, Ken Brown, for many discussions and ideas; and Keith Dennis for helping with some of my group theory questions. Thanks also to Yoav Segev, Anders Björner, and the anonymous referees for their helpful comments. Though no computations appear in this paper, GAP [11] and the XGAP package often helped in searching for examples that furthered understanding.

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