an involutory automorphism whose set of fixed points is exactly J_2 . Its order is:

$|S| = 448345497600$.

(f) The group M of McLaughlin. This group is constructed in terms of a graph and contains ${}^2A_3(9)$ as a subgroup of index 275. Its order is:

$|M| = 898128000$.

Theorem 38: Among all the finite simple groups above (i.e., all that are currently known), the only coincidences in the orders which do not come from isomorphisms are:

(a) $B_n(q)$ and $C_n(q)$ for $n \geq 3$ and q odd .

(b) $A_2(4)$ and $A_3(2) \sim \mathcal{a}_8$.

(c) H and H' if they aren't isomorphic.

That the groups in (a) have the same order and are not isomorphic has been proved earlier. The orders in (b) are both equal to 20160 by Theorem 25, and the groups are not isomorphic since relative to the normalizer B of a 2-Sylow subgroup the first group has six double cosets and the second has 24. The proof that (a), (b) and (c) represent the only possibilities depends on an exhaustive analysis of the group orders which can not be undertaken here.

§12. Representations. In this section we consider the irreducible representations of the infinite Chevalley groups. As we shall see, here the theory is quite complete. All representations are assumed to be finite-dimensional and the standard terminology is used. In particular 1 must act as the identity, and the trivial 0-dimensional (but not the trivial 1-dimensional) representation is excluded from the list of irreducible representations. We start with a general lemma.

Lemma 68: Let K be an algebraically closed field, B and C associative algebras with 1 over K, and $A = B \otimes C$.

(a) If (β, V) and (γ, W) are (finite-dimensional) irreducible modules for B and C, then $(\alpha, U) = (\beta \otimes \gamma, V \otimes W)$ is one for A.

(b) Conversely, every irreducible A-module (α, U) is realizable, uniquely, as a tensor product as in (a).

Proof: (a) By Burnside's Theorem (see, e.g., Jacobson, Lectures in Abstract Algebra, Vol. 2), $\beta B = \text{End } V$ and $\gamma C = \text{End } W$, whence $\alpha A = \text{End } U$ and (α, U) is irreducible.

(b) Let V be an irreducible B-submodule of U. Such exist since U is finite-dimensional. Let L be the space of B-homomorphisms of V into U. This is nonzero and is a C-module under the rule $c\iota = \alpha(c) \circ \iota$. (Check this.) Let (γ, W) be an irreducible submodule. The map $\varphi: V \otimes W \longrightarrow U$ defined by $v \otimes f \longrightarrow f(v)$ is easily checked to be an A-homomorphism. $V \otimes W$ is irreducible by (a), and U is by assumption. Hence by Schur's

Lemma (see loc. cit.) φ is an isomorphism. If $\alpha = \beta' \otimes \gamma'$ is a second decomposition of the required form, then restriction to B yields $\beta \otimes 1 \cong \beta' \otimes 1$, i.e. multiples of β and β' are isomorphic, so that by the Jordan-Holder or Krull-Schmidt theorems β and β' are also. Similarly γ and γ' are isomorphic, which proves the uniqueness in (b).

Corollary: (a) If K is an algebraically closed field and $G = \prod G_i$ is a direct product of a finite number of groups, then the tensor product V of irreducible KG_i-modules V_i is an irreducible KG-module, and every irreducible KG-module is uniquely realizable in this way.

(b) Similarly for a direct sum $\mathcal{L} = \Sigma \mathcal{L}_i$ of Lie algebras over K.

Proof: We apply Lemma 68, extended to several factors, in (a) to group algebras, in (b) to enveloping algebras.

Exercise: If the direct product above is one of algebraic groups over K (of topological groups, of Lie groups,...), then V is rational (continuous, analytic,...) if and only if each V_i is.

Remark: If we are interested in the irreducible representations of a Chevalley group G, we may as well assume it is universal. The corollary then implies that we may as well also assume that G (i.e. that Σ) is indecomposable. This we will do whenever it is convenient.

Now we take up the study of rational representations for

Chevalley groups over algebraically closed fields viewed as algebraic groups. In such representations the coordinates of the representative matrix are required to be rational functions of the original coordinates. Whether this requirement is to be taken locally (e.g. as in the proof of Theorem 7, Cor. 1) or globally is immaterial, in view of the following result.

Lemma 69: Let G be a Chevalley group viewed as an algebraic group as above, and $f: G \longrightarrow k$ a function. Then the following conditions are equivalent.

(a) f is expressible as a rational function locally.

(b) f is expressible as a rational function globally.

(c) f is expressible as a polynomial.

Proof: It will be enough to show that (a) implies (c). Let A be the algebra of polynomial functions on G. By assumption there exists an open covering $\{U_i\}$ of G, which may be taken finite by the maximal condition on the open subsets of G (which holds by Hilbert's basis theorem in A), and elements g_i, h_i in A such that $f = g_i/h_i$ and $h_i \neq 0$ on U_i for all i. Since the h_i don't all vanish together, by Hilbert's Nullstellensatz there exist elements a_i in A such that $1 = \Sigma a_i h_i$ on G. Let $U_0 = \cap U_i$; it is nonempty, in fact dense, since G is irreducible. On U_0 we have $f = \Sigma a_i f h_i = \Sigma a_i g_i$, a polynomial, hence by density also on each U_i and on G, as required.

Presently we will need the following result.

<u>Lemma 70</u>: The algebra A of polynomial functions on G is integrally closed (in its quotient field).

<u>Proof</u>: We observe first that A is an integral domain (since G is irreducible as an algebraic set, the polynomial ideal defining it is prime), so that it really has a quotient field. Assume $f = p_1/p_2$ ($p_i \in A$) is integral over A: $f^n + a_1 f^{n-1} + \dots + a_n = 0$ for some $a_i \in A$. On restriction to the open set $U^- H U$ of G the p_i and a_i become, by Theorem 7(b), polynomials in the coordinates $\{t_\alpha, t_i, t_i^{-1}\}$. Since such polynomials form a unique factorization domain, we see by the above equation that f itself is such a polynomial, on $U^- HU$. The same being true on each of the translates of $U^- HU$ by elements of G, we conclude that f is a polynomial on G, i.e. f is in A, by Lemma 69.

Two more lemmas and then the main theorem.

<u>Lemma 71</u>: The rational characters of H (homomorphisms into k^*) are just the elements of the lattice L generated by the global weights of the representation defining G.

<u>Proof</u>: Let λ be a character. Then it is a polynomial in the diagonal elements of H (written as a group of diagonal matrices), i.e. a linear combination of elements of L. Being multiplicative, it equals some element of L. (Prove this.) Conversely, if $\lambda \in L$, then λ is a power product of weights in the representation defining G, and all exponents may be taken positive since the product of the latter weights (as functions) is 1; so that λ is a polynomial on H.

Now for any rational G-module V we may define the weights λ and the corresponding weight spaces V_λ, relative to H, in the obvious way.

Lemma 72: Let V be a rational G-module, λ a weight, v an element of V_λ, and α a root. Then there exist vectors $v_i \in V_{\lambda+i\alpha}$ $(i = 1,2,\ldots)$ so that $x_\alpha(t)v = v + \Sigma\, t^i v_i$ for all $t \in k$.

Proof: Since V is rational and $t \longrightarrow x_\alpha(t)$ is an isomorphism, $x_\alpha(t)$ is a polynomial in t: $x_\alpha(t)v = \Sigma\, t^i v_i$. If we apply h to this equation and compare the result with the equation got by replacing $x_\alpha(t)$ by $hx_\alpha(t)h^{-1} = x_\alpha(\alpha(h)t)$, we get $v_i \in V_{\lambda+i\alpha}$. Setting $t = 0$, we get $v = v_0$, whence the lemma.

Theorem 39 (Compare with Theorem 3): Let G be a Chevalley group over an algebraically closed field k (i.e. a semisimple algebraic group over k); and assume the notations as above.

(a) Every nonzero rational G-module V contains a nonzero element v^+ which belongs to some weight $\lambda \in L$ and is fixed by all $x \in U$.

(b) Assume $V = kGv^+$ with v^+ as in (a). Then $V = kU^- v^+$. Further $\dim V_\lambda = 1$; every weight μ on V has the form $\lambda - \Sigma\,\alpha$ (α positive root), and $V = \Sigma\, V_\mu$.

(c) In (a) $\langle\lambda,\alpha\rangle \in \mathbb{Z}^+$ for every positive root α.

(d) If V is irreducible, then the weight λ (the "highest weight") and the line kv^+ of (a) are uniquely determined.

(e) Given any character λ on H satisfying (c), there exists a unique irreducible rational G-module V in which λ is realized as in (a).

<u>Proof:</u> (a) The proof is the same as that of Theorem 3(a) with Lemma 72 in place of Lemma 11.

(b) Since U^-B is dense in G (Theorem 7) and V is rational, any linear function on V which vanishes on U^-Bv^+ also vanishes on Gv^+. Thus $V = kU^-Bv^+ = kU^-v^+$. The other assertions of (b) follow from this equation and Lemma 72.

(c) $w_\alpha\lambda$ is a weight on V (with $w_\alpha v^+$ a corresponding weight vector). Since $w_\alpha\lambda = \lambda - \langle\lambda,\alpha\rangle\alpha$, it follows from (b) that $\langle\lambda,\alpha\rangle \in \mathbb{Z}^+$.

(d) This follows from the second and third parts of (b).

(e) We will use the correspondence between local weights (on $\mathcal{L}$) and global weights (on G) (see p. 60). Let λ be as in (e). By Lemma 71, $\lambda \in L$. Let λ also denote the corresponding weight on $\mathcal{L}$, so that $\lambda(H_\alpha) = \langle\lambda,\alpha\rangle \in \mathbb{Z}^+$ for all $\alpha > 0$. Let (ρ, V') be an irreducible $\mathcal{L}$-module with λ as its highest weight, v^+ a corresponding weight vector, and G' a corresponding Chevalley group over k (constructed from $\rho\mathcal{L}$ and some choice of the lattice M in V'). Since λ is in the lattice generated by the weights of the representation of $\mathcal{L}$ used to construct G, it follows (Theorem 7, Cor. 1) that there exists a rational homomorphism $\varphi: G \longrightarrow G'$ such that $x_\alpha(t) \longrightarrow x'_\alpha(t)$ or 1 for all α and t, in the usual notation. The resulting representation of G on V' need not be irreducible (and its representation class may vary with the choice of M), but at least it contains the vector v^+ which is of weight λ and is fixed by every $x \in U$. Let V'' be the submodule of V' generated by v^+, and V''' a maximal submodule of V''. (V''' is in fact unique as follows from the equation $V'' = kU^- v^+$ of (b). Check this.) The G-module $V = V''/V'''$ meets the existence requirements of (e). For the uniqueness, let $V_i, v_i^+ (i = 1,2)$ satisfy the conditions on V, v^+ in (e). Let $v^+ = v_1^+ + v_2^+ \in V_1 \dot{+} V_2$, and $V = kGv^+$. Then $V_\lambda = kv^+$ by (b), so that $v_2^+ \notin V$. Consider the G-homomorphism $p_1 : V \longrightarrow V_1$, projection on the first factor. Since v_1^+ generates V_1, p_1 is onto. Since also $\ker p_1 \subseteq V_2 \cap V$, which is 0 because V_2 is irreducible and $v_2^+ \notin V$, it follows that p_1 is an isomorphism. Thus V is isomorphic to V_1, and similarly to V_2, so that V_1 and V_2 are isomorphic, as required.

<u>A complement:</u> If char $k = 0$, then in the existence proof above V' is itself irreducible, i.e. $V' = V''$ and $V''' = 0$. In other words, if the Chevalley group G is constructed from an irreducible $\mathcal{L}$-module V and a field k of characteristic 0, then as a linear group it is irreducible.

<u>Proof:</u> Recall that V was originally an $\mathcal{L}_{\mathbb{C}}$-module (irreducible by assumption), and that a lattice M as in Theorem 2, Cor. 1 was then used to shift the coefficients to k. Clearly $V_{\mathbb{Q}}$ is irreducible relative to $\mathcal{L}_{\mathbb{Q}}$. It follows that V_k is irreducible

relative to $\mathcal{L}_k$: otherwise there would be a proper invariant subspace V_1, excluding kv^+ since $kv^+ \cap V_{\mathbb{Q}} \neq 0$, then some nonzero $v \in V_1$ such that $X_\alpha v = 0$ for all $\alpha > 0$, so that writing $v = \Sigma t_i m_i$ ($m_i \in M$, $t_i \in k$ and linearly independent over $\mathbb{Q}$) and choosing α so that $X_\alpha m_i \neq 0$ for some i, we would arrive at the contradiction $\Sigma t_i X_\alpha m_i = 0$. Since we can recover each X_α from G by using $x_\alpha(t) = 1 + tX_\alpha + \ldots$ for several values of t and the X_α's generate $\mathcal{L}$, we conclude that V_k is irreducible for G.

In contrast to the case just considered, if char $k \neq 0$, then $V' \neq V''$ and $V''' \neq 0$ in general and the exact situation is not at all understood, except in a few scattered cases (types A_1, A_2, B_2 or when char k is large "compared" to λ). However, the following is true.

__Exercise:__ (a) For the lattice M of Theorem 2, Cor. 1 (with V there assumed to be irreducible) assume that $\mathbb{C}v^+ \cap M$ is prescribed. Prove that there is a unique minimal choice for M (contained in all others) and a unique maximal choice.

Assume now as in the complement, except that char $k \neq 0$.

(b) If M is maximal, then $V''' = 0$, i.e. V'' is irreducible.

(c) If M is minimal, then $V' = V''$.

__Example:__ If $\mathcal{L}$ is of type A_1, char $k = 2$, and the adjoint representation is used, then (b) holds for $M_{max} = \langle X, H/2, Y\rangle$ and (c) holds for $M_{min} = \langle X, H, Y\rangle$, but not vice versa.

The proof given above for the existence in Theorem 39(e) brings out the connection between the representations of G and those of $\mathcal{L}$ and shows that every irreducible rational representation of a Chevalley group in characteristic $p \neq 0$ can be constructed by the reduction mod p of a corresponding representation of a group in characteristic 0. It depends, however, on the existence of representations of $\mathcal{L}$, which we have not proved here, thus in its entirety is very long. We shall now develop an alternate, more intrinsic, proof.

We start with the connection between a G-module V and its dual V^*, on which G acts by the rule $(xf)(v) = f(x^{-1}v)$ for all $x \in G$, $f \in V^*$, and $v \in V$. We recall that w_o is the element of the Weyl group which makes all positive roots negative.

<u>Lemma 73:</u> Let V be an irreducible rational G-module, λ its highest weight, v^+ a corresponding weight vector, $\lambda^* = -w_o\lambda$, and f^+ the element of V^* defined thus: if we write $v^- = w_o v^+$ and $v \in V$ as $v = cv^-$ + terms of other (hence higher) weights, then $f^+(v) = c$. Then λ^* and f^+ are highest weight and highest weight vector for V^*.

<u>Proof:</u> In the definition of f^+ we have used the fact that $\dim V_{w_o\lambda} = \dim V_\lambda = 1$. Here, and also in similar situations later, we extend λ to B by the rule $\lambda(b) = \lambda(h)$ if $b = uh$ $(u \in U,\ h \in H)$, and similarly for λ^*. If we write $v \in V$ as in the lemma and use Lemma 72, we see that $bv = c(w_o\lambda)(h)v^-$ + higher terms. Since $c = f^+(v)$ and

$(w_o\lambda)(h) = \lambda^*(b^{-1})$, we have $f^+(bv) = \lambda^*(b^{-1})f^+(v)$. On replacing b by b^{-1} we get $bf^+ = \lambda^*(b)f^+$, as required.

Theorem 40: For $\lambda \varepsilon L$ let A_λ be the space of polynomial functions a on G such that $a(yb) = a(y)\lambda(b)$ for all $y \varepsilon G$, $b \varepsilon B$, made into a G-module in the obvious way.

(a) If V, λ, v^+ are as in Lemma 73, then the map $\varphi: V^* \longrightarrow A_\lambda$ defined by $(\varphi f)(x) = f(xv^+)$ for $f \varepsilon V^*$ and $x \varepsilon G$ is a G-isomorphism into.

(b) Conversely, if λ is such that $A_\lambda \neq 0$, then A_λ contains a unique irreducible G-submodule. The latter is finite-dimensional and rational and its highest wieght is λ^* .

Proof: (a) The points to be checked here will be left as an exercise.

(b) We observe first that as a G-module A_λ is locally finite-dimensional (in fact, it is finite-dimensional, but we shall not prove this), since the set of polynomials of a given degree is. Thus there exist irreducible submodules and all of them are finite-dimensional and rational. Let μ be the highest weight of any one of them and a^+ a corresponding nonzero weight vector. We have (*) $a^+(bxb') = \mu(b^{-1})a^+(x)\lambda(b')$ for all $x \varepsilon G$, $b, b' \varepsilon B$. Since Bw_oB is dense in G, $a^+(w_o) \neq 0$. Since also $a^+(bw_o) = a^+(w_o.w_o^{-1}bw_o)$, we get from the above equation that $\mu(b^{-1}) = \lambda(w_o^{-1}bw_o)$, so that $\mu = \lambda^*$. Since μ is uniquely determined by λ , the function a^+ is determined by its value at w_o by

(*) with $x = w_o$ and the density of Bw_oB in G, proving the uniqueness in (b).

Remarks: (a) In characteristic 0 it easily follows from the theorem of complete reducibility that A_λ itself is irreducible.

(b) The representation of G on A_λ is, in the context of polynomial representations, the one induced by the character λ on B. The fact that it contains a representation of highest weight λ^*, is, in view of Theorem 39(a), a form of Frobenius reciprocity.

Lemma 74: Let f^+ be as in Lemma 73 and $a^+ = \varphi f^+$ with φ as in Theorem 40(a) so that $xv^+ = a^+(x)w_ov^+ +$ higher terms. Let W_λ be the stabilizer of λ in W, and for $w \in W_\lambda$ assume that the corresponding representative $w \in G$ has been chosen so that $wv^+ = v^+$. Then if $x \in G$ is written uhw_owu_1 (see Theorem 4')

we have $a^+(x) = \lambda^*(h^{-1})$ if $w \in W_\lambda$,

$= 0$ otherwise.

Proof: A choice for $w \in G$ as above is always possible: if $\lambda = 0$, then V is trivial since $G = \mathcal{D}G$, while if $\lambda \neq 0$, then wv^+ has weight $w\lambda = \lambda$, hence is a multiple of v^+, so that by modifying it by a suitable element of H we can achieve $wv^+ = v^+$. From the definitions and Lemma 72 we have $a^+(x)w_ov^+ = hw_owv^+$. If $w \in W_\lambda$, then $a^+(x) = \lambda^*(h^{-1})$ by the choice of w, while if $a^+(x) \neq 0$, then w fixes kv^+ by the equation so that $w \in W_\lambda$.

This brings us to the

Second proof of the existential part of Theorem 39(e):

Proof: Let λ be as in Theorem 39(c). It will be enough to prove that the function defined by the last equations of Lemma 74 is rational on G. The existence will then follow from Theorem 40(b) with λ^* in place of λ. By Lemma 70 any power of this function will do, so that by Lemma 74 it will be enough to construct an irreducible representation whose highest weight is some positive power (positive multiple if we write characters on H additively) of λ. This we will do, using the following interesting result.

Lemma 75 (Chevalley): Let G be a linear algebraic group and P a closed subgroup. Then there exists a rational G-module V and a line L in V whose stabilizer in G is P.

Proof: Let A be the algebra of polynomials in the matric entries and I the ideal defining P. By Hilbert's basis theorem I is generated by a finite number of its elements, so that there exists a finite-dimensional G-invariant subspace B of A such that $B \cap I$, say C, generates I. For $x \in G$ we have the following equivalent conditions: $x \in P$; $f(x^{-1}y) = 0$ for all $f \in I$, $y \in P$; $xI \subseteq I$; $xC \subseteq C$. If now $c = \dim C$, $V = \bigwedge^c B$, v is the product in V of a basis for C, and $L = kv$, it follows that the stabilizer of L is exactly P.

We resume the proof of existence. Let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ be the set of simple roots. For $i = 1, 2, \ldots, \ell$ let P_i be the parabolic subgroup of G corresponding to $\Pi - \{\alpha_i\}$ (see Lemma

30), $L_i = kv_i$ the corresponding line of Lemma 75, μ_i the corresponding rational character on P_i, hence also on B and H, and $V_i = kGv_i$. If $j \neq i$, then w_j is represented in P_i, so that $w_j\mu_i = \mu_i$ and $\langle\mu_i,\alpha_j\rangle = 0$. Since w_i does not fix L_i, by choice, it follows from parts (b) and (c) of Theorem 39 applied to V_i that $-\langle\mu_i,\alpha_i\rangle$ is a positive integer, say d_i. If now λ is as before so that $\langle\lambda,\alpha_i\rangle = c_i \in \mathbb{Z}^+$, it follows that $d\lambda = \Sigma e_i\mu_i$ with $d = \prod d_i$ and $e_i = c_i d/d_i$. If we form the tensor product $\prod V_i^{e_i}$, then $\prod v_i^{e_i}$ is a vector of weight $d\lambda$ for B, so that we may extract an irreducible component whose highest weight is $d\lambda$, and thus complete our second existence proof.

<u>Remark:</u> We are indebted to G. D. Mostow for the proof just given.

The extra problems that arise when char $k \neq 0$ are compensated for by the fact that only a finite number of representations has to considered in this case, as we shall now see.

<u>Lemma 76:</u> Assume char $k = p \neq 0$. Let Fr (for Frobenius) denote the operation of replacing the matric entries of the elements of G by their p^{th} powers. If ρ is an irreducible rational representation of G, then so is $\rho \circ \mathrm{Fr}$. If the highest weight of ρ is λ, that of $\rho \circ \mathrm{Fr}$ is $p\lambda$.

<u>Proof:</u> Exercise.

<u>Theorem 41:</u> Assume that G above is universal (i.e. G is a simply connected algebraic group), and that char $k = p \neq 0$. Let

$\mathcal{R}$ be the set of p^{ℓ} irreducible rational representations of G for which the highest weight λ satisfies $0 \le \langle\lambda,\alpha_i\rangle \le p-1$ (α_i simple). Then every irreducible rational representation of G can be written uniquely $\bigotimes_{j=0}^{\infty} \rho_j \circ Fr^j$ ($\rho_j \in \mathcal{R}$).

Sketch of proof: We observe first that since G is universal $L = L_1$, so that all λ's with all $\langle\lambda,\alpha_i\rangle \in \mathbb{Z}^+$ occur as highest weights, in particular those used to define $\mathcal{R}$. Consider $\rho = \otimes \rho_j \circ Fr^j$. Let λ_j be the highest weight of ρ_j. The product of the corresponding weight vectors yields for ρ a highest weight vector of weight $\lambda = \Sigma\, p^j\lambda_j$, by Lemma 76. If we vary the ρ_j's in $\mathcal{R}$, we obtain, in view of the uniqueness of the expansion of a number in the scale of p, each possible highest weight λ exactly once. Thus to prove the theorem we need only show that each ρ above is irreducible. The proof of this fact depends eventually on the linear independence of the distinct automorphisms $Fr^j (j = 0,1,\ldots)$ of k. We omit the details, referring the reader to R. Steinberg, Nagoya Math. J. 22 (1963), or to P. Cartier, Sém. Bourbaki 255 (1963).

Corollary: Assume that one of the special situations of Theorem 28 holds. Let $\mathcal{R}_\ell$ (resp. $\mathcal{R}_s$) be the subsets of $\mathcal{R}$ defined by $\langle\lambda,\alpha_i\rangle = 0$ for all i such that α_i is long (resp. short). Then every element of $\mathcal{R}$ can be written uniquely $\rho_\ell \otimes \rho_s$ with $\rho_\ell \in \mathcal{R}_\ell$ and $\rho_s \in \mathcal{R}_s$.

Proof: Given $\rho \in \mathcal{R}$, write the corresponding highest weight λ

as $\lambda_\ell + \lambda_s$ so that the corresponding irreducible representations ρ_ℓ and ρ_s are in $\mathcal{R}_\ell$ and $\mathcal{R}_s$. We have to show that $\rho_\ell \otimes \rho_s$ is irreducible. If we define φ as in Theorem 28 but with G and G^* interchanged, and set $\rho_\ell^* = \varphi \circ \rho_\ell$, $\rho_s^* = \varphi \circ \rho_s$, we have to show that the representation $\rho_\ell^* \otimes \rho_s^*$ of G^* is irreducible. Since the corresponding highest weights satisfy

$$\langle \lambda_\ell^*, \alpha^* \rangle = \langle \lambda_\ell, \alpha \rangle \quad \text{if } \alpha \text{ is short}$$
$$= 0 \quad \text{if not}$$

$$\langle \lambda_s^*, \alpha^* \rangle = p\langle \lambda_s, \alpha \rangle \quad \text{if } \alpha \text{ is long}$$
$$= 0 \quad \text{if not,}$$

we see that λ_ℓ^* and λ_s^*/p correspond to elements of $\mathcal{R}^*$, so that the corollary follows from Theorem 41 applied to G^*.

<u>Examples:</u> (a) SL_2. Here there are p representations ρ_i $(i = 0,1,\ldots,p-1)$ in $\mathcal{R}$, the i^{th} being realized on the space of polynomials homogeneous of degree i over k^2.

(b) Sp_4, $p = 2$. Here there are 4 representations in $\mathcal{R}$. If φ is the graph automorphism of Theorem 28 then $\mathcal{R}_\ell \circ \varphi = \mathcal{R}_s$ so that by the above corollary, these 4 are, in terms of the defining representation ρ, just 1 (trivial), ρ, $\rho \circ \varphi$, and $\rho \otimes (\rho \circ \varphi)$.

The results we have obtained can easily be extended to the case that k is infinite (but perhaps not algebraically closed). We consider representations on vector spaces over K, some algebraically closed field containing k, and call them rational if the coordinates of the image are polynomial functions over K in the coordinates of the source. The preceding theory is then applicable almost word for word because of the following two facts both coming from the denseness of G in G_K (this is G with k extended to K).

(a) Every irreducible polynomial representation of G extends uniquely to one of G_K.

(b) On restriction to G every irreducible rational representation of G_K remains irreducible.

Exercise: Prove (a) and (b).

The structure of arbitrary irreducible representations is given in terms of the polynomial ones by the following general theorem. Given an isomorphism φ of k into K, we shall also write φ for the natural isomorphism of G onto the group φG obtained from G by replacing k by φk.

Theorem 42 (Borel, Tits): Let G be an indecomposable universal Chevalley group over an infinite field k, and let σ be an arbitrary (not necessarily rational) irreducible representation of G on a finite-dimensional vector space V over an algebraically closed field K. Assume that σ is nontrivial. Then there exist finitely-many isomorphisms φ_i of k into K and corresponding irreducible rational representations ρ_i of $\varphi_i G$ over K such that $\sigma = \bigotimes_i \rho_i \circ \varphi_i$.

Remarks: (a) As a corollary we see that k is necessarily imbeddable as a subfield of K. In other words, if k and K are such that no such imbedding exists, e.g. if $\text{char}\, k \neq \text{char}\, K$, then every irreducible representation of G on a finite-dimensional vector space over K is necessarily trivial. (Deduce that the same is true even if the representation is not irreducible.) If k is finite, these statements are, of course, false.

(b) The theorem can be completed by statements concerning the uniqueness of the decomposition and the condition for irreducibility if the factors are prescribed. Since these statements are a bit complicated we shall omit them.

(c) The theorem was conjectured by us in Nagoya Math. J. 22 (1963). The proof to follow is based on an as yet unpublished paper by A. Borel and J. Tits in which results of a more general character are considered.

<u>Lemma 77:</u> Let G, G' be indecomposable Chevalley groups over fields k, k' with k infinite and k' algebraically closed, and $\sigma : G \longrightarrow G'$ a homomorphism such that σG is dense in G'.

(a) There exists an isomorphism φ of k into k' and a rational homomorphism ρ of φG into G' such that $\sigma = \rho \circ \varphi$.

(b) If G is universal, then ρ can be lifted, uniquely, to the universal covering group of G'.

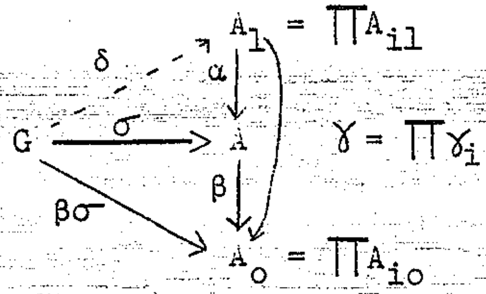
<u>Proof:</u> (a) If the reader will examine the proof of Theorem 31 he will observe that what is shown there is that σ can be normalized so that $\sigma x_\alpha(t) = x_{\alpha'}(\varepsilon_\alpha \varphi(t)^{q(\alpha)})$ with $\alpha \longrightarrow \alpha'$ an

angle-preserving map of Σ on Σ', $\varepsilon_\alpha = \pm 1$, φ an isomorphism of k onto k', and $q(\alpha) = 1$ on p. Since we are assuming only that σG is dense in G', not that $\sigma G = G'$, the proof of the corresponding result in the present case is somewhat harder. However, the main ideas are quite similar. We omit the proof. From the above equations and the corresponding ones on H, it follows from Theorem 7 that σ has the form of (a).

(b) From these equations we see also, e.g. by considering the relations (A), (B), (C) of Theorem 8, that ρ can be lifted to any covering of G', uniquely since $G = \mathcal{D}G$.

<u>Proof of Theorem 42:</u> Let $A = \overline{\sigma G}$, the smallest algebraic subgroup of $GL(V)$ containing σG. We claim A is a connected semisimple group, hence a Chevalley group. As in the proof of Theorem 30, step (12), $\overline{\sigma U}$ is connected, and similarly for $\overline{\sigma U^-}$, so that A, being generated by these groups, is also. Let R be a connected solvable normal subgroup of A. By the Lie-Kolchin theorem R has weights on V, finite in number. A permutes the corresponding weight spaces, and being connected, fixes them all. Since V is irreducible, there is only one such space and it is all of V, so that R consists of scalars, of determinant 1 since $A = \mathcal{D}A$, so that R is finite. Since R is connected, $R = 1$, so that A is semisimple, as claimed. Let $A_1 = \prod A_{i1}$ be the universal covering group of A written as a product of its indecomposable components, $A_o = \prod A_{io}$ the corresponding factorization of the adjoint group, and $\alpha, \beta, \gamma = \prod \gamma_i$ the

corresponding natural maps as shown:



By Lemma 77 we can lift $\beta\sigma$ componentwise to get a homomorphism $\delta\colon G \longrightarrow A_1$ of the form $\delta(x) = \prod \varepsilon_i\varphi_i(x)$ with each φ_i an isomorphism of k into K and ε_i a rational homomorphism of $\varphi_i G$ into A_{i1}. We have $\alpha\delta = \sigma$ since otherwise we would have a homomorphism of G into the center of A. By Lemma 68, Cor. (a), α, interpreted as an irreducible rational representation of A_1, may be factored $\bigotimes_i \alpha_i$ with α_i an irreducible rational representation of A_{i1}. On setting $\rho_i = \alpha_i\varepsilon_i$, we see that $\sigma = \alpha\delta = \bigotimes \alpha_i\varepsilon_i\varphi_i$ $= \bigotimes_i \rho_i\varphi_i$, as required.

<u>Corollary:</u> (a) Every absolutely irreducible real representation of a real Chevalley group G is rational.

(b) Every holomorphic irreducible representation of a semisimple complex Lie group is rational.

(c) Every continuous irreducible representation of a simply connected semisimple complex Lie group is the tensor product of a holomorphic one and an antiholomorphic one.

<u>Proof:</u> (a) If G is universal, this follows from the theorem and the fact that the only isomorphism of $\mathbb{R}$ into $\mathbb{R}$ is id. The

transition to the nonuniversal case is an easy exercise.

(b) The proof is similar to that of (a).

(c) The only continuous isomorphisms of $\mathbb{C}$ into $\mathbb{C}$ are the identity and complex conjugation.

<u>Exercise:</u> Prove that the word "absolutely" in (a) and the words "simply connected" in (c) may not be removed.

Now we shall touch briefly on some additional results.

<u>Characters.</u> As is customary in representation theory, the characters (i.e. the traces of the representative matrices) play a vital role. We state the principal results in the form of an exercise.

<u>Exercise:</u> (a) Prove that two irreducible rational G-modules are isomorphic if and only if their characters are equal. (Consider the characters on H.)

(b) Assume that $\operatorname{char} k = 0$ and that the theorem of complete reducibility has been proved in this case. Prove (a) for representations which need not be irreducible.

(c) Assume $\operatorname{char} k = 0$. Prove Weyl's formulas: Let V, λ be as in Theorem 39(e), χ the corresponding character, and δ one-half the sum of the positive roots, a character on H. Set $S_\lambda = \sum_{w \in W} \det w . w(\lambda + \delta)$, a sum of functions on H. Then

(1) $\chi(h) = S_\lambda(h)/S_0(h)$ at all $h \in H$ where $S_0(h) \neq 0$.

(2) $\dim V = \prod_{\alpha > 0} \langle \lambda + \delta, \alpha \rangle / \langle \delta, \alpha \rangle$.

(Hint: use the corresponding formulas for Lie algebras (see, e.g., Jacobson, Lie Algebras) and the complement to Theorem 39).

Remark: The formula (1) determines χ uniquely since it turns out that the elements of G which are conjugate to those elements of H for which $S_o \neq 0$ form a dense open set in G.

The unitarian trick. The basic results about the irreducible complex representations of a compact semisimple Lie group K, i.e. a maximal compact subgroup of a complex Chevalley group G as in §8, can be deduced from those of G because of the following important fact: (*) K is Zariski-dense in G. Because of Lemmas 43(b) and 45 (K is generated by the groups $\varphi_2 SU_2$) this comes down to the fact that SU_2 is Zariski-dense in $SL_2(\mathbb{C})$, whose proof is an easy exercise. By (*) the rational irreducible representations of G remain distinct and irreducible on restriction to K. That a complete set of continuous representations of K is so obtained then follows from the fact that the corresponding characters form a complete set of continuous class functions on K. The proof of this uses the formula for Haar measure on K and the orthogonality and completeness properties of complex exponentials, and yields as a by-product Weyl's character formula itself. This is how Weyl proved his formula in Math. Zeit. 24 (1926) and it is still the best way. The theorem of complete reducibility can be proved as follows. Given any rational representation space V for G and an invariant subspace V', we can, by averaging over K, relative to Haar measure, any projection of V onto V' and taking the kernel of the result, get a complementary subspace invariant under K, thus also invariant under

G because of (*). It is then not difficult to replace the complex field by any field of characteristic 0 .

Invariant bilinear forms. G denotes an indecomposable infinite Chevalley group, V an irreducible rational G-module, and λ its highest weight.

Lemma 78: The following conditions are equivalent.

(a) There exists on V a (nonzero) invariant bilinear form.

(b) V and its dual V^* are isomorphic.

(c) $-w_o\lambda = \lambda$.

Proof: Exercise (see Lemma 73).

Exercise: Prove that $-w_o$ is the identity for all simple types except $A_n (n \geq 2)$, D_{2n+1}, E_6, and for these types it comes from involutory automorphism of the Dynkin diagram. (Hint: for all of the unlisted cases except for D_{2n} the Dynkin diagram has no symmetry.)

Exercise: If there exists an invariant bilinear form on V , then it is unique up to multiplication by a scalar and is either symmetric or skew-symmetric. (Hint: use Schur's Lemma.)

Lemma 79: Let $h = \prod h_\alpha(-1)$, the product over the positive roots.

(a) h is in the center of G and $h^2 = 1$.

(b) If V possesses an invariant bilinear form then it is symmetric if $\lambda(h) = 1$,
skew-symmetric if $\lambda(h) = -1$.

Proof: (a) Since $h_\alpha(-1) = h_{-\alpha}(-1)$ (check this), h is fixed by all elements of W. This implies that h is in the center, as easily follows from Theorem 4'. Since $h_\alpha(-1)^2 = h_\alpha(1) = 1$, we have $h^2 = 1$.

(b) We have an isomorphism $\varphi : V \longrightarrow V^*$, $v^+ \longrightarrow f^+$ with v^+ and f^+ as in Lemma 73; the corresponding bilinear form on V is given by $(v,v') = (\varphi v)(v')$. It follows that $(xv^+,yv^+) = f^+(x^{-1}yv^+)$ for all $x, y \in G$. Thus $(v^+,w_0v^+) = f^+(w_0v^+) \neq 0$ by the definition of f^+, and $(w_0v^+,v^+) = f(w_0^{-1}v^+)$. If $w_0 = w_\alpha w_\beta w_\gamma \ldots$ is a minimal product of simple reflections in W, then for definiteness we pick $w_0 = w_\alpha(1)w_\beta(1)\ldots$ in G, so that $w_0^{-1} = \ldots w_\gamma(-1)w_\beta(-1)w_\alpha(-1)$. We have $w_\alpha(-1) = w_\alpha(1)h_\alpha(1)$, and similarly for $\beta, \gamma, \ldots$. Substituting into the expression for w_0^{-1} and bringing all the h's to the right, by repeated conjugation by w's, we get to the right h by Appendix II (25) and to the left $\ldots w_\gamma(1)w_\beta(1)w_\alpha(1)$ which is just w_0 by a lemma to be proved in the last section. Thus $w_0^{-1} = w_0h$, and (w_0v^+,v^+) becomes $\lambda(h)f^+(w_0v^+) = \lambda(h)(v^+,w_0v^+)$, as required.

Observation: h as in Lemma 79 is 1 in each of the following cases, since the center is of odd order.

(a) G is adjoint.

(b) Char $k = 2$.

(c) G is of type A_{2n}, E_6, E_8, F_4, G_2.

Exercise: In the remaining cases find h, as a product $\prod h_\alpha(-1)^{n_\alpha}$ over the simple roots.

<u>Example:</u> SL_2 . For every V there is an invariant bilinear form. Assume char $k = 0$, so that for each $i = 1,2,3,\ldots$ there is exactly one V of dimension i , viz. the space of polynomials homogeneous of degree $i-1$. Then the invariant form is symmetric if i is odd, skew-symmetric if i is even.

<u>Invariant Hermitean forms.</u> Assume now that G is complex, σ is the automorphism of Theorem 16, $K = G_\sigma$ is the corresponding maximal compact subgroup, V and v^+ are as before, and $f: G \longrightarrow \mathbb{C}$ is defined by $xv^+ = f(x)v^+$ + terms of other weights.

(a) Prove that $f(\sigma x^{-1}) = \overline{f(x)}$. (First prove it on U^-HU , then use the density of U^-HU in G .)

(b) Prove that there exists a unique form $(\ ,\ )$ from $V \times V$ to $\mathbb{C}$ which is linear in the second position, conjugate linear in the first, and satisfies $(xv^+, yv^+) = f(\sigma x^{-1} y)$, and that this form is Hermitean.

(c) Prove that $(\ ,\ )$ is positive definite and invariant under K .

<u>Dimensions.</u> Assume now that G is a Chevalley group over an infinite field k , that V and λ are as before, and that $\mathcal{U}_{\mathbb{Z}}$ is the universal algebra of Theorem 2, written in the form $\mathcal{U}_{\mathbb{Z}}^- \mathcal{U}_{\mathbb{Z}}^0 \mathcal{U}_{\mathbb{Z}}^+$ of page 16.

(a) Prove that there exists an antiautomorphism σ of $\mathcal{U}_{\mathbb{Z}}$ such that $\sigma X_\alpha = X_{-\alpha}$ and $\sigma H_\alpha = H_\alpha$ for all α .

(b) Define a bilinear form (u,u') from $\mathcal{U}_{\mathbb{Z}}$ to $\mathcal{U}_{\mathbb{Z}}^0$ thus:

write $\sigma u.u'$ in the above form and then set every $X_\alpha = 0$. Prove that this form is symmetric.

(c) Now define a bilinear form from $\mathcal{U}_{\mathbb{Z}}$ to $\mathbb{Z}$ thus: $(\ , \ )_\lambda = \lambda \circ (\ , \ )$ (interpreting λ as a linear form on $\mathcal{H}$ such that $\lambda(H_\alpha) \in \mathbb{Z}^+$ for all $\alpha > 0$) . Assuming now that this form is reduced modulo the characteristic of k , prove that its rank is just the dimension of V .

§ 13. Representations continued. In this section the irreducible representations of characteristic p (the characteristic of the base field k) of the finite Chevalley groups and their twisted analogues will be considered. The main result is as follows.

Theorem 43: Let G be a finite universal Chevalley group or one of its twisted analogues constructed as in §11 as the set of fixed points of an automorphism of the form $x_\alpha(t) \to x_{\rho\alpha}(\pm t^{q(\alpha)})$. Then the $\prod_{\alpha \text{ simple}} q(\alpha)$ irreducible polynomial representations of the including algebraic group (got by extending the base field k to its algebraic closure) for which the highest weights λ satisfy $0 \leq \langle \lambda, \alpha \rangle \leq q(\alpha) - 1$ for all simple α remain irreducible and distinct on restriction to G and form a complete set.

By Theorem 41 we also have a tensor product theorem with the product $\prod_0^\infty$ suitably truncated, for example to $\prod_0^{n-1}$ if G is a Chevalley group over a field of p^n elements.

Exercise: Deduce from Theorem 43 the nature of the truncations for the various twisted groups.

Instead of proving the above results (see Nagoya Math. J 22 (1963)), which would take too long, we shall give an a priori development, similar to the one of the last section.

We start with a group of twisted rank 1 (i.e. of type A_1, ${}^2A_2$, ${}^2C_2$, or ${}^2G_2$, the degenerate cases $A_1(2)$, $A_1(3)$, ... not being excluded). The subscript σ on G_σ, W_σ, ... will

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We start with a group of twisted rank 1 (i.e. of type A_1, ${}^2A_2$, ${}^2C_2$, or ${}^2G_2$, the degenerate cases $A_1(2)$, $A_1(3)$, ... not being excluded). The subscript σ on G_σ, W_σ, ... will henceforth be omitted. We write X, Y, w, K for $\mathfrak{X}_a (a > o)$, $\mathfrak{X}_{-a}$, the nontrivial element of the Weyl group realized in G , and an algebraically closed field of characteristic p . We observe that $U = X$ and $U^- = Y$ in the present case. In addition, $\overline{X}$ will denote the sum of the elements of X in KG .

Definition: An element v in a KG-module V is said to be a highest weight vector if it is nonzero and satisfies

(a) $xv = v$ for all $x \in U$.

(b) $hv = \lambda(h)v$ for all $h \in H$ and some character λ on H .

(c) $\overline{X}wv = \mu v$ for some $\mu \in K$.

The couple (λ, μ) is called the corresponding weight.

Remark: The refinement (c) of the usual definition is due to C. W. Curtis (Ill. J. Math. 7(1963) and J. für Math. 219 (1965)). That such a refinement is needed is already seen in the simplest case $G = SL_2(p)$. Here there are p representations realized on the spaces of homogeneous polynomials of degrees $i = 0, 1, \ldots, p - 1$, with the highest weight in the usual sense being $i\lambda_1$. Since the group H is cyclic of order $p - 1$, the weights 0 and $(p - 1)\lambda_1$ are identical on H , hence do not distinguish the corresponding representations from each other.

Theorem 44: Let G (of rank 1) and the notations be as above.

(a) Every nonzero KG-module V contains a highest weight vector v . For such a v we have $KGv = KYv = KU^-v$.

(b) If V is irreducible, then it determines Kv as the unique line of V fixed by U, hence it also determines the corresponding highest weight.

(c) Two irreducible KG-modules are isomorphic if and only if their highest weights are equal.

The proof depends on the following two lemmas.

Lemma 80: Let Y and K be as above, or, more generally, let Y be any finite p-group and K any field of characteristic p.

(a) Every nonzero KY-module V contains nonzero vectors invariant under Y.

(b) Every irreducible KY-module is trivial.

(c) $\text{Rad } KY = \{\Sigma c(y)y \mid \Sigma c(y) = 0\}$ is the unique maximal (one-sided or two-sided) ideal of KY. It is nilpotent.

(d) $K\overline{Y}$ is the unique minimal ideal of KY.

Proof: (a) By induction on $|Y|$. Assume $|Y| > 1$. Since Y is a p-group, it has a normal subgroup Y_1 of index p, and the subspace V_1 of invariants of Y_1 on V is nonzero by the inductive assumption. Choose $y \in Y$ to generate Y/Y_1, and v in V_1 and nonzero. Then $(1 - y)^p v = 0$. Now choose r maximal so that $(1 - y)^r v \neq 0$. The resulting vector is fixed by Y, whence (a).

(b) By (a).

(c) By inspection Rad KY is an ideal, maximal because its codimension in KY is 1. Bring the kernel of the trivial representation, which is the only irreducible one by (b), it is the unique maximal left ideal; and similarly for right ideals. On each factor of a composition series for the left regular representation of KY on itself Y acts trivially by (b), hence Rad KY acts as 0, so that Rad KY is nilpotent.

(d) By (b).

Lemma 81: For $x \in X - 1$, write $wxw = f(x)h(x)wg(x)$ with $f(x), g(x) \in X$ and $h(x) \in H$.

(a) f and g are permutations of $X - 1$.

(b) $(\bar{X}w)^2 = \bar{X}w^2 + \sum_{x \in X-1} h(x)\bar{X}wg(x)$.

Proof: (a) If $f(x) = 1$, we get the contradiction $xw \in B$, while if $f(x') = f(x)$, we see that $w^{-1}x^{-1}x'w \in B$, so that $x' = x$. Similarly for g.

(b) $(\bar{X}w)^2 = \bar{X}w^2 + \sum_{x \in X-1} \bar{X}f(x)h(x)wg(x)$.

Here $f(x)$ gets absorbed in $\bar{X}$ and $h(x)$ normzlizes $\bar{X}$, whence (b).

Proof of Theorem 44: (a) By Lemma 80(b) the space of fixed points of $U = X$ on V is nonzero. Since H normalizes U and is Abelian, that space contains a nonzero vector v such that (a) and (b) of the definition of highest weight vector hold. Let

$\bar{X}wv = v_1$. If $v_1 = 0$, then (c) holds with $\mu = 0$. If not, we replace v by v_1. Then (a) and (b) of the definition still hold with $w\lambda$ in place of λ, and by Lemma 81(b) so does (c) with $\mu = \Sigma\lambda(h(x))$. Now to prove that $KGv = KYv$ it is enough, because of the decomposition $G = YB \cup wB$, to prove that $wv \in KYv$. By the two parts of Lemma 81 we may write $\bar{X}wv = \mu v$, after some simplification, in the form

$$(*) \qquad wv + \sum_{x \in X-1} \lambda(wh^{-1}w)y(x)v = \mu v ,$$

with $y(x) = wxw^{-1} \in Y$, whence our assertion.

(b) Let $V' = Kv$ and $V'' = \text{Rad } KY \cdot v$. It follows from Lemma 80(c) that the sum $V = V' + V''$ is direct. Now assume there exists some $v_1 \in V$, $v_1 \notin V'$, fixed by X. We may assume that $v_1 \in V''$ and also that v_1 is an eigenvector for H since H is Abelian. We have $\bar{X}wv_1 = w\bar{Y}v_1 = 0$; since $\bar{Y}(1-y) = 0$ for any $y \in Y$. Thus v_1 is a highest weight vector. By (a), $V = KYv_1 \subseteq KYV'' = V''$, a contradiction, whence (b).

(c) By (b) an irreducible KG-module determines its highest weight (λ,μ) uniquely. Conversely, assume that V_1 and V_2 are irreducible KG-modules with highest weight vectors v_1 and v_2 of the same weight (λ,μ). Set $v = v_1 + v_2 \in V_1 \dot{+} V_2$ and then $V = KGv = KYv$. Now $v_2 \notin V$, since otherwise we could write $v_2 = cv + v''$ with $c \in K$ and $v'' \in \text{Rad } KY.v$ and then projecting on V_1 and V_2 get that $c = 0$ and $c = 1$, a contradiction. Thus we may complete the proof as in the proof of Theorem 39(e).

Theorem 45: Let (λ,μ) be the highest weight of an irreducible KG-module V.

(a) If $\lambda \neq 1$, then $\mu = 0$. If $\lambda = 1$, then $\mu = 0$ or -1.

(b) Every weight as in (a) can be realized. Thus the number of possibilities is $|H| + 1$.

Proof: (1) Proof of (b). In KG let $H_\lambda = \sum_{h\varepsilon H} \lambda(h^{-1})h$, then $u = \bar{X}H_\lambda w\bar{X}$ and $v = \bar{X}H_\lambda$. As in the proof of Theorem 44(a), a simple calculation yields $\bar{X}w(u+cv) = \sum_{x\varepsilon X-1} \lambda(h(x))u + c\bar{X}wH_\lambda\bar{X}$. Here H acts, from the left, according to the characters $w\lambda, \lambda, w\lambda$ on the respective terms. Thus if $w\lambda \neq \lambda$, we may realize the weight $(\lambda,0)$ by taking $c = 0$. If $w\lambda = \lambda$, we take $c = -\Sigma\lambda(h(x))$ instead. Finally if $\lambda = 1$, then $\Sigma\lambda(h(x)) = -1$ and we get $\mu = -1$ by taking $c = 0$. To achieve (λ,μ) in an irreducible module we simply take $KG(u+cv)$ modulo a maximal submodule.

(2) If $\dim V = 1$, then V is trivial and $(\lambda,\mu) = (1,0)$. Since X and Y are p-groups, they act trivially by Lemma 80(b), whence (2).

(3) If $\dim V \neq 1$, then λ determines μ. Write $V = V' + V''$ as in the proof of Theorem 44(b). We have $V'' \neq 0$. Since Y fixes V'' it fixes some line in it, uniquely determined in V, by Theorem 44(b) with Y in place of X. Since Y clearly fixes wv, we conclude that $wv \varepsilon V''$. Projecting (*) of the proof

of Theorem 44(a) onto V', we get

(**) $\Sigma\, \lambda(wh^{-1}w) =$ [illegible],

whence (3).

(4) Proof of (a). Combine (1), (2) and (3).

<u>Corollaries to Theorems 44 and 45</u>:

(a) If $\lambda \neq 1$, then $\sum_{x\varepsilon X-1} \lambda(h(x)) = 0$. The number of solutions $n(h)$ of $h(x) = h$ with h given is, modulo p, independent of h, in particular for each h is at least 1 (cf. Lemma 64, Step (1)).

(b) The irreducible representation of weight (λ, μ) can be realized in the left ideal generated by $\bar{X}H_\lambda w\bar{X} + c\bar{X}H_\lambda$ with

$c = 1$ for the trivial representation (1,0)

$= 0$ otherwise.

(c) If $L \subset K$ is a splitting field for H, it is one for G.

(d) $\text{Dim } V = |X|$ if $(\lambda, \mu) = (1,-1)$

$< |X|$ if not.

(e) The number of p-regular conjugacy classes of G is $|H| + 1$.

<u>Proof</u>: (a) If $\lambda \neq 1$, then $\mu = 0$ by Theorem 45(a), so that $\Sigma\lambda(h(x)) = 0$ by (**) above applied with λ replaced by $w\lambda$. Then $\Sigma n(h)\lambda(h) = 0$ for every $\lambda \neq 1$. By the orthogonality relations for the characters on H (which are valid since $p \nmid |h|$), we conclude that $n(h)$, as an element of K, is independent of h. If $n(h)$ were 0 for some h, we would get $|H| = \Sigma\, n(h) = 0 \pmod p$, a contradiction.

(b) Let V be an irreducible module whose dual V^* has the highest weight (λ,μ). We consider, as in Theorem 40 the isomorphism φ of V^* into the induced representation space of functions $a : G \longrightarrow K$ such that $a(yb) = a(y)\lambda^*(b)$ for all $y \in G$, $b \in B$ defined by $(\varphi f)(x) = f(xv^+)$ with v^+ a highest weight vector for V. Using the decomposition $V = Kwv^+ + \text{Rad } KX\cdot wv^+$, we may define f^+ as in Lemma 73, prove that it is a highest weight vector, and that $a^+ = \varphi f^+$ is given by the equations of Lemma 74, with λ^* in place of λ. Coverting functions on G to elements of KG in the usual way, $a \sim \Sigma\, a(x)x$, we see that a^+ becomes the element of (b), whence (b). At the same time we see that V may be realized in the induced module $B_{\lambda^*} \longrightarrow G$, as the unique irreducible submodule in case $\lambda \neq 1$, as one of two in case $\lambda = 1$.

(c) By (b).

(d) If $\mu = 0$, then $\overline{X}wv = 0$, whence $\overline{Y}v = 0$ and $\dim V < |X|$ by Theorem 44(a). Conversely if $\dim V < |X|$, then the annihilator of v in KY contains $\overline{Y}$ by Lemma 80(d), so that $\mu = 0$.

(e) By a classical theorem of Brauer and Nesbitt (University of Toronto Studies, 1937) the number in question equals the number of irreducible KG-modules, hence equals $|H| + 1$ by Theorem 45(b).

<u>Example</u>: $G = SL_2(q)$. Here $|H| = q - 1$, so that $|H| + 1 = q$.

<u>Remarks</u>. (a) We see that the extra condition $\overline{X}v = \mu v$ serves two purposes. First it distinguishes the smallest module $\sim (1,0)$

from the largest $(1,-1)$. Secondly, in the proof of the key relation $KGv = KU^-v$ it takes the place of the density argument (U^-B dense in G) used in the infinite case.

(b) The preceding development applies to a wide class of doubly transitive permutation groups (with B the stabilizer of a point, H of two points), since it depends only on the facts that H is Abelian and has in B a normal complement U which is a p-Sylow subgroup of G.

Now we consider groups of arbitrary rank. W ($= W_\sigma$) will be given the structure of reflection group as in Theorem 32 with Σ/R (see p. 177), projected into V_σ and scaled down to a set of unit vectors, the corresponding root system. For each simple root a, we write Y_a for X_{-a} and choose $\overline{w}_a$ in $\langle X_a, Y_a \rangle$ to represent w_a in W. If $w \in W$ is arbitrary, we choose a minimal expression $w = w_a w_b \cdots$ as a product of simple reflections, and set $\overline{w} = \overline{w}_a \overline{w}_b \cdots$. Then $\overline{w}$ is independent of the minimal expression chosen. We postpone the proof of this fact, which could (and probably should) have been given much earlier, to the end of the section so as not to interrupt the present development. As a consequence we have:

<u>Lemma 82</u>: If $w = w_a w_b \cdots$ is any minimal expression, then $\overline{X}_w = \overline{X}_a \overline{w}_a \cdot \overline{X}_b \overline{w}_b \cdots$.

<u>Proof</u>: Since $\overline{w} = \overline{w}_a \overline{w}_b \cdots$ this easily follows by induction on $N(w)$ or by Appendix II.25.

We extend the earlier definition of highest weight vector by the new requirement:

(c) $\overline{X}_a \overline{w}_a v = \mu_a v$ ($\mu_a \in K$) for every simple root a.

Theorem 46: Let G be a (perhaps twisted) finite Chevalley group (of arbitrary rank).

(a), (b), (c) Same as (a), (b), (c) of Theorem 44.

(d) Let $H_a = H \cap \langle X_a, Y_a \rangle$. If (λ, μ_a) is the highest weight of some irreducible module then $\mu_a = 0$ if $\lambda | H_a \neq 1$, and $\mu_a = 0$ or -1 if $\lambda | H_a = 1$.

(e) Every weight as in (a) can be realized on some irreducible KG-module.

Proof: We shall prove this theorem in several steps.

(a1) There exists in V a nonzero eigenvector v for B. This is proved as in Theorem 44(a).

(a2) If v is as in (a1), then so is $v_1 = \overline{X}_a \overline{w}_a v$ (a simple), unless it is 0.

Proof: Let x be any element of U. Write $x = x_a' x_a$ with $x_a \in X_a$ and $x_a' \in X_a'$, the subgroup of elements of U whose X_a components are 1. We recall that X_a and $\overline{w}_a$ normalize X_a' (see Appendix I.11). Thus $x v_1 = x_a' \overline{X}_a \overline{w}_a v = v_1$, since $Uv = v$. Since H normalizes X_a and is normalized by $\overline{w}_a$, we see that v_1 is also an eigenvector for H.

(a3) Choose v as in (a1), then $w \in W$ so that $N(w)$ is maximal subject to $v_1 = \overline{X}_w \overline{w} v \neq 0$. Then v_1 is a highest weight vector.

<u>Proof</u>: By Lemma 82 and (a2), v_1 is an eigenvector for B. Let a be any simple root. If $w^{-1}a > 0$, then $N(w_a w) = N(w) + 1$ by Appendix II.19, so that $\bar{X}_{w_a w}\overline{w_a w} = \bar{X}_a \bar{w}_a \bar{X}_w \bar{w}$ by Lemma 82, and $\bar{X}_a \bar{w}_a v_1 = 0$ by the choice of w. If $w^{-1}a < 0$, then we may choose a minimal expression $w = w_a w_b \ldots$ starting with w_a. Then

$$\bar{X}_a \bar{w}_a v_1 = (\bar{X}_a \bar{w}_a)^2 \bar{X}_b \bar{w}_b \ldots v \quad \text{by Lemma 82}$$
$$= \mu \bar{X}_a \bar{w}_a \bar{X}_b \bar{w}_b \ldots v, \quad \text{with } \mu \in K \text{ by Lemma 81(b)}$$
$$= \mu v_1 .$$

By (a1) and (a3) we have the first statement in (a).

(a4) $KGv = KU^- v$.

<u>Proof</u>: We have $G = \bar{w}_0 G \subseteq \bigcup_w \bar{w} U^- B$ by Theorem 4. Thus it is enough to show each $\bar{w}$ fixes $KU^- v$, and for this we may assume $w = w_a$ with a simple. Assume $y \in U^-$. Write $y = y_a y_a'$ as above, but using negative roots instead. Then y_a and $\bar{w}_a$ normalize Y_a', so that $\bar{w}_a y v = \bar{w}_a y_a y_a' v \in U^- \bar{w}_a y_a v \subseteq KU^- v$ by Theorem 44(a) applied to $\langle X_a, Y_a \rangle$.

(b1) If $V = V' + V''$ with $V' = Kv$ and $V'' = \text{Rad } KU^- v$ as before, then V'' is fixed by every $\bar{X}_a \bar{w}_a$.

<u>Proof</u>: Write $y = y_a y_a'$ as before.

Then $\bar{X}_a \bar{w}_a y v = \sum_{x \in X_a} y(x) x \bar{w}_a v$ with $y(x) \in U^-$.

Thus $\bar{X}_a \bar{w}_a (y-1) v = \sum_{x \in X_a} (y(x)-1) x \bar{w}_a v \in V''$.

(b2) Proof of (b). If this is false, there exists

$v_1 \in V''$, $v_1 \neq 0$, $\bar{v}_1$ fixed by X. As usual we may choose v_1 as an eigenvector for H, and then by (b1) and (a3) also an eigenvector for each $\bar{X}_a \bar{w}_a$. Then, as before, $V = KGv_1 = KU^- v_1 \subseteq V''$, a contradiction.

(c) Same proof as for Theorem 44(c).

(d) By Theorem 45(a) applied to $\langle X_a, Y_a \rangle$.

(e) Let π be the set of simple roots a such that $\mu_a = 0$, and W_π the corresponding subgroup of W. The reader should have no trouble in proving that the left ideal of KG generated by $\sum_{w \in W_\pi w_0} \bar{U} H_\lambda \bar{w} \bar{X}_w$ is an irreducible KG-module whose highest weight is (λ, μ_a).

<u>Corollary</u>: (a) A splitting field for H is also one for G.

(b) Let V be irreducible, of highest weight (λ, μ_a).

Then $\dim V = |U|$ if $\lambda = 1$ and all $\mu_a = -1$.

$< |U|$ if not.

(c) For each set π of simple roots, let H_π be the group generated by all H_a ($a \in \pi$). Then the number of irreducible KG-modules, or, equivalently, of p-regular conjugacy classes of G is $\sum_\pi |H/H_\pi|$.

<u>Proof</u>: (a) Clear.

(b) Write $\bar{U}\bar{w}_0 v = \bar{X}_{w_0} \bar{w}_0 v = \bar{X}_a \bar{w}_a \bar{X}_b \bar{w}_b \cdots v$, with $w_0 = w_a w_b \cdots$ as in Lemma 82, and then proceed as in the proof of Cor. (d) to Theorems 44 and 45.

(c) The given sum counts the number of possible weights (λ, μ_a) according to the set π of simple roots a such that

$\mu_a = -1$.

Exercise: If G is universal, the above number is

$\prod_{a \text{ simple}} (|H_a| + 1) = \prod_{\alpha \text{ simple}} q(\alpha)$, in the notation of Theorem 43. If in addition G is not twisted, then the number is q^ℓ .

It remains to prove the following result used (inessentially) in the proof of Lemma 82.

Lemma 83: (a) If $w \in W$, then any two minimal expressions for w as a product of simple reflections can be transformed into each other by the relations

(*) $w_a w_b w_a \cdots = w_b w_a w_b \cdots$ (n terms on each side, n = order $w_a w_b$, with a and b distinct simple roots).

(b) Assume that for each simple root a the corresponding element $\overline{w}_a$ of G (any Chevalley group) is chosen to lie in $\langle X_a, X_{-a} \rangle$. Let $w = w_a w_b \cdots$ be a minimal expression for $w \in W$. Then $\overline{w} = \overline{w}_a \overline{w}_b \cdots$ is independent of the minimal expression chosen.

Proof: (a) This is a refinement of Appendix IV.38 since the relations $w_\alpha^2 = 1$ are not required. It is an easy exercise to convert the proof of the latter result into a proof of the former, which we shall leave to the reader.

(b) Because of (a) we only have to prove (b) when w has the form of the two sides of (*). For this we can refer to the proof of Lemma 56 since the extra restrictions there, that G is untwisted and that $\overline{w}_a = w_a(1)$ for each a, are not essential for the proof.

Remark: It would be nice if someone could incorporate in the elementary development just given the tensor product theorem mentioned after Theorem 43 or at least a proof that every irreducible K-module for G can be extended to the including algebraic group, hence also to the other finite Chevalley groups contained in the latter group.

§14. Representations concluded. Now we turn to the complex representations of the groups just considered. Here the theory is in poor shape. Only GL_n (Green, T.A.M.S. 1955) and a few groups of low rank have been worked out completely, then only in terms of the characters. Here we shall consider a few general results which may lead to a general theory.

Henceforth K will denote the complex field. Given a (one-dimensional) character λ on a subgroup B of a group G, realized on a space V_λ, we shall write V_λ^G for the induced module for G. This may be defined by $V_\lambda^G = KG \otimes_{KB} V_\lambda$ (this differs from our earlier version in that we have not switched to a space of functions), and may be realized in KG in the left ideal generated by $B_\lambda = \sum_{b \in B} \lambda(b^{-1})b$ (and will be used in this form). Its dimension is $|G/B|$.

Exercise: Check these assertions.

Lemma 84: Let B, C be subgroups of a finite group G, let λ, μ be characters on B, C, and let V_λ^G, V_μ^G be the corresponding modules for G.

(a) If $x \in G$, then $B_\lambda x C_\mu$ in KG is determined up to multiplication by a nonzero scalar by the (B,C) double coset to which x belongs.

(b) $\mathrm{Hom}_G(V_\lambda^G, V_\mu^G)$ is isomorphic as a K-space to the one generated by all $B_\lambda x C_\mu$.

(c) If $B = C$ and $\lambda = \mu$, then the isomorphism in (b) is one of algebras.