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ON THE POSET OF NON-TRIVIAL PROPER SUBGROUPS OF A FINITE GROUP

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In this paper we describe the connected components of $\mathcal{L}(G)$, the partially ordered set of non-trivial proper subgroups of a finite group G. This result is related to the study of the simple connectivity of the coset poset of a finite group.

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1. Introduction

There is a covariant functor from the category of finite posets to the category of finite simplicial complexes. In finite group theory, this procedure has been applied to the study of posets of subgroups ordered by inclusion (see [4, 5, 1, 2]).

The coset poset $\mathcal{C}(G)$ of a finite group G is the poset of all left cosets of all proper subgroups of G, ordered by inclusion. It was first introduced by K. S. Brown [2] in connection with the probabilistic zeta function $P(G, s)$. The question of the simply connectivity of $\mathcal{C}(G)$ is still open, but there are some results in [6]. In particular in [6] some relations between the coset poset and the poset $\mathcal{L}(G)$ of all non-trivial proper subgroups of G have been proved.

In this paper we answer question 6.9 of [6], namely when a simple group has a connected subgroup poset. We characterize the finite groups in which $\mathcal{L}(G)$ is not connected, describe its connected components $\mathcal{L}_1, \ldots, \mathcal{L}_r$ and the action of G by conjugation on these components. In particular we prove that every simple group has a connected subgroup poset.

We denote by p, q two prime numbers. We denote by C_p the cyclic group of order p.

Proposition 1.1. *If G is a finite group, then $\mathcal{L}(G)$ is not connected if and only if G is isomorphic to one of the following*:

(i) $G \cong C_p \times C_q$, $\mathcal{L}_1 = \{C_p\}$ *and* $\mathcal{L}_2 = \{C_q\}$, $p \neq q$;
(ii) $G \cong C_p \times C_p$ *and* $\mathcal{L}_i = \{W_i\}$, $i = 1, \ldots, p+1$ *where* W_i *are the subgroups of order* p *of* G; *or*
(iii) G *has a unique minimal normal subgroup* N, G *is a Frobenius group, with Frobenius kernel* N *and a cyclic Frobenius complement* H *of order* p. *The connected components of* $\mathcal{L}(G)$ *are* $\mathcal{L}_1 = \mathcal{L}(N) \cup \{N\}$ *and* $\mathcal{L}_i = \{H^g\}$ *for* $g \in N$, $i = 2, \ldots, |N| + 1$.

Moreover G *acts by conjugation on the connected components of* $\mathcal{L}(G)$. *The action is trivial in cases* (i) *and* (ii), *while in case* (iii), G *fixes* $\mathcal{L}_1$ *and transitively permutes the other components.*

2. The Proof

Let $\mathcal{L}(G)$ be the set of all proper non-trivial subgroups of a finite group G, ordered by inclusion. We define the following relation in $\mathcal{L}(G)$: $R \sim S$ if, for some $n \in \mathbb{N}$, there exist $R_0, R_1, \ldots, R_n$ such that $R_0 = R$, $R_n = S$ and either $R_i \leq R_{i+1}$ or $R_i \geq R_{i+1}$. This is an equivalence relation. A *connected component* of $\mathcal{L}(G)$ is an equivalence class for the relation $\sim$. We denote by $\mathcal{L}_1, \ldots, \mathcal{L}_r$ the connected components of $\mathcal{L}(G)$. We observe that G acts by conjugation on $\mathcal{L}(G)$.

We consider separately the cases in which G is simple or not. We first consider the case in which G is not a simple group, showing that G has exactly the structure described in Proposition 1.1. Then we prove that if G is a simple group, then $\mathcal{L}(G)$ is connected.

Lemma 2.1. *If G is a finite non-simple group, then one of the three conclusions of Proposition* 1.1 *holds.*

Proof. If G is one of the groups described in (i), (ii), (iii), then it is easily seen that $\mathcal{L}(G)$ is not connected.

We suppose that $\mathcal{L}(G)$ is not connected. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two distinct connected components of $\mathcal{L}(G)$. Let N be a proper normal subgroup of G. We can suppose that N is in $\mathcal{L}_1$. Then every $H \in \mathcal{L}_2$ is a complement for N in G. It follows that H is cyclic of prime order, for any $H \in \mathcal{L}_2$. Since $\mathcal{L}_2$ is connected, this implies that $\mathcal{L}_2 = \{H\}$. We observe that H acts on N and every proper H-invariant subgroup of N is trivial. Since H fixes a q-Sylow subgroup of N, N is an elementary abelian q-group, with q a prime. Moreover we have either $C_N(H) = \{1\}$ or $C_N(H) = N$. If $C_N(H) = N$, then G is abelian and N is also a cyclic group of order q, q a prime not necessarily distinct from p. Therefore $G \cong C_p \times C_q$ and we obtain case (i) or (ii), according to whether $p = q$ or $p \neq q$. Finally suppose that $C_N(H) = \{1\}$. Then G is a Frobenius group, and we get (iii). □

We can now suppose that G is a simple group, but first we begin with a general lemma concerning the 2-subgroups of G.

Lemma 2.2. *Let G be a finite group of even order. If there exist two 2-subgroups which are not connected in $\mathcal{L}(G)$, then G is isomorphic to D_{2q}, the dihedral group of order $2q$, with q a prime. In particular, G is not a simple group.*

Proof. Let P, Q be two different 2-subgroups and $x \in P$, $y \in Q$ be involutions. We consider the subgroup $D = \langle x, y\rangle$. H is a dihedral group. If $D < G$, then

$$P \geq \langle x\rangle \leq D \geq \langle y\rangle \leq Q\,,$$

which proves that P and Q lie in the same connected component of $\mathcal{L}(G)$. Hence $G = D$ is a dihedral group. Therefore by Lemma 2.1 either G is an elementary abelian group of order 4, that is $G \cong D_4$, or G is a Frobenius group with kernel N and complement $H = \langle x\rangle$ of order 2. Since G is dihedral and N is minimal normal in G, N is cyclic of order q for some odd prime q, completing the proof. □

We now restrict to the simple group case and refer to [3] for the notation on simple groups of Lie type.

We define the graph $\Gamma^*(G)$ as follows: the set of vertices of $\Gamma^*(G)$ is the set of primes dividing $|G|$ and vertices p and q are joined when there exists a proper subgroup of G of order divisible by pq.

Proposition 2.1. *Let G be a finite simple group. Then $\mathcal{L}(G)$ is connected.*

Proof. We may assume that G is a non-abelian simple group. It is enough to prove that for any $x, y \in G$, x, y of prime order we have that $\langle x\rangle$ and $\langle y\rangle$ lie in the same connected component of $\mathcal{L}(G)$. By Lemma 2.2, we can also reduce to the case in which $|x| = 2$ and $|y| = p$, p an odd prime.

If $\Gamma^*(G)$ is connected, then $\mathcal{L}(G)$ is connected. In fact if $(2, p)$ is an edge in $\Gamma^*(G)$, then there exist a subgroup H of order divisible by $2p$, a Sylow 2-subgroup D, and a Sylow p-subgroup P such that $H \cap P$ and $H \cap D$ are non-trivial subgroups of H. By Sylow's Theorem, there exist $g, h \in G$ with $y \in P^g$ and $x \in D^h$. Then by Lemma 2.2, we have

$$\langle y\rangle \leq P^g \geq (H \cap P)^g \leq H^g \geq (H \cap D)^g \leq D^g \sim D^h \geq \langle x\rangle\,.$$

If p is connected to 2 in $\Gamma^*(G)$, we can prove similarly that $\langle x\rangle \sim \langle y\rangle$, using induction on the shortest chain connecting p to 2.

By the preceding remark, we may assume that $\Gamma^*(G)$ is not connected and let p be the smallest prime not connected to 2. Then p is not connected to r for any prime $r < p$. Let P be a Sylow p-subgroup of G. If P is cyclic, by Burnside's Transfer Theorem, $N_G(P) > C_G(P)$ and therefore there must exist a prime r dividing $|N_G(P)/C_G(P)|$ with $r < p$, a contradiction. Hence P is not cyclic.

If G is a sporadic group, then [3, Tables 5.3] shows that whenever p^2 divides $|G|$, p divides $|C_G(z)|$ for some involution z of G. Hence p is connected to 2, a contradiction.

If $G = A_n$, then G has a proper subgroup isomorphic to A_4. Hence p is not connected to 3 and so $p \geq n-2$. But then P is cyclic, a contradiction.

Finally suppose that G is a simple group of Lie type in characteristic r of level r^m (see [3, definition 2.1.9]) If $G = PSL(2, r^m)$, then $\Gamma^*(G)$ is connected by Dickson's Theorem ([3, 6.5.1]). If G is not a Suzuki group, then G has a subgroup isomorphic to $PSL(2, r^m)$ or $SL(2, r^m)$. Therefore any prime divisor of $r(r^{2m}-1)$ is connected to 2. In particular, p is not connected to r and p does not divide $r^m - 1$ or $r^m + 1$. Hence p does not divide the canonical part of the Schur multiplier of G ([3, Table 6.1.2]).

We now argue that G contains an element of order pr, which will contradict the fact that p is not connected to r. For this argument, by the above, we may replace G by $\hat{G}$, where $Z(\hat{G})$ is the canonical part of the Schur multiplier of G, and then regard $\hat{G}$ as a subgroup of $\bar{G}$, the simply connected algebraic group associated with $\hat{G}$. Since P is not cyclic, $\bar{G}$ contains an elementary abelian subgroup E of order p^2. By Steinberg [3, 4.1.16], E lies in a maximal torus of $\hat{G}$. Then by [3, 4.2.5], there exists a non-identity element x of E such that r divides $|C_G(x)|$, completing the proof. $\square$

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