essential for the proof.

Remark: It would be nice if someone could incorporate in the elementary development just given the tensor product theorem mentioned after Theorem 43 or at least a proof that every irreducible K-module for G can be extended to the including algebraic group, hence also to the other finite Chevalley groups contained in the latter group.

§14. Representations concluded. Now we turn to the complex representations of the groups just considered. Here the theory is in poor shape. Only GL_n (Green, T.A.M.S. 1955) and a few groups of low rank have been worked out completely, then only in terms of the characters. Here we shall consider a few general results which may lead to a general theory.

Henceforth K will denote the complex field. Given a (one-dimensional) character λ on a subgroup B of a group G, realized on a space V_λ, we shall write V_λ^G for the induced module for G. This may be defined by $V_\lambda^G = KG \otimes_{KB} V_\lambda$ (this differs from our earlier version in that we have not switched to a space of functions), and may be realized in KG in the left ideal generated by $B_\lambda = \sum_{b \varepsilon B} \lambda(b^{-1})b$ (and will be used in this form). Its dimension is $|G/B|$.

Exercise: Check these assertions.

Lemma 84: Let B, C be subgroups of a finite group G, let λ, μ be characters on B, C, and let V_λ^G, V_μ^G be the corresponding modules for G.

(a) If $x \varepsilon G$, then $B_\lambda x C_\mu$ in KG is determined up to multiplication by a nonzero scalar by the (B,C) double coset to which x belongs.

(b) $\mathrm{Hom}_G(V_\lambda^G, V_\mu^G)$ is isomorphic as a K-space to the one generated by all $B_\lambda x C_\mu$.

(c) If $B = C$ and $\lambda = \mu$, then the isomorphism in (b) is one of algebras.

(d) The dimension of $\mathrm{Hom}_G(V_\lambda^G, V_\mu^G)$ is the number of (B,C) double cosets D such that $B_\lambda\, x\, C_\mu \neq 0$ for some, hence for every x in D, or, equivalently, such that the restrictions of λ and $x\mu$ to $B \cap xCx^{-1}$ are equal.

<u>Proof</u>: (a) This is clear.

(b) Assume $T \in \mathrm{Hom}_G(V_\lambda^G, V_\mu^G)$. Since B_λ generates V_λ^G as a KG-module, TB_λ determines T. Let $TB_\lambda = \sum_{x \in G/C} c_x\, x\, C_\mu$. Since $bB_\lambda = \lambda(b)B_\lambda$, we get by averaging over B that $TB_\lambda = \sum_{x \in B\backslash G/C} c_x B_\lambda\, x\, C_\mu$. Thus T is realized on B_λ, hence on all of V_λ^G, by right multiplication by $|B|^{-1} \Sigma\, c_x B_\lambda\, x\, C_\mu$. Conversely, any such right multiplication yields a homomorphism, which proves (b).

(c) By discussion in (b).

(d) The first statement follows from (a) and (b). Let $B_1 = B \cap xCx^{-1}$ and $C_1 = x^{-1}Bx \cap C$, and $\{y_i\}$ and $\{z_j\}$ systems of representatives for $B_1\backslash B$ and C/C_1. Set $B'_\lambda = \Sigma\, \lambda(y_i^{-1})y_i$ and $C'_\mu = \Sigma\, \mu(z_j^{-1})z_j$. Then $B_\lambda\, x\, C_\mu = B'_\lambda B_{1\lambda} B_{1,x\mu}\, x C'_\mu$ with $(x\mu)(xcx^{-1}) = \mu(c)$ since $xC_1x^{-1} = B_1$. If $\lambda \neq x\mu$ on B_1, then $B_{1\lambda}B_{1,x\mu} = 0$. If $\lambda = x\mu$, this product is $|B_1|B_{1\lambda}$, and then $B_\lambda\, x\, C_\mu \neq 0$ since the elements $y_i b_1\, x\, z_j$ are all distinct.

<u>Remarks</u>: (a) This is a special case of a theorem of G. Mackey. (See, e.g., Feit's notes.)

(b) The algebra of (c) is also called the commuting algebra since it consists of all endomorphisms of V_λ^G that commute with the action of G.

<u>Theorem 47</u>: Let G be a (perhaps twisted) finite Chevalley group.

(a) If λ is a character on H extended to B in the usual way, then V_λ^G is irreducible if and only if $w\lambda \neq \lambda$ for every $w \in W$ such that $w \neq 1$.

(b) If λ, μ both satisfy the conditions of (a), then V_λ^G is isomorphic to V_μ^G if and only if $\lambda = w\mu$ for some $w \in W$.

<u>Proof</u>: (a) V_λ^G is irreducible if and only if its commuting algebra is one-dimensional (Schur's Lemma), i.e., by (c) and (d) of Lemma 84, if and only if λ and $w\lambda$ agree on $B \cap wBw^{-1}$, hence on H, for exactly one $w \in W$, i.e. for only $w = 1$.

(b) Since V_λ^G and V_μ^G are irreducible, they are isomorphic if and only if $\dim \mathrm{Hom}_G(V_\lambda^G, V_\mu^G) = 1$, which, as above, holds exactly when $\lambda = w\mu$ for some (hence for exactly one) $w \in W$.

<u>Exercise</u>: (a) $\dim \mathrm{Hom}_G(V_\lambda^G, V_\mu^G) = |W_\lambda|$ if $\lambda = w\mu$ for some w,
$= 0$ otherwise.

(b) In Theorem 47 the conclusion in (b) holds even if the condition in (a) doesn't.

Here W_λ is the stabilizer of λ in W. We see, in particular, that if $\lambda = 1$ then the commuting algebra of V_λ^G is $|W|$-dimensional. But more is true.

<u>Theorem 48</u>: Let V be the KG-module induced by the trivial one-dimensional KB-module. Then the commuting algebra $\mathrm{End}_G(V)$

is isomorphic to the group algebra KW.

Reformulations:

(a) The multiplicities of the irreducible components of V are just the degrees of the irreducible KW-modules.

(b) The subalgebra, call it A, of KG spanned by the double coset sums $\bar{B}w\bar{X}_w$ is isomorphic to KW.

(c) The algebra of functions $f: G \to K$ biinvariant under B ($f(bxb') = f(x)$ for all $b, b' \in B$) with convolution as multiplication is isomorphic to KW.

<u>Proof</u>: The theorem is equivalent to (a) by Schur's Lemma and to (b) by Lemma 84, while (b) and (c) are clearly equivalent. We shall give a proof of (b), due to J. Tits. $\hat{w}$ will denote the average in KG of the elements of the double coset BwB. The elements $\hat{w}$ form a basis of A and $\hat{1}$ is the unit element. If a is a simple root, c_a will denote $|X_a|^{-1}$.

(1) A is generated as an algebra by $\{\hat{w}_a \mid a \text{ simple}\}$ subject to the relations

(α) $\hat{w}_a^2 = c_a\hat{1} + (1-c_a)\hat{w}_a$ for all a.

(β) $\hat{w}_a\hat{w}_b\hat{w}_a \cdots = \hat{w}_b\hat{w}_a\hat{w}_b \cdots$, as in Lemma 83(a).

<u>Proof</u>: We observe that if each c_a is replaced by 1 then these relations go over into a defining set for KW, by Appendix II.38. Since $B \cup Bw_aX_a$ is a group, we have $(\bar{B}w_a\bar{X}_a)^2 = r\bar{B} + s\bar{B}w_a\bar{X}_a$ with $r, s \in K$. Since Bw_aX contains with each of its elements its inverse, we get $r = |Bw_aX_a|$, and then from the total coefficient $s = |Bw_aX_a| - |B|$. Thus (α) holds in A, and so does (β) by

Lemma 25, Cor., which also shows that the $\hat{w}_a$ generate A. Conversely, let A_1 be the abstract associative algebra (with $\hat{1}$) generated by symbols $\hat{w}_a$ subject to (α) and (β). For each $w \in W$ choose a minimal expression $w = w_a w_b \cdots$ and set $\hat{w} = \hat{w}_a \hat{w}_b \cdots$. By Lemma 83(a) and ($\beta$) this is independent of the expression chosen. By (α) it follows that

$$\hat{w}_a \hat{w} = \widehat{w_a w} \qquad \text{if} \quad w^{-1}a > 0,$$

$$= c_a \widehat{w_a w} + (1 - c_a)\hat{w} \quad \text{if} \quad w^{-1}a < 0.$$

Thus the $\hat{w}$ form a basis for A_1, which, having the same dimension as A, is therefore isomorphic to it.

(2) There exists a positive number $c = c(G)$ such that $c_a = c^{n_a}$ with n_a a positive integer depending only on the type of G. The multiplication table of A in terms of the basis $\{\hat{w}\}$ is given by polynomials in c depending only on the type.

<u>Proof</u>: Consider ${}^2A_4(q^2)$, for example. Here the two possibilities for $|X_a|$ are q^2 and q^3 by Lemma 63(c). If we set $c = q^{-1}$, then the corresponding values of n_a are 2 and 3, which depend only on the type. For each of the other types the verification is similar. From the first statement of (2) and the equations of the proof of (1) the second statement follows.

(3) An associative algebra A_c with multiplication table given by the polynomials of (2) exists for every complex number c. In particular $A_{c(G)} = A$ and $A_1 = KW$.

<u>Proof</u>: Since the type of the group G contains an infinite number

of members, the multiplication table is associative for an infinite set of values of c, hence for all values.

(4) A_c is semisimple for $c = c(G)$, for $c = 1$, and for all but a finite number of values of c.

<u>Proof</u>: A_c is for $c = c(G)$ the commuting algebra of a KG-module and for $c = 1$ a group algebra KW, hence semisimple in both cases. The discriminant of A_c is a polynomial in c, nonzero at $c = 1$ since then A_c is semisimple, hence nonzero for all but a finite number of values of c.

(5) Completion of proof. Since A is semisimple and K is an algebraically closed field, A is a direct sum of complete matric algebras, of certain degrees over K (see, e.g., Jacobson's Structure of Rings or Feit's notes), and similarly for KW. We have to show that the degrees are the same in the two cases. If $\mathfrak{a}$ is any finite-dimensional associative algebra which is separable, i.e. which is semisimple when the base field is extended to its algebraic closure, we define the numerical invariants of $\mathfrak{a}$ to be the degrees of the resulting matric algebras. The proof of Theorem 48 will be completed by the following lemma.

<u>Lemma 85</u>: Let R be an integral domain, F its field of quotients, and f a homomorphism of R onto a field K. Let $\mathfrak{a}$ be a finite-dimensional associative algebra over R, and $\mathfrak{a}_F$ and $\mathfrak{a}_K$ the resulting algebras over F and K. If $\mathfrak{a}_F$ and $\mathfrak{a}_K$ are separable, then they have the same numerical

invariants.

In fact, from (3) and the lemma with $R = K[c]$, $\mathcal{A} = A_c$, and first $f : c \longrightarrow c(G)$ and then $f : c \longrightarrow 1$, it follows that A and KW have the same numerical invariants, hence that they are isomorphic.

<u>Proof of the lemma</u>:

(a) Assume that $\mathcal{B}$ is a finite-dimensional semisimple associative algebra over an algebraically closed field L, that $b_1, b_2, \ldots, b_n$ form a basis for $\mathcal{B}/L$, that $x_1, x_2, \ldots, x_n$ are independent indeterminates over L, that $b = \Sigma\, x_i b_i$, and that $P(t)$ is the characteristic polynomial of b acting from the left on $\mathcal{B}_{L(x_1,\ldots,x_n)}$, written as $P(t) = \prod P_i(t)^{p_i}$ with the P_i distinct monic polynomials irreducible over $L(x_1,\ldots,x_n)$. Then:

(a1) The p_i are the numerical invariants of $\mathcal{B}$.

(a2) $p_i = dg_t P_i$ for each i.

(a3) If $P(t) = \prod Q_j(t)^{q_j}$ is any factorization over $L(x_1,\ldots,x_n)$ such that $q_j = dg_t Q_j$ for each j, then it agrees with the one above so that the q_j are the numerical invariants of $\mathcal{B}$.

<u>Proof</u>: For (a1) and (a2) we may assume that $\mathcal{B}$ is the complete matric algebra $\mathrm{End}\, L^p$ and that $b = \Sigma\, x_{ij} E_{ij}$ in terms of the matric units E_{ij}. If $X = [x_{ij}]$, then $P(t) = \det(tI - X)^p$, so that we have to show that $\det(tI - X)$ is irreducible over $L(x_{ij})$. This is so since specialization to

the set of companion matrices
$$\begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ x_{p1} & x_{p2} & \cdots & x_{pp} \end{bmatrix}$$
yields the general equation of degree p. In (a3) if some Q_j were reducible or equal to some Q_k with $k \neq j$, then any irreducible factor P_i of Q_j would violate (a2).

(b) Let R^* be the integral closure of R in $\overline{F}$ (consisting of all elements satisfying monic polynomial equations over R), and $x_1,x_2,\ldots,x_n$ indeterminates over $\overline{F}$. Then $R^*[x_1,\ldots,x_n]$ is the integral closure of $R[x_1,\ldots,x_n]$ in $\overline{F}(x_1,\ldots,x_n)$.

__Proof__: See, e.g., Bourbaki, Commutative Algebra, Chapter V, Prop. 13.

(c) If R^* is as in (b), then any homomorphism of R into K can be extended to one of R^* into $\overline{K}$.

__Proof__: By Zorn's lemma this can be reduced to the case $R^* = R[\alpha]$, where it is almost immediate since $\overline{K}$ is algebraically closed.

(d) Completion of proof. Let $\{a_i\}$ be a basis for $\mathcal{a}/R$, hence also for $\mathcal{a}_F/F$, and $\{x_i\}$ independent indeterminates over $\overline{F}$ and also over $\overline{K}$. The given homomorphism $f : R \longrightarrow K$ defines a homomorphism $f : \mathcal{a} \longrightarrow \mathcal{a}_K$. By (c) it extends to a homomorphism of R^* into $\overline{K}$ and then naturally to one of $R^*[x_1,\ldots,x_n]$ into $\overline{K}[x_1,\ldots,x_n]$. If $a = \Sigma\, x_i a_i$ and $P(t) = \prod P_i(t)^{p_i}$ is its characteristic polynomial, factored over $\overline{F}(x_1,\ldots,x_n)$ as before, then the coefficients of each P_i are integral polynomials in its roots, hence integral over the coefficients of P, hence integral over $R[x_1,x_2,\dots,x_n]$, hence belong to $R^*[x_1,\dots,x_n]$ by (b). Thus if $f(a)=\Sigma x_i f(a_i)$, then its character polynomial has a corresponding factorization $P_f(t) = \prod P_{if}(t)^{p_i}$ over $\overline{K}(x_1,\dots,x_n)$. By (a1) the p_i are the numerical invariants of $\mathcal{A}_F$, and by (a2) they satisfy $p_i = dg_t P_i = dg_t P_{if}$, so that by (a3) with $\mathcal{B} = \mathcal{A}_{\overline{K}}$ they are also the numerical invariants of $\mathcal{A}_K$, which proves the lemma.

Exercise: If λ is a character on H extended to B in the usual way, then $\mathrm{End}_G(V_\lambda^G)$ is isomorphic to KW_λ. (Observe that this result includes both Theorem 47 and Theorem 48.)

Remark: Although A is isomorphic to KW there does not seem to be any natural isomorphism and no one has succeeded in decomposing the module V of Theorem 48 into its irreducible components, except for some groups of low rank. We may obtain some partial results, in terms of characters, by inducing from the parabolic subgroups and using the following simple facts.

Lemma 86: Let π be a set of simple roots, W_π and G_π the corresponding subgroups of W and G (see Lemma 30), and V_π^W and V_π^G the corresponding trivial modules induced to W and G; and similarly for π'.

(a) A system of representatives in W for the $(W_\pi, W_{\pi'})$ double cosets becomes in G a system of representatives for $(G_\pi, G_{\pi'})$ double cosets.

(b) $\dim \mathrm{Hom}_G(V_\pi^G, V_{\pi'}^G) = \dim \mathrm{Hom}_W(V_\pi^W, V_{\pi'}^W)$.

Proof: (a) Exercise.

(b) By Lemma 84(d) and (a).

Corollary 1: Let χ_π^G denote the character of V_π^G and similarly for W. If $\{n_\pi\}$ is a set of integers such that $\chi^W = \Sigma n_\pi \chi_\pi^W$ is an irreducible character of W, then $\chi^G = \Sigma n_\pi \chi_\pi^G$ is, up to sign, one of G.

Proof: Let $(\chi, \psi)_G$ denote the average of $\chi\overline{\psi}$ over G. We have $(\chi_\pi^G, \chi_{\pi'}^G)_G = \dim \mathrm{Hom}_G(V_\pi^G, V_{\pi'}^G)$, and similarly for W. Since χ^W is irreducible, $(\chi^W, \chi^W)_W = 1$, it follows that $(\chi^G, \chi^G)_G = 1$, so that $\pm\chi^G$ is irreducible, by the orthogonality relations for finite group characters.

Remarks: (a) In a beautiful paper in Berliner Sitzungsberichte, 1900, Frobenius has constructed a complete set of irreducible characters for the symmetric group S_n, i.e. the Weyl group of type A_{n-1}, as a set of integral combinations of the characters χ_π^W. Using his method and the preceding corollary one can decompose the character of V in Theorem 48 in case G is of type A_{n-1}. (See R. Steinberg T.A.M S. 1951).

(b) The situation of (a) does not hold in general. Consider, for example, the group W of type B_2, i.e. the dihedral group of order 8. It has five irreducible modules (of dimensions 1,1,1,1,2), while there are only four χ_π^W's to work with.

(c) A result of a general nature is as follows.

Corollary 2: If the notation is as above and $(-1)^\pi$ is as in Lemma 66(d), then $\chi^G = \Sigma(-1)^\pi \chi_\pi^G$ is an irreducible character of G and its degree in $|U|$.

<u>Proof</u>: Consider $\chi^W = \Sigma(-1)^\pi \chi_\pi^W$. By (8) on p. 142, extended to twisted groups (check this using the hints given in the proof of Lemma 66), $\chi^W = \det$, an irreducible character. Hence $\pm\chi^G$ is also one by Cor. 1 above. We have $\chi^G(1) = \Sigma(-1)^\pi |G/G_\pi|$. If G is untwisted and the base field has q elements, then by Theorem 4' applied to G and to G_π this can be continued $\Sigma(-1)^\pi W(q)/W_\pi(q) = q^N = |U|$, as in (4) of the proof of Theorem 26. If G is twisted, the proof is similar.

We continue with some remarks on the algebra A of Theorem 48(b).

<u>Lemma 87</u>: The homomorphisms of A onto K are given by: $f(\hat{w}_a) = 1$ or $-c_a$ for each simple root a, subject to the condition that $f(\hat{w}_a)$ is constant on each W-orbit.

<u>Proof</u>: For a and b simple, let $n(a,b)$ denote the order of $w_a w_b$ in W. We claim that (*) a and b belong to the same W-orbit if and only if there exists a sequence of simple roots $a = a_0, a_1, \ldots, a_r = b$ such that $n(a_i, a_{i+1})$ is odd for every i. The equation $(w_{a_i} w_{a_{i+1}})^n = 1$ with n odd can be rewritten to show that w_{a_i} and $w_{a_{i+1}}$ are conjugate, so that if the sequence exists then a and b are conjugate. If a and b are conjugate, then so are w_a and w_b, and this remains true when we project into the reflection group obtained by imposing on W the additional relations: $(w_c w_d)^2 = 1$ whenever $n(c,d)$ is even. In this new group w_a and w_b must belong to the same component, so that the

required sequence exists. By $(*)$ the condition of the Lemma holds exactly when $f(\hat{w}_a) = f(\hat{w}_b)$ whenever $n(a,b)$ is odd, i.e. exactly when f preserves the relations (β) (of the proof of Theorem 48). Since $f(\hat{w}_a) = 1$ or $-c_a$ exactly when f preserves (α) (solve the quadratic), we have the lemma.

Remark: By finding the annihilator in A of the kernel of each of the homomorphisms of Lemma 87, we get a one-dimensional ideal I in A. This corresponds to an irreducible submodule of multiplicity 1 in V, realized in the left ideal KGI of KG. By working out the corresponding idempotent, the degree of the submodule can be found.

Exercise: (a) If $f(\hat{w}_a) = 1$ for all a, show that $I = K\Sigma q_w \hat{w}$ with $q_w = |X_w|$, and that the corresponding KG-module is the trivial one. (Hint: by writing W as a union of right cosets relative to $\{1, w_a\}$ and writing (α) in the form $(\hat{w}_a - 1)(\hat{w}_a + c_a) = 0$, show that I is as indicated.)

(b) If $f(\hat{w}_a) = -c_a$ for all a, show that $I = K\Sigma (\det w)\hat{w}$, and that in this case the dimension is $|U|$. (Hint: if e is the given sum and $c_w = q_w^{-1}$, show that $e^2 = me$ with $m = \Sigma c_w = |G/B|/|U|$.)

(c) For $G = B_2(q)$ work out all (four) cases of Lemma 87, hence obtain the degrees of all (five) irreducible components of V.

(d) Same for A_2 and for G_2.

Remark: It can be shown that the module of (b) is isomorphic to the one with character χ^G as in Lemma 86, Cor. 2. If we reduce the element $\Sigma(\det w)|U/X_w|\overline{B}\, w\, \overline{X}_w$ (which is $|B||U|e$) of I mod p, we see from the proof of Theorem 46(e) that the given module reduces to the one of that theorem for which $\lambda = 1$ and every $\mu_a = -1$. This latter module can itself be shown to be isomorphic to the one in Theorem 43 for which $\langle\lambda, \alpha\rangle = q(\alpha) - 1$ for every α. From these facts and Weyl's formula the character of the original module can be found up to sign and the result used to prove that the number of p-elements (of order a power of p) of G is $|U|^2$. For the details see R. Steinberg, Endomorphisms of linear algebraic groups, to appear.

Finally, we should mention that the algebra A admits an involution given by $\hat{w}_a \longrightarrow 1 - c_a - \hat{w}_a$ for all a (which in case $c_a \longrightarrow 1$ and $A \longrightarrow KW$ reduces to $w \longrightarrow (\det w)w$).

The preceding discussion points up the following

<u>Problem</u>: Develop a representation theory for finite reflection groups and use it to decompose the module V (or the algebra A) of Theorem 48.

It is natural that in studying the complex representations of G we have considered first those induced by characters on B since for representations of characteristic p this leads to a complete set. In characteristic 0, however, this is not the case, as even the simplest case $G = SL_2$ shows. One must delve deeper. Therefore, we shall consider representations of G induced by (one-dimensional) characters λ on U. We can not expect such a representation ever to be irreducible since its degree $|G/U|$ is too large (larger than $|G|^{1/2}$), but what we shall show is that if λ is sufficiently general then at least it is multiplicity-free.

In other words, $\mathrm{End}_G(V_\lambda^G)$ is Abelian, hence a direct sum of fields. (If λ is not sufficiently general, we can expect the Weyl group to play a role, as in Theorem 48.)

Before stating the theorem, we prove two lemmas.

Lemma 88: Let k be a finite field and λ a nontrivial character from the additive group of k into K^*. Then every character can be written uniquely $\lambda_c : t \longrightarrow \lambda(ct)$ for some $c \in k$.

Proof: The map $c \longrightarrow \lambda_c$ is a homomorphism of k into its dual, and its kernel is clearly 0.

Lemma 89: For $w \in W$ the following conditions are equivalent.

(a) If a and wa are positive roots and one of them is simple, then so is the other.

(b) If a is simple and wa is positive, then wa is simple.

(c) $w = w_0 w_\pi$ for some set π of simple roots, with w_0 as usual and w_π the corresponding object of W_π.

There are 2^ℓ possibilities for w.

Proof: (c) $\Rightarrow$ (a) Because w_π maps π onto $-\pi$ and (*) permutes the positive roots with support not in π (same proof as for Appendix I.11).

(a) $\Longrightarrow$ (b) Obvious.

(b) $\Longrightarrow$ (c) Let π be the set of simple roots kept positive, hence simple, by w. We claim: (**) if $a > 0$ and $\mathrm{supp}\, a \not\subseteq \pi$, then $wa < 0$. Write $a = b + c$ with $\mathrm{supp}\, b \subseteq \pi$, $\mathrm{supp}\, c \subseteq \Pi - \pi$. Then $wa = wb + wc$. Here $wc < 0$ by the choice of π, and supp $wc \not\subseteq w\pi \supseteq$ supp wb. Thus $wa < 0$. If a is a simple root not in π then $ww_\pi a < 0$ by (*) and (**), while if a is in π this holds by the definition of π. Thus $ww_\pi = w_0$, whence (c).

<u>Theorem 49</u>: Let G be a finite, perhaps twisted, Chevalley group and $\lambda : U \longrightarrow K^*$ a character such that $\lambda|X_a \neq 1$ if a is simple, $= 1$ if a is positive but not simple. Then V_λ^G is multiplicity-free. In other words, $\mathrm{End}_G(V_\lambda^G)$ is Abelian, or, equivalently, the subalgebra A of KG spanned by the elements $U_\lambda h\bar{w}U_\lambda$ $(h \in H,\ w \in W)$ is Abelian.

Here $U_\lambda = \sum_{u \in U} \lambda(u^{-1})u$, and we assume that the $\bar{w}$ are chosen as in Lemma 83(b).

<u>Remarks</u>: (a) If a is not simple, then usually $X_a \subseteq \mathcal{D}U$, so that the assumption $\lambda|X_a = 1$ is superfluous, but this is not always the case, e.g. for B_2 or F_4 with $|k| = 2$ or for G_2 with $|k| = 3$. In these latter groups, there are other possibilities, which because of their special nature will not be gone into here.

(b) The proof to follow is suggested by that of Gelfand and Graev, Doklady, 1963, who have given a proof for SL_n and announced the general result for the untwisted groups. T. Yokonuma, C. Rendues, Paris, 1967, has also given a proof for these latter groups, but his details are unnecessarily complicated.

<u>Proof of Theorem 49</u>: The fact that A is Abelian will follow from the existence of an (involutory) antiautomorphism f of G such that

(a) $fU = U$.

(b) $\lambda f = \lambda$ on U.

(c) For each double coset UnU such that $U_\lambda n U_\lambda \neq 0$, we have $f_n = n$ (here $n \in N = \Sigma H\bar{w}$).

For since f extended to KG and then restricted to A is an antiautomorphism and at the same time the identity (by (a), (b), (c)) it is clear that A is Abelian. The existence of f will be proved in several steps.

(1) If $U_\lambda n U_\lambda \neq 0$ and $n \in H\bar{w}$, then $w = w_0 w_\pi$ for some set π of simple roots.

<u>Proof</u>: By Lemma 89 we need only prove that if a is simple and wa positive then wa is simple. Writing the first U_λ above with the X_{wa} component on the right, and the second with the X_a component on the left, we get $X_{wa,\lambda} n X_{a,\lambda} n^{-1} \neq 0$. Since λ is nontrivial on X_a it is also so on X_{wa}, whence wa is simple by the assumptions on λ, which proves (1).

The condition in (1) essentially forces the correct definition of f. We set $a^* = -w_0 a$. If a is simple, so is a^*. In order to simplify the discussion in one or two spots we assume henceforth that G (i.e. its root system) is indecomposable. If G is untwisted, we start with the graph automorphism corresponding to $*$ (see the Corollary on p. 156), compose it with the inversion $x \longrightarrow x^{-1}$, and finally with a diagonal automorphism so that the result f satisfies, not only $fU = U$ but also $\lambda f = \lambda$ on U. This is possible because of Lemma 89 and the assumptions on λ in the theorem. If G is twisted, then we may omit the graph automorphism (because $*$ is then the identity), and use the explicit isomorphism $X_a/\mathcal{D}X_a \cong k$ of (2) of the proof of Theorem 36 in combination with Lemma 89 to achieve the second condition. We see that

(2) f is an involutory antiautomorphism which satisfies the required conditions (a) and (b). We must prove that it also satisfies (c). As consequences of the construction we have:

(3) $fh = \bar{w}_o h \bar{w}_o^{-1}$ for every $h \in H$.

(4) If $a^* = a$, then f is the identity on $X_a/\mathcal{D}X_a$.

(5) If $a^* = a$, there exists a nontrivial element of X_a fixed by f.

Proof: For X_a of type A_1 this follows from (4). For X_a of type ${}^2A_2$ we choose the element (t,u) of Lemma 63(c) with $t = 2$, $u = 2$ if $p \neq 2$, and $t = 0$, $u = 1$ if $p = 2$, since $f(t,u) = (t, -tt^\theta u)$ (check this, referring to the construction of f). For types ${}^2C_2$ and ${}^2G_2$ we may choose $(0,1)$ and $(0,1,0)$ since f is the identity on $\{(0,u)\}$ and $\{(0,u,0)\}$.

(6) The elements $\bar{w}_a \in G$ may be so chosen that:

(6a) $f\bar{w}_a = \bar{w}_{a^*}$ for every simple root a.

(6b) If a and b are simple and $n \in N$ is such that $nX_an^{-1} = X_b$ and $\lambda(nxn^{-1}) = \lambda(x)$ for all $x \in X_a$, then $n\bar{w}_an^{-1} = \bar{w}_b$.

Proof: Under the action of f and the inner automorphisms i_n by elements n as in (6b) the X_a (a simple) form orbits. From each orbit we select an element X_a. If $a^* = a$, we choose $x_a \in X_a$ as in (5), write it as (*) $x_a = x_1\bar{w}_ax_2$ with $x_1, x_2 \in X_{-a}$, and choose $\bar{w}_a$ accordingly. Since f is an antiautomorphism and fixes X_{-a}, it also fixes $\bar{w}_a$ by the uniqueness of the above form. If $a^* \neq a$, we choose $x_a \in X_a, x_a \neq 1$, arbitrarily. We then use the equations $f\overline{w}_a = \overline{w}_{a^*}$ and $i_n\overline{w}_a = \overline{w}_b$ of (6a) and (6b) to extend the definition of $\overline{w}$ to the orbit of a. We must show this can be done consistently, that we always return to the same value. Let $a_1, a_2, \ldots, a_n$ be a sequence of simple roots such that $a_1 = a_n = a$ and for each j either $a_{j+1} = a_j^*$ or else there exists n_j such that the assumptions in (6b) holds with a_j, a_{j+1}, n_j in place of a, b, n. Let g denote the product of the corresponding sequence of f's and i_{n_j}'s. We must show that g fixes $\overline{w}_a$. We have $gX_a = X_a$, $gX_{-a} = X_{-a}$, and in fact g acts on $X_a/\mathcal{D}X_a$, identified with k, by multiplication by a scalar c as follows from the definition of f and the usual formulas for i_n. Since $\lambda g = \lambda$ by the corresponding condition on f and each i_n, it follows from Lemma 89 that $c = 1$, so that g is the identity on $X_a/\mathcal{D}X_a$. If $\mathcal{D}X_a = 0$, then g fixes the element x_a above, hence also $\overline{w}_a$ by (*), whether g is an automorphism or an antiautomorphism. If $\mathcal{D}X_a \neq 0$, then G is twisted so that $a^* = a$. If g is an automorphism, then by the proof of Theorem 36 from (5) on its restriction to $\langle X_a, X_{-a} \rangle$ is the identity so that it fixes $\overline{w}_a$, while if g is an antiautomorphism then by the same result its restriction coincides with that of f so that it fixes $\overline{w}_a$ by the choice of $\overline{w}_a$.

<u>Remark</u>: If G is untwisted, the above proof is quite simple.

We assume henceforth that the $\overline{w}_a$ are as in (6).

(7) If $\overline{w} = \overline{w}_a\overline{w}_b \ldots$ as in Lemma 83(b) and $w^* = w_0 w^{-1} w_0^{-1}$,

then $f\bar{w} = \bar{w}^*$.

__Proof__: Since $w_a w_b \cdots$ is minimal, $\cdots w_{b^*} w_{a^*}$ is also. (Check this.) Since f is an antiautomorphism it follows from (6a) that $f\bar{w} = \cdots \bar{w}_{b^*} \bar{w}_{a^*} = \overline{\cdots w_{b^*} w_{a^*}} = \bar{w}^*$.

(8) If w is as in (1) then $f\bar{w} = \bar{w}$.

__Proof__: $w^* = w$ in this case (see (7)).

(9) If n is as in (1) then $fn = n$.

__Proof__: By (1), $n \in \bar{w}H$ with $w = w_o w_\pi$. Assume $a \in \pi$. Then wa is simple and $\lambda(n x n^{-1}) = \lambda(x)$ for all $x \in X_a$, by the inequality in the proof of (1). Thus $n\bar{w}_a n^{-1} = \bar{w}_{wa}$ by (6b), from which we get, on picking a minimal expression for w_π , that (*) $n\bar{w}_\pi n^{-1} = \overline{w_{\pi^*}}$. Since $N(w) = N(w_o w_\pi) = N(w_o) - N(w_\pi)$, it follows that if we put together minimal expressions for w and w_π we will get one for w_o . Thus $\bar{w}_o = \bar{w}\bar{w}_\pi$ by Lemma 83(b), and similarly $\bar{w}_o = \bar{w}_{\pi^*}\bar{w}$, so that (**) $\bar{w}\bar{w}_\pi \bar{w}^{-1} = \bar{w}_{\pi^*}$. If now we write $n = \bar{w}h$, then h commutes with $\bar{w}_\pi$ by (*) and (**). Hence

$fn = fh \cdot f\bar{w}$ since f is an antiautomorphism

$= \bar{w}_o h \bar{w}_o^{-1} w$ by (3) and (8)

$= \bar{w}\bar{w}_\pi h \bar{w}_\pi^{-1}$ since $\bar{w}_o = \bar{w}\bar{w}_\pi$

$= \bar{w}h$ since h commutes with w_π

$= n$.

Thus f satisfies condition (c) and the proof of Theorem 49 is complete.

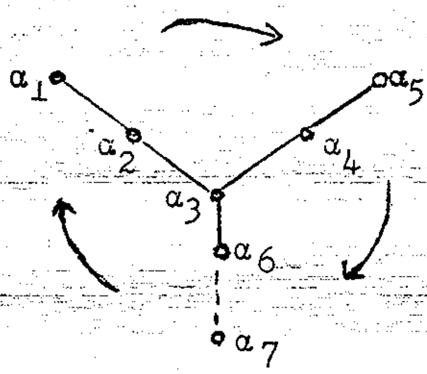
<u>Exercise</u>: (a) Prove that if $\{\overline{w}_a\}$ is as in (6) and w as in (1), then $U_\lambda \overline{w} U_\lambda \neq 0$.

(b) Deduce that if H_π denotes the kernel of the set of simple roots π then the dimension of A, hence the number of irreducible components of V_λ^G, in Theorem 49 is $\Sigma|H_\pi|$.

<u>Remark</u>: The natural group for the preceding theorem seems to be the adjoint group extended by the diagonal automorphisms, a group of the same order as the universal group, but with something extra at the top instead of at the bottom. For this group, G', prove that the dimension above is just $\prod(|H_a|+1) = \prod q(\alpha)$ in the notation of the exercise just before Lemma 83. Prove also that in this case V_λ^G is independent of λ.

<u>Remark</u>: The problem now is to decompose the algebra A of Theorem 49 into its simple (one-dimensional) components. If this were done, it would be a major step towards a representation theory for G. As far as we know this has been done only for the group A_1 (see Gelfand and Graev, Doklady, 1962). It would not, however, be the complete story. For not every irreducible G-module is contained in one induced by a character on U, i.e., by Frobenius reciprocity, contains a one-dimensional U-module, as the following, our final, example, due to M. Kneser, shows (although it is for some types of groups such as A_n).

As remarked earlier, reduction mod 3 yields an isomorphism of the subgroup W^+ of elements of determinant 1 of the Weyl group W of type E_6 onto the group $G = SO_5(3)$, the adjoint group of type $B_2(3)$. If we reverse this isomorphism and extend the scalars we obtain a representation of G on a complex space V. The assertion is that U, i.e. a 3-Sylow subgroup of G, fixes no line of V. Consider the following diagram.



This is the Dynkin diagram of E_6 with the lowest root α_7 adjoined (α_7 is the unique root in $-D$ (see Appendix III.33), unique because all roots are conjugate in the present case. It is connected as shown because of symmetry and the fact that each proper subdiagram must represent a finite reflection group.) We choose as a basis for V the α's with α_3 omitted, a union of three bases of mutually orthogonal planes. w_1w_2 acts as a rotation of 120° in the plane $\langle\alpha_1,\alpha_2\rangle$ and as the identity in the other two planes, and similarly for w_4w_5 and w_6w_7. The group W^+ also contains an element permuting the three planes cyclically as shown, because of the conjugacy of simple systems and the uniqueness of lowest roots, and the four elements generate a 3-subgroup of W^+. It is now a simple matter to prove that this subgroup fixes no line of V.

APPENDIX ON FINITE REFLECTION GROUPS

The results (and some of the terminology in what follows) are motivated by the theory of semisimple Lie algebras, but no kmowledge of this theory is assumed. The main results are starred.

I Preliminaries

V will be a finite-dimensional real or rational Euclidean space. By a reflection (on V) is meant a reflection in some hyperplane H. If α is a nonzero vector orthogonal to H, the reflection, denoted σ_α, is given by

(1) $\sigma_\alpha \rho = \rho - 2(\rho,\alpha)/(\alpha,\alpha).\alpha \qquad (\rho \in V).$

We observe that σ_α is an automorphism of V, of order 2.

A useful fact is:

(2) If w is an automorphism of V, then $w\sigma_\alpha w^{-1} = \sigma_{w\alpha}$.

To prove this, apply both sides to $\rho \in V$, then use (1) and the invariance of $(\ ,\ )$ under w.

Σ will denote a finite set of nonzero elements of V such that:

(3) $\alpha \in \Sigma \Longrightarrow -\alpha \in \Sigma$ and $k\alpha \notin \Sigma$ if $k \neq \pm 1$. credo che questo chiarisca che il sistema di radici è ridotto, secondo le convenzioni di Serre

(4) $\alpha \in \Sigma \Longrightarrow \sigma_\alpha \Sigma = \Sigma$.

The elements of Σ will be called roots, and W will denote the group generated by all σ_α $(\alpha \in \Sigma)$.

(5) Lemma. The restriction of W to Σ is faithful.

For, each $w \in W$ fixes pointwise the orthogonal complement of Σ.

(6) Cor. W is finite.

(7) Examples. (a) If Σ is the root system of a semisimple Lie algebra over the complex field, then W is the corresponding Weyl group. (b) If W is any finite group generated by reflections, e.g. the group of symmetries of a regular solid, Σ may be taken as the set of unit normals to the hyperplanes in which reflections of W take place.

(8) Definitions. A subset of roots is called a positive system if it consists of the roots which are positive relative to some ordering of V. (Recall that this involves the specification of a subset V^+ of V which is closed under addition and under multiplication by positive scalars and satisfies trichotomy.) A subset of roots, say Π, is a simple system if (a) Π is a linearly independent set, and (b) every root is a linear combination of the elements of Π in which all nonzero coefficients are either all positive or all negative.

* (9) Proposition. (a) Each simple system is contained in a unique positive system. (b) Each positive system contains a unique simple system.

If Π is a simple system, then clearly the "all positive" roots in (8b) form the unique positive system containing Π, whence (a). Now let P be any positive system. Let Π be a subset of P which generates P under positive linear combinations and is minimal relative to this property. Then

(*) $\alpha, \beta \in \Pi, \alpha \neq \beta \Longrightarrow (\alpha,\beta) \leq 0$. Assume not, so that $\sigma_\alpha\beta = \beta - c\alpha$ with $c > 0$. Assume $\sigma_\alpha\beta \in P$, so that $\sigma_\alpha\beta = \Sigma c_\gamma\gamma \quad (\gamma \in \Pi , c_\gamma \geq 0)$. If $c_\beta < 1$, the last equation, written suitably, expresses β as a positive combination of the other elements of Π , a contradiction to the minimality of Π , while if $c_\beta \geq 1$, it expresses 0 as a positive combination of elements of Π , hence of P , equally a contradiction. Similarly $-\sigma_\alpha\beta \in P$ leads to a contradiction, whence (*) . Now a linear relation on Π may be written $\Sigma a_\alpha\alpha = \Sigma b_\beta\beta$ with the two sums over disjoint parts of Π and $a_\alpha, b_\beta \geq 0$. Writing this as $\rho = \sigma$ and using (*), we get $(\rho,\rho) = (\rho,\sigma) \leq 0$, whence $\rho = 0$ and then every $a_\alpha = 0$ because the α's are all positive. Similarly every $b_\beta = 0$. Thus Π is independent, is a simple system. From the definition of a simple system any simple system contained in P consists of those elements of P which are not positive combinations of others, hence is uniquely determined by P .

(10) Lemma. Let Π be a simple system and P the positive system containing Π . (a) $\alpha, \beta \in \Pi , \alpha \neq \beta \Longrightarrow (\alpha,\beta) \leq 0$. (b) $\rho \in P \Longrightarrow$ there exists $\alpha \in \Pi$ so that $(\rho,\alpha) > 0$.

For (b) write $\rho = \Sigma c_\alpha\alpha (\alpha \in \Pi, c_\alpha \geq 0)$ as in (8b), and then use $0 < (\rho,\rho) = \Sigma c_\alpha(\rho,\alpha)$.

*(11) Main lemma. Let Π , P be as in (10) and $\alpha \in \Pi$. Then $\sigma_\alpha\alpha = -\alpha$ and $\sigma_\alpha(P \setminus \alpha) = P \setminus \alpha$.

Pick $\rho \in P \setminus \alpha$, $\rho = \Sigma c_\beta\beta \ (c_\beta \geq 0 ; \beta \in \Pi)$. By (3) some

$c_\beta > 0$ $(\beta \neq \alpha)$. Application of σ_α does not change this c_β. Hence $\sigma_\alpha \rho \in P \setminus \alpha$.

(12) Theorem. Any two simple (or positive) systems are conjugate under W.

By (9) we need only consider two positive systems, say P and P'. We use induction on $n = |P \cap (-P')|$. If $n = 0$, $P = P'$. Assume $n > 0$. Then there is a root α simple relative to P such that $\alpha \in P \cap (-P')$. By (11), $|\sigma_\alpha P \cap (-P')| = n-1$, whence $|P \cap -\sigma_\alpha P'| = n-1$. By the inductive assumption $\sigma_\alpha P'$ is conjugate to P ; hence so is P'.

Henceforth Π, P will be as in (10) and fixed.

(13) Definition. If $\rho \in \Sigma$, $\rho = \sum_{\alpha \in \Pi} c_\alpha \alpha$ as in (8b), then Σc_α is called the height of ρ and written ht ρ.

e.g. $\alpha \in \Pi \Rightarrow$ ht $\alpha = 1$.

(14) Lemma. Let W_o be the group generated by $\{\sigma_\alpha | \alpha \in \Pi\}$. If $\rho \in P$, the minimum value of ht on the set $W_o\rho \cap P$ is 1 and is taken on only on $W_o \rho \cap \Pi$.

Let ρ' be a minimum point and assume, if possible, that $\rho' \notin \Pi$. By (10b) there is $\alpha \in \Pi$ so that $(\rho', \alpha) > 0$, whence by (1) ht $\sigma_\alpha \rho' <$ ht ρ' and by (11) $\sigma_\alpha \rho' > 0$, a contradiction to the choice of ρ'.

(15) Corollary. (a) If $\rho \in P$, $\rho \notin \Pi$, then ht $\rho > 1$.

(b) $W_o \Pi = \Sigma$. i.e. Every root ρ is conjugate under W_o to a simple root.

By (14), (a) is clear and so is (b) if $\rho > 0$. If $\rho < 0$, then $-\rho > 0$, whence $-\rho = w\alpha$ ($w \in W_o$, $\alpha \in \Pi$), so that $\rho = (w\sigma_\alpha)\alpha$.

(16) Theorem. W is generated by $\{\sigma_\alpha | \alpha \in \Pi\}$. i.e. $W = W_o$ in (14).

If ρ is a root we have $\rho = w\alpha$ ($w \in W_o$, $\alpha \in \Pi$), by (15b), whence $\sigma_\rho = w\sigma_\alpha w^{-1}$ (see (2)), an element of W_o. Hence $W \subseteq W_o$ and $W = W_o$.

II The function N

(17) Definition. For $w \in W$, $N(w)$ will denote the number of roots ρ such that $\rho > 0$ and $w\rho < 0$. In other words, $N(w) = |P \cap w^{-1}(-P)|$.

e.g. $N(1) = 0$, $N(\sigma_\alpha) = 1$ if $\alpha \in \Pi$, by (11).

(18) Lemma. $w \in W \Rightarrow N(w^{-1}) = N(w)$.

Prove this.

(19) Lemma. Assume $w \in W$, $\alpha \in \Pi$.

(a) If $w^{-1}\alpha > 0$, then $N(\sigma_\alpha w) = N(w) + 1$.

(a') If $w^{-1}\alpha < 0$, then $N(\sigma_\alpha w) = N(w) - 1$.

(b) If $w\alpha > 0$, then $N(w\sigma_\alpha) = N(w) + 1$.

(b') If $w\alpha < 0$, then $N(w\sigma_\alpha) = N(w) - 1$.

Let $S(w) = P \cap w^{-1}(-P)$. Then $S(\sigma_\alpha w) = w^{-1}\alpha \cup S(w)$, whence (a). To get (a') replace w by $\sigma_\alpha w$ in (a), and to get (b) and (b') replace w by w^{-1} and use (18).

(20) Problem. (a) $N(ww') \leq N(w) + N(w')$ and $N(ww') \equiv N(w) + N(w') \bmod 2$. (b) $\det w = (-1)^{N(w)}$. $(w, w' \in W)$.

(21) Lemma Assume $w = w_1 w_2 \cdots w_n$ $(w_i = \sigma_{\alpha_i}, \alpha_i \in \Pi)$. If $N(w) < n$, then for some i, j $(1 \leq i \leq j \leq n-1)$, we have:

(a) $\alpha_i = w_{i+1} w_{i+2} \cdots w_j \alpha_{j+1}$.

(b) $w_{i+1} w_{i+2} \cdots w_{j+1} = w_i w_{i+1} \cdots w_j$.

(c) $w = w_1 w_2 \cdots ' \cdots ' \cdots w_n$, with w_i and w_{j+1} missing.

By (19b) and $N(w) < n$, $w_1 w_2 \cdots w_j \alpha_{j+1} < 0$ for some $j \leq n-1$. Since $\alpha_{j+1} > 0$, we have $w_i (w_{i+1} \cdots w_j \alpha_{j+1}) < 0$ and $w_{i+1} \cdots w_j \alpha_{j+1} > 0$ for some $i \leq j$, whence $w_{i+1} \cdots w_j \alpha_{j+1} = \alpha_i$ by (11), which is (a). Using (2) with $w = w_{i+1} \cdots w_j$ and $\alpha = \alpha_{j+1}$, we get (b), and then replacing the left side of (b) by the right side in the product for w and using $w_i^2 = 1$, we get (c).

Problem. Prove, conversely, that (a), (b) or (c) implies $N(w) < n$.

(22) Cor. If $w \in W$, then $N(w)$ is the number of terms in a minimal expression of w as a product of reflections corresponding to simple roots.

Let $w = w_1 w_2 \cdots w_n$ be a minimal expression. By (19) ((a) or (b)), $N(w) \leq n$, and by (2|c), $N(w) \geq n$.

* (23) Theorem. For $w \in W$, if $wP = P$ or $w\Pi = \Pi$ or $N(w) = 0$, then $w = 1$.

The three assumptions are clearly equivalent. Now $N(w) = 0$

implies that the minimal expression of w in (22) is empty, whence $w = 1$.

* (24) Theorem. W is simply transitive on the positive systems, and also on the simple systems.

By (12) and (23).

(25) Problem. (a) For $w \in W$, choose a minimal expression as in (22), $w = w_1 w_2 \cdots w_n$ ($w_i = \sigma_{\alpha_i}$, $\alpha_i \in \Pi$) (so that $n = N(w)$), and set $\rho_i = w_1 w_2 \cdots w_{i-1}\alpha_i$. Prove that $\rho_i (1 \leq i \leq n)$ is a complete list of all roots ρ such that $\rho > 0$ and $w^{-1}\rho < 0$.

(b) Since $-P$ is a positive system, there exists by (24) a unique $w_o \in W$ such that $w_o P = -P$. Write $w_o = w_1 w_2 \cdots w_n$ ($n = N(w_o) = |P|$) as above. Prove that $\rho_i (1 \leq i \leq n)$ is a complete list of all positive roots. (Hint: (19), (21)).

III A fundamental domain for W.

(26) Definition. D will denote the region $\{\rho \in V \mid (\rho, \alpha) \geq 0, \alpha \in \Pi\}$, Thus D is a closed convex cone, and if Π spans V it is even a simplicial cone with vertex at 0.

(27) Lemma. Every $\rho \in V$ is conjugate to some $\rho' \in D$, in fact to some ρ' in D such that $\rho' - \rho$ is a nonnegative combination of the elements of Π.

Let S be this set of nonnegative combinations (in other words, the dual cone of D). We introduce a partial order in V by the definition $\gamma \succ \gamma'$ if and only if $\gamma - \gamma' \in S$. Among the conjugates ρ' of ρ under W such that $\rho' \succ \rho$, we pick one which is maximal relative to this partial order. Then $\alpha \in \Pi \Rightarrow \sigma_\alpha \rho' \not> \rho' \Rightarrow (\rho', \alpha) \geq 0$ (by (1)), whence $\rho' \in D$ and (27) follows.

(28) <u>Theorem</u>. Assume $w \in W$, $\rho \in V$, ρ not orthogonal to any root, and $w\rho = \rho$. Then $w = 1$. (Restatement: $w \neq 1, w\rho = \rho \Rightarrow \rho$ is orthogonal to some root.)

By (27) we may assume $\rho \in D$. For $\alpha \in P$, $(w\alpha, \rho) = (\alpha, w^{-1}\rho) = (\alpha, \rho) > 0$. Hence $w\alpha \in P$, for all $\alpha \in P$, so that $wP = P$. Then $w = 1$ by (23).

(29) <u>Cor</u>. If $\rho \in V$ is not orthogonal to any root, its conjugates under W are all distinct. (And conversely, of course.)

(30) <u>Cor</u>. The only reflections in W are those in hyperplanes orthogonal to roots, i.e. those of the form $\sigma_\rho (\rho \in \Sigma)$.

Let u be any reflection in a hyperplane H not orthogonal to any root. The roots being finite in number, there exists $\rho \in H$, ρ not orthogonal to any root. Then $u \neq 1$, $u\rho = \rho \Rightarrow u \notin W$, by (28).

(31) <u>Problem</u>. Let S be a set of roots such that $\{\sigma_\alpha \mid \alpha \in S\}$ generates W. Prove that every root is conjugate, under W, to some $\alpha \in S$, and every reflection in W to some $\sigma_\alpha (\alpha \in S)$.

(32) <u>Lemma</u>. Assume $\rho, \sigma \in D$, $w \in W$, $w\rho = \sigma$. Then (a) w is a product of simple reflections (i.e relative to simple roots) fixing ρ. (b) $\rho = \sigma$.

For (a) we use induction on $N(w)$. If $N(w) = 0$, then

$w = 1$ by (23). Assume $N(w) > 0$. Pick $\alpha \in \Pi$ so that $w\alpha < 0$. Then $0 \geq (\sigma, w\alpha) = (\rho, \alpha) \geq 0$, whence $(\rho, \alpha) = 0$ and $\sigma_\alpha \rho = \rho$. Since $(w\sigma_\alpha)\rho = \sigma$, and $N(w\sigma_\alpha) = N(w) - 1$ by (19b'), the inductive assumption applied to $w\sigma_\alpha$ yields (a). Clearly (a) implies (b).

* (33) Theorem. D is a fundamental domain for W on V. In other words, each element of V is conjugate to exactly one element of D.

By (27) and (32b).

(34) Problem. If $\rho \in D$ and $w \in W$, show that $\rho - w\rho$ is a nonnegative combination of positive roots.

(35) Restatement. The reflecting hyperplanes (those orthogonal to roots) partition V into closed chambers, each of which is a fundamental domain for W. For a given chamber, the roots normal to the walls and inwardly directed form a simple system, and each simple system is obtained in this way. Prove these assertions and also that the angle between two walls of a chamber is always a submultiple of π.

* (36) Theorem. If S is any subset of V, the subgroup of W which fixes S pointwise is a reflection group. In other words, every $w \in W$ which fixes S pointwise is a product of reflections which also do.

Remark. (36) is an extension of (28). Verify this.

For the proof of (36) we may assume that S is indepedent, hence finite, and using induction, reduce to the case where S has a single element, say ρ, which may be taken in D by (27). Then (32a) with $\sigma = \rho$ yields our result.

(37) Problem. For each subset Π' of Π let $W(\Pi')$ denote the group generated by $\{\sigma_\alpha \mid \alpha \in \Pi'\}$. Prove that $W(\Pi' \cap \Pi'') = W(\Pi') \cap W(\Pi'')$.

IV Generators and relations for W.

* (38) Theorem. For $\alpha, \beta \in \Pi$, let $n(\alpha,\beta)$ denote the order of $\sigma_\alpha\sigma_\beta$ in W. (So $n(\alpha,\alpha) = 1$, while $n(\alpha,\beta) > 1$ if $\alpha \neq \beta$.) Then the group W is defined by the generators $\{\sigma_\alpha \mid \alpha \in \Pi\}$ subject to the relations $\{(\sigma_\alpha\sigma_\beta)^{n(\alpha,\beta)} = 1 \mid \alpha,\beta \in \Pi\}$. In other words, the given elements generate W and the given relations imply all others in W.

By (16) the given elements generate W. Suppose the relation (*) $w_1 w_2 \cdots w_r = 1$ ($w_i = \sigma_{\alpha_i}$, $\alpha_i \in \Pi$) holds in W. We will show it is a formal consequence of the given relations, by induction on r. By (20b) or by (19) r is even, say $r = 2s$. If $s = 0$ there is nothing to show. Suppose $s > 0$. We start with the observation:

(0) (*) is equivalent to $w_{i+1} w_{i+2} \cdots w_r w_1 w_2 \cdots w_i = 1$ $(1 \leq i \leq r)$.

Case 1. Suppose $\alpha_1 \neq w_2 w_3 \cdots w_s \alpha_{s+1}$. We have $N(w_1 w_2 \cdots w_{s+1}) = N(w_{2s} w_{2s-1} \cdots w_{s+2}) < s+1$, by (19a). Hence by (21) we have (21a) and (21b) for some i,j such that $1 \leq i \leq j \leq s$. Since $i,j = 1,s$ is excluded in the present case, both sides of (21b) have length $< s$. By our inductive assumption we may replace the left side of (21b) by the right side in (*). If w_i^2 is then replaced by 1, the inductive assumption can be applied to the resulting relation to complete the proof, in the present case.

Case 2. Suppose $\alpha_1 \neq \alpha_3$. (If $s = 1$, this case doesn't occur.) By the first case we may assume $\alpha_1 = w_2 w_3 \cdots w_s \alpha_{s+1}$ and then by (0) also $\alpha_2 = w_3 w_4 \cdots w_{s+1} \alpha_{s+2}$, whence

(**) $w_2 w_3 \cdots w_{s+1} = w_3 w_4 \cdots w_{s+2}$ by (2).

If (**) is substituted into (*), we can shorten (*), as above. Thus we are reduced to showing that (**) is a consequence of the original relations in (38), i.e. that

$w_3 \underbrace{w_2 w_3 \cdots w_{s+1}} w_{s+2} w_{s+1} \cdots w_4 = 1$ is. Since

$\alpha_3 \neq \alpha_1 = w_2 w_3 \cdots w_s \alpha_{s+1}$, we are back in Case 1.

Because of (0) above the only case that remains is:

Case 3. Suppose $\alpha_1 = \alpha_3 = \alpha_5 = \ldots$ and $\alpha_2 = \alpha_4 = \alpha_6 = \ldots$. Then (*) has the form $(\sigma_\alpha \sigma_\beta)^s = 1$ with $\alpha = \alpha_1$, $\beta = \alpha_2$. Here s must be a multiple of $n(\alpha,\beta)$, the order of $\sigma_\alpha \sigma_\beta$, so that (*) is a consequence of the relation $(\sigma_\alpha \sigma_\beta)^{n(\alpha,\beta)} = 1$.

(39) Examples. (a) $W = S_n$. The symmetric group of degree n acts on an n-dimensional space by permuting the coordinates relative to an orthonormal basis ε_i $(1 \leq i \leq n)$. The transposition (ij) corresponds to the reflection in the hyperplane orthogonal to $\varepsilon_i - \varepsilon_j$. We may take $\Sigma = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$, and relative to a lexicographic ordering, $\Pi = \{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n-1\}$.

Thus S_n is generated by the transpositions $w_i = (i\ i+1)$ $(1 \leq i \leq n-1)$ subject to the relations $w_i^2 = 1$, $(w_i w_{i+1})^3 = 1$, and $(w_i w_j)^2 = 1$ if $|i-j| > 1$. (b) $\underline{W = Oct_n}$. The octahedral group includes sign changes as well as permutations of the coordinates, so has order $2^n n!$. Here we may take $\Sigma = \{\pm\varepsilon_i \pm \varepsilon_j, \pm\varepsilon_i \mid i \neq j\}$ and $\Pi = \{\varepsilon_i - \varepsilon_{i+1}, \varepsilon_n \mid 1 \leq i \leq n-1\}$. So, comparing with (a), we have one more generator w_n and n more relations $w_n^2 = 1$, $(w_{n-1}w_n)^4 = 1$, $(w_i w_n)^2 = 1$ if $i \leq n-2$. We observe that S_n and Oct_n are the groups of symmetries of the regular simplex and the regular cube.

* (40) Problem. Prove that W is defined by the generators $\{\sigma_\gamma \mid \alpha \in \Sigma\}$ subject to the relations

(A) $\{\sigma_\alpha^2 = 1 \mid \alpha \in \Sigma\}$, (B) $\{\sigma_\alpha \sigma_\beta \sigma_\alpha^{-1} = \sigma_\alpha \mid \alpha, \beta \in \Sigma, \gamma = \sigma_\alpha \beta\}$.

(Hint. Using (15b) and (16) show that the group so defined is generated by $\{\sigma_\alpha \mid \alpha \in \Pi\}$ as a consequence of (B), and then using (21) show that any nontrivial relation $w_1 w_2 \cdots w_n = 1$ $(w_i = \sigma_{\alpha_i}, \alpha_i \in \Pi)$ which holds in W can be shortened as a consequence of (A) and (B).)

(V) Appendix.

We consider some refinements in our results which occur in the crystalographic case, when $2(\alpha,\beta)/(\beta,\beta)$ is an integer for all $\alpha, \beta \in \Sigma$, which we henceforth assume. (This case occurs when we have root systems of Lie algebras.)

(41) Refinement of (8). In the present case, all coefficients

in (8b) are integers.

Prove this, by induction on ht ρ $(\rho \in \Sigma)$ (see (13)).

(42) Cor. ht ρ is always an integer.

(43) Problem. Under the assumptions of (34), assume also that $(2\rho, \alpha)/(\alpha,\alpha)$ is an integer for every $\alpha \in \Pi$. Show that $\rho - w\rho$ is a nonnegative integral combination of the elements of Π.

(44) Problem. If α and β are roots, $\alpha \neq -\beta$ and $(\alpha,\beta) < 0$, prove that $\alpha + \beta$ is a root. (Hint: prove that $\alpha + \beta$ equals $\sigma_\alpha \beta$ or $\sigma_\beta \alpha$.)